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Submit the solutions in groups of two at the lecture on Tuesday, 2018-04-24

Exercise 1. The *weak L^p norm* of a function f is the quantity

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \lambda |\{|f| > \lambda\}|^{1/p}, \qquad 0 < p < \infty.$$

(a) Show that $\|f\|_{p,\infty} \le \|f\|_p$ for all $0 < p < \infty$.

(b) Show that the functional $\|f\|_{p,\infty}$ satisfies the quasi-triangle inequality.

(c) Let $0 < r < p < \infty$. Show that the expression

$$\|f\|_{p,\infty,r} := \sup_{0<|E|<\infty} |E|^{-\frac{1}{r}+\frac{1}{p}} (\int_E |f|^r)^{1/r}$$

is equivalent to the weak L^p norm in the sense that there exist constants $0 < c_{p,r} \le C_{p,r} < \infty$ such that $c_{p,r}\|f\|_{p,\infty} \le \|f\|_{p,\infty,r} \le C_{p,r}\|f\|_{p,\infty}$ holds for all f. Hint: in order to show the second inequality use the layer cake formula

$$\int_E |f|^r = r \int_{\lambda=0}^{\infty} \lambda^{r-1} |E \cap \{|f| > \lambda\}|.$$

(d) Suppose that $1 = r < p < \infty$. Show that $\|f\|_{p,\infty,r}$ is a norm (that is, it satisfies the genuine triangle inequality).

Exercise 2. Recall

$$Mf(x) = \sup_{B \ni x, B \text{ ball}} \frac{1}{|B|} \int_B |f(y)| \mathrm{d}y.$$

For a measurable set $E \subset \mathbb{R}^n$, denote by 1_E the function such that $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$.

(a) Let E be a set of finite measure. Prove

$$\int_E M(1_E f)(x) \mathrm{d}x \le C \left(|E| + \int_E |f(x)| \log_+ |f(x)| \mathrm{d}x \right)$$

for a constant C independent of f and E. (Hint: use the layer cake formula. Here $\log_+ |f| = 1_{\{\log |f| > 0\}} \log |f|$.)

(b) Let

$$f(x) = \frac{1_{\{|x|<10^{-1}\}}}{|x|(\log |x|)^2}.$$

Show that $f \in L^1(\mathbb{R})$ and

$$Mf(x) > -c \frac{1}{|x| \log |x|}$$

for $|x| < 10^{-1}$. Conclude that $Mf \notin L^1_{loc}(\mathbb{R})$.