McKay quivers and the deformation and resolution theory of Kleinian singularities

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)
der Mathematische-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn.

Vorgelegt von

Nguyen Son Do aus Vietnam

Bonn
2005

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

1. Referent: Prof. Dr. C.-F. Bödigheimer, Universität Bonn
2. Referent: P.D. Dr. L. Hille, Universität Hamburg

Tag der Promotion: 25. 7. 2005

TABLE OF CONTENTS

0. Introduction

The theory of **Kleinian singularities** is in relationship with the conjugacy classes of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and with **Dynkin diagrams** of type ADE. This relationship can be described by the following scheme :

$$\begin{tikzcd}
\begin{array}{c}\text{Finite subgroups}\\ \Gamma \subset SL(2,\mathbb{C})\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Dynkin diagrams}\\ \text{of type ADE}\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{c}\text{Kleinian singularities}\\ \mathbb{C}^2/\Gamma\end{array} \arrow[r, leftrightarrow] & \begin{array}{c}\text{minimal resolutions of}\\ \mathbb{C}^2/\Gamma.\end{array}
\end{tikzcd}$$

Klein [24] determined the structure of the quotient space $\mathbb{C}^2/\Gamma$, where Γ is any finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. For each such subgroup Γ the algebra of invariant polynomials has three generators which are related by single equation. Thus $\mathbb{C}^2/\Gamma$ can be realized as a subspace of $\mathbb{C}^3$ defined by a single equation. Here the origin is the unique singular point. We list these equations below:

$$\begin{array}{lll}
\mathbb{A}_n & : x^2+y^2+z^{n+1} & n \geq 1 \\
\mathbb{D}_n & : x^2+y^{n-1}+yz^2 & n \geq 4 \\
\mathbb{E}_6 & : x^2+y^3+z^4 & \\
\mathbb{E}_7 & : x^2+y^3+yz^3 & \\
\mathbb{E}_8 & : x^2+y^3+z^5. &
\end{array}$$

They correspond to Γ being a cyclic group, a dihedral group and the groups of the tetrahedron, the octahedron, and the icosahedron respectively. These singularities are usually called the Kleinian singularities.

Let $\pi : \widetilde{X} \longrightarrow X$ be the **minimal resolution** of Kleinian singularity X. The exceptional fiber E, the fiber of π over the singular point of X, is known to be union of projective lines meeting transversally, and the graph whose vertices correspond to the irreducible components of E, with two vertices joined if and only if the components intersect, is a Dynkin diagram of type ADE [21].

In 1979, McKay [26] constructed directly via representation theory the bijection in the first row of the diagram. Let $\{R_0,\ R_1,\ldots,R_n\}$ be the set of the isomorphism classes of the complex irreducible representations of Γ with R_0 the trivial representation. We construct a graph $\Delta(\Gamma)$ as follows. The vertices of the graph $\Delta(\Gamma)$ are indexed by the irreducible representations R_0, R_1,...,R_n. Let

$$N \otimes R_i = \bigoplus_{j=0}^{n} R_j \otimes \mathbb{C}^{m_{ij}}$$

be the decomposition of $N \otimes R_i$ into irreducible representations, where N is the natural 2-dimensional representation. Then we connect the vertex i to the vertex j by m_{ij} arrows. Two arrows going in opposite directions are then replaced by an undirected edge. Then the corresponding graph is the extended Dynkin diagram. We call this graph the **McKay graph**.

A new approach to the **deformation and resolution theory of Kleinian singularities** was given by P. B. Kronheimer [12]. His construction starts directly from the finite group Γ and uses hyper-Kähler quotient constructions. In 1996 Cassens and Slodowy [7] reformulated Kronheimer's results using geometric invariant theory.

Let $Q = (Q_0, Q_1, s, t)$ be a finite **quiver**, i.e. a finite set $Q_0 = \{1, 2, \ldots, n\}$ of vertices and a finite set Q_1 of arrows $a : s(a) \to t(a)$, where $s(a)$ and $t(a)$ denote the starting and terminating vertex of a, respectively.

A **representation** X of Q over $\mathbb{C}$ is a collection $(X_i)_{i\in Q_0}$ of finite dimensional $\mathbb{C}$-vector spaces together with a collection $(X(a) : X_{s(a)} \to X_{t(a)})_{a\in Q_1}$ of $\mathbb{C}$-linear maps. A **morphism** $\varphi : X \to Y$ between two representations is a collection $(\varphi_i : X_i \to Y_i)_{i\in Q_0}$ of $\mathbb{C}$-linear maps such that $\varphi_{t(a)}X(a) = Y(a)\varphi_{s(a)}$. The **dimension vector** of a representation X of Q is the vector

$$\underline{\dim} X = (\dim X_1, \ldots, \dim X_n) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\mathrm{Rep}(Q)$. If we identify any complex vector space of dimension k with $\mathbb{C}^k$, then for any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{Q_0}$ we may view

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(a)}}, \mathbb{C}^{\alpha_{t(a)}})$$

as the set of all representations of the quiver Q of given dimension vector α.

Let $GL(\alpha)$ be the algebraic group defined by

$$\prod_{i\in Q_0} GL(\alpha_i, \mathbb{C}).$$

This group acts on $\mathrm{Rep}(Q,\alpha)$ by conjugation:

$$(g\cdot X(a))_{a\in Q_1} = (g_{t(a)}\cdot X(a)\cdot g_{s(a)}^{-1})_{a\in Q_1}.$$

Since the scalar subgroup $\mathbb{G}_m$ acts trivially, it is more natural to consider the quotient group $G(\alpha) := GL(\alpha)/\mathbb{G}_m$.

The set of isomorphism classes of representations of Q with dimension vector α is in 1-1 correspondence with the set of $G(\alpha)$-orbits in $\mathrm{Rep}(Q,\alpha)$. We consider the quotient space $\mathrm{Rep}(Q,\alpha)/G(\alpha)$. It is equipped with a geometric structure by King [11] using the geometric invariant theory. We note that the characters of $G(\alpha)$ are given by

$$\begin{array}{rcl} \chi_\theta : G(\alpha) & \longrightarrow & \mathbb{C}^* \\ (g_i)_{i\in Q_0} & \longmapsto & \prod\limits_{i\in Q_0} [\det(g_i)]^{\theta_i}, \end{array}$$

for $\theta\in\mathbb{Z}^{Q_0}$ with $\sum\limits_{i\in Q_0}\theta_i\alpha_i = 0$. Such a vector θ is considered as a function $\theta : \mathbb{Z}^{Q_0}\longrightarrow\mathbb{Z}$ by $\theta(\beta) := \sum\limits_{i\in Q_0}\theta_i\beta_i$.

We consider two types of quotient of $\mathrm{Rep}(Q,\alpha)$ by $G(\alpha)$. Let $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]$ be the coordinate ring of the variety $\mathrm{Rep}(Q,\alpha)$. Then $\mathrm{Rep}(Q,\alpha)//\mathrm{G}(\alpha)$ is the variety whose coordinate ring is the invariant part of $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]$:

$$\mathrm{Rep}(Q,\alpha)//G(\alpha) := \mathrm{Specm}(\mathbb{C}[\mathrm{Rep}(Q,\alpha)])^{G(\alpha)}.$$

Then we consider the graded ring

$$\mathbb{C}[\mathrm{Rep}(Q,\alpha)]^{G(\alpha),\lambda} := \bigoplus_{m=0}^{\infty}\mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{\chi^m}$$

where $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{\chi^m}$ denotes the subspace of $G(\alpha)$-semiinvariant with respect to χ^m. Hence we define

$$\mathrm{Rep}(Q,\alpha)//^{\chi}G(\alpha) := \mathrm{Projm}(\bigoplus_{m=0}^{\infty}\mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{\chi^m}).$$

This quotient is a projective variety [11]. There is a natural projective morphism

$$\mathrm{Rep}(Q,\alpha)//^{\chi}G(\alpha)\longrightarrow\mathrm{Rep}(Q,\alpha)//G(\alpha).$$

This map is called the **χ-linear modification** of $\mathrm{Rep}(Q,\alpha)//G(\alpha)$.

By the general result in the geometric invariant theory, the quotient $\mathrm{Rep}(Q,\alpha)//^{\chi}G(\alpha)$ has a geometric description as follows. We say that $X \in \mathrm{Rep}(Q,\alpha)$ is **χ-semistable** if there exists $f \in \mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{\chi^m}$ with $m > 0$ such that $f(X) \neq 0$. A point $X \in \mathrm{Rep}(Q,\alpha)$ is **χ-stable** if there exists $f \in \mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{\chi^m}$ with $m > 0$ such that $f(X) \neq 0$; the isotropy group $G(\alpha)_X$ is finite, and the orbit $G(\alpha) \cdot X$ is closed in the affine open subset $\{Y \in \mathrm{Rep}(Q,\alpha) \mid f(Y) \neq 0\}$. Then the variety $\mathrm{Rep}(Q,\alpha)//^{\chi}G(\alpha)$ may be identified with the set of closed orbits of $G(\alpha)$ in the set of χ-semistable points.

Using the Hilbert-Mumford criterion [11, Prop. 3.1], the notion χ_θ-semistable (resp.χ_θ-stable) can be translated into the language of the representation of the quiver: A representation X is χ_θ-semistable (resp. χ_θ-stable) if and only if any non-trivial proper subrepresentation N of X satisfies $\theta(\underline{\dim} N) \leq 0$ (resp. $\theta(\underline{\dim} N) < 0$).

The Lie algebra of $GL(\alpha)$ is given by

$$\mathrm{End}(\alpha) := \prod_{i\in Q_0} \mathrm{Mat}_{\mathbb{C}}(\alpha_i,\alpha_i).$$

We may identify $\mathrm{Lie}(GL(\alpha))$ with its dual via the trace pairing. Under this pairing the dual to $\mathrm{Lie}(GL(\alpha))$ is identified with the trace zero matrices in $\mathrm{End}(\alpha)$. We denote the variety of trace zero matrices by $\mathrm{End}(\alpha)_0$:

$$\mathrm{End}(\alpha)_0 := \{A \in \mathrm{End}(\alpha) \mid \sum_{i\in Q_0} \mathrm{tr}(A_i) = 0\}.$$

Let $\overline{Q}$ be the double quiver of Q. Thus $\overline{Q}$ has the same vertices as Q but the set of the edges is given by $\{a, a^* \mid a \in Q\}$, where $s(a^*) = t(a)$ and $t(a^*) = s(a)$. Then we consider the map

$$\mu_\alpha : \mathrm{Rep}(\overline{Q},\alpha) \longrightarrow \mathrm{End}(\alpha)_0$$

given by

$$(X(a), X(a^*))_{a\in Q_1} \longmapsto (\sum_{\substack{a\in Q_1\\ s(a)=i}} X(a)X(a^*) - \sum_{\substack{a\in Q_1\\ t(a)=i}} X(a^*)X(a))_{i\in Q_0}.$$

Let $Q(\Gamma)$ be the quiver obtained by choosing any orientation of the McKay graph $\Delta(\Gamma)$. It is an extended Dynkin quiver with minimal positive imaginary root $\delta \in \mathbb{N}^{Q_0}$ given by $\delta_i = \dim R_i$. We call the double quiver $\overline{Q}(\Gamma)$ of $Q(\Gamma)$ the **McKay quiver of type $\Delta(\Gamma)$**. Let $Z_\delta = [\mathrm{End}(\alpha)_0]^{G(\delta)}$ be center of $\mathrm{End}(\alpha)_0$.

For any $z \in Z_\delta$ the reductive group $G(\delta)$ acts on the fiber $\mu_\delta^{-1}(z)$ since the $G(\delta)$-equivariance of μ_δ. By the general result in the geometric invariant theory [19] we can form an algebraic quotient $\mu_\delta^{-1}(Z_\delta)//G(\delta)$. Let Φ be the morphism

$$\Phi : \mu_\delta^{-1}(Z_\delta)//G(\delta) \longrightarrow Z_\delta$$

which is obtained from the universal property of the quotient

$$\begin{tikzcd}
\mu_\delta^{-1}(Z_\delta) \arrow[rr, "\rho"] \arrow[dr, "\mu_\delta"'] & & \mu_\delta^{-1}(Z_\delta)//\ G(\delta) \arrow[dl, "\Phi"] \\
& Z_\delta. &
\end{tikzcd}$$

The morphism Φ is a deformation of the Kleinian singularity.([12], [7])

Now we apply a linear modification to the construction above, i.e. for any character χ_θ we obtain a diagram

$$\begin{tikzcd}
\mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta) \arrow[r, "\pi_\chi"] \arrow[d, "\Phi_\chi"'] & \mu_\delta^{-1}(Z_\delta)//G(\delta) \arrow[d, "\Phi"] \\
Z_\delta \arrow[r, phantom, "="] & Z_\delta \quad .
\end{tikzcd}$$

By Cassens and Slodowy ([7],[8]) for generic χ_θ (i.e. $\theta(\beta) \neq 0$ for all β statistfying $0 \neq \delta - \beta \in \mathbb{N}^{Q_0}$) this diagram is a simultaneous resolution of Φ. In particular

$$\pi : \mu_\delta^{-1}(0)//^{\chi_\theta}G(\delta) \longrightarrow \mu_\delta^{-1}(0)//G(\delta)$$

is a minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$.

The strategy for the proof of the above result does not give an analysis of exceptional components of π. It is the purpose of this work to give a detailed description of the exceptional set of the minimal resolution π.

A point in $\mu_\delta^{-1}(0)$ can be considered as a representation of the quotient algebra of the corresponding path algebra by the relation given by the equation $\mu_\delta = 0$:

$$\Pi(Q(\Gamma)) = \mathbb{C}\overline{Q}(\Gamma)/(\sum_{\substack{a\in Q(\Gamma)_1\\ s(a)=i}} aa^* - \sum_{\substack{b\in Q(\Gamma)_1\\ t(b)=i}} b^*b)_{i\in Q(\Gamma)_0}.$$

Such an algebra is called **preprojective algebra of the quiver** $Q(\Gamma)$. Therefore we would like write $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)$ for $\mu_\delta^{-1}(0)$ and define $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)^\theta$ as the χ_θ-semistable part of $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)$. Our starting point is the following commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)^\theta \arrow[r, "\rho_\theta"] \arrow[d] & \mathrm{Rep}(\Pi(Q(\Gamma)),\delta)^\theta//G(\delta) \arrow[d, "\pi"] & \\
\mathrm{Rep}(\Pi(Q(\Gamma)),\delta) \arrow[r, "\rho"] & \mathrm{Rep}(\Pi(Q(\Gamma)),\delta)//G(\delta) & \simeq \mathbb{C}^2/\Gamma
\end{tikzcd}$$

where the maps ρ and ρ_θ are the quotient maps, and π is the minimal resolution.

An element $X = (X(a))_{a\in\overline{Q}(\Gamma)} \in \mathrm{Rep}(\Pi(Q(\Gamma)),\delta)$ is **nilpotent** if there exists an integer $N \geq 2$ such that the composition $X(a_{i_N})\cdots X(a_{i_2})X(a_{i_1})$ is equal to zero for any sequence of arrows $a_{i_1}, a_{i_2}, \ldots, a_{i_N} \in \overline{Q}(\Gamma)$ such that $t(a_{i_1}) = s(a_{i_2}), t(a_{i_2}) = s(a_{i_3}), \ldots, t(a_{i_{N-1}}) = s(a_{i_N})$. We denote $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)_{\mathrm{nil}}$ the subvariety consisting of all nilpotent elements of $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)$. The following theorem plays an important part in our work.

Theorem 12. *The exceptional set of the resolution π is given by*

$$\pi^{-1}(0) = \rho_\theta(\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)^\theta_{\mathrm{nil}}).$$

Thus, to describe the exceptional set $\pi^{-1}(0)$ of the resolution π we need to consider the nilpotent variety $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)^\theta_{\mathrm{nil}}$. This variety is a particular case of the nilpotent variety $\mathrm{Rep}(\Pi(Q),\alpha)_{\mathrm{nil}}$ which is introduced by Lusztig [13]. It has pure dimension equal to $\frac{1}{2}\dim\,\mathrm{Rep}(\overline{Q},\alpha)$. Hille [8] gave a formula determining the number of irreducible components of $\mathrm{Rep}(\Pi(Q),\alpha)_{\mathrm{nil}}$. In particular, this formula allows us to determine explicitly the number of irreducible components of the variety $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)_{\mathrm{nil}}$ except the case $\Delta(\Gamma) = \widetilde{\mathbb{E}_8}$.

The central result of the thesis is the description of the variety $\mathrm{Rep}(\Pi(Q(\Gamma)),\delta)_{\mathrm{nil}}$ in the cases $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_4$ and $\widetilde{\mathbb{D}}_5$. Furthermore we also describe explicitly the action of the Weyl group on the space of weights $\mathbb{H}(\delta)$ in the case $\widetilde{\mathbb{A}}_n$.

We say that two weights θ and θ' in the space of weights with respect to the dimension vector δ

$$\mathbb{H}(\delta) := \{\theta \in \mathbb{Z}^I \mid \theta(\delta) := \sum \theta_i \delta_i = 0\}$$

are **δ-equivalent** if for any X in $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)_{\mathrm{nil}}$ X is θ-stable if and only if X is θ'-stable and for any X in $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)_{\mathrm{nil}}$ X is θ-semistable if and only if X is θ'-semistable. We can also define the **wall system** with respect to dimension vector δ. That is the minimal set of hyperplanes $\{W_i\}_{i\in I}$ in $\mathbb{H}(\delta)$, where I is a finite index set, with the following property: whenever two generic weights θ and θ' in $\mathbb{H}(\delta)$ lie on the same side of each of these hyperplanes W_i, then they are δ-equivalent.

Let X and Y be two varieties of the same dimension. We say that the variety X **meets** the variety Y if and only if codim(X$\cap$Y) = 1. Let X be a reducible variety which has pure dimension n. We define the **intersection diagram** Γ(X) with respect to X as follows: Associate to each irreducible component X_i a vertex $i \in \Gamma(\mathrm{X})$. Vertices i and j are connected by an edge if the component X_i meets the component X_j.

We first consider the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$. We investigate the action of symmetric group on the space of weights $\mathbb{H}(\delta)$ and construct the wall system in $\mathbb{H}(\delta)$. We describe the intersection behavious of the irreducible components of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ and put it in relationship with Kleinian singularity of type $\mathbb{A}_n$.

Let $\overline{Q}(\Gamma)$ be the McKay quiver of type $\widetilde{\mathbb{A}}_{n-1}$ with $n \geq 2$. We assume that $\overline{Q}(\Gamma)$ has the set of vertices $\overline{Q}(\Gamma)_0 = \mathbb{Z}/n = \{1, 2, \ldots, n\}$, and consists of arrows: $\alpha_i : i+1 \to i$, $\alpha_i^* : i \to i+1$, $i \in I$. Let Ω be the set of all arrows α_i. The nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ has $2^n - 2$ irreducible components. Each component has the form

$$C_I := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \bigoplus_{i=1}^{n} \mathbb{C}^2 \mid a_i^* = 0 \text{ if } \alpha_i \in I;\ a_j = 0 \text{ if } \alpha_j \notin I\})$$

where I is a subset of Ω such that $\varnothing \neq I \neq \Omega$. We say that C_I is a **component of type k** if $\mid I \mid$ = k and write $C_{\{i_1, i_2, \ldots, i_k\}}$ for $C_{\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}}$.

Let $\delta'_{i,j}$ be the non-trivial proper subdimension vectors of δ given by

$$\delta'_{i,j} := (\overbrace{\underbrace{0, \ldots, 0}_{i}, 1, \ldots, 1}^{j}, 0, \ldots, 0), \quad 1 \leq i < j \leq n.$$

Then we define hyperplanes

$$W_{i,j} := \{\theta \in \mathbb{H}(\delta) \mid \theta(\delta'_{i,j}) = 0\}.$$

The action of symmetric group on $\mathbb{H}(\delta)$, the δ-equivalence classes and the wall system are described in the following theorem:

Theorem 13. *1) The symmetric group S_n acts on the space of weights $\mathbb{H}(\delta)$ by reflections with respect to the hyperplanes $W_{i,j}$ dividing $\mathbb{H}(\delta)$ in n! chambers:*

$$\Theta(\sigma) := \{\theta \in \mathbb{H}(\delta) \mid \sum_{l=1}^{\sigma(1)} \theta_l < \sum_{l=1}^{\sigma(2)} \theta_l < \cdots < \sum_{l=1}^{\sigma(n)} \theta_l\}, \quad \sigma \in S_n.$$

The resulting action of S_n on the set of these n! chambers is simply transitive.

2) For each generic $\theta \in \Theta(\sigma)$ there are exactly (n-1) θ-stable components of the nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ which are

$$C_{\{\sigma(1),\sigma(2),\ldots,\sigma(i)\}}, \quad i = 1, 2, \ldots, n-1.$$

3) For each generic $\theta \in \Theta(\sigma)$ the corresponding δ-equivalence class of θ is the chamber $\Theta(\sigma)$.

4) The set of hyperplanes $\{W_{i,j}\}_{1 \le i < j \le n}$ forms the wall system in $\mathbb{H}(\delta)$.

Let $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ denote the intersection diagram of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$. Then $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ relates with the intersection diagram of the resolution of the Kleinian singularity of type $\mathbb{A}_{n-1}$ by the following theorem:

Theorem 14. *1) For each permutation $\sigma \in S_n$ the map φ_σ given by*

$$\begin{array}{rcl} \varphi_\sigma : \overset{1}{\bullet} - \overset{2}{\bullet} \cdots \bullet - \overset{n-1}{\bullet} & \longrightarrow & \Gamma(\widetilde{\mathbb{A}}_{n-1}) \\ \overset{k}{\bullet} & \longmapsto & \varphi_\sigma(k) := C_{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}, \end{array}$$

gives an embedding of the Dynkin diagram $\mathbb{A}_{n-1}$ into the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$.

2) For each $\theta \in \Theta(\sigma)$ the intersection diagram of the exceptional set of the resolution

$$\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)^\theta // G(\delta) \longrightarrow \mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta) // G(\delta)$$

is the Dynkin diagram $\mathbb{A}_{n-1}$.

We denote $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)_{\text{w-nil}}$ the subvariety consisting of all **locally nilpotent elements** of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)$ (see definition in Sect. 5.1). Let $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}}$ be the intersection diagram of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)_{\text{w-nil}}$. It is obtained from the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ by adding two vertices which correspond to the components $C_{\varnothing}$ and C_{Ω}. Then the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}}$ is described as follows:

Theorem 16. *The intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{w}$ is a skeleton of a n-dimensional cube H^n. Its vertices are the vertices of the cube H^n and its edges are the edges of the cube H^n.*

Let $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{\mathrm{nil}}^{\mathrm{stab}}$ denote the stable part of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{\mathrm{nil}}$. The $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{\mathrm{nil}}^{\mathrm{stab}}$ has 48 irreducible components. Then the intersection diagrams $\Gamma(\widetilde{\mathbb{D}}_4)$ of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{\mathrm{nil}}^{\mathrm{stab}}$ is described as follows:

Theorem 18. *The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$ has 48 vertices. They are arranged in a 4-dimensional cube H^4 as follows.*

The 16 vertices correspond to the vertices O^1_i, $i=1,\dots,16$, of H^4. The 24 vertices correspond to the centers O^2_j, $j=1,\dots,24$, of the 2-facets of H^4. The 8 vertices correspond to the centers O^3_k, $k=1,\dots,8$, of the 3-facets of H^4.

Vertices O^1_i and O^2_j are connected by an edge if the vertex O^1_i belongs to a 2-facet whose center is O^2_j.

Vertices O^3_k and O^2_j are connected by an edge if the 2-facet whose center is O^2_j belongs to a 3-facet whose center is O^3_k.

The $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_5),\delta)_{\mathrm{nil}}^{\mathrm{stab}}$ has 162 irreducible components. The intersection diagrams $\Gamma(\widetilde{\mathbb{D}}_5)$ of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_5),\delta)_{\mathrm{nil}}^{\mathrm{stab}}$ is described as follows:

Theorem 20. *The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$ has 162 vertices. They are arranged in a 5-dimensional cube H^5 as follows.*

The 32 vertices corresponds to the vertices O^1_i, $i=1,\dots,32$, of H^5. The 80 vertices correspond to the centers O^2_j, $j=1,\dots,80$, of the 2-facets, of H^5. The 40 vertices correspond to the centers O^3_k, $k=1,\dots,40$, of the 3-facets, of H^5. The 10 vertices correspond to the centers O^4_h, $h=1,\dots,10$, of the 4-facets, of H^5.

Vertices O_i^1 and O_j^2 are connected by an edge if the vertex O_i^1 belongs to a 2-facet whose center is O_j^2.

Vertices O_k^3 and O_j^2 are connected by an edge if the 2-facet whose center is O_j^2 belongs to a 3-facet whose center is O_k^3.

Vertices O_h^4 and O_k^3 are connected by an edge if the 3-facet whose center is O_k^3 belongs to a 4-facet whose center is O_h^4.

CONJECTURE. One may hope that the following statements might be true:

1) The number of stable components of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_n), \delta)_{\mathrm{nil}}$ is

$$\sum_{\substack{k=0 \\ k\neq 1}}^{n-1} \binom{k}{n} 2^{n-k}$$

where $\binom{k}{n} 2^{n-k}$ is the number of k-facets in a n-dimensional cube H^n.

2) One can describe the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_n)$ by induction on n the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$.

The thesis is organized as follows. In Section 1 we shall give a quick review of the definition of Kleinian singularities. In Section 2 we shall introduce the deformation and resolution theory of sigularities. In Section 3 we collect the basic notations and definitions of the representation theory of quivers. In Section 4 we present the main results from the deformation and resolution theory of the Kleinian singularities which are related to quiver varieties. These results are given by Kronheimer, Cassens and Slodowy. In Section 5 we say about the nilpotent and stable representations of quivers and related results. These results are given by Lusztig and Hille. We explain how these results apply to a description of the exceptional set of the minimal resolution of Kleinian singularity. In Section 6 we describe explicitly the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ and consider action of the Weyl group on the space of weights $\mathbb{H}(\delta)$. In Section 7 we shall describe the intersecion diagrams $\Gamma(\widetilde{\mathbb{D}}_4)$ and $\Gamma(\widetilde{\mathbb{D}}_5)$.

NOTATIONS. We list a few notations that will be used throughout this thesis.

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^*$	multiplicativve group of complex numbers $\neq 0$
$\mathbb{C}^n$	numerical complex vector space of dimension n
$\subset$	strict inclusion
$\mathbb{C}[x_1, x_2, \ldots, x_n]$	polynomial ring over $\mathbb{C}$
$GL(n, \mathbb{C})$	general linear group
$SL(n, \mathbb{C})$	special linear group
$\mathbb{I}_n$	unit matrix of dimension n
$\mathbb{G}_m$	scalar subgroup
$\mathbb{A}^n$	affine n-space over $\mathbb{C}$
$\mathbb{P}^n$	projective n-space over $\mathbb{C}$
$\mathbb{C}[X]$	coordinate ring of X
R^G	ring of invariants
$X//G$	quotient of X by G
$X//^{\chi}G$	quotient the χ-semistable part of X by G
$\overline{X}, \mathrm{Cl}(X)$	closure of X
X^{stab}	stable part of X
Specm R	maximal spectrum of R
Projm R	homogeneous maximal spectrum of the graded ring R
Hom(M,N)	homomorphism group
End(R)	endomorphism ring = Hom(R,R)
$\mathbb{C}Q$	path algebra of the quiver Q
$\Pi(Q)$	preprojective algebra of the quiver Q
Rep(Q)	category of representations of the quiver Q
$\mathbb{C}Q$-mod	category of finite dimensional $\mathbb{C}Q$-modules
$\Delta(\Gamma)$	McKay graph associated to a finite subgroup Γ of $SL(2, \mathbb{C})$
$\overline{Q}(\Gamma)$	McKay quiver of type $\Delta(\Gamma)$
$\mathbb{H}(d)$	the space of weights with respect to the dimension vector d
$\Gamma(X)$	intersection diagram of X
$G \cdot X$	orbit of X by G
G_X	stabilizer of X

The author is grateful to C.-F. Bödigheimer and L. Hille for their great help to the thesis. The author also thanks KAAD for financial support.

1 Kleinian Singularities

1.1 Singularities

Let $X \subset \mathbb{C}^n$ be an algebraic or analytic variety defined by the vanishing of polynomial or analytic functions

$$X = V(f_1, f_2, \ldots, f_k) = \{z \in \mathbb{C}^n | f_1(z) = f_2(z) = \ldots = f_k(z) = 0\}.$$

Definition. A point $x \in X$ is called **regular** if there exists a neighborhood of $x \in X$ is isomorphic to a complex manifold (isomorphism in the category of analytic varieties).

A point $x \in X$ is called **singular** if it is not regular.

A point $x \in X$ is called **isolated singular** if it is singular and all near-by points are regular.

Theorem 1 [18] *Let $X \subset \mathbb{C}^3$ is a hypersurface*

$$X = V(f) = \{(x, y, z) \in \mathbb{C}^3 | f(x, y, z) = 0\}.$$

f: $\mathbb{C}^3 \longrightarrow \mathbb{C}$ a polynomial or analytic, f(0) = 0, f square-free, then 0 $\in X$ is a isolated singularity iff

(i)

$$\frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = \frac{\partial f}{\partial z}(0) = 0$$

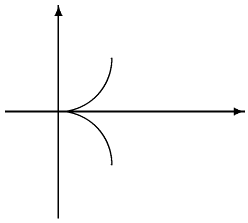
(ii)

$$\dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) < \infty$$

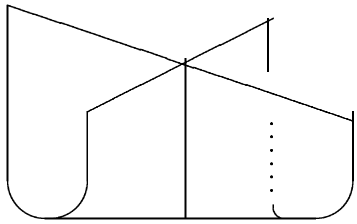
where $\mathbb{C}\{x, y, z\}$ denotes the ring of the convergent power series in x, y, z.

Example. Let $S = \{y^2 - zx^2 = 0\}$ be the variety in $\mathbb{C}^3$. This variety has clearly singularity in 0. It is non-isolated singularity because $\mathbb{C}\{x, y, z\}/(y, x^2, xz)$ has a infinite basis $\{1, x, z, z^2, \ldots, z^n, \ldots\}$.

Example. Let $C = \{y^2 - x^3 = 0\}$ be the variety in $\mathbb{C}^2$. This variety has isolated singularity in 0.



$C = \{y^2 - x^3 = 0\}$



$S = \{y^2 - zx^2 = 0\}$

1.2 Kleinian singularities

Let Γ be a nontrivial finite subgroup of $SL(2, \mathbb{C})$. The action of the group Γ on $\mathbb{C}^2$ by matrix multiplication induces an action on the $\mathbb{C}$-algebra $\mathbb{C}[u, v]$ of all polynomials in 2 variables by means of

$$(\gamma, f(z)) \longmapsto f(\gamma.z) \qquad \text{for} \quad \gamma \in \Gamma \quad \text{and} \quad z \in \mathbb{C}^2.$$

Theorem 2 [23] *The $\mathbb{C}$-algebra $\mathbb{C}[u, v]^{\Gamma}$ of Γ-invariant polynomials on $\mathbb{C}^2$ is generated by three fundamental generators X, Y, Z, satisfying a relation $R(X, Y, Z) = 0$, where R is a polynomial on $\mathbb{C}^3$.*

Here is the list of the finite subgroups $\Gamma \in SL_2(\mathbb{C})$ and the corresponding relations

Γ	$\mathcal{C}_n$	$\mathcal{D}_n$	$\mathcal{T}$	$\mathcal{O}$	$\mathcal{I}$
$\|\Gamma\|$	n	$4n$	24	48	120
R	$X^n + YZ$	$X(Y^2 - X^n) + Z^2$	$X^4 + Y^3 + Z^2$	$X^3 + XY^3 + Z^2$	$X^5 + Y^3 + Z^2$

where $\mathcal{C}_n$ is cyclic group of order n, and $\mathcal{D}_n$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ are binary dihedral, tetrahedral, octahedral, icosahedral groups.

Geometrically, the theorem means that the quotient variety $\mathbb{C}^2/\Gamma$ may be viewed as a hypersurface in $\mathbb{C}^3$ given by the equation $R(X, Y, Z) = 0$:

$$\mathbb{C}^2/\Gamma = \{(X, Y, Z) \in \mathbb{C}^3 | R(X, Y, Z) = 0\}.$$

The hypersurface $\mathbb{C}^2/\Gamma$ has an isolated singularity at the origin. This singularity is called **Kleinian singularity**.

Example. Let Γ be the cyclic of order n which is generated by the matrix $\mathrm{diag}(\epsilon, \epsilon^{-1})$, where ϵ is a n-th primitive root of 1. The basis invariants are expressed in terms of the variables (u,v) in $\mathbb{C}^2$ as follows: $X = uv$, $Y = u^n$, $Z = v^n$. They are related by $X^n = YZ$.

2 Deformation and resolution theory of singularities

2.1 Deformation of singularities

Definition. Let S be a variety. A **deformation** of S consists of a pair (χ, i) where $\chi : X \longmapsto (U, u)$ is flat morphism (i.e. all fibers have the same dimension) of varieties and i: $S \longmapsto \chi^{-1}(u)$ is an isomorphism of S onto the special fibre of X over u.

X is called the **total space** and U the **base** of the deformation χ.

Let (χ, i) be a deformation of S, and let $\varphi : (U, u) \longmapsto (U_1, u_1)$ be a morphism of pointed varieties (i.e. $\varphi(u_1) = u$). Then the pull-back

$$\begin{tikzcd} X \times_U U_1 \arrow[r, "pr_1"] \arrow[d, "\varphi^*(\chi)"'] & \mathrm{X} \arrow[d, "\chi"] \\ (U_1, u_1) \arrow[r, "\varphi"] & (U, u) \end{tikzcd}$$

gives another deformation of S which is called the **induced deformation** by the base change φ.

Two deformations (χ, i) and (χ', i') of S over the same base (U, u) are called **isomorphism** if there exist an isomorphism $\phi : X \longrightarrow X'$ of the total spaces such that the following diagram is commutative

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\chi"'] & & X' \arrow[dl, "\chi'"] \\ & (U, u). & \end{tikzcd}$$

Definition. A deformations (χ, i) of S is called **semi-universal** if the following conditions hold:

i) Up to isomorphism any deformation of S can be induced from (χ, i) by means of some base change $\varphi : (U', u') \longrightarrow (U, u)$.

ii) The differential $D_{u'} : T_{u'}U' \longrightarrow T_u U$ of φ at u' is uniquely determined.

Theorem 3 [24] *Let $S = \{f = 0\} \in \mathbb{C}^n$ be a hypersurface with an isolated singularity at 0. Then a semi-universal deformation of S is given by*

$$\begin{array}{l} X = \{(z,u) \in \mathbb{C}^n \times \mathbb{C}^r \,|\, f(z) = \sum\limits_{i=0}^{r} u_i b_i(z)\} \\ \Big\downarrow \quad \chi = pr_2 \\ (\mathbb{C}^r, 0) \end{array}$$

$$i : S \hookrightarrow X, \quad z \longmapsto (z,0)$$

where $b_1, b_2, \ldots, b_r$ are a basis of space

$$\mathbb{C}[z_1, z_2, \ldots, z_n] / < f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} > .$$

Example. Let $S = \{x^{n+1} - (y^2 + z^2)\}$ be the singularity of type $\mathbb{A}_n$. Then $\mathbb{C}[x,y,z]/ < x^n, y, z >$ has a basis $\{1, x, x^2, \ldots, x^{n-1}\}$.
Thus, a semi-universal deformation of S is given by

$$\begin{array}{l} \mathrm{X} = \{(x, y, z, u_1, u_2, \ldots, u_n) \in \mathbb{C}^{3+n} | x^{n+1} + \sum\limits_{i=1}^{n} u_i x^{n-i} - y^2 - z^2 = 0\} \\ \Big\downarrow \chi = pr_2 \\ (\mathbb{C}^n, 0). \end{array}$$

2.2 Resolution of singularities

Definition. Let S be a variety with set of regular points S^{reg} dense in S and set of singularities $S^{sing} = S \setminus S^{\mathrm{reg}}$. A **resolution of singularities** or **desingularization** of S is a morphism π: $\tilde{S} \longrightarrow$ S (in the category of algebraic or analytic varieties) with the following properties

a) $\tilde{S}$ is smooth, i.e. $\tilde{S} = \tilde{S}^{reg}$,
b) π is proper, i.e. $\pi^{-1}(K)$ is compact if $K \subset S$ is compact,
c) π: $\pi^{-1}(S^{reg}) \longrightarrow (S^{reg})$ is an isomorphism.

The set E $= \pi^{-1}(S^{sing})$ is called the **exceptional set** of the resolution π.

A resolution π: $\tilde{S} \longrightarrow$ S is called **minimal** if it has the property that any other resolution $\pi^{'} : \tilde{S}^{'} \longrightarrow S$ there is a unique morphism $\varphi : \tilde{S}^{'} \longrightarrow \tilde{S}$ with $\pi \circ \varphi = \pi^{'}$

$$\begin{tikzcd} \tilde{S}' \arrow[rr, "\varphi"] \arrow[dr, "\pi'"'] & & \tilde{S} \arrow[dl, "\pi"] \\ & \mathrm{S} & \end{tikzcd}$$

It follows that the minimal resolution is unique up to isomorphism.

Using the blowing-up procedure we shall describe resolutions of singularities in some simple cases. Let

$$B(0, \mathbb{C}^n) = \{(x, [y] \in \mathbb{C}^n \times \mathbb{P}^{n-1} | x \in [y]\}$$

$$= \{(x_1, \ldots, x_n), (y_1 : \ldots : y_n) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid x_i y_j - x_j y_i, i, j = 1, \ldots, n\}.$$

The first projection φ: $B(0, \mathbb{C}^n) \longrightarrow \mathbb{C}^n$ is called the **blow-up** of $0 \in \mathbb{C}^n$. One has

$$\varphi^{-1}(x) = \{(x, [x] \in \mathbb{C}^n \times \mathbb{P}^{n-1}\} \simeq \{(x)\} \in \mathbb{P}^{n-1} \quad \text{for} \quad x \neq 0.$$

and

$$\varphi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}.$$

Thus φ induces an isomorphism

$$\varphi : B(0, \mathbb{C}^n) \setminus \varphi^{-1}(0) \longrightarrow \mathbb{C}^n \setminus \{0\}.$$

The point $0 \in \mathbb{C}^n$ is replaced in $B(0, \mathbb{C}^n)$ by the projective space $\mathbb{P}^{n-1}$ of all lines through 0.

Let $S \subset \mathbb{C}^n$ be a subvariety with $0 \in S$. The restriction

$$\varphi_S := \varphi|_{\varphi^{-1}(S)} : \varphi^{-1}(S) \longrightarrow S$$

is called the blow-up of $0 \in$ S. The φ_S^{-1}(S) is called the **total transform** of S. The (Zariski-)closure of $\varphi^{-1}(S \setminus \{0\})$ is called the **proper transform** of S. The $\varphi_S^{-1}(0)$ is called the **exceptional divisor** of φ_S.

To calculate the blowing-up we give a local form of φ. We cover $\mathbb{P}^{n-1}$ by n standard affine charts

$$U_i = \{(x_1 : \ldots : 1 : \ldots : x_n) \in \mathbb{P}^{n-1}\} \simeq \mathbb{C}^{n-1}, i = 1, 2, \ldots, n.$$

This covering induces a covering of $B(0, \mathbb{C}^n)$

$$B(0, \mathbb{C}^n) = \bigcup_{i=1}^{n} \varphi^{-1}(U_i) := \bigcup_{i=1}^{n} \widetilde{U}_i.$$

We construct an isomorphism $\mathbb{C}^n \simeq \widetilde{U}_i$. Then restriction $\varphi_i = \varphi|_{\widetilde{U_i}}$ of φ on $\widetilde{U}_i$ is described as follows

$$\begin{tikzcd}
(y_1, \ldots, y_n) \arrow[r] \arrow[dr] & ((y_1 : \ldots : 1 : \ldots : y_n)y_i, (y_1 : \ldots : 1 : \ldots : y_n)) \in \widetilde{U}_i \arrow[d, "\varphi_i"] \\
& ((y_1, \ldots, 1, \ldots, y_n)y_i).
\end{tikzcd}$$

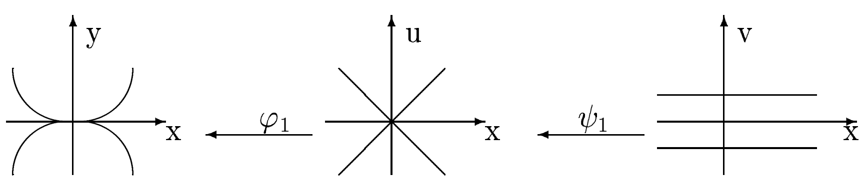
Theorem 4 *(Hironaka) Any variety may be desingularized by succesive blow-up of point and normalization.*

Examples. 1) Let C be the variety in $\mathbb{C}^2$

$$C = \{x^4 - y^2 = 0\}.$$

We choose the chart $\widetilde{U_1}$. In this chart the blowing up φ_1 is given by

$$\varphi_1(x, u) = (x, xu)$$



We have

$$\begin{aligned}
\varphi_1^{-1}(C) &= \{(x, u) \in \widetilde{U_1} | x^4 - x^2u^2 = 0\} \\
&= \{x = 0\} \cup \{u = x\} \cup \{u = -x\}
\end{aligned}$$

for the total transform of C. The proper transform $C_1 = \varphi_1^{-1}(C \setminus \{0\})$ consists of the two lines $u = x$ and $u = -x$. The C_1 has a isolated singularity in 0. We blow again up. We choose also chart $\widetilde{U_1}$ as before. In this chart the blowing up ψ_1 is given by

$$\psi_1(x, v) = (x, xv)$$

The total transform of C_1 is

$$\begin{aligned}\psi_1^{-1}(C_1) &= \{(x, v) \in \widetilde{U_1} | x^2 - x^2v^2 = 0\} \\ &= \{x = 0\} \cup \{v = 1\} \cup \{v = -1\}.\end{aligned}$$

The proper transform $\psi_1^{-1}(C_1 \setminus \{0\})$ consists of the two lines v = 1 and v = -1. It has no more singularity. Thus, by iterated application of the blowing-up procedure the curve $C \in \mathbb{C}^2$ can be transformed to a smooth curve.

2) Let S be the variety in $\mathbb{C}^3$

$$S = \{4x^4 - 5x^2y^2 + y^4 + z^2 = 0\}$$

We choose the chart $\widetilde{U_1}$. In this chart the blowing up φ_1 is given by

$$\varphi_1(x, v, w) = (x, xv, xw)$$

The total transform of S is

$$\begin{aligned}\varphi_1^{-1}(S) &= \{(x, v, w) \in \widetilde{U_1} | 4x^4 - 5x^4v^2 + x^4v^4 + x^2w^2 = 0\} \\ &= \{x = 0\} \cup \{4x^2(4 - 5v^2 + v^4) + w^2 = 0\}.\end{aligned}$$

The proper transform $S_1 = \varphi_1^{-1}(S \setminus \{0\})$ has a non-isolated singularity along the exceptional divisor $\varphi_1^{-1}(S) \cap \{x = 0\}$. Here we have to use normalization. Note that t = w/x is an element of the field of fractions of

$$\mathbb{C}[S_1] = \mathbb{C}[x, v, w]/(x^2(4 - 5v^2 + v^4) + w^2)$$

which is integral over $\mathbb{C}[S_1]$:

$$t^2 = -(v^4 - 5v^2 + 4).$$

We define the map α by

$$\alpha : \mathbb{C}^3 \longrightarrow \widetilde{U}_i, (x, v, t) \longmapsto (x, v, tx).$$

Then we have

$$\begin{aligned}\alpha^{-1}(S_1) &= \{x^2(4 - 5v^2 + v^4 + t^2) = 0\} \\ &= \{x^2 = 0\} \cup \{4 - 5v^2 + v^4 + t^2 = 0\} \\ &= \{x^2 = 0\} \cup \widetilde{S}_1\end{aligned}$$

The smoothness of $\widetilde{S}_1$ follows that $\alpha|_{\widetilde{S_1}} : \widetilde{S}_1 \to S_1$ is a normalization of S_1. Thus, the map $\pi = \varphi_1 \circ \alpha : \widetilde{S}_1 \to S$ give a resolution of singularity of S.

The Kleinian singularities and the Dynkin diagrams are closely related. The following theorem says about that due to Du Val.

Theorem 5 [21] *1) Let $\pi : \widetilde{S} \longrightarrow S$ be the minimal resolution of a Kleinian singularity. Then the exceptional set of π is a union of smooth rational curves*

$$\pi^{-1}(0) = C_1 \cup \cdots \cup C_r, C_i \simeq \mathbb{P}^1.$$

All self-intersection C_iC_j are -2 (i.e the normal bundle $N_{C_i|\widetilde{S}}$ of C_i in $\widetilde{S}$ is isomorphic to $O_{\mathbb{P}}$(-2)) and pairwise intersections are transversal. The intersection matrix $(C_iC_j)_{i,j}, i,j = 1,2,\ldots,n$, is the negative of a cartan matrix of A_n, D_n and E_n.

2) Conversely, let $\pi : \widetilde{S} \longrightarrow S$ be the minimal resolution of a isolated surface singularity and assume that $\pi^{-1}(0)$ satisfies the properties as in i). Then S is a Kleinian singularity.

Hier is list of Kleinian singularities $\mathbb{C}^2/\Gamma$, and their Dynkin diagrams $\Delta(\Gamma)$.

Γ	Type of $\Delta(\Gamma)$	$\Delta(\Gamma)$
$\mathcal{C}_n$	$\mathbb{A}_{n-1}$	$\bullet - \bullet \cdots \bullet - \bullet$
$\mathcal{D}_n$	$\mathbb{D}_{n-1}$	$\begin{array}{ccccc} & \bullet & & & \\ & \vert & & & \\ \bullet - & \bullet & \cdots & \bullet - & \bullet \end{array}$
$\mathcal{T}$	$\mathbb{E}_6$	$\begin{array}{ccccc} & & \bullet & & \\ & & \vert & & \\ \bullet - & \bullet - & \bullet - & \bullet - & \bullet \end{array}$
$\mathcal{O}$	$\mathbb{E}_7$	$\begin{array}{cccccc} & & \bullet & & & \\ & & \vert & & & \\ \bullet - & \bullet - & \bullet - & \bullet - & \bullet - & \bullet \end{array}$
$\mathcal{I}$	$\mathbb{E}_8$	$\begin{array}{ccccccc} & & \bullet & & & & \\ & & \vert & & & & \\ \bullet - & \bullet - & \bullet - & \bullet - & \bullet - & \bullet - & \bullet \end{array}$

The Dynkin diagrams (see Sect. 3.4) has a direct intepretation as the **intersection diagram** of the exceptional set of the resolution π which consisder of vertices representing the components of $\pi^{-1}(0)$ and edges representing transversal intersections of these components.

2.3 Simultaneous resolutions

Let $f : X \longrightarrow$ S be a morphism of varieties. A **simultaneous resolution** of the morphism f is a commutative diagram of varieties and morphisms

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "\varphi"] & S \end{tikzcd}$$

with the following properties

(i) The morphism $g : Y \longrightarrow T$ is flat;

(ii) φ is surjective; ψ is proper and surjective,

(iii) each fiber Y_t of g is non-singular and

$$\Psi|_{Y_t} : Y_t \longrightarrow X_{\varphi(t)}$$

is a resolution of singularities in the sense of 2.2.

Example.[10] Let $f : X \longrightarrow S = \mathbb{C}^n$ be the deformation of the singularity of type A_n

$$\begin{aligned} &\mathrm{X} = \{(x, y, z, u_1, u_2, \ldots, u_n) \in \mathbb{C}^{3+n} | x^2 + y^2 + z^{n+1} + \sum_{j=1}^{n} s_j z^{n-j} = 0\} \\ &\Big\downarrow f = pr \\ &(\mathbb{C}^n, 0). \end{aligned}$$

Let $T \subset \mathbb{C}^n$ be the hyperplane

$$t_1 + t_2 + \cdots + t_{n+1} = 0.$$

Define a map $\varphi : T \longrightarrow S$ by setting

$$s_j = \varphi_j(t) := \sigma_{j+1}, j = 1, 2, \ldots, n$$

where σ_{j+1} is the (j+1)-th elementary symmetric function of $t_1, t_2, \ldots, t_{n+1}$.

The base change φ give an induced deformation by pulling back X to T, which has the equation

$$\widetilde{X}: \qquad x^2 + y^2 + \prod_{j=1}^{n+1}(z - t_j) = 0.$$

Blow $\widetilde{X}$ up by taking the closure of graph of the mapping

$$\mu : \widetilde{X} \longrightarrow \mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(n)}$$

given by ($\mathbb{P}^1_{(j)}$ is the j-th factor, $\mathbb{P}^1_{(j)} = \mathbb{P}^1$)

$$(X_j : Y_j) := ((ix - y) : \prod_{\nu=1}^{j}(z - t_\nu)), j = 1, \ldots, n$$

where $(X_j : Y_j)$ are homogeneous coordinates in $\mathbb{P}^1_{(j)}$ and i $= \sqrt{-1}$.

We going to show that the closure of graph G_μ of this mapping, together with its obvious projective on hyperplane T, is a simultaneous resolution of the morphism $f : X \longrightarrow S$.

The graph $G_\mu \subset \mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(n)} \times \mathbb{C}^3 \times T$ is defined by the equations

$$x^2 + y^2 + \prod_{\nu=1}^{n+1}(z - t_\nu) = 0$$

$$X_j \prod_{\nu=1}^{j}(z - t_\nu) = X_j(ix - y), j = 1, 2, \ldots, n.$$

Let

$$Y = \overline{G}_\mu \setminus \{t_r = t_s, r \neq s\}.$$

The projection

$$\mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(n)} \times \mathbb{C}^3 \times T \longrightarrow T$$

induces a map $g : Y \longrightarrow T$.

The composition

$$\mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(n)} \times \mathbb{C}^3 \times T \longrightarrow \mathbb{C}^3 \times T \xrightarrow{1\times\varphi} \mathbb{C}^3 \times S$$

induces a map $\psi : Y \longrightarrow X$.

Then we have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\varphi} & S \end{array}$$

We show that this diagram is a simultaneous resolution of the morphism $f : X \longrightarrow S$.

Let $U_0, U_1, \ldots, U_n$ be the open subsets of $\mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(n)} \times \mathbb{C}^3 \times T$ defined by

$$\begin{array}{ll} U_0 & = \{X_1 \neq 0\} \\ U_\rho & = \{Y_\rho \neq 0, X_{\rho+1} \neq 0\}, \rho = 1, \ldots, n-1 \\ U_n & = \{Y_n \neq 0\} \end{array}$$

in U_ρ, we let

$$\sigma_\rho = \frac{X_\rho}{Y_\rho} \qquad \text{and} \qquad \tau_\rho = \frac{Y_{\rho+1}}{X_{\rho+1}}$$

By an elementary commutation we can take a parametrization of Y in U_ρ as follows

$$\begin{array}{lll} (X_j : Y_j) & = \sigma_\rho \prod_{\nu=j+1}^{\rho}(z - t_\nu : 1) & \text{for} \quad j \leq \rho - 1 \\ (X_j : Y_j) & = (1 : \tau_{\rho+1} \prod_{\nu=\rho+2}^{j}(z - t_\nu) & \text{for} \quad j \geq \rho + 2 \\ (X_\rho : Y_\rho) & = (\sigma_\rho : 1) & \\ (X_{\rho+1} : Y_{\rho+1}) & = (1 : \tau_{\rho+1}) & \\ ix + y & = \tau_{\rho+1} \prod_{\nu=\rho+2}^{n}(z - t_\nu) & \\ ix - y & = \sigma_\rho \prod_{\nu=1}^{\rho}(z - t_\nu) & \\ z + t_{\rho+1} & = \sigma_\rho \tau_{\rho+1} & \end{array}$$

In U_ρ the simultaneous resolution can be written as follows

$$\begin{array}{ccccc} \mathbb{C}^2 \times T & \xrightarrow{\simeq} & Y \cap U_\rho & \xrightarrow{\psi} & X \\ (\sigma_\rho, \tau_{\rho+1}, t) & \longmapsto & ((X_j : Y_j)_j, x, y, z, t) & \longmapsto & (x, y, z, \varphi(t)) \\ & & \downarrow g & & \downarrow f \\ & & t \in T & \overset{\varphi}{\longmapsto} & \varphi(t) \in S. \end{array}$$

It follows that Y is non-singular, and $g : Y \longrightarrow T$ has maximal rank, hence g is flat. It follows also from the definitions of φ, ψ that φ is surjective and ψ is proper and surjective.

3 Quivers

3.1 Definition of quivers

A **quiver** $Q = (Q_0, Q_1; s, t : Q_1 \longrightarrow Q_0)$ consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows and two maps s, t which send an arrow $a \in Q_1$ to its starting vertex $s(a)$ and its terminating vertex $t(a)$. We shall write $a : i \to j$ or $i \xrightarrow{a} j$ for an arrow starting in i and ending in j.

3.2 Representations of quivers

One defines a category Rep(Q) of representations of the quiver Q as follows.

A **representation** M of Q consists of a collection of finite dimensional vector space $(M_q)_{q \in Q_0}$ for each vertex $q \in Q_0$, together with linear maps $M(a) : M_{s(a)} \longrightarrow M_{t(a)}$ for each arrow $a \in Q_1$. The **dimension vector** of the representation M is the vector $\underline{\dim} M \in \mathbb{N}^{Q_0}$, with $(\underline{\dim} M)_i = \dim M_i$.

A **morphism** φ between representations M and N of quiver Q is given by linear maps $\varphi_i : M_i \longrightarrow N_i$, for each $i \in Q_0$, such that the diagram

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[d, "M(a)"'] & N_i \arrow[d, "N(a)"] \\ M_j \arrow[r, "\varphi_j"] & N_j \end{tikzcd}$$

commutes for each arrow $a : i \to j \in Q_1$.

The **composition** of the morphism $\varphi : M \longrightarrow N$ with the morphism $\psi : M \longrightarrow P$ is given by $(\psi \circ \varphi)_i = \psi_i \circ \varphi_i$.

A **subrepresentation** of a representation $(M_i, M(a))_{i \in Q_0, a \in Q_1}$ is a representation $(N_i, N(a))_{i \in Q_0, a \in Q_1}$ with N_i is a vector subspace of M_i and $N(a)$ is the restriction of $M(a)$ to the vector space $N_{s(a)}$.

The **direct sum** of two representations $(M_i, M(a))$ and $(N_i, N(a))$ is the representation $(M_i \oplus N_i, M_i(a) \oplus N_i(a))$.

A representation $(M_i, M(a))$ is called **indecomposable** if it is not zero and can not be represented as a direct sum of two non-trivial representations.

3.3 Path algebras

A **path** of length l $\geqslant 1$ in a quiver Q has the form

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \cdots\cdots \bullet \xrightarrow{a_m} \bullet$$

with $t(a_{i+1}) = s(a_i)$ for $1 \leq$ i $\leq$ n. This path starts at $s(a_1)$ and ends at $t(a_m)$. For each vertex i we denote by e_i the path of length 0 which starts and ends at i. We also use the notation $s(x)$ and $t(x)$ to denote the starting and ending vertex of the path x.

The **path algebra** $\mathbb{C}$Q is the $\mathbb{C}$-algebra with basis the paths in Q, and with product of two paths

$$x = \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$$

$$y = \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$$

given by

$$xy = \overset{s(y)}{\bullet} \to \cdots \bullet \to \overset{s(x)=t(y)}{\bullet} \to \bullet \to \cdots \bullet \quad \text{if s(x) = t(y)}$$

$$xy = 0 \quad \text{if s(x)} \neq \text{t(y)}.$$

This is an associative multiplication.

Lemma 1 *The category* Rep(Q) *of representations of the quiver Q is equivalent to the category* $\mathbb{C}Q$-mod *of finite dimensional* $\mathbb{C}Q$*-modules.*

PROOF. If X is a $\mathbb{C}$Q-mod, we define a representation M with

$$M_i = e_i X$$

$$M(a) : M_{s(a)} \longrightarrow M_{t(a)}, \quad x \longmapsto ax = e_{t(a)}$$

If $M = (M_i, M(a))_{i \in Q_0, a \in Q_1}$ is a representation, then we define a module X via

$$X = \bigoplus_{i \in Q_0} M_i.$$

Next, the $\mathbb{C}$Q-mod structure on X is given by

$$(\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \cdots\cdots \bullet \xrightarrow{a_m} \bullet) \sum_{i \in Q_0} x_i = M(a_m) \circ \cdots \circ M(a_1)(x_{s(a_1)}). \qquad \square$$

REMARK. 1) $\mathbb{C}Q$ is finitely generated. The path algebra $\mathbb{C}Q$ is finite dimensional if and only if Q has no oriented cycles, i.e paths of length ≥ 1 from a vertex i to itself.

2) The e_i are orthogonal idempotents, i.e. $e_i e_j = 0$ for i $\neq$ j and $e_i^2 = e_i$.

3) If Q consists of one vertex and one loop, then $\mathbb{C}Q \simeq \mathbb{C}[T]$. If Q consists of one vertex and n loops, then $\mathbb{C}Q$ is the free associative algebra on n letters.

3.4 Dynkin and extended Dynkin quivers

Give a quiver $Q = (Q_0, Q_1, s, t)$. The **Ringel form** for Q is the bilinear on $\mathbb{Z}^{Q_0}$ defined by

$$< \alpha, \beta > = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{s(a)}.$$

The **Tits form** is the quadratic form $q(\alpha) = < \alpha, \alpha >$.

The corresponding symmetric bilinear form is

$$(\alpha, \beta) = < \alpha, \beta > + < \beta, \alpha >$$

We say q is **positive definite** if $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^{Q_0}$. The form q is called **semi-positive definite** if $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^{Q_0}$.

The **radical** of the Tit's form q is

$$\mathrm{Rad}(q) = \{\alpha \in \mathbb{Z}^{Q_0} \mid (\alpha, \epsilon_i) = 0, \forall \epsilon_i\},$$

where ϵ_i is the i-th coordinate vector.

We have a partial ordering on $\mathbb{Z}^{Q_0}$ given by $\alpha \leq \beta$ if $\beta - \alpha \in \mathbb{N}^{Q_0}$. We say that $\alpha \in \mathbb{Z}^{Q_0}$ is **sincere** if each component is non-zero.

The **underlying graph** of Q is the graph obtained by replacing an arrow in Q with a simple edge.

Lemma 2 *If Q is connected and $\beta \geq 0$ is a non-zero radical vector, then β is sincere and q is positive semi-definite. For $\alpha \in \mathbb{Z}^{Q_0}$ we have*

$$q(\alpha) = 0 \Longleftrightarrow \alpha \in \mathbb{Q}\beta \Longleftrightarrow \alpha \in \mathrm{Rad}(q).$$

PROOF. Let n_{ij} be the number of edges $i - j$ in the underlying graph of Q (loops count twice). The condition that β is in the radical translates as

$$(2 - n_{ii})\beta_i = \sum_{j \neq i} n_{ij}\beta_j.$$

If $\beta_i = 0$ then $\sum_{j \neq i} n_{ij}\beta_j = 0$. Since each term in this sum is ≥ 0 we have $\beta_j = 0$ whenever there is an edge $i - j$. Since Q is connected it follows that $\beta = 0$. This gives a contradiction. Thus, β is sincere.

We have

$$\begin{aligned}
\sum_{i<j} n_{ij} \tfrac{\beta_i\beta_j}{2}\left(\tfrac{\alpha_i}{\beta_i} - \tfrac{\alpha_j}{\beta_j}\right)^2 &= \sum_{i<j} n_{ij} \tfrac{\beta_j}{2\beta_i}\alpha_i^2 - \sum_{i<j} n_{ij}\alpha_i\alpha_j + \sum_{i<j} n_{ij} \tfrac{\beta_i}{2\beta_j}\alpha_j^2 \\
&= \sum_{i \neq j} n_{ij} \tfrac{\beta_j}{2\beta_i}\alpha_i^2 - \sum_{i<j} n_{ij}\alpha_i\alpha_j \\
&= \sum_i (2 - 2n_{ii})\beta_i \tfrac{1}{2\beta_i}\alpha_i^2 - \sum_{i<j} n_{ij}\alpha_i\alpha_j = q(\alpha).
\end{aligned}$$

It follows that q is positive semi-definite.

If $q(\alpha) = 0$ then $\frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j}$ whenever there is an edge $i - j$, and since Q is connected it follows that $\alpha \in \mathbb{Q}\beta$.

If $\alpha \in \mathbb{Q}\beta$ then $\alpha \in \mathrm{Rad}(q)$ since $\beta \in \mathrm{Rad}(q)$ by assumption.

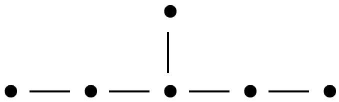
Finally if $\alpha \in \mathrm{Rad}(q)$ then certainly $q(\alpha) = 0$. □

We say that, Q is **Dynkin (resp. extended Dynkin) quiver** if its underlying graph is one of the Dynkin (resp. extended Dynkin) diagrams. By definition the Dynkin diagrams are

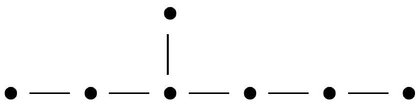
$\mathbb{A}_n$ $\bullet - \bullet - \bullet \cdots\cdots \bullet - \bullet - \bullet$

$\mathbb{D}_n$

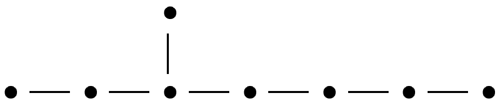
$\mathbb{E}_6$



$\mathbb{E}_7$

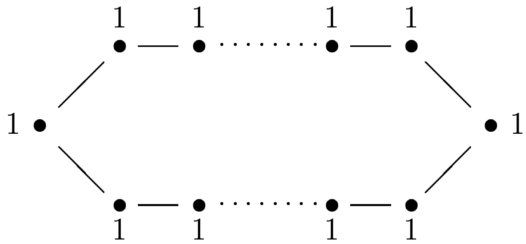


$\mathbb{E}_8$

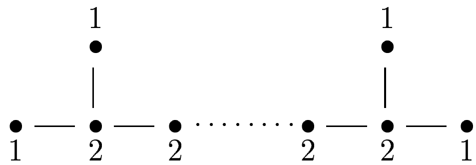


By definition the extended Dynkin diagrams are as below. Here we have marked each vertex i with the value of δ_i. Note that δ is sincere and $\delta \geq 0$.

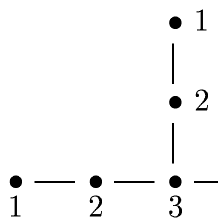
$\widetilde{\mathbb{A}_n}$



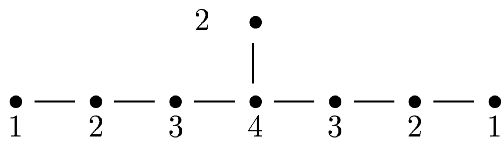
$\widetilde{\mathbb{D}_n}$



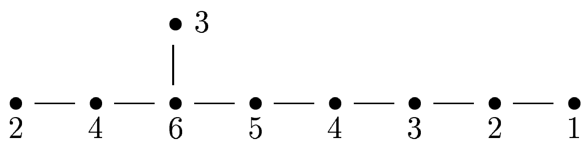
$\widetilde{\mathbb{E}_6}$



$\widetilde{\mathbb{E}_7}$



$\widetilde{\mathbb{E}_8}$



Note that $\widetilde{\mathbb{A}_0}$ has one vertex and one loop, and $\widetilde{\mathbb{A}_1}$ has two vertices joined by two edges.

Lemma 3 *Let Q be a connected quiver.*

1) If Q is an extended Dynkin quiver, then q is positive semi-definite and $\mathrm{Rad}(q) = \mathbb{Z}\delta$.
2) If Q is the Dynkin quiver then q is positive definite.
3) If Q is neither Dynkin nor extended Dynkin, then there is a vector $\alpha \geq 0$ with $q(\alpha) < 0$.

PROOF. 1) It is easy to check that the vector δ satisfy

$$2\delta_i \quad = \sum_{j-i} \delta_j$$

Thus the vector δ is radical. Note that δ is sincere and $\delta \geq 0$. Now q is positive semi-definite by the Lemma 2. Finally, since there always is a vertex i with δ_i $= 1$ we have $\mathrm{Rad}(q) = \mathbb{Z}\delta$.

2) Embed the Dynkin quiver in the corresponding extended Dynkin quiver $\widetilde{Q}$, and note that the quadratic form for $\widetilde{Q}$ is strictly positive on non-zero, non-sincere vectors.

3) If Q is neither Dynkin nor extended Dynkin then Q contains an extended Dynkin quiver Q'. If all vertices of Q are in Q' we take $\alpha = \delta$. Otherwise choose i be a vertex not in Q', connected to Q' by an edge. Then we take $\alpha = 2\delta_i + \epsilon_i$. $\square$

3.5 Roots

Given a quiver $Q = (Q_0, Q_1, s, t)$ we introduce the associated root system $\Delta(Q)$ as subset in $\mathbb{Z}^{Q_0}$ as follows. If k is a loopfree vertex, then there is a reflection

$$s_k : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_0}, s_k(\alpha) = \alpha - (\alpha, \epsilon_k)\epsilon_k.$$

The group $\mathrm{W} \subset \mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by all reflections is called the **Weyl group** of the quiver Q. For an element $\alpha = \sum k_i \epsilon_i \in \mathbb{Z}^{Q_0}$ we call the **height** of α the number $\sum\limits_{i \in Q_0} k_i$.

We call the support of α (write: supp α) the subquiver of Q consisting of those vertices i for which $k_i \neq 0$ and the arrows joining these vertices. Define the fundamental set $\mathrm{F} \subset \mathbb{Z}^{Q_0}$ by

$$F = \{\alpha \in \mathbb{N}^{Q_0} \mid \alpha \neq 0, \text{with connected support}, (\alpha, \epsilon_i) \leq 0, \forall i\}.$$

The **real roots** are the orbits of ϵ_k (k loopfree) under W. The **imaginary roots** are the orbits of $\pm\alpha$ ($\alpha \in F$) under W.

The real roots have $q(\alpha) = 1$. The imaginary roots have $q(\alpha) \leq 0$. Hence

$$\{\text{real roots}\} \cap \{\text{imaginary roots}\} = \varnothing$$

Then the **root system** $\Delta(Q)$ is defined by

$$\Delta(Q) = \{\text{real roots}\} \cup \{\text{imaginary roots}\}.$$

Lemma 4 *If Q is Dynkin or extended Dynkin quiver, then*

$$\Delta(Q) = \{\alpha \in \mathbb{Z}^{Q_0} \mid \alpha \neq 0, q(\alpha) \leq 1\}.$$

In particular, if Q is Dynkin quiver, then

$$\Delta(Q) = \{\alpha \in \mathbb{Z}^{Q_0} \mid q(\alpha) = 1\},$$

and if Q is extended Dynkin quiver, then

$$\{\textit{real roots}\} = \{\alpha \in \mathbb{Z}^{Q_0} \mid q(\alpha) = 1\}, \{\textit{imaginary roots}\} = (\mathbb{Z}^{Q_0} \setminus 0)\delta.$$

PROOF. Let $\alpha \in \mathbb{Z}^{Q_0} \setminus \{0\}$ be such that $q(\alpha) \leq 1$. We have to show that $\alpha \in \Delta(Q)$. Note that supp α is connected , for if in the contrary case $\alpha = \beta + \gamma$, where supp β and supp γ are unions of subgraphs of Dynkin type and $(\beta, \gamma) = 0$, but then $(\alpha, \alpha) \geq 2$.

Either α or $\alpha \in \mathbb{Z}_+^{Q_0}$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Indeed, in the contrary case, $\alpha = \beta - \gamma$, where γ, $\beta \in \mathbb{Z}_+^{Q_0}$, supp $\gamma \cap$ supp $\beta = \varnothing$, supp β is an union of subgraphs of type Dynkin or is a subgraph of extended Dynkin type. But $(\alpha, \alpha) = (\beta, \beta) + (\gamma, \gamma) - 2(\beta, \gamma) \leq 1$ and $(\beta, \gamma) \leq 0$. Hence the only possiblility is that $(\beta, \beta) = 1$, $(\gamma, \gamma) = 0$ and $(\beta, \gamma) = 0$. But then supp γ is a subgraph of extended Dynkin type and $(\beta, \gamma) < 0$, a contradiction.

So, supp α is connected and we can assume that $\alpha \in \mathbb{Z}_+^{Q_0}$. We can assume that $\mathrm{W}(\alpha)$ is not simple root. Then $\mathrm{W}(\alpha) \in \mathbb{Z}_+^{Q_0}$. Taking in $\mathrm{W}(\alpha)$ an element of minimal height, we can assume that $(\alpha, \alpha_i) \leq 0$ for simple α_i. Since supp α is connected, it follows that α lies in the fundamental domain. $\square$

3.6 Reflection functors

If k is a loopfree vertex of the quiver Q, define reflections

$$r_k : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_0}, r_k(\lambda)_j := \lambda_j - (\epsilon_k, \epsilon_j)\lambda_k.$$

dual to s_k (see Sect. 3.5) via $r_k(\lambda) \cdot \alpha = \lambda \cdot s_k(\alpha)$.

A vertex $k \in Q_0$ is called a **sink** (resp. a **source**) if there are no arrows starting (resp. terminating) at k.

Given a sink (or a source) $k \in Q_0$. Let Q' be the quiver obtained by reversing the direction of every arrow connected to k. We say that Q' is obtained from Q by reflecting at the vertex k. The two categories $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(Q')$ are closely related by means of so-called reflection functors.

We define the reflection functor

$$\mathcal{F}_k^+ : \mathrm{Rep}(Q) \longrightarrow \mathrm{Rep}(Q')$$

associated with a sink k as follows.

Let $M = (M_i, M(a))$ be a representation of Q.
If $j \neq k$ and $a : i \to j \in \mathrm{Q}_1$, then we take

$$\mathcal{F}_k^+(M)_j = M_j, \mathcal{F}_k^+(M)(a) = M(a)$$

If $j = k$ and $a : i \to k \in \mathrm{Q}_1$, then we take

$$\mathcal{F}_k^+(M)_k = \mathrm{Ker}(\bigoplus_{i \to k} M_i \xrightarrow{\alpha_k} M_k)$$

and

$$\mathcal{F}_k^+(M)_k \longrightarrow M_k \mathcal{F}_k^+(M)_i = M_i$$

be the composition

$$\mathcal{F}_k^+(M)_k \hookrightarrow \bigoplus_{i \to k} M_i \longrightarrow M_i.$$

Let $\varphi = (\varphi_i) : M \longrightarrow N$ be a morphism in $\mathrm{Rep}(Q)$. Then $\mathcal{F}_k^+(\varphi)$ is defined as follows:

If $j \neq k$, then we take $\mathcal{F}_k^+(\varphi)_j = \varphi_j$.

If $j = k$, then we take $\mathcal{F}_k^+(\varphi)_k$ be the restriction to $\mathcal{F}_k^+(M)_k$ of the mapping

$$\bigoplus_{i \to k} \varphi_i : \bigoplus_{i \to k} M_i \longrightarrow \bigoplus_{i \to k} N_i.$$

Dually, we can also define the reflection functor

$$\mathcal{F}_k^- : \mathrm{Rep}(Q) \longrightarrow \mathrm{Rep}(Q')$$

associated with a source k.

Note that the short exact sequence

$$0 \longrightarrow \mathcal{F}_k^+(M)_k \longrightarrow \bigoplus_{i \to k} M_i \xrightarrow{\alpha_k} M_k \longrightarrow 0$$

gives

$$\mathrm{dim}\mathcal{F}^+(M)_k = -\mathrm{dim}M_k + \sum_{i \to k} \mathrm{dim}M_i = -r_k(\underline{\mathrm{dim}}M)_k.$$

Hence $\quad \underline{\mathrm{dim}}\mathcal{F}_k^+(M) = r_k(\underline{\mathrm{dim}}M)$.

3.7 The variety of representations

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\alpha \in \mathbb{N}^{Q_0}$. We define

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(a)}}, \mathbb{C}^{\alpha_{t(a)}}).$$

This is an affine space and isomorphic to $\mathbb{A}^r$ where

$$r = \sum_{a \in Q_1} \alpha_{t(a)} \alpha_{s(a)}.$$

Identifying any complex vector space of dimension k with $\mathbb{C}^k$, we may be view $\mathrm{Rep}(Q, \alpha,)$ as the set of all representations of the quiver Q of given dimension vector α. We also say that, $\mathrm{Rep}(Q, \alpha)$ is the **variety of representations of Q of dimension vector** $\alpha \in \mathbb{N}^{Q_0}$**.**

We define $GL(\alpha) := \prod_{i \in Q_0} GL(\alpha_i, \mathbb{C})$. The linear algebraic group $GL(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ by conjugation. Explicitly

$$(g \cdot X(a))_{a \in Q_1} = (g_{t(a)} \cdot X(a) \cdot g_{s(a)}^{-1})_{a \in Q_1}$$

for $g \in$GL(α) and X $= (X(a))_{a \in Q_1} \in$ Rep(Q, α).

If $X, Y \in \mathrm{Rep}(Q, \alpha)$, then the set of isomorphisms of representation $X \longrightarrow Y$ can be identified with $\{g \in GL(\alpha) \mid g \cdot X = Y\}$. It follows that, there is a 1-1 correspondence between isomorphism classes of representation X with dimension vector α and $GL(\alpha)$-orbits O_X. In particular, the stabilizer $GL(\alpha)_X$ of X in $GL(\alpha)$ is identified with the set $\mathrm{Aut}_{\mathbb{C}Q}(X)$ of automorphisms of X.

Lemma 5

$$\begin{aligned} \dim \mathrm{Rep}(Q, \alpha) - \dim O_X &= \dim \mathrm{End}_{\mathbb{C}Q}(X) - q(\alpha) \\ &= \dim \mathrm{Ext}^1(X, X). \end{aligned}$$

PROOF. We have

$$\begin{aligned} \dim O_X &= \dim GL(\alpha) - GL(\alpha)_X \qquad \text{(Dimension formula)} \\ &= \dim GL(\alpha) - \dim \mathrm{Aut}_{\mathbb{C}Q}(X). \end{aligned}$$

Now dim $GL(\alpha)$ is non-empty and open in $\mathbb{A}^s$, so dense, so dim $GL(\alpha) = s$. Similarly $\mathrm{Aut}_{\mathbb{C}Q}$(X) is non-empty and open in $\mathrm{End}_{\mathbb{C}Q}(X)$, so dim $\mathrm{Aut}_{\mathbb{C}Q}(X)$ = dim $\mathrm{End}_{\mathbb{C}Q}(X)$.

It follows that

$$\begin{aligned} \dim \mathrm{Rep}(Q, \alpha) &= \sum_{a \in Q_1} \alpha_{t(a)} \alpha_{s(a)} - \sum_{i \in Q_0} \alpha_i^2 + \dim \mathrm{End} X \\ &= \dim \mathrm{End} X - q(\alpha) \\ &= \dim \mathrm{Ext}^1(X, X) \qquad \text{(Ringel's formula).} \end{aligned}$$

$\square$

4 Kleinian singularities and quiver varieties

4.1 McKay correspondence

Let Γ be a nontrivial finite subgroup of $GL(2, \mathbb{C})$. Let

$$\mathrm{Irr}(\Gamma) := \{R_0, R_1, R_2, \ldots, R_r\}$$

denote the set of the isomorphism classes of the complex irreducible representations of Γ, R_0 the trivial representation, and N is the natural two-dimensional representation on $\mathbb{C}^2$ obtained from the inclusion $\Gamma \subset GL(2, \mathbb{C})$. We decompose $R_j \otimes N$ into irreducible representations

$$R_j \otimes N = \bigoplus_{i=0}^{r} R_i \otimes \mathbb{C}^{a_{ij}}.$$

McKay's observation may be formulated in the following way:[26]

Let Γ be a finite subgroup of $SL(2, \mathbb{C})$. One attaches to Γ a quiver $\overline{Q}(\Gamma)$ by associating to each representation of Γ a vertex and connecting the i-th vertex with j-th vertex by a_{ij} arrows.

If in the quiver $\overline{Q}(\Gamma)$ each double arrow $\bullet \rightleftarrows \bullet$, i.e. two arrows in opposite direction, is replaced by a simple line $\bullet - \bullet$, and the dimension δ_i of the corresponding representation R_i are inserted, then the resulting graph $\Delta(\Gamma)$ is one of the extended Dynkin diagram $\widetilde{A_r}$,$\widetilde{D_r}$, $\widetilde{E_6}$, $\widetilde{E_7}$, $\widetilde{E_8}$, which occur respectively for cyclic, binary dihedral, binary tetrahedral, binary octahedral, binary icosahedral groups, with extended Cartan matrix $C = 2\mathbb{I}_{r+1} - (a_{ij})$.

Example. Let $\mathcal{C}_n$ be the cyclic group of order n which is generated by the matrix $a = \mathrm{diag}(\epsilon, \epsilon^{-1})$, where ϵ is the n-th primitive root of 1. In this case the group $\mathcal{C}_n$ is commutative. Hence its irreducible representations are one-dimensional and are determinded by

$$R_k(a^m) = \exp(\frac{2\pi i k m}{n}), \qquad k = 0, 1, \ldots, n-1.$$

The natural two-dimensional representation on $\mathbb{C}^2$ is isomorphic to $R_1 \oplus R_{n-1}$. Then $\mathbb{C}^2 \otimes R_k = R_{k+1} \oplus R_{k-1}$, for $k = 0, 1, \ldots, n-1$.

Henceforth, we shall call the graph $\Delta(\Gamma)$ as above the **McKay graph** and call the quiver $\overline{Q}(\Gamma)$ the **McKay quiver of type** $\Delta(\Gamma)$. For example, the McKay quiver of type $\widetilde{\mathbb{D}}_n$ is

$$\begin{tikzcd}
& \overset{1}{\bullet} \arrow[d, shift left] & & & & \overset{1}{\bullet} \arrow[d, shift left] & \\
\underset{1}{\bullet} \arrow[r, shift left] & \underset{2}{\bullet} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \underset{2}{\bullet} \arrow[l, shift left] & \cdots & \underset{2}{\bullet} \arrow[r, shift left] & \underset{2}{\bullet} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \underset{1}{\bullet}. \arrow[l, shift left]
\end{tikzcd}$$

4.2 Kronheimer's construction

A new approach to the deformation and resolution theory of Kleinian singularities was given by P. B. Kronheimer [12]. His construction starts directly from the finite group Γ of $SL(2, \mathbb{C})$ and uses hyper-Kähler quotient constructions. However he thought also about algebraic approach using McKay correspondence. Cassens and Slodowy used this idea to reformulate his results in terms of geometric invariant theory.

Let N denote the natural 2-dimensional and R the regular representation of Γ. Then consider a representation of Γ

$$M := \mathrm{End}(R) \otimes N$$

where $\mathrm{End}(R) = \mathrm{Hom}_{\mathbb{C}}(R, R)$. The group Γ acts on $\mathrm{End}(R)$ by conjugation and acts on N as obvious. We denote by

$$M(\Gamma) = (\mathrm{End}(R) \otimes N)^{\Gamma}$$

the set of invariant elements under the action of Γ.

Taking an orthonormal basis for N, we represent an element of $\mathrm{End}(R) \otimes N$ as a pair (α, β) of the endomorphisms $\alpha, \beta \in \mathrm{End}(R)$.

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathrm{I} + \mathbb{R}\mathrm{J} + \mathbb{R}\mathrm{K}$ be the quaternions. We introduce a $\mathbb{H}$-module structure on M by giving a action of $\mathbb{H}$ on M as follows.

$$\begin{aligned} I(\alpha, \beta) &= (i\alpha, i\beta) \\ J(\alpha, \beta) &= (-\alpha^*, \beta^*) \\ K(\alpha, \beta) &= (-i\alpha^*, i\beta^*) \end{aligned}$$

where z^* is the conjugation transpose to z.

Let $U(R)$ denote the unitary subgroup of End(R) and let

$$U(\Gamma) = \{g \in U(R) \mid g\gamma = \gamma g, \forall \gamma \in \Gamma\}.$$

The action of $U(\Gamma)$ on $M(\Gamma)$ is given by

$$(\alpha, \beta) \longmapsto (u\alpha u^{-1}, u\beta u^{-1}), u \in U(\Gamma).$$

If we decompose R into irreducible representations

$$R = \bigoplus_{i=0}^{r} R_i \otimes \mathbb{C}^{\delta_i}, \quad \delta_i = \dim R_i,$$

then we have $U(\Gamma) \simeq \prod_{i=0}^{r} U(\delta_i)$, where $U(\delta_i)$ is the unitary matrix group.

Since the subgroup $\mathbb{G}_m$ of scalar

$$\mathbb{G}_m = \{(\lambda_0 \mathbb{I}_{\delta_0}, \lambda_1 \mathbb{I}_{\delta_1}, \ldots, \lambda_r \mathbb{I}_{\delta_r}) \in \prod_{i=0}^{r} U(\delta_i) | \lambda_i \in U(1)\}$$

acts trivially we have an action of $U(\Gamma)/\mathbb{G}_m$ on $M(\Gamma)$ which preserves $\mathbb{H}$-module structure.

Let μ be the corresponding hyper-Kähler moment map for this action. Its explicit form is given by

$$\begin{aligned} \mu : M(\Gamma) &\longrightarrow \mathfrak{u} \otimes \mathbb{H}_0 \\ (\alpha, \beta) &\longmapsto \mu_I(\alpha, \beta) \otimes I + \mu_J(\alpha, \beta) \otimes J + \mu_K(\alpha, \beta) \otimes K \end{aligned}$$

where

$$\begin{aligned} \mu_I(\alpha, \beta) &= \tfrac{i}{2}([\alpha, \alpha^*] + [\beta, \beta^*]) \\ \mu_J(\alpha, \beta) &= \tfrac{1}{2}([\alpha, \beta] + [\alpha^*, \beta^*]) \\ \mu_K(\alpha, \beta) &= \tfrac{i}{2}(-[\alpha, \beta] + [\alpha^*, \beta^*]), \end{aligned}$$

and $\mathbb{H}_0 = \mathbb{R}\mathrm{I} + \mathbb{R}\mathrm{J} + \mathbb{R}\mathrm{K}$ denotes the pure quaternions, $\mathfrak{u}$ is the Liealgebra of $U(\Gamma)/\mathbb{G}_m$ which is identified with its dual space $\mathfrak{u}^*$ via the trace form (A,B) $\mapsto$ tr(AB). Let

$$\mathfrak{C} = \{(\mu_0, \mu_1, \ldots, \mu_n) \in \prod_{i=0}^{r} U(\delta_i) \mid \mu_i \in U(1), \sum_{i=0}^{r} \delta_i \mu_i = 0\}$$

denote the r-dimensional center of $\mathfrak{u}^*$. Taking scalars, we can consider $\mathfrak{C}$ as a subspace of $\mathbb{R}^r$.

Let

$$\begin{aligned} R_+ &= \{\theta \in \mathbb{Z}^{r+1} \mid \theta^t C \theta \leq 2\} \setminus \{0\} \\ D_\theta &= \{x = (x_k) \in \mathbb{R}^r \mid \sum_{k=1}^{r+1} x_k \theta_k = 0\}, \text{for } \theta \in R_+, \end{aligned}$$

where $C = 2\mathbb{I}_{r+1} - (a_{ij})$ is the extended Cartan matrix of type $\Delta(\Gamma)$ in McKay's observation.

If we choose an element

$$\zeta = \zeta_1 I + \zeta_2 J + \zeta_3 K \in \mathfrak{C} \otimes \mathbb{H}_0,$$

then $\mathrm{U}(\Gamma)$ acts on the fibre $\mu^{-1}(\zeta)$. The real differential-geometric quotient $\mu^{-1}(\zeta)/\mathrm{U}(\Gamma)$ is a **hyper-Kähler quotient**.

In general, $\mu^{-1}(\zeta)/\mathrm{U}(\Gamma)$ has singularities. By the general theory of hyper-Kähler quotient the set of smooth points of $\mu^{-1}(\zeta)/\mathrm{U}(\Gamma)$ is a **hyper-Kähler manifold**.

Kronheimer's work may be formulated as follows.

Theorem 6 [12] *For all $\zeta \in \mathfrak{C} \otimes \mathbb{H}_0$, the quotient $\mu^{-1}(\zeta)/U(\Gamma)$ is a complex analytic surface with at most isolated (Keinian) singularities. In particular*

$$\mu^{-1}(0)/U(\Gamma) \simeq \mathbb{C}^2/\Gamma.$$

The complex r-parameter family

$$\mathit{\Phi} : \mu^{-1}(\mathfrak{C} \otimes \mathbb{C})/U(\Gamma) \longrightarrow \mathfrak{C} \otimes \mathbb{C} \qquad (1)$$

realizes a semiuniversal deformation of $\mathbb{C}^2/\Gamma$.

For any fixed **generic** *$\zeta_1 \in \mathfrak{C}$, i.e. the ζ_1 not lying on any D_θ, one obtains a simultaneous resolution of the family (1) given by the following diagram*

$$\begin{tikzcd} \mu^{-1}(\zeta_1 I + \mathfrak{C} \otimes \mathbb{C})/U(\Gamma) \arrow[r] \arrow[d, "\mathit{\Phi}"'] & \mu^{-1}(\mathfrak{C} \otimes \mathbb{C})/U(\Gamma) \arrow[d] \\ \zeta_1 I + \mathfrak{C} \otimes \mathbb{C} \arrow[r, "\simeq"] & \mathfrak{C} \otimes \mathbb{C}. \end{tikzcd}$$

REMARK. Kronheimer used McKay's observation to interpret $M(\Gamma)$ in terms of representation of quivers as follows

$$\begin{aligned}
M(\Gamma) &= (\mathrm{End}(R) \otimes N)^{\Gamma} \\
&= (R^* \otimes R \otimes N)^{\Gamma} \\
&= \mathrm{Hom}(R, R \otimes N)^{\Gamma} \\
&= \mathrm{Hom}_{\Gamma}(R, R \otimes N) \\
&= \mathrm{Hom}_{\Gamma}(\bigoplus_{i=0}^{n} R_i \otimes \mathbb{C}^{\delta_i}, \bigoplus_{j=0}^{n} R_j \otimes \mathbb{C}^{\delta_i} \otimes N) \\
&= \bigoplus_{i,j=0}^{n} \mathrm{Hom}_{\Gamma}(R_i, R_j \otimes N) \otimes \mathrm{Hom}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j}) \\
&= \bigoplus_{i-j}^{n} \mathrm{Hom}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j}) \\
&= \mathrm{Rep}(\overline{Q}(\Gamma), \delta),
\end{aligned}$$

where $\overline{Q}(\Gamma)$ is the McKay quiver of type $\Delta(\Gamma)$.

Due to observation above, Kronheimer's original results are reformulated in terms of the geometric invariant theory by Cassens and Slodowy [7].

4.3 Notations and constructions

Let $\overline{Q}$ be the **double quiver** of $Q = (Q_0, Q_1, s, t)$, i.e. $\overline{Q}$ has the same vertices as Q but the arrows are given by $\{a, a^* \mid a \in Q_1\}$, where $s(a^*) = t(a)$ and $t(a^*) = s(a)$.

Recall that, representations of Q of dimension vector α are given by elements of the vector space

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(a)}}, \mathbb{C}^{\alpha_{t(a)}}).$$

Representation of $\overline{Q}$ of dimension vector α are given by elements of the vector space

$$\mathrm{Rep}(\overline{Q}, \alpha) = \mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q^{\mathrm{op}}, \alpha),$$

where Q^{op} is the opposite quiver to Q with an arrow $a^* : j \to i$ for each arrow $a : i \to j \in Q_1$.

Let $GL(\alpha)$ be the group $\prod_{a \in Q_0} GL(\alpha_i, \mathbb{C})$. This group acts on the space of representations $\mathrm{Rep}(\overline{Q},\ \alpha)$ by conjugation:

$$(g \cdot X(a))_{a \in \bar{Q}_1} = (g_{t(a)} \cdot X(a) \cdot g_{s(a)}^{-1})_{a \in \overline{Q}_1}.$$

Let $G(\alpha) = GL(\alpha)/\mathbb{G}_m$, where $\mathbb{G}_m$ is the diagonal scalar subgroup of $GL(\alpha)$. This group acts effectively on $\mathrm{Rep}(\overline{Q}, \alpha)$, i.e. $\mathbb{G}_m$ acts trivially.

The Lie algebra of $GL(\alpha)$ is given by

$$\mathrm{End}(\alpha) = \prod_{i \in Q_0} \mathrm{Mat}_{\mathbb{C}}(\alpha_i, \alpha_i).$$

We may identify $\mathrm{End}(\alpha)$ with its dual via the trace pairing:

$$\begin{array}{rl} \mathrm{End}(\alpha) & \longrightarrow (\mathrm{End}(\alpha))^* \\ (A_i)_i & \longmapsto [(B_i)_i \longmapsto \sum_{i \in Q_0} \mathrm{tr}(A_i B_i)]. \end{array}$$

Under the pairing the dual to $\mathrm{Lie}(G(\alpha))$ is identified with the trace zero matrices in $\mathrm{End}(\alpha)$:

$$\mathrm{Lie}(G(\alpha))^* \simeq \mathrm{End}(\alpha)_0 := \{A \in \mathrm{End}(\alpha) \mid \sum_{i \in Q_0} \mathrm{tr}(A_i) = 0\}.$$

The space of representations $\mathrm{Rep}(\overline{Q}, \alpha)$ can be identified with the cotangent bundle $\mathrm{T}^*\mathrm{Rep}(Q, \alpha)$ of $\mathrm{Rep}(Q, \alpha)$. It has a symplectic form given by

$$\omega(X, Y) = \frac{1}{2} \sum_{a \in Q_1} \mathrm{tr}(X(a^*)Y(a)) - \mathrm{tr}(X(a)Y(a^*)).$$

The group $G(\alpha)$ acts on $\mathrm{Rep}(\overline{Q}, \alpha)$ preserving ω with moment map

$$\mu_\alpha : \mathrm{Rep}(\overline{Q}, \alpha) \longrightarrow \mathrm{End}(\alpha)_0$$

given by

$$(X(a), X(a^*))_{a\in Q_1} \longmapsto (\sum_{\substack{a\in Q_1\\ s(a)=i}} X(a)X(a^*) - \sum_{\substack{a\in Q_1\\ t(a)=i}} X(a^*)X(a))_{i\in Q_0}.$$

The center Z_α of $\mathrm{Lie}(G(\alpha))^* = \mathrm{End}(\alpha)_0$ is

$$\begin{aligned} Z_\alpha &= \mathrm{End}(\alpha)_0^{G(\alpha)} = \{(\lambda_1\mathbb{I}_{\alpha_1}, \lambda_2\mathbb{I}_{\alpha_2}, \ldots, \lambda_n\mathbb{I}_{\alpha_n}) \in \mathrm{End}(\alpha)_0 \mid \sum_{i\in Q_0} \alpha_i\lambda_i = 0\} \\ &= \{\lambda \in \mathbb{C}^{Q_0} | \alpha \cdot \lambda := \sum_{i\in Q_0} \alpha_i\lambda_i = 0\}, \end{aligned}$$

where n $= \mid Q_0 \mid$.

The **preprojection algebra** $\Pi(Q)$ associated to a quiver Q is defined as

$$\Pi(Q) = \mathbb{C}\overline{Q}/(\sum_{\substack{a\in Q_1\\ s(a)=i}} aa^* - \sum_{\substack{b\in Q_1\\ t(b)=i}} b^*b)_{i\in Q_0}.$$

$\Pi(Q)$-modules correspond to representations $(X_i, X(a))$ of the quiver $\overline{Q}$ in which the linear maps satisfy the relations

$$\sum_{\substack{a\in Q_1\\ s(a)=i}} X(a)X(a^*) - \sum_{\substack{b\in Q_1\\ t(b)=i}} X(b^*)X(b) = 0, \forall i \in Q_0.$$

It is clearly that $\mu_\alpha^{-1}(0)$ is identified with the space of representations of $\Pi(Q)$ of dimension vector α

$$\mu_\alpha^{-1}(0) = \mathrm{Rep}(\Pi(Q), \alpha).$$

4.4 Deformation of the Kleinian singularities

Let $Q(\Gamma)$ be the McKay quiver of type $\Delta(\Gamma)$ and let δ be the vector with $\delta_i =$ dim R_i, where the R_i are the irreducible representations of the group Γ. The vector δ is also the minimal imaginary root for the corresponding extended Dynkin diagram (see Sect. 3.4).

For any $z \in Z_\delta = \{\lambda \in \mathbb{C}^{Q_0} \mid \lambda \cdot \delta = 0\}$ the reductive group $G(\delta)$ acts on the fiber $\mu_\delta^{-1}(z)$ since the $G(\delta)$-equivariance of μ_δ. By general geometric invariant theory [19] we can form an algebraic quotient $\mu_\delta^{-1}(Z_\delta)//\ G(\delta)$. Let Φ be the morphism

$$\Phi : \mu_\delta^{-1}(Z_\delta)//G(\delta) \longrightarrow Z_\delta$$

which is obtained from the universal property of the quotient

$$\begin{tikzcd}
\mu_\delta^{-1}(Z_\delta) \arrow[rr, "\rho"] \arrow[dr, "\mu_\delta"'] & & \mu_\delta^{-1}(Z_\delta)//\mathrm{G}(\delta) \arrow[dl, "\Phi"] \\
& Z_\delta. &
\end{tikzcd}$$

We going to show that the morphism Φ is a deformation of the Kleinian singularity.

Lemma 6 [1, Lemma 4.2] *Let Q be a quiver and let $X = (X(a))_{a\in Q} \in Rep(Q,\alpha)$ be a representation of dimension α. Then there is an exact sequence*

$$0 \to \mathrm{Ext}^1(X,X)^* \to \mathrm{Rep}(Q^{op},\alpha) \xrightarrow{c} \mathrm{End}(\alpha) \xrightarrow{t} \mathrm{End}(X)^* \to 0$$

where c sends $(X(a^*)) \in \mathrm{Rep}(Q^{op})$ to

$$\Big(\sum_{\substack{a\in Q_1\\ s(a)=i}} X(a)X(a^*) - \sum_{\substack{a\in Q_1\\ t(a)=i}} (X(a^*)X(a))_i$$

and t sends (A_i) to the linear map

$$\begin{array}{lll} \mathrm{End}(X) & \longrightarrow \mathbb{C} \\ (B_i)_i & \longmapsto \sum_{i\in Q_0} \mathrm{tr}(A_iB_i). \end{array}$$

PROOF. There is an exact sequence

$$0 \to \mathrm{End}(X) \to \mathrm{End}(\alpha) \xrightarrow{f} \mathrm{Rep}(Q,\alpha) \to \mathrm{Ext}^1(X,X) \to 0$$

where the map f sends (A_i) to $(A_{s(a)}X(a) - X(a)A_{t(a)})_{a\in Q}$.

Indeed, it is clear that the kernel of the map f is $\mathrm{End}(X)$, and by the Ringel formula

$$\begin{aligned} \dim\,\mathrm{End}(X) - \dim\,\mathrm{Ext}^1(X,X) &= <\alpha,\alpha> \\ &= \dim\,\mathrm{End}(\alpha) - \dim\,\mathrm{Rep}(Q,\alpha). \end{aligned}$$

the cokernel has the same dimension as dim Ext^1(X,X).

The required exact sequence is the dual of this one, using the trace pairing

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s) \times \mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r) &\longrightarrow \mathbb{C} \\ (\varphi, \psi) &\longmapsto \mathrm{tr}(\varphi\psi) \end{aligned}$$

to identify

$$\mathrm{Rep}(Q^{op}, \alpha) \simeq \mathrm{Rep}(Q, \alpha),\ \mathrm{End}(\alpha) \simeq \mathrm{End}(\alpha)^*. \qquad \square$$

Lemma 7 [1, Lemma 8.3] *For an extended Dynkin quiver Q and the minimal imaginary root δ the moment map*

$$\mu_\delta : \mathrm{Rep}(\overline{Q}, \delta) \longrightarrow \mathrm{End}(\delta)_0.$$

is surjective and every irreducible component of every fiber has dimension 1 + $\sum\limits_{i \in Q_0} \delta_i^2$. Thus, μ_δ is flat.

PROOF. One knows the following result (see [1], Lemma 4.4)

If Q is an extended Dynkin quiver and β is the imaginary root, then there is a representation of Q of dimension vector β whose endomorphism algebra is $\mathbb{C}$. Furthermore the representations with endomorphism algebra $\mathbb{C}$ form a dense open subset in Rep(Q, β),and its complement is the union of only finitely many GL(β)-orbits.

Suppose $X \in \mathrm{Rep}(Q, \delta)$ has endomorphism algebra $\mathrm{End}(X) = \mathbb{C}$. By the Ringel formula and $< \delta, \delta > = 0$ we have

$$\dim \mathrm{End}(X) = \dim \mathrm{Ext}^1(X, X) = 1.$$

Then the map (see Lemma 6)

$$c : \mathrm{Rep}(Q^{op}, \delta) \longrightarrow \mathrm{End}(\delta)$$

has 1-dimensional cokernel. Thus the map c has image $\mathrm{End}(\delta)_0$; for, in any case, its image is contained in this subspace.

For any $A \in \mathrm{End}(\delta)_0$ we have $c^{-1}(A) \neq \emptyset$. Let $X^* =(X(a^*))$ be an element of $c^{-1}(A)$. Then we have

$$\mu_\delta(X, X^*) = c(X^*) = A.$$

It follows that μ_δ is surjective.

Thus, every irreducible component of every fiber has dimension at least

$$\dim \operatorname{Rep}(\bar{Q}, \delta) - \dim \operatorname{End}(\delta)_0 = 1 + \sum_{i \in Q_0} \delta_i^2.$$

Now for any $A \in \operatorname{End}(\delta)_0$, we consider the projection

$$\pi : \mu_\delta^{-1}(A) \longrightarrow \operatorname{Rep}(Q, \delta).$$

Then the fiber $\pi_\delta^{-1}(A) \simeq c^{-1}(A)$.

The representations with endomorphism algebra $\mathbb{C}$ form a dense open set $B \subseteq \operatorname{Rep}(Q, \delta)$. The fibers of the map $\pi^{-1}(B) \to B$ all have dimension 1, so

$$\begin{aligned} \dim \pi^{-1}(B) &= 1 + \dim B & &= 1 + \dim \operatorname{Rep}(Q, \delta) \\ &= 1 + \sum_{a: i \to j \in Q_1} \delta_i \delta_j & &= 1 + \sum_{i \in Q_0} \delta_i^2. \end{aligned}$$

On the other hand, the complement $\operatorname{Rep}(Q, \alpha) \setminus B$ consists of a finite number of $G(\delta)$-orbits $O_1, O_2, \ldots, O_k$. If $X_i \in O_i$, then

$$\dim \pi^{-1}(O_i) = \dim O_i + \dim \operatorname{End}(X_i)$$

Since $\dim \operatorname{End}(X_i) = \dim \operatorname{Aut}(X_i)$ and the stabilizer of X is identified with $\operatorname{Aut}(X_i)$ we have

$$\dim \pi^{-1}(O_i) = \sum_{i \in Q_0} \delta_i^2.$$

Thus, $\dim \mu_\delta^{-1}(A) = 1 + \sum_{i \in Q_0} \alpha_i^2$. It follows that μ is flat. □

Lemma 8 [1, Lemma 8.6] *The morphism* $\Phi : \mu_\delta^{-1}(Z)//G(\delta) \longrightarrow Z_\delta$ *is flat.*

PROOF. By Lemma 7 the morphism $\mu_\delta^{-1}(Z_\delta) \longrightarrow Z_\delta$ is flat since it is the pullback of μ_δ.

Let $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)//G(\delta)] = \mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)}$ and $\mathbb{C}[Z_\delta]$ be the coordinate rings of $\mu_\delta^{-1}(Z_\delta)//G(\delta)$ and Z_δ.

Since the group $G(\delta)$ is linear reductive the ring $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)}$ is a direct summand of $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]$. Since $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]$ is flat over $\mathbb{C}[Z_\delta]$ it follows that the ring $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)}$ is also flat over $\mathbb{C}[Z_\delta]$. Equivalently, the morphism

$\Phi : \mu_\delta^{-1}(Z)//G(\delta) \longrightarrow Z_\delta$ is flat. $\square$

Lemma 9 *The algebraic quotient $\mu_\delta^{-1}(0)//G(\delta)$ is isomorphic to $\mathbb{C}^2/\Gamma$.*

PROOF. If $X = (X_i, X(a)) \in \mathrm{Rep}(Q, \delta)$ and $p = a_1 a_2 \ldots a_m$ is a path in $\mathbb{C}\overline{Q}$ that starts and ends at the same vertex, then the function tr_p given by

$$\mathrm{tr}_p(X) = \mathrm{tr}(X(a_m) \cdots X(a_2) X(a_1))$$

is invariant under the action of $G(\delta)$. By [4] the algebra $\mathbb{C}[\mathrm{Rep}(Q,\delta)]^{G(\delta)}$ is generated by the trace functions tr_p. By Cassen's calculation [7] for the extended Dynkin quivers $\widetilde{A}_n$,$\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ one obtains

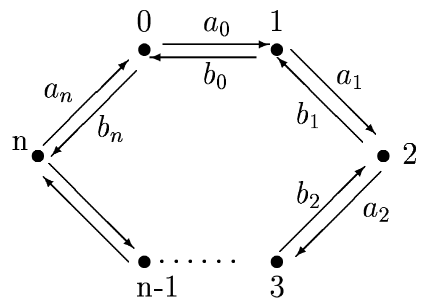
$$\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)} \simeq \mathbb{C}[u,v]^\Gamma.$$

$\square$

Theorem 7 *The flat morphism $\Phi : \mu_\delta^{-1}(Z_\delta)]//G(\delta) \longrightarrow Z_\delta$ is a deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$.*

PROOF. According to the definition of deformation (see Sect. 2.1) the result follows from Lemmas 8, 9. $\square$

Example. Let $\Gamma = \mathcal{C}_{n+1}$ be the cyclic group of order $n+1$ then the McKay quiver $\overline{Q}(\Gamma)$ of type $\widetilde{\mathbb{A}}_n$ has the following form :



Since $\delta = (1, 1, \ldots, 1)$ we have

$$\mathrm{Rep}(\overline{Q}(\Gamma), \delta) = \{(a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n) \in \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}\}$$

and

$$
\begin{aligned}
G(\delta) &= {\mathbb{C}^*}^{n+1}/\mathbb{C}^* \simeq {\mathbb{C}^*}^n\} \\
Z_\delta &= \mathrm{Lie}(G(\delta))^* = \{(\mu_0, \dots, \mu_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} \mu_i = 0\} \\
\mu_\delta^{-1}(Z_\delta) &= \mathrm{Rep}(\overline{Q}(\Gamma), \delta).
\end{aligned}
$$

The action of $G(\delta)$ on $\mathrm{Rep}(\overline{Q}(\Gamma), \delta)$ is given by

$$
(t_1 t_0^{-1} a_0, t_2 t_1^{-1} a_1, \dots, t_0 t_n^{-1} a_n, t_0 t_1^{-1} b_0, t_1 t_2^{-1} b_1, \cdots, t_n t_0^{-1} b_n)
$$

The fundamental invariants for this action are

$$
\begin{aligned}
z_i &= a_i b_i, \qquad i = 0, 1, 2, \dots, n \\
x &= a_0 a_1 \cdots a_n \\
y &= b_0 b_1 \cdots b_n.
\end{aligned}
$$

These invariants satisfy relations

$$
xy = z_0 z_1 \cdots z_n.
$$

Thus $\mathrm{Rep}(Q(\Gamma), \delta)//GL(\delta)$ is given as the hypersurface

$$
\{(z_0, z_1, \dots, z_n, x, y) \in \mathbb{C}^{n+3} \mid xy = z_0 z_1 \cdots z_n\}
$$

If we introduce new coordinates on $\mathrm{Rep}(Q(\Gamma), \delta)//GL(\delta)$ by putting

$$
z = \frac{1}{n+1} \sum_{i=0}^{n} z_i; \qquad t_i = z - z_i,
$$

then we obtain the standard form of the semiuniversal deformation of the Kleinian singularity $\mathrm{S} = \{xy = z^{n+1}\}$ of type $\mathbb{A}_n$ (see Theorem 3, Sect. 2.1).

$$
\begin{array}{ccl}
\mu_\delta^{-1}(Z)//G(\delta) & = & \{(x, y, z, t_0, \dots, t_n) \in \mathbb{C}^3 \times Z \mid xy = \prod_{i=0}^{n} (z - t_i)\} \\
\big\downarrow & & \\
\mathrm{Z} & = & \{(t_0, t_1, \dots, t_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} t_i = 0\}.
\end{array}
$$

4.5 Simultaneous resolution

Let V denote an affine algebraic variety over $\mathbb{C}$. We will consider two types of quotient of V by a reductive algebraic group G.

Let $\mathbb{C}[V]$ be the coordinate ring of the affine algebraic variety V. Then $V//G$ is defined as the variety whose coordinate ring is the invariant part of $\mathbb{C}[V]$:

$$V//G := \mathrm{Specm}(\mathbb{C}[V])^G.$$

By the geometric invariant theory [19], this is an affine algebraic variety. It is also known that the geometric points of $V//G$ are closed G-orbits.

Definition. A point $v \in V$ is called **semistable** if there exists $f \in \mathbb{C}[V]^G$ such that $f(v) \neq 0$.

A point $v \in V$ is called **stable** if there exists $f \in \mathbb{C}[V]^G$ such that $f(v) \neq 0$, the isotropy group $G_v = \{g \in G | gv = v\}$ is finite, and the orbit $G \cdot v = \{gv, v \in V\}$ is closed in the affine open subset V_f =$\{w \in V | f(w) \neq 0\}$.

A point $v \in V$ is called **unstable** if it is not stable. By the geometric invariant theory, in the case V is a G-module, $v \in V$ is unstable if and only if $0 \in \overline{G \cdot v}$ (the closure of the orbit of v).

Let $\chi : G \longrightarrow \mathbb{C}^*$ denote a multiplicaticative character of G (an element of additively written group $X^*(G) = \mathrm{Hom}(G, \mathbb{C}^*)$. Then we consider the graded ring

$$\mathbb{C}[V]^{G,\chi} := \bigoplus_{i=0}^{\infty} \mathbb{C}[V]_{\chi^m}$$

where $\mathbb{C}[V]_{m_\chi}$ denotes the subspace of G-semi-invariants with respect to χ^m

$$\mathbb{C}[V]_{\chi^m} := \{f \in \mathbb{C}[V] \mid f(gv) = \chi(g)^m f(v), \forall g \in G, \forall v \in V\}.$$

Let

$$V//^\chi G := \mathrm{Projm}\ \mathbb{C}[V]^{G,\chi}$$

be the homogeneous maximal spectrum of $\mathbb{C}[V]^{G,\chi}$.

Definition. The natural projective morphism

$$V//^\chi G \longrightarrow V//G.$$

is called the χ**-linear modification** of V//G.

A point $v \in V$ is called χ**-semistable** if there exists $f \in \mathbb{C}[V]_{m\chi}$ with $m > 0$ such that $f(v) \neq 0$. Let us denote by V$^{\chi-ss}$ the set of χ- semistable points.

A point $v \in V$ is called χ**-stable** if there exists $f \in \mathbb{C}[V]_{m\chi}$ with $m > 0$ such that $f(v) \neq 0$; the isotropy group $G_v = \{g \in G | gv = v\}$ is finite; and the orbit

$G\cdot v = \{gv, v \in V\}$ is closed in the affine open subset $V_f = \{w \in V \mid f(w) \neq 0\}$. We write $V^{\chi-s}$ for the set of χ-stable points.

We lift the G-action to the trivial line bundle $L = V \times \mathbb{C}$ by

$$g \cdot (x, z) := (g \cdot z, \chi(g)z).$$

Then the set of G-invariant section of $L^{\otimes m}$ can be identified with $\mathbb{C}[V]_{m_\chi}$. Thus, the set $V^{\chi-ss}$ is the set of the semi-stable points in the sense of Mumford [19]. Hence, there exist for the action on $V^{\chi-ss}$ a algebraic quotient $V^{\chi-ss}//G$ that is a quasi-projective variety (see [19], theorem 1.10). In this way, $V//^{\chi}$G can be described as the algebraic quotient $V^{\chi-ss}$ //G.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and let $G(\alpha) = \prod_{a\in Q_0} GL(\alpha_i, \mathbb{C})/\mathbb{G}_m$ be the algebraic group acting on the space of representations $\operatorname{Rep}(Q, \alpha)$ by conjugation. We note that the characters of $G(\alpha)$ are given by

$$\begin{array}{rcl} \chi_\theta : G(\alpha) & \longrightarrow & \mathbb{C}^* \\ (g_i)_{i\in Q_0} & \longmapsto & \prod_{i\in Q_0} [\det(g_i)]^{\theta_i}, \end{array}$$

for $\theta \in \mathbb{Z}^{Q_0}$ with $\sum_{i\in Q_0} \theta_i\alpha_i = 0$. Such a vector θ is considered as function

$$\begin{array}{rcl} \theta : \mathbb{Z}^{Q_0} & \longrightarrow & \mathbb{Z} \\ \beta & \longmapsto & \theta(\beta) := \sum_{i\in Q_0} \theta_i\beta_i. \end{array}$$

Using the Hilbert-Mumford criterion [19] the notion of χ_θ-stability (resp. χ_θ-semistability) can be translated into language of the representation of the quiver.

Lemma 10 ([11], 3.2) *Let $X \in \operatorname{Rep}(Q, \alpha)$ be a representation of quiver Q of dimension δ. Then X is χ_θ-stable (resp. χ_θ-semistable) with respect to the action of $G(\alpha)$ on $\operatorname{Rep}(Q, \alpha)$ if and only if*

$$\theta(\underline{\dim}N) < 0 \quad (\textit{resp.} \quad \theta(\underline{\dim}N) \leq 0)$$

for all non-trivial proper subrepresentations N of X.

Definition. We say that a character χ_θ is **generic** if $\theta(\delta) = 0$ but $\theta(\beta) \neq 0$ for all $0 < \beta < \alpha$.

Lemma 11 *If the character χ_θ is generic then the notions χ_θ-stability and χ_θ-semistability coincide.*

PROOF. Let $X \in \mathrm{Rep}(Q, \alpha)$ be χ_θ-semistable but not χ_θ-stable. Then there is a proper subrepresentation X' of X such that $\theta(\underline{\dim}\ \mathrm{X'}) = 0$. This gives the contradiction to the definition of genericity for χ_θ. $\square$

Let $\overline{Q}(\Gamma)$ be the McKay quiver of type $\Delta(\Gamma)$ and δ be the minimal imaginary root as before (see Sect. 4.4). For any character χ_θ we have the following diagram by applying a linear modification to Kronheimer's construstion:

$$\begin{tikzcd}
\mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta) \arrow[rr, "\pi_\chi"] \arrow[d, "\Phi_\chi"'] & & \mu_\delta^{-1}(Z_\delta)//G(\delta) \arrow[d, "\Phi"] \\
Z_\delta & = & Z_\delta
\end{tikzcd}$$

Theorem 8 [17] *For generic χ_θ diagram above is a simultaneous resolution of the morphism Φ. In particular, $\mu_\delta^{-1}(0)//^{\chi_\theta}G(\delta)$ is a minimal resolution of Kleinian singularity $\mathbb{C}^2/\Gamma$.*

PROOF. 1) Since the character χ_θ is generic, we have by Lemma 11

$$\begin{aligned}
\mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta) &= \mu_\delta^{-1}(Z_\delta)^{\chi-ss}//G(\delta)) \\
&= \mu_\delta^{-1}(Z_\delta)^{\chi-s}//G(\delta)).
\end{aligned}$$

We note that if $X \in \mu_\delta^{-1}(Z_\delta)^{\chi-s}$ then $\mathrm{End}(X) = \mathbb{C}$. Thus $G(\delta)$ acts on $\mu_\delta^{-1}(Z_\delta)^{\chi-s}$ with trivial isotropy group, i.e. it acts freely. By ([17], Lemma 3) μ_δ is smooth in X. Thus retriction of μ_δ to $\mu_\delta^{-1}(Z_\delta)^{\chi-s}$ is smooth. Since Z_δ is smooth it follows that $\mu_\delta^{-1}(Z_\delta)^{\chi-s}$ is also smooth. Therefore $\mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta)$ is smooth. Finally, the smoothness of the map

$$\mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta) \longrightarrow Z_\delta$$

follows from the following commutative diagram

$$\begin{tikzcd}
\mu_\delta^{-1}(Z_\delta)^{\chi-s} \arrow[rr] \arrow[dr] & & \mu_\delta^{-1}(Z_\delta)//^{\chi_\theta}G(\delta) \arrow[dl] \\
& Z_\delta. &
\end{tikzcd}$$

2) All the fibers $\mu_\delta^{-1}(z)//^{\chi_\theta}G(\delta)$ are minimal resolution of the singular fibers $\mu_\delta^{-1}(z)//G(\delta)$. This follows from an interpretation of the $\mu_\delta^{-1}(z)//^{\chi_\theta}G(\delta)$ as

symplectic quotient of the symplectic manifold $\mu_\delta^{-1}(Z_\delta)^{\chi-s}$ by the freely acting group $G(\delta)$. We will show that the resolution is minimal, that is the canonical bundle of $\mu_\delta^{-1}(z)//^{\chi_\theta}G(\delta)$ is trivial. By [39], Lemma 5.4, the tangent space of $\mu_\delta^{-1}(z)//^{\chi_\theta}G(\delta)$ has a symplectic structure. This implies that the canonical bundle is trivial, since the cotangent bundle is isomorphic to the tangent bundle via the symplectic structure. $\square$

REMARK. The strategy for above proof does not give an analysis of exceptional component. In [5] Cassens has tried to describe the exceptional set of the resolution $\mu_\delta^{-1}(z)//^{\chi}G(\delta) \longrightarrow \mu_\delta^{-1}(z)//G(\delta)$. Unfortunately, it was not clear.

Example. Let $\Gamma = \mathcal{C}_{n+1}$ be the cyclic group of order $n+1$ then the McKay quiver $\overline{Q}(\Gamma)$ is of type $\widetilde{\mathbb{A}}_n$ (see the example in Sect. 4.4). Let us choose a character of $G(\delta)$ as follows

$$\begin{array}{rcl} \chi : G(\delta) & \longrightarrow & \mathbb{C}^* \\ (t_0, t_1, \cdots, t_n) & \longmapsto & t_0^{-n} t_1 \cdots t_n. \end{array}$$

Let

$$\begin{array}{ll} X_i & = a_0 a_1 \cdots a_i \\ Y_i & = b_{i+1} b_{i+2} \cdots b_n \qquad i = 0, 1, 2, \ldots, n-1. \end{array}$$

Then X_i and Y_i satisfy the following relations

$$\begin{array}{ll} z_0 z_1 \cdots z_i Y_i & = X_i y \\ z_{i+1} z_{i+2} \cdots z_n X_i & = Y_i x \quad i = 0, 1, \ldots, n-1 \\ X_j Y_i & = X_i z_{i+1} \cdots z_j Y_j, \qquad (i < j). \end{array}$$

The action $G(\delta)$ on the space

$$\{X_0, X_1, \ldots X_{n-1}, Y_0, Y_1, \ldots Y_{n-1}\}$$

is given by

$$(t_1 t_0^{-1} X_0, t_2 t_0^{-1} X_1, \ldots, t_n t_0^{-1} X_{n-1}, t_1 t_0^{-1} Y_0, t_2 t_0^{-1} Y_1, \ldots, t_n t_0^{-1} Y_{n-1})$$

There are 2^n χ-semi-invariants which have the form $X_A Y_{\overline{A}}$
where

$$A \subseteq \{0, 1, \ldots, n-1\}, \overline{A} = \{0, 1, \ldots, n-1\} \setminus A$$

and

$$X_A = X_{i_0} \dots X_{i_k}, Y_A = Y_{i_0} \dots Y_{i_k}, \text{if } A = \{i_0, \dots, i_k\}$$

There are only the following (n+1) χ-semi-invariants which are independent from $\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)}$

$$\begin{array}{ll} f_0 & = X_0 X_1 \cdots X_{n-1} \\ f_1 & = X_0 X_1 \cdots X_{n-2} Y_{n-1} \\ \vdots & \\ f_{n-j-2} & = X_0 \cdots X_{j+1} Y_{j+2} \cdots Y_{n-1} \\ \vdots & \\ f_n & = Y_0 Y_1 \cdots Y_{n-1}. \end{array}$$

We have

$$\begin{array}{ll} \mu_\delta^{-1}(Z_\delta)//^\chi G(\delta) & = \mathrm{Projm}(\mathbb{C}[\mu_\delta^{-1}(Z_\delta)]^{G(\delta)}[f_0, f_1, \dots, f_n]) \\ & \subset \mu_\delta^{-1}(Z_\delta)//G(\delta) \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1. \end{array}$$

Let $U_0, U_1, \dots, U_n$ be the open subsets of $\mathbb{C}^3 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ defined by

$$\begin{array}{ll} U_0 & = \{Y_0 \neq 0\} \\ U_\rho & = \{X_{\rho-1} \neq 0, Y_\rho \neq 0\}, \rho = 1, \dots, n-1 \\ U_n & = \{X_{n-1} \neq 0\} \end{array}$$

In U_ρ, we let

$$\sigma_\rho = \frac{Y_{\rho-1}}{X_{\rho-1}} \quad \text{and} \quad \tau_{\rho+1} = \frac{X_\rho}{Y_\rho}$$

We can take a parameterization of $\mu_\delta^{-1}(Z_\delta)//^\chi \mathrm{G}(\delta) \cap \mathrm{U}_\rho$ as follows

$$\begin{array}{lll} (Y_i : X_i) & = (z_{i+1} \cdots z_{\rho-1}\sigma_\rho : 1) & \text{for} \quad 0 \leq i < \rho - 1 \\ (Y_i : X_i) & = (1 : z_{\rho+1} \cdots z_i \tau_{\rho+1}) & \text{for} \quad \rho < i \leq n-1 \\ (Y_{\rho-1} : X_{\rho-1}) & = (\sigma_\rho : 1) & \\ (Y_\rho : X_\rho) & = (1 : \tau_{\rho+1}) & \\ x & = \tau_{\rho+1} z_{\rho+1} \cdots z_n & \\ y & = \sigma_\rho z_0 \cdots z_{\rho-1} & \\ z_\rho & = \sigma_\rho \tau_{\rho+1} & . \end{array}$$

By parameter change $z = \frac{1}{n+1} \sum_{i=0}^n z_i$ and $t_i = z - z_i$ we have

$$
\begin{aligned}
(Y_i : X_i) &= \sigma_\rho \textstyle\prod_{\nu=i+1}^{\rho-1}(z - t_\nu : 1) \qquad \text{for} \quad 0 \le i < \rho - 1 \\
(Y_i : X_i) &= (1 : \tau_{\rho+1} \textstyle\prod_{\nu=\rho+1}^{i}(z - t_\nu) \qquad \text{for} \quad \rho < i \le n - 1 \\
(Y_{\rho-1} : X_{\rho-1}) &= (\sigma_\rho : 1) \\
(Y_\rho : X_\rho) &= (1 : \tau_{\rho+1}) \\
x &= \tau_{\rho+1} \textstyle\prod_{\nu=\rho+1}^{n}(z - t_\nu) \\
y &= \sigma_\rho \textstyle\prod_{\nu=0}^{\rho-1}(z - t_\nu) \\
z - t_\rho &= \sigma_\rho \tau_{\rho+1}.
\end{aligned}
$$

Finally, put $x = i\bar{x} + \bar{y}$ and $y = i\bar{x} - \bar{y}$ then we obtain all equations in Kas's construction (see the example in Sect. 2.3).

5 Nilpotent and stable representations of quivers

5.1 Nilpotent representations

In this section we use the notations as in Sect. 4.3.

Definition. An element $X = (X_i, X(a)) \in \mathrm{Rep}(Q, \alpha)$ is called to be **nilpotent** if there exists an integer $N \geq 2$ such that the composition

$$X(a_{i_N})X(a_{i_{N-1}}) \cdots X(a_{i_2})X(a_{i_1})$$

is equal to zero for any sequence of arrows $a_{i_1}, a_{i_2}, \ldots, a_{i_N} \in Q_1$ such that $t(a_{i_1}) = s(a_{i_2}), t(a_{i_2}) = s(a_{i_3}), \ldots, t(a_{i_{N-1}}) = s(a_{i_N})$.

Let us define a subvariety $\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$ of $\mathrm{Rep}(\Pi(Q), \alpha)$ by

$$\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}} := \{X \in \mathrm{Rep}(\Pi(Q), \alpha) \mid X \text{ is nilpotent}\}.$$

This variety is introduced by Lusztig [13].

Theorem 9 ([13], Theorem 8.7)

$\mathrm{Rep}(\Pi(Q), \alpha)_{nil}$ *has pure dimension equal to* $\frac{1}{2}\mathrm{dim}\mathrm{Rep}(\overline{Q}, \alpha)$.

Let Ω denote an orientation of the quiver $\overline{Q}$ (i.e. a subset $\Omega \subset \overline{Q}_1$ such that $\Omega^{op} \cup \Omega = \overline{Q}_1$, $\Omega^{op} \cap \Omega = \varnothing$, where Ω^{op} consists of reversed arrows $a* : j \to i$ for all arrows $a : i \to j$ in Ω). Let

$$C_\Omega = \{(X(a))_{a \in \overline{Q}_1} \in \mathrm{Rep}(\overline{Q}, \alpha) \mid X(a) = 0 \text{ for all } a \notin \Omega\}$$

If Q is a Dynkin quiver or an extended Dynkin quiver and Ω is an orientation of $\overline{Q}$ (non-cyclic for the case $\widetilde{\mathbb{A}}_n$), then C_Ω is an irreducible component of $\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$. In the case of type $\widetilde{\mathbb{A}}_n$, all irreducible components of $\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$ are of this form. However, in all other cases there are more components of $\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$ which are different from the C_Ω. Hille [8] gave a formula determining the number of irreducible components of $\mathrm{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$. In particular, this formula allows us to determine explicitly the number of irreducible components of $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)_{\mathrm{nil}}$ for the McKay quivers $\overline{Q}(\Gamma)$ of type $\Delta(\Gamma)$ except the case $\Delta(\Gamma) = \widetilde{\mathbb{E}}_8$.

5.1.1 The standard diagram for the Lusztig's nilpotent variety

We consider the following diagram [8]

$$\begin{array}{ccc} \times_{w}\operatorname{Rep}(\widetilde{\mathbb{A}}_0, \alpha_{s(w)}) & \xleftarrow{\pi^c} \operatorname{Rep}(\Pi(Q), \alpha) \xrightarrow{\pi^a} & \times_{u\in Q_1} \operatorname{Rep}(\widetilde{\mathbb{A}}_1, (\alpha_{s(u)}, \alpha_{t(u)})) \\ & \downarrow \pi^p & \\ & \times_{q\in Q_0} \operatorname{Rep}(F_{n(q)-1}, \alpha_q)), & \end{array}$$

where the first product is over all cycles in $\overline{Q}$.

Let $w = a_{i_1}a_{i_2}\cdots a_{i_r}$ is a such cycle, then we define

$$\pi^c(X) = (\pi^c_w(X))_w := (X(a_{i_r})\cdots X(a_{i_2})X(a_{i_1}))_w.$$

We define

$$\pi^a(X) = (\pi^a_u(X))_{u\in Q_1},$$

where the map π^a_u sends a representation X to its restriction on the quiver $\widetilde{\mathbb{A}}_1$ with arrows u and u^*.

Finally, the map π^p sends the representation X to the representation of the free associative algebra with $n(q)-1$ generators, where

$$n(q) := \sharp\{u \in \overline{Q}_1 \mid s(u) = q\}.$$

The representation X is maped to the vector space X_q with the linear map $X(u^*)X(u)$ (for $u \in \overline{Q}_1$ and $(u^*)^* = u$). Because the linear map $X(u^*)X(u)$ statisfies exactly one linear relation (the relation of preprojection algebra $\Pi(Q)$, see Sect. 4.3) $\pi^p(X)$ is a representation of $F_{n(q)-1}$.

If we restrict the maps π^a, π^c, π^p to Lusztig's nilpotent variety $\operatorname{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}}$, then the images are again nilpotent. So we obtain the diagram

$$\begin{array}{ccc} \times_{w}\operatorname{Rep}(\widetilde{\mathbb{A}}_0, \alpha_{s(w)})_{\mathrm{nil}} & \xleftarrow{\pi^c} \operatorname{Rep}(\Pi(Q), \alpha)_{\mathrm{nil}} \xrightarrow{\pi^a} & \times_{u\in Q_1} \operatorname{Rep}(\widetilde{\mathbb{A}}_1, (\alpha_{s(u)}, \alpha_{t(u)}))_{\mathrm{nil}} \\ & \downarrow \pi^p & \\ & \times_{q\in Q_0} \operatorname{Rep}(F_{n(q)-1}, \alpha_q))_{\mathrm{nil}}. & \end{array}$$

5.1.2 Nilpotent classes

Let n be a natural number. We write a **partition** of n in non-increasing order, that is a sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s), \quad \sum_{i=1}^{s} \lambda_i = n,$$

where the λ_i non-increasing

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0.$$

If $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of n, then we write $\lambda \vdash n$.

Each partition $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n corresponds to a conjugacy class of nilpotent $n \times n$-matrices. If A is such a matrix, then A is equivalent to a matrix in Jordan normal form, where the Jordan blocks have lengths $\lambda_1, \lambda_2, \ldots, \lambda_s$. Then the **zero-partition** is the partition $\lambda = (1, 1, \ldots, 1)$, it corresponds to the zero-matrix . The **regular partition** is the partition $\lambda = (n)$, it corresponds to the regular nilpotent class, and the **subregular partition** is the partition $\lambda = (n-1, 1)$, it corresponds to the subregular nilpotent class. We denote $C(\lambda)$ the class of nilpotent matrices corresponding to the partition λ.

Each partition $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n corresponds to a Young diagram, that is an array of n boxes, arranged in s rows with λ_i boxes in the i-th row, so that the left most boxes of each rown form a column.

Definition. Let λ, μ be two partitions. Then we define

$$NA(\lambda, \mu) := \begin{cases} 0 & \text{if} \quad \mid \lambda_i - \mu_i \mid > 1 \quad \text{for some i} \\ \prod_{a=1}^{\infty} (\sharp\{i \mid \lambda_i = a = \mu_i\} + 1) & \text{else.} \end{cases}$$

Example. Let

$$\begin{array}{lll} \lambda = (4,3,2,2,2,2,1) & \mu_1 = (4,2,2,2,1,1,1,1) & \\ \mu_2 = (4,3,2,2,1,1,1) & \alpha = (2,2,2,1,1) & \beta = (4,3,2). \end{array}$$

Then $NA(\lambda, \mu_1) = 2.3 = 6$, $NA(\lambda, \mu_2) = 2.3.2.2 = 24$, $NA(\lambda, \alpha) = 0$ (because $\mid \lambda_1 - \alpha_1 \mid = 2$), $NA(\lambda, \beta) = 0$ (because $\mid \lambda_4 - \beta_4 \mid = 2$).

Let λ^i be partitions of a fixed natural number d. Then we define the following variety

$$\mathrm{Rep}(F_n; \lambda^0, \ldots, \lambda^n) := \{(A_0, \ldots, A_n) \mid \exists \mathfrak{b}_{\mathfrak{u}} : A_i \in \mathfrak{b}_{\mathfrak{u}} \cap C(\lambda^i), \sum_{i=0}^{n} A_i = 0\}.$$

This variety consists of all $(n+1)$-tuples of nilpotent matrices which are simultaneously triagonalizable, whose sum is zero and lying in given nilpotent classes. Note that the relation corresponds to the relation $\sum a^*a = \sum aa^*$ in the preprojective algebra.

Definition. For $n+1$ partitions $\lambda^0, \cdots, \lambda^n$ of the natural number d we define $NP(\lambda^0, \cdots, \lambda^n)$ as the number of irreducible components of $\mathrm{Rep}(F_n; \lambda^0, \cdots, \lambda^n)$ of dimension $\frac{1}{2}\sum_{i=0}^n \dim\, C(\lambda^i)$.

In general it is not easy to count the irreducible components of the variety $\mathrm{Rep}(F_n; \lambda^0, \cdots, \lambda^n)$. However in some special cases the number NP are easy to caculate.

Lemma 12 *Let d be a natural number and let $\lambda^0, \ldots, \lambda^n)$ be partitions of d. Then*

1)

$$NP(\lambda^0) = \begin{cases} 0 & \textit{for} \quad \lambda^0 \neq (1, 1, \ldots, 1) \\ 1 & \textit{for} \quad \lambda^0 = (1, 1, \ldots, 1) \end{cases}$$

2)

$$NP(\lambda^0, \lambda^1) = \begin{cases} 0 & \textit{for} \quad \lambda^0 \neq \lambda^1 \\ 1 & \textit{for} \quad \lambda^0 = \lambda^1 \end{cases}$$

3)

$$NP((1, 1, \ldots, 1)\lambda^1, \ldots, \lambda^n) = NP(\lambda^1, \ldots, \lambda^n).$$

PROOF. 1) Since the equality $\sum\limits_{i=0}^n A_i = 0$ it follows $A_0 = 0$. Thus λ^0 must be $(1, 1, \ldots, 1)$. 2) Since $A_0 + A_1 = 0$ we have to $\lambda^0 = \lambda^1$. 3) Since $A_0 = 0$ the equality $\sum\limits_{i=0}^n A_i = 0$ is equvalent to $\sum\limits_{i=1}^n A_i = 0$. The result follows. $\square$

5.1.3 The irreducible components of the nilpotent variety

We say that a representation of $\mathrm{Rep}(\Pi(Q), \alpha)$ is **locally nilpotent** if its image under the morphism π^p is nilpotent. Then we define a subvariety $\mathrm{Rep}(\Pi(Q), \alpha)_{\text{w-nil}}$ of $\mathrm{Rep}(\Pi(Q), \alpha)$ by

$$\mathrm{Rep}(\Pi(Q), \alpha)_{\text{w-nil}} := \{X \in \mathrm{Rep}(\Pi(Q), \alpha) \mid X \quad \text{is locally nilpotent}\}.$$

We shall denote $NC(Q, \alpha)$ the number of elements of irreducible components of $\mathrm{Rep}(\Pi(Q), \alpha)_{\text{nil}})$ and write $NC(Q, \alpha)_w$ for the number of irreducible components of $\mathrm{Rep}(\Pi(Q), \alpha)_{\text{w-nil}}$ of dimension $\dim \mathrm{Rep}(Q, \alpha)$.

Theorem 10 [8] *The variety* $Rep(\Pi(Q),\alpha)_{w\text{-}nil}$ *has dimension* $\sum_{u\in Q_1}\alpha_{s(u)}\alpha_{t(u)}$. *The number of irreducible components of* $Rep(\Pi(Q),\alpha)_{w\text{-}nil}$ *of dimension* $\sum_{u\in Q_1}\alpha_{s(u)}\alpha_{t(u)}$ *is equal to*

$$NC(Q,\alpha)_w=\sum_{\underline{\lambda}}\prod_{q\in Q_0}NP(\lambda^{q,v}\mid s(v)=q)\prod_{u\in Q_1}NA(\lambda^{s(u),u},\lambda^{s(u^*),u^*}),$$

where the sum is over all tuples of partitions $\underline{\lambda}=(\lambda^{s(u),u}\mid u\in\overline{Q}_1)$ *with* $\lambda^{s(u),u}\vdash\alpha_{s(u)}$.

Theorem 11 [8] *1) Let* Q *be an extended Dynkin quiver and let* δ *be the minimal imaginary root with respect to* Q. *Then*

$$NC(Q,\delta)_w \quad = \quad \begin{cases} 2^{n+1} & \text{for} \quad Q=\widetilde{\mathbb{A}}_n \\ 25d(n-4) & \text{for} \quad Q=\widetilde{\mathbb{D}}_n \quad n>4 \\ 1805 & \text{for} \quad Q=\widetilde{\mathbb{E}}_6 \\ 52410 & \text{for} \quad Q=\widetilde{\mathbb{E}}_7, \end{cases}$$

here d *is the function* $\mathbb{N}\longrightarrow\mathbb{N}$ *satisfying*

$$\begin{aligned} d(2i) &= 2\cdot 5^i f(2i+1)+5^i f(2i) \\ d(2i+1) &= 7\cdot 5^i f(2i+1)+4\cdot 5^i f(2i), \end{aligned}$$

where f *denotes the Fibonacci-sequence, i.e.*

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(i+1) &= f(i)+f(i-1). \end{aligned}$$

2) If Q *is a star (e.g. of type* $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, $\widetilde{\mathbb{D}}_4$*), then*

$$NC(Q,\delta)=NC(Q,\delta)_w.$$

3) If Q *is not star (i.e. of type* $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_n$, $n>4$*), then*

$$NC(Q,\delta) \quad = \quad \begin{cases} NC(Q,\delta)_w-2 & \text{for} \quad Q=\widetilde{\mathbb{A}}_n \\ NC(Q,\delta)_w-1 & \text{for} \quad Q=\widetilde{\mathbb{D}}_n. \end{cases}$$

Example. We check the formula in the above theorem for the McKay quivers of type $\widetilde{\mathbb{D}}_4$ and $\widetilde{\mathbb{D}}_5$. The result is given in the tables below.

$2-1$	$2-1$	$2-1$	$2-1$	$\sharp$	$\prod NA$	$\sum$
(2)	(2)	(2)	(2)	1	$1.1.1.1 = 1$	1
(2)	(2)	(2)	(1,1)	4	$1.1.1.2 = 2$	8
(2)	(2)	(1,1)	(1,1)	6	$1.1.2.2 = 4$	24
(1,1)	(1,1)	(1,1)	(1,1)	1	$2.2.2.2 = 16$	16
						49

The calculation of $NC(\widetilde{\mathbb{D}}_4, \delta)_w = 49$

$2-1$	$2-1$	$2-2$	$2-1$	$2-1$	$\sharp$	$\prod NA$	$\sum$
(2)	(2)	(2)−(2)	(2)	(2)	1	$1.1.2.1.1 = 2$	2
(1,1)	(2)	(2)−(2)	(2)	(2)	4	$2.1.2.1.1 = 4$	16
(2)	(2)	(1,1)−(2)	(2)	(2)	2	$1.1.1.1.1 = 1$	2
(1,1)	(2)	(2)−(2)	(1,1)	(2)	4	$2.1.2.2.1 = 8$	32
(2)	(2)	(1,1)−(2)	(1,1)	(2)	4	$1.1.1.2.1 = 2$	8
(2)	(2)	(1,1)−(1,1)	(2)	(2)	1	$1.1.3.1.1 = 3$	3
(1,1)	(1,1)	(1,1)−(2)	(2)	(2)	2	$2.2.1.1.1 = 4$	8
(1,1)	(1,1)	(1,1)−(1,1)	(2)	(2)	2	$2.2.3.1.1 = 12$	24
(1,1)	(1,1)	(1,1)−(2)	(1,1)	(2)	4	$2.2.1.2.1 = 8$	32
(1,1)	(1,1)	(1,1)−(1,1)	(1,1)	(1,1)	1	$2.2.3.2.2 = 48$	48
							175

The calculation of $NC(\widetilde{\mathbb{D}}_5, \delta)_w = 175$

5.2 Stable representations of quivers

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. We define the space of weights $\mathbb{H}(d)$ with respect to the dimension vector $d \in \mathbb{N}^{Q_0}$ by

$$\mathbb{H}(d) := \{\theta \in \mathbb{Z}^{Q_0} \mid \theta(d) = \sum \theta_i d_i = 0\}.$$

Definition. Let $X = (X_i, X(a)) \in \mathrm{Rep}(Q, d)$ be a representation of dimension vector d and $\theta \in \mathbb{H}(d)$ be a weight with respect to a dimension vector d. The representation X is **θ-stable** if for all proper non-zero subrepresentation N of X we have $\theta(\underline{\dim} N) < 0$. It is said to be **$\theta$-semistable** if for each subrepresentation N we have $\theta(\underline{\dim} N) \leq 0$.

A representation X is **stable** (resp. **semistable**) if there exists a weight θ, so that X is θ-stable (resp. θ-semistable). We denote

$$\mathrm{Rep}(Q, d)^{\mathrm{stab}} := \{X \in \mathrm{Rep}(Q, d) \mid X \text{ is stable}\}$$

the stable part of $\mathrm{Rep}(Q, d)$.

Let θ and θ' be two weights in $\mathbb{H}(d)$, we say that θ is **d-equivalent** to θ' if for any X of $\mathrm{Rep}(Q, d)$, X is θ-stable if and only if X is θ'-stable and for any X of $\mathrm{Rep}(Q, d)$ X is θ-semistable if and only if X is θ'-semistable.

We can also define the **wall system** with respect to dimension vector d. That is the minimal set of hyperplanes $\{W_i\}_{i\in I}$ in $\mathbb{H}(d)$, where I is a finite index set, with the following property: whenever two weights θ and θ' in $\mathbb{H}(d)$ lie on the same open side of each of these hyperplanes W_i, then they are d-equivalent.

5.3 Relation between the exceptional set and the nilpotent variety $\mathbf{Rep}(\Pi(Q), \delta)_{\mathbf{nil}}$

Lemma 13 *An element $0 \neq X = (X_i, X(a))_{i\in Q_0, a\in Q_1} \in \mathrm{Rep}(Q, \alpha)$ is unstable under the action of $G(\alpha)$ if and only if it is nilpotent.*

PROOF. Assume that $X = (X_i, X(a))_{i\in Q_0, a\in Q_1} \in \mathrm{Rep}(Q, \alpha)$ is unstable. Then $0 \in \overline{G(\alpha)\cdot X}$. By Hilbert's criterion, there is a 1-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow G(\alpha)$ such that

$$\lim_{t\to 0} \lambda(t)\cdot X = 0. \qquad (1)$$

For each $k \in Q_0$ we make the λ-weight decomposition of X_k

$$X_k = \bigoplus_{m\in\mathbb{Z}} X_k^m,$$

where $X_k^m := \{v \in X_k \mid \lambda(t)\cdot v = t^m v\}$.

Let

$$X_k^{(n)} := \bigoplus_{m\geq n} X_k^m.$$

Then we have a filtration of X_k.

Let ι_i be the inclusion of X_k^i into $\bigoplus_{m\in\mathbb{Z}} X_k^m$, and let π_i be the projection of $\bigoplus_{m\in\mathbb{Z}} X_k^m$ to X_k^i.

For $a \in Q$ the linear map

$$X(a) = \bigoplus_{m\in\mathbb{Z}} X_{s(a)}^m \longrightarrow \bigoplus_{m\in\mathbb{Z}} X_{t(a)}^m$$

has the form

$$X(a) = (\pi_k X(a) \iota_i)_{i,k} = (X(a)_{ik})_{i,k},$$

where $X(a)_{ik}$ are the maps

$$X(a)_{ik} : X^i_{s(a)} \longrightarrow X^k_{t(a)}.$$

$$\begin{tikzcd}
\bigoplus\limits_{m\in\mathbb{Z}} X^m_{s(a)} \arrow[r, "\text{X(a)}"] & \bigoplus\limits_{m\in\mathbb{Z}} X^m_{t(a)} \arrow[d, "\pi_k"] \\
X^i_{s(a)} \arrow[u, "\iota_i"'] \arrow[r, "X(a)_{ik}"] & X^k_{t(a)}
\end{tikzcd}$$

We have

$$\lambda(t) \cdot X(a) = (\lambda(t) \cdot X(a)_{ik} \cdot \lambda^{-1}(t)) = (t^{k-i} X(a)_{ik}). \qquad (2)$$

$$\begin{tikzcd}
X^i_{s(a)} \arrow[r, "X(a)_{ik}"] & X^k_{t(a)} \arrow[d, "\lambda(t)"] \\
X^i_{s(a)} \arrow[u, "\lambda^{-1}(t)"] \arrow[r, "t^{k-i}X(a)_{ik}"] & X^k_{t(a)}
\end{tikzcd}$$

By existing of the limit $\lim\limits_{t\to 0} \lambda(t) \cdot X(a)$ it must have

$$X(a)_{ik} = 0 \qquad \text{for all} \quad k < i.$$

Thus $X(a)$ gives a map $X^{(n)}_{s(a)} \longrightarrow X^{(n)}_{t(a)}$, for all n. Therefore the subspaces $X^{(n)}_k$ determine subrepresentations X^n of X. These subrepresentations form a filtration of X:

$$0 = X^{(N)} \subseteq \ldots \subseteq X^{(n)} \subseteq X^{(n-1)} \subseteq \ldots \subseteq X^{(M)} = X$$

for $N \gg 0$ and $M \ll 0$.

This may be rewritten as

$$\begin{array}{lllll} 0 & = X_{s(a)}^{(N)} & \subseteq \ldots \subseteq X_{s(a)}^{(n)} \subseteq X_{s(a)}^{(n-1)} & \subseteq \ldots \subseteq X_{s(a)}^{(M)} & = X_{s(a)} \\ & & & & X(a) \downarrow \\ 0 & = X_{t(a)}^{(N)} & \subseteq \ldots \subseteq X_{t(a)}^{(n)} \subseteq X_{s(t)}^{(n-1)} & \subseteq \ldots \subseteq X_{t(a)}^{(M)} & = X_{t(a)}. \end{array}$$

From (1) and (2) it follows that

$$\begin{aligned} \lim_{t \to 0} \lambda(t) \cdot X(a) &= \bigoplus_{n=M}^{N-1} (X_k^n, X(a)_{nn}) \\ &= \bigoplus_{n=M}^{N-1} X^n / X^{n+1} = 0. \end{aligned}$$

Hence it must have

$$X_{s(a)}^n / X_{s(a)}^{n+1} = 0 \quad \text{or} \quad X_{t(a)}^n / X_{t(a)}^{n+1} = 0$$

$$\text{i.e.} \qquad X_{s(a)}^n = X_{s(a)}^{n+1} = 0 \quad \text{or} \quad X_{t(a)}^n = X_{t(a)}^{n+1} = 0$$

for all $n = M, \ldots, (N-1)$ and $a \in Q_1$.

If $\quad X_{s(a)}^n = X_{s(a)}^{n+1} = 0$, then

$$X(a)(X_{s(a)}^n) = X(a)(X_{s(a)}^{n+1}) \subseteq X_{t(a)}^{n+1}.$$

If $\quad X_{t(a)}^n = X_{t(a)}^{n+1} = 0$, then

$$X(a)(X_{s(a)}^n) \subseteq X(a)(X_{t(a)}^n) \subseteq X_{t(a)}^{n+1}.$$

Thus we have always

$$X(a)(X_{s(a)}^n) \subseteq X_{t(a)}^{n+1}.$$

Now, for any sequence $a_1, a_2, \ldots, a_N$ in Q_1 such that $t(a_1) = s(a_2), t(a_2) = s(a_3), \ldots, t(a_{N-1}) = s(a_N)$ we see that

$$\begin{array}{lll} X(a_1)(X_{s(a_1)}^1) & \subseteq X_{t(a_1)}^2 = X_{s(a_2)}^2 & \\ X(a_2)X(a_1)(X_{s(a_1)}^1) & \subseteq X(a_2)(X_{s(a_2)}^2) & \subseteq X_{t(a_2)}^3 = X_{s(a_3)}^3 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \\ X(a_N)\cdots X(a_1)(X_{s(a_1)}^1) & = X_{t(a_N)}^N = 0. & \end{array}$$

Thus, the representation $X = (X_i, X(a))$ is nilpotent.

Conversely, assume $X = (X_i, X(a))$ is nilpotent. If $p = a_1 \cdots a_m$ is a path in $\mathbb{C}$Q that starts and ends at the same vertex, then the function tr_p given by

$$\mathrm{tr}_p(X) = \mathrm{tr}_p(X(a_m) \cdots X(a_2)X(a_1))$$

is invariant under the action of $G(\alpha)$. By Le Bruyn and Procesi([4]) the algebra $\mathbb{C}[\mathrm{Rep(Q,}\ \alpha)]^{G(\alpha)}$ is generated by the trace functions tr_p.

Since X is nilpotent, $\mathrm{tr}_p X = 0$ for all cyclic paths p in $\mathbb{C}Q$. Thus, all invariants vanish at X and therefore it is unstable. $\square$

Now, let $\overline{Q}(\Gamma)$ be the McKay quiver of type $\Delta(\Gamma)$ and δ be the minimal imaginary root as before. Let θ be a weight in $\mathbb{H}(\delta)$. We denote

$$\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta := \{X \in \mathrm{Rep}(\Pi(Q(\Gamma)), \delta) \mid \text{X is } \theta\text{-stable}\}$$

the θ-stable part of $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)$.

For generic weight θ the minimal resolution π of the Kleinian singularity $\mathbb{C}^2/\Gamma$ can be put in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta \arrow[r, "\rho_\theta"] \arrow[d] & \mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta // G(\delta) \arrow[d, "\pi"] & \\
\mathrm{Rep}(\Pi(Q(\Gamma)), \delta) \arrow[r, "\rho"] & \mathrm{Rep}(\Pi(Q(\Gamma)), \delta) // G(\delta) & \simeq \mathbb{C}^2/\Gamma
\end{tikzcd}$$

where the maps ρ and ρ_θ are the quotient morphisms.

The following theorem plays an important part in our work.

Theorem 12 *The exceptional set of the minimal resolution π is given by*

$$\pi^{-1}(0) = \rho_\theta(\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta_{nil}).$$

PROOF. It is known that the zero-fiber consists of the unstable elements. By Lemma 13 the unstable elements of $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)$ are nilpotent. Thus we have

$$\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)_{\mathrm{nil}} = \rho^{-1}(0).$$

So by commutativity of the diagram above it follows

$$\pi^{-1}(0) = \rho_\theta(\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta_{\mathrm{nil}}).$$

$\square$

The theorem shows that to describe the exceptional set $\pi^{-1}(0)$ of the resolution π we need to consider the nilpotent variety $\mathrm{Rep}(\Pi(Q(\Gamma)), \delta)^\theta_{\mathrm{nil}}$.

Definition. Let X and Y be two varieties of the same dimension. We say that X **meets** Y if and only if codim(X$\cap$Y) = 1.

Let X be a reducible variety which has pure dimension n. We define the **intersection diagram** Γ(X) with respect to X as follows: Associate to each irreducible component $\mathrm{X}_i \in \mathrm{Irr}(\mathrm{X})$ a vertex $i \in \Gamma(\mathrm{X})$. Vertices i and j are connected by an edge if the component X_i meets the component X_j.

It is known that the exceptional set of the minimal resolution of the Kleinian singularities of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{E}}_n$ is the union of the complex projective lines and its intersection diagram is the Dynkin diagram of the same type (see Theorem 5, Sect. 2.2). In the next section, we show how this can be observed in terms of representations of quivers.

6 The nilpotent variety $\mathbf{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathbf{nil}}$

6.1 Notations and definitions

Let $\overline{Q}(\Gamma) = (\overline{Q}_0, \overline{Q}_1, s, t)$ be the McKay quiver of type $\widetilde{\mathbb{A}}_{n-1}$ with n ≥ 2. We assume that $\overline{Q}_0 = \mathbb{Z}/n = \{1, 2, \ldots, n\}$, and that $\overline{Q}_1$ consist of the arrows

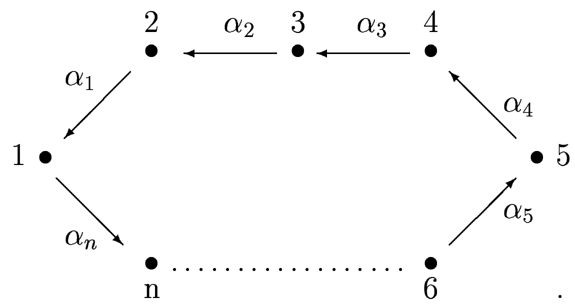
$$\alpha_i : i+1 \to i \qquad i \in Q_0$$

$$\alpha_i^* : i \to i+1 \qquad i \in Q_0.$$

Since $\delta = (1, 1, \ldots, 1)$ we have

$$\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta) = \{(a_i, a_i^*)_i \in \bigoplus_{i=1}^{n} \mathbb{C}^2 \mid a_i a_i^* = a_j a_j^*; \forall i, j \in \overline{Q}_0\}.$$

Let Ω be the subset of Q_1 consisting of all arrows α_i, $i \in \overline{Q}_0$. It corresponds to the following cyclic orientation of $\overline{Q}(\Gamma)$:

$\Omega \quad =$ 

The nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ has $2^n - 2$ irreducible components (see Theorem 11, 5.1). Let I be a subset of Ω such that $\varnothing \neq I \neq \Omega$, and let

$$C_I := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \bigoplus_{i=1}^{n} \mathbb{C}^2 \mid a_i^* = 0 \text{ if } \alpha_i \in I;\ a_j = 0 \text{ if } \alpha_j \notin I\})$$

Then C_I is a vector space of dimension n. It is a irreducible component of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$. All irreducible components of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ are of this form:

$$\mathrm{Irr}(\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}) = \bigcup_{\substack{I \subsetneq \Omega \\ I \neq \varnothing}} C_I.$$

We say that C_I is a **component of type k** if $\mid I \mid =$ k. In the next sections we shall write $C_{\{i_1, i_2, \ldots, i_k\}}$ for $C_{\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}}$.

6.2 The action of the Weyl group on $\mathbb{H}(\delta)$

In this part we define the equivalence for two weights θ and θ' in $\mathbb{H}(\delta)$ as folows. We say that θ and θ' are δ-equivalent if for any $X \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$, X is θ-stable if and only if X is θ'-stable and for any $X \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$, X is θ-semistable if and only if X is θ'-semistable. We shall investigate the action of the symmetric group S_n on $\mathbb{H}(\delta)$, the δ-equivalence classes and the system of walls associated to the equivalence above.

Lemma 14 *Let θ be a weight in $\mathbb{H}(\delta)$ and let $C_{\{i_1,i_2,\ldots,i_k\}}$ be a component of type k of the nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$. Then $C_{\{i_1,i_2,\ldots,i_k\}}$ is θ-stable if only if for each $s = 1, 2, \ldots, k$ the following condition holds:*

$$\begin{cases} \sum\limits_{l=j}^{i_s} \theta_l < 0 & \forall j \in \{1, \ldots, i_s\} \setminus \{i_1+1, \ldots, i_k+1\} \\ \sum\limits_{l=1}^{i_s} \theta_l + \sum\limits_{l=j}^{n} \theta_l < 0 & \forall j \in \{i_s+1, \ldots, n\} \setminus \{i_1+1, \ldots, i_k+1\}. \end{cases}$$

PROOF. For each $s = 1, 2, \ldots, k$ the dimension vectors of possible indecomposable proper non-trivial subrepresentations of $C_{\{i_1,i_2,\ldots,i_k\}}$ are:

$$(\overbrace{\underbrace{0, 0, \ldots, 0}_{j-1}, 1, 1, \ldots, 1}^{i_s}, 0, 0, \ldots, 0),$$

for $j \in \{1, 2 \ldots, i_s\} \setminus \{i_1+1, i_2+1, \ldots, i_k+1\}$

$$(\overbrace{\underbrace{1, 1, \ldots, 1}_{i_s}, 0, 0, \ldots, 0}^{j-1}, 1, 1, \ldots, 1),$$

for $j \in \{i_s+2, i_s+3, \ldots, n\} \setminus \{i_1+1, i_2+1, \ldots, i_k+1\}$.

Then the assertion follows from definition of θ-stability. $\square$

Lemma 15 *Let θ be a weight in $\mathbb{H}(\delta)$ and let $C_{\{i_1,i_2,\ldots,i_k\}}$ be a component of type k of the nilpotent variety Rep($\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{nil}$. Then $C_{\{i_1,i_2,\ldots,i_k\}}$ is θ-stable if only if for each $s = 1, 2, \ldots, k$ the following condition holds:*

$$\sum_{l=1}^{i_s} \theta_l < \sum_{l=1}^{m} \theta_l \qquad (*)$$

for all $m \in \{1, 2, 3, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$.

PROOF. For each $s = 1, 2, \ldots, k$ the condition (*) can be written in the form

$$\sum_{l=1}^{i_s} \theta_l - \sum_{l=1}^{m} \theta_l < 0, \quad \forall m \in \{1, 2, 3, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$$

$$\iff \begin{cases} \sum\limits_{l=m+1}^{i_s} \theta_l < 0 & \forall m \in \{1, 2 \ldots, i_s\} \setminus \{i_1, \ldots, i_k\} \\ -\sum\limits_{l=i_s+1}^{m} \theta_l < 0 & \forall m \in \{i_s, \ldots, n\} \setminus \{i_1, \ldots, i_k\}. \end{cases}$$

Since $\sum\limits_{l=1}^{n} \theta_l = 0$ the inequalities above are equivalent to

$$\begin{cases} \sum\limits_{l=m+1}^{i_s} \theta_l < 0 & \forall m \in \{1, 2 \ldots, i_s\} \setminus \{i_1, \ldots, i_k\} \\ \sum\limits_{l=1}^{i_s} \theta_l + \sum\limits_{l=m+1}^{n} \theta_l < 0 & \forall m \in \{i_s, \ldots, n\} \setminus \{i_1, \ldots, i_k\}. \end{cases}$$

By setting j = m + 1, we can write the inequalities above as

$$\begin{cases} \sum\limits_{l=j}^{i_s} \theta_l < 0 & \forall j \in \{2, \ldots, i_s + 1\} \setminus \{i_1 + 1, \ldots, i_k + 1\} \\ \sum\limits_{l=1}^{i_s} \theta_l + \sum\limits_{l=j}^{n} \theta_l < 0 & \forall j \in \{i_s + 1, \ldots, n\} \cup \{1\} \setminus \{i_1 + 1, \ldots, i_k + 1\} \end{cases}$$

or

$$\begin{cases} \sum\limits_{l=j}^{i_s} \theta_l < 0 & \forall j \in \{1, 2, \ldots, i_s + 1\} \setminus \{i_1 + 1, \ldots, i_k + 1\} \\ \sum\limits_{l=1}^{i_s} \theta_l + \sum\limits_{l=j}^{n} \theta_l < 0 & \forall j \in \{i_s + 1, \ldots, n\} \setminus \{i_1 + 1, \ldots, i_k + 1\}. \end{cases}$$

Hence, our lemma follows from Lemma 14. $\square$

Let δ' be the non-trivial proper subdimension vector of δ. We define a hyperplane by

$$W(\delta') := \{\theta \in \mathbb{H}(\delta) \mid \theta(\delta') = 0\}.$$

It is clear that $W(\delta') = W(\delta - \delta')$. Thus, without loss of generality we need only to consider subdimension vectors of δ which have the form

$$\delta'_{i,j} := (\overbrace{\underbrace{0,\ldots,0}_{i},1,\ldots,1}^{j},0,\ldots,0), \quad 1 \le i < j \le n.$$

We shall write $W_{i,j}$ for the hyperplane $W(\delta'_{i,j})$. The following lemma is obvious:

Lemma 16 *The assignment $W_{i,j} \longmapsto (i,j)$ gives a bijection between the set of hyperplanes $\{W_{i,j}\}_{1\le i<j\le n}$ and the set of transpositions of the symmetric group S_n.*

Given any pair (i,j) of distinct integers in $\{1,2,\ldots,n\}$, the corresponding transpositions of S_n act on $\mathbb{H}(\delta)$ as the reflections with respect to the hyperplanes

$$W_{i,j} := \{(\theta_1,\theta_2,\ldots,\theta_n) \in \mathbb{H}(\delta) \mid \sum_{l=1}^{i}\theta_l = \sum_{l=1}^{j}\theta_l\},$$

dividing $\mathbb{H}(\delta)$ into n! cones which will be called **chambers**.

If we choose a so-called **fundamental chamber**, for example

$$\Theta(1) := \{\theta \in \mathbb{H}(\delta) \mid \sum_{l=1}^{1}\theta_l < \sum_{l=1}^{2}\theta_l < \ldots < \sum_{l=1}^{n}\theta_l = 0\},$$

then the others chambers may be labeled by the elements of S_n:

$$\Theta(\sigma) := \{\theta \in \mathbb{H}(\delta) \mid \sum_{l=1}^{\sigma(1)}\theta_l < \sum_{l=1}^{\sigma(2)}\theta_l < \ldots < \sum_{l=1}^{\sigma(n)}\theta_l\},$$

The group S_n acts on the set of chambers by

$$\gamma \cdot \Theta(\sigma) := \Theta(\gamma\sigma), \quad \gamma,\sigma \in S_n.$$

We shall describe the action of symmetric group S_n on the space of weights $\mathbb{H}(\delta)$, the δ-equivalence classes and the wall system in the following theorem:

Theorem 13 *1) The symmetric group S_n acts on the space of weights $\mathbb{H}(\delta)$ by reflections with respect to the hyperplanes $W_{i,j}$ dividing $\mathbb{H}(\delta)$ in n! chambers:*

$$\Theta(\sigma) := \{\theta \in \mathbb{H}(\delta) \mid \sum_{l=1}^{\sigma(1)}\theta_l < \sum_{l=1}^{\sigma(2)}\theta_l < \ldots < \sum_{l=1}^{\sigma(n)}\theta_l\}, \quad \sigma \in S_n.$$

The resulting action of S_n on the set of these n! chambers is simply transitive.

2) For each generic $\theta \in \Theta(\sigma)$ there are exactly (n-1) θ-stable components of the nilpotent variety Rep$(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ which are

$$C_{\{\sigma(1),\sigma(2),\cdots,\sigma(i)\}}, \quad i = 1, 2, \ldots, n-1.$$

3) For each generic $\theta \in \Theta(\sigma)$ the corresponding δ-equivalence class of θ is the chamber $\Theta(\sigma)$.

4) The set of hyperplanes $\{W_{i,j}\}_{1\leq i<j\leq n}$ forms the wall system in $\mathbb{H}(\delta)$.

PROOF. 1) We have to show that $\Theta(\sigma) \cap \Theta(\gamma) = \varnothing$ for distinct elements $\sigma, \gamma \in S_n$. Indeed, in the contrary case, there exists $\theta \in \Theta(\sigma) \cap \Theta(\gamma)$ satisfying

$$\begin{cases} \sum\limits_{l=1}^{\sigma(1)} \theta_l < \sum\limits_{l=1}^{\sigma(2)} \theta_l < \ldots < \sum\limits_{l=1}^{\sigma(n)} \theta_l \\ \\ \sum\limits_{l=1}^{\gamma(1)} \theta_l < \sum\limits_{l=1}^{\gamma(2)} \theta_l < \ldots < \sum\limits_{l=1}^{\gamma(n)} \theta_l. \end{cases}$$

We assume that the integer i to be maximal with the property $\sigma(i) \neq \gamma(i)$. Then there exist two integers $j, k > i$ such that $\sigma(k) = \gamma(i)$ and $\gamma(j) = \sigma(i)$.

Hence we have the contradiction:

$$\begin{cases} \sum\limits_{l=1}^{\sigma(i)} \theta_l < \sum\limits_{l=1}^{\sigma(k)} \theta_l = \sum\limits_{l=1}^{\gamma(i)} \theta_l \\ \\ \sum\limits_{l=1}^{\gamma(i)} \theta_l < \sum\limits_{l=1}^{\gamma(j)} \theta_l = \sum\limits_{l=1}^{\sigma(i)} \theta_l. \end{cases}$$

2) For $\theta \in \Theta(\sigma)$ and $k \in \{1, 2, \ldots, n-1\}$ the component $C_{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}$ is stable by Lemma 15.

Now suppose there exists a component $C_{\{i_1,i_2,\ldots,i_k\}}$ which is θ-stable for $\theta \in \Theta(\sigma)$. We show that

$$C_{\{i_1,i_2,\ldots,i_k\}} = C_{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}.$$

Indeed, in the contrary case, there exists a positive integer $i \leq k$ such that $\sigma(i) \notin \{i_1, i_2, \ldots, i_k\}$. Since $C_{\{i_1,i_2,\ldots,i_k\}}$ is θ-stable we have by Lemma 15:

$$\sum_{l=1}^{i_s}\theta_l < \sum_{l=1}^{\sigma(i)}\theta_l < \sum_{l=1}^{\sigma(i+1)}\theta_l < \ldots < \sum_{l=1}^{\sigma(n)}\theta_l, \forall s = 1,\ldots,k.$$

Because $i \leq k$ there exists an integer $t \in \{1,2,\ldots,n\}$ such that $t > i$ and $\sigma(t) = i_r$ for some $r \in \{1,2,\ldots,k\}$. This gives the contradiction:

$$\sum_{l=1}^{\sigma(i)}\theta_l < \sum_{l=1}^{\sigma(t)}\theta_l = \sum_{l=1}^{i_r}\theta_l.$$

3) We have to show that θ' is δ-equivalent to θ for all $\theta' \in \Theta(\sigma)$. Indeed, if θ and θ' were not δ-equivalent, then there exists a dimension vector

$$\delta'_{i,j} := (\overbrace{\underbrace{0,\ldots,0}_{i},1,\ldots,1}^{j},0,\ldots,0), \quad 1 \leq i < j \leq n$$

with $\theta(\delta'_{i,j})\theta'(\delta'_{i,j}) < 0$.

Hence we have the contradiction to the definition of $\Theta(\sigma)$:

$$(\sum_{l=1}^{j}\theta_l - \sum_{l=1}^{i}\theta_l)(\sum_{l=1}^{j}\theta'_l - \sum_{l=1}^{i}\theta'_l) < 0.$$

Conversely, if the weight θ' is equivalent to the weight θ, then $\theta' \in \Theta(\sigma)$. Indeed, in the contrary case, there exitsts a component C_I such that C_I is θ-stable but not θ'-stable. Since C_I is θ-stable it contains a θ-stable representation $X \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)_{\mathrm{nil}}$. Because $\theta \sim \theta'$ the representation X is also θ'-stable. This gives a contradiction to that C_I is not θ'-stable.

4) If two generic weights θ and θ' in $\mathbb{H}(\delta)$ lie on the same side of each of these hyperplanes $W_{i,j}$, then they must belong to $\Theta(\sigma)$ for some $\sigma \in S_n$. So by 3) they are equivalent. □

Let us illustrate the action for the small values of n.

Case n = 3. There are three transpositions in S_3 which are

$$s_1 = (1,2), s_2 = (2,3) \quad \text{and} \quad (1,3) = s_1s_2s_1 = s_2s_1s_2.$$

The three transpositions act by reflections with respect to three lines dividing $\mathbb{H}(\delta)$ in six chambers as in Figure 1.

Case n = 4. We shall illustrate the action of S_4 on the 2-sphere $S^2 = S^3 \cap \mathbb{H}(\delta) \subset \mathbb{R}^4$, represented via stereographic projection as $\mathbb{R}^2 \cup \{\infty\}$. There are six transpositions in S_4, including

$$s_1 = (1,2), s_2 = (2,3), s_3 = (3,4).$$

The transpositions of S_4 act as reflections with respect to six great circles of S^2, three of them being lines in the stereographic projection as in figure 2 (namely $s_1 = (1,2)$, $(2,4) = s_2 s_3 s_2 = s_2 s_3 s_2$, $(1,4) = s_1 s_2 s_3 s_2 s_1$). These six great circles divide S^2 into 24 chambers, which are spherical triangles.

Figure 1

Figure 2

6.3 The intersection diagram of the variety $\mathbf{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathbf{nil}}$

We denote $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ be the intersection diagram of the nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\mathrm{nil}}$ (see definitions in Sect. 5.3).

Lemma 17 *Let C_I and C_J be two irreducible components of the nilpotent variety* $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{nil}$. *Then C_I meets C_J if and only if the following condition holds:*

$$(I \subset J \text{ and } \mid J \setminus I \mid = 1) \text{ or } (J \subset I \text{ and } \mid I \setminus J \mid = 1)$$

PROOF. Let Q be the quiver $\bullet \rightleftarrows \bullet \cdots \bullet \rightleftarrows \bullet$ and let X and Y be the varieties of representations of Q given by

$$X = \mathbb{C} \underset{a}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{y_1}{\overset{x_1}{\rightleftarrows}} \mathbb{C} \cdots \mathbb{C} \underset{y_k}{\overset{x_k}{\rightleftarrows}} \mathbb{C} \underset{d}{\overset{0}{\rightleftarrows}} \mathbb{C}$$

$$Y = \mathbb{C} \underset{0}{\overset{b}{\rightleftarrows}} \mathbb{C} \underset{y_1}{\overset{x_1}{\rightleftarrows}} \mathbb{C} \cdots \mathbb{C} \underset{y_k}{\overset{x_k}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{c}{\rightleftarrows}} \mathbb{C}.$$

We have $\mathrm{codim}(X \cap Y) = 2$ because $X \cap Y = \{0\}$.

If $I \nsubseteq J$ and $J \nsubseteq I$, then there exists a subquiver

$$\bullet \rightleftarrows \bullet \cdots \bullet \rightleftarrows \bullet$$

such that restrictions of C_I and C_J to this subquiver are the varieties of representations X and Y as above. So $\mathrm{codim}(C_I \cap C_J) \geq 2$.

We use similar arguments for the case : $\mid J \setminus I \mid \geq 2$.

Now suppose that $I \subset J$ and $J = I \cup \{k\}$. Then we have

$$\begin{aligned} C_I \cap C_J &= \{(a, a^*) \mid a_i^* = 0 \quad \forall i \in I \cup J, \quad a_j = 0 \quad \forall j \notin I \cap J\} \\ &= \{(a, a^*) \mid a_i^* = 0 \quad \forall i \in I \cup \{k\} \quad a_j = 0 \quad \forall j \notin I\}. \end{aligned}$$

Thus dim $(C_I \cap C_J) = n - 1$. This gives codim $(C_I \cap C_J) = 1$. $\square$

The intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ relates to the intersection diagram of resolution of the Kleinian singularity of type $\mathbb{A}_{n-1}$ by the following theorem:

Theorem 14 *For each permutation $\sigma \in S_n$ the map φ_σ given by*

$$\begin{array}{rcl} \varphi_\sigma : \overset{1}{\bullet} - \overset{2}{\bullet} \ldots \bullet - \overset{n-1}{\bullet} & \longrightarrow & \Gamma(\mathbb{A}_{n-1}) \\ \overset{k}{\bullet} & \longmapsto & \varphi_\sigma(k) := C_{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}, \end{array}$$

gives an embedding of the Dynkin diagram $\mathbb{A}_{n-1}$ into the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$.

2) For each $\theta \in \Theta(\sigma)$ the intersection diagram of the resolution

$$Rep(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)^\theta//G(\delta) \longrightarrow Rep(\Pi(\widetilde{\mathbb{A}}_{n-1}),\delta)//G(\delta)$$

is the Dynkin diagram $\mathbb{A}_{n-1}$.

PROOF. 1) From Lemma 17 it follows that the component $C_{\{\sigma(1),\sigma(2),\ldots,\sigma(i)\}}$ meets the component $C_{\{\sigma(1),\sigma(2),\ldots,\sigma(j)\}}$ if and only if the verex i and the vertex j are adjacent in $\mathbb{A}_{n-1}$. Therefore φ_σ is an embedding of the Dynkin diagram $\mathbb{A}_{n-1}$ into the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})$.

2) We note that the assignment $\varphi_\sigma \longmapsto \Theta(\sigma)$ gives a bijection between the set of embeddings $\{\varphi_\sigma, \sigma \in S_n\}$ and the set of equivalence classes $\{\Theta(\sigma), \sigma \in S_n\}$. By Theorem 12, Section 5.3, the claim follows. $\square$

REMARK. Let $\pi : \widetilde{S} \longrightarrow \mathbb{C}^2/\Gamma$ be the resolution of a singularity of type $\mathbb{A}_n$ and $C = \bigcup_{i=1}^{n} C_i$ be its exceptional set (see Theorem 5, Sect. 2.2). It is known that, for sufficiently small $\widetilde{X} \supset C$ there is a smooth curve C_0 which intersects C_1 transversally in one point without meeting any of the other curves C_i. Similarly, there is such a curve C_{n+1} intersecting C_n transversally in one point which does not intersect any of the curves $C_0, \ldots, C_n$. Thus we have the so-called extended intersection diagram

$$\overset{C_0}{\circ} - \overset{C_1}{\bullet} - \overset{C_2}{\bullet} \cdots \bullet - \overset{C_n}{\bullet} - \overset{C_{n+1}}{\circ}$$

Let $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1},\delta)_{\text{w-nil}}$ be the subvariety consitsting of all locally nilpotent representations of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1},\delta)$. We denote $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}}$ the intersection diagram of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1},\delta)_{\text{w-nil}}$. It is obtained from $\Gamma(\widetilde{\mathbb{A}}_{n-1})$ by adding two vertices corresponding to the components $C_\varnothing$ and C_Ω. Then we have a similar statement as the above theorem.

Theorem 15 *1) For each permutation $\sigma \in S_n$ the map φ^0_σ given by*

$$
\begin{array}{rcl}
\varphi^0_\sigma : \overset{0}{\circ} - \overset{1}{\bullet} - \overset{2}{\bullet} \cdots \bullet - \overset{n-1}{\bullet} - \overset{n}{\circ} & \longrightarrow & \Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}} \\
k \neq 0, n & \longmapsto & \varphi_\sigma(k) := C_{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}} \\
0 & \longmapsto & C_\varnothing \\
n & \longmapsto & C_\Omega,
\end{array}
$$

gives an embedding of the diagram

$$\mathbb{A}^0_{n-1} : \overset{0}{\circ} - \overset{1}{\bullet} - \overset{2}{\bullet} \cdots \bullet - \overset{n-1}{\bullet} - \overset{n}{\circ}$$

into the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}}$.

2) For each $\theta \in \Theta(\sigma)$ *the extended intersection diagram of the resolution*

$$\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}})_{n-1}), \delta)^\theta // G(\delta) \longrightarrow \mathrm{Rep}(\Pi(\widetilde{\mathbb{A}})_{n-1}), \delta) // G(\delta)$$

is the diagram $\mathbb{A}^0_{n-1}$.

Now, we take a n-dimensional cube H^n. This cube has 2^n vertices. Then the intersection diagram $\Gamma(\widetilde{\mathbb{A}}_{n-1})_{\mathrm{w}}$ of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_{n-1}), \delta)_{\text{w-nil}}$ is described as follows:

Theorem 16 *The intersection diagram* $\Gamma^0(\widetilde{\mathbb{A}}_{n-1})_{\text{w-nil}}$ *is the skeleton of a* n*-dimensinal cube* H^n*. Its vertices are the vertices of the cube* H^n *and its edges are the edges of the cube* H^n.

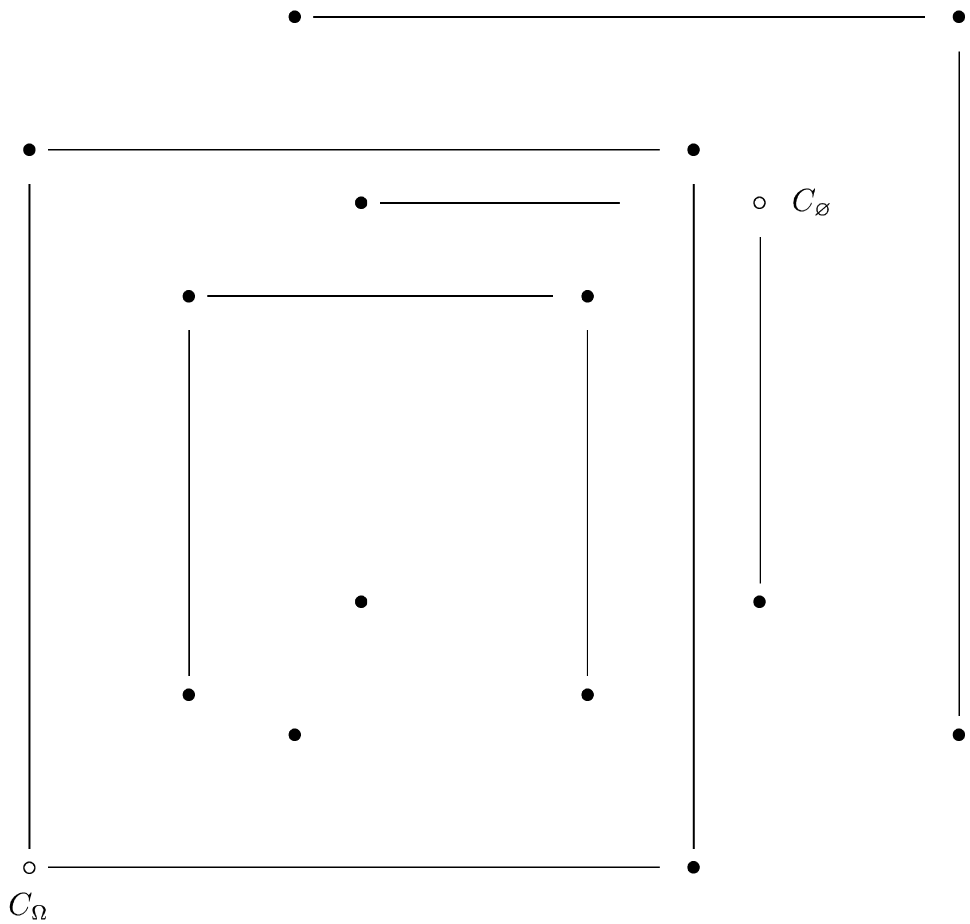
PROOF. We suppose that $H^n = [-1, 1]^n$. Then H^n has the set of vertices $H^n_0 = \{(\pm 1, \pm 1, \ldots, \pm 1)$. We note that the set of irreducible components in $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}})_{n-1}, \delta)_{\text{w-nil}}$ is in 1-1 correspondence with H^n_0 given by

$$C_I \longleftrightarrow M_I$$

where M_I denotes a vertex in H^n_0 whose i-th coordinate is equal 1 if $\alpha_i \in I$, and is equal -1, if $\alpha_i \notin I$.

Let C_I and C_J be the components in $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}})_{n-1}, \delta)_{\text{w-nil}}$. Without loss of generality we assume that $I \subset J$. By Lemma 17 the component C_I meets the component C_J if and only if there exists $k \in \{1, 2, \ldots, n-1\}$ such that $J = I \cup \{k\}$. This means the coresponding vertices M_I and M_J have the same i-th coordinates, for $i \neq k$ and have the k-th coordinates with different signs. In other word, M_I and M_J belong to an edge of the cube H^n. $\square$

We will illustrate the intersection diagram $\Gamma^0(\widetilde{\mathbb{A}}_3)$ of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{A}}_3), \delta)_{\text{w-nil}}$ as in Figure 3 below.

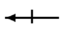
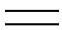
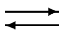
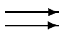
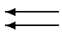
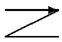
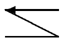
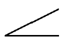
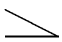


The intersection diagram $\Gamma^0(\widetilde{\mathbb{A}}_3)$

Figure 3

7 The nilpotent variety $\mathbf{Rep}(\Pi(\widetilde{\mathbb{D}_n}), \delta)_{\mathbf{nil}}$

Let us fix the following notations for representations of quivers used in this section.

	$\equiv$	$\mathbb{C} \underset{B}{\overset{0}{\rightleftarrows}} \mathbb{C}^2$	
	$\equiv$	$\mathbb{C} \underset{0}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	
	$\equiv$	$\mathbb{C} \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	either $A = 0$ or $B = 0$
	$\equiv$	$\mathbb{C} \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	$AB = BA = 0$, $\text{rank}(A) \leq 1$, $\text{rank}(B) \leq 1$
	$\equiv$	$\mathbb{C}^2 \underset{0}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{0}{\rightleftarrows}} \mathbb{C}^2$	
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	$AB, BA \in C((2))$, $\text{rank}(A) = 2$, $\text{rank}(B) = 1$
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	$AB, BA \in C((2))$, $\text{rank}(B) = 2$, $\text{rank}(A) = 1$
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	$AB \in C((2))$, $BA = 0$
	$\equiv$	$\mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2$	$BA \in C((2))$, $AB = 0$

7.1 The nilpotent variety $\mathbf{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta)_{\mathbf{nil}}$

Let $\overline{Q}(\Gamma) = (\overline{Q}_0, \overline{Q}_1, s, t)$ be the McKay quiver of type $\widetilde{\mathbb{D}_4}$. We assume that $\overline{Q}_0 = \{0, 1, 2, 3, 4\}$ and that $\overline{Q}_1$ consists of the arrows

$$\begin{aligned} \alpha_i &: \; i \to 0 \\ \alpha_i^* &: \; 0 \to i, \quad i = 1, 2, 3, 4. \end{aligned}$$

Since $\delta = (2, 1, 1, 1, 1)$ we have

$$\begin{aligned} \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta) \; &= \{(a_i, a_i^*)_i \in \bigoplus_{i=1}^{4} \mathbb{C}^2 \oplus \mathbb{C}^2 \mid \; a_i^* a_i = 0, i = 1, 2, 3, 4 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sum_{i=1}^{4} a_i a_i^* = 0\}. \end{aligned}$$

Let Ω be the subset of $\overline{Q}_1$ consisting of all arrows α_i, $i = 1, 2, 3, 4$. This corresponds to the following orientation of $\overline{Q}((\Gamma)$.

$$\Omega = \begin{tikzcd} & 2 \arrow[d, "\alpha_2"] & \\ 1 \arrow[r, "\alpha_1"] & 0 & 3 \arrow[l, "\alpha_3"'] \\ & 4 \arrow[u, "\alpha_4"'] & \end{tikzcd}$$

According to Theorem 11, Sect. 5.1, the nilpotent variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta)_{\mathrm{nil}}$ has 49 irreducible components. We shall describe these components in detail.

1) **The components of type 1.** Let I be a subset of Ω and let

$$\begin{aligned} C_I \; := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta) \mid \; & a_i^* = 0 \quad \text{for} \quad \alpha_i \in I, \\ & a_j = 0 \quad \text{for} \quad \alpha_j \notin I\}) \end{aligned}$$

For example, if $I = \{\alpha_1, \alpha_3\}$ then

$$C_I = \mathrm{Cl}(\{(a_1, 0, a_3, 0, 0, a_2^*, 0, a_4^*) \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4})\})$$

$$= \begin{tikzcd} & \bullet & \\ \bullet \arrow[r, "a_1"] & \bullet \arrow[u, "a_2^*"'] \arrow[d, "a_4^*"] & \bullet \arrow[l, "a_3"'] \\ & \bullet & \end{tikzcd}$$

Then C_I is a vector space of dimension 8 and is an irreducible component of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta)_{\mathrm{nil}}$. We say that C_I is the **component of type 1**. It is clear that there are 16 components of type 1.

2) **The components of type 2.** Let J_1 be a subset of Ω with $\mid J_1 \mid= 2$ and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$C_{J_1,J_2} = \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta) \mid \begin{array}{lll} a_i^* & = 0 & \text{for} \quad \alpha_i \in J_2, \\ a_j & = 0 & \text{for} \quad \alpha_j \notin J_1 \cup J_2\}) \end{array}$$

For example, if $J_1 = \{\alpha_1, \alpha_2\}$ and $J_2 = \{\alpha_3, \alpha_4\}$ then

$$C_{\{\alpha_1,\alpha_2\},\{\alpha_3,\alpha_4\}} = \mathrm{Cl}(\{(a_1, a_2, a_3, a_4, a_1^*, a_2^*, 0, 0) \mid \begin{array}{l} a_2^* a_2 = 0 = a_1^* a_1\}, \\ a_2 a_2^* + a_1 a_1^* = 0\}) \end{array}$$

$$= \begin{tikzcd} & \bullet \arrow[d, shift left, "a_2"] & \\ \bullet \arrow[r, shift left, "a_1"] & \bullet \arrow[u, shift left, "a_2^*"] \arrow[l, shift left, "a_1^*"] & \bullet \arrow[l, "a_3"'] \\ & \bullet \arrow[u, "a_4"'] & \end{tikzcd}$$

We shall call C_{J_1,J_2} as above the **component of type 2**. There are 24 components of type 2.

3) **The components of type 3.** Let J_1 be a subset of Ω with $\mid J_1 \mid = 3$ and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$C_{J_1,J_2} := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta) \mid \begin{array}{lll} a_i^* & = 0 & \text{for} \quad \alpha_i \in J_2, \\ a_j & = 0 & \text{for} \quad \alpha_j \notin J_1 \cup J_2 \}) \end{array}$$

For example, if $J_1 = \{\alpha_1, \alpha_3, \alpha_4\}$ and $J_2 = \{\alpha_2\}$ then

$$C_{\{\alpha_1,\alpha_3,\alpha_4\},\{\alpha_2\}} = \mathrm{Cl}(\{(a_1, a_2, a_3, a_4, a_1^*, 0, a_3^*, a_4^*) \mid \begin{array}{l} a_3^* a_3 = 0 = a_1^* a_1 = a_4^* a_4, \\ a_1 a_1^* + a_3 a_3^* + a_4 a_4^* = 0 \}) \end{array}$$

$$= \begin{tikzcd}
& \bullet \arrow[d, "a_2"] & \\
\bullet \arrow[r, shift left, "a_1"] & \bullet \arrow[l, shift left, "a_1^*"] \arrow[r, shift left, "a_3^*"] \arrow[d, shift left, "a_4^*"] & \bullet \arrow[l, shift left, "a_3"] \\
& \bullet \arrow[u, shift left, "a_4"] &
\end{tikzcd}$$

Such a component C_{J_1,J_2} is called the component of type 3. There are 8 components of type 3.

4) **Ghost component.** Let

$$C_G = \mathrm{Cl}(\{(a_1, a_2, a_3, a_4, a_1^*, 0, a_3^*, a_4^*) \mid \begin{array}{l} a_1^* a_1 = 0 = a_2^* a_2 = a_3^* a_3 = a_4^* a_4, \\ a_1 a_1^* + a_2 a_2^* + a_3 a_3^* + a_4 a_4^* = 0 \}) \end{array}$$

$$= \begin{tikzcd}
& \bullet \arrow[d, shift left, "a_2"] & \\
\bullet \arrow[r, shift left, "a_1"] & \bullet \arrow[u, shift left, "a_2^*"] \arrow[l, shift left, "a_1^*"] \arrow[r, shift left, "a_3^*"] \arrow[d, shift left, "a_4^*"] & \bullet \arrow[l, shift left, "a_3"] \\
& \bullet \arrow[u, shift left, "a_4"] &
\end{tikzcd}$$

The component C_G is not stable because C_G has two subrepresentations whose dimension vectors are:

$$\delta_1' = (1,0,0,0,0) \text{ and } \delta_2' = (1,1,1,1,1).$$

Such a non-stable component will be called the ghost component.

7.2 The intersection diagram of $\mathbf{Rep}(\Pi(\widetilde{\mathbb{D}_4}), \delta)_{\mathbf{nil}}$

In the lemmas below we need following matrices

$$a = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad c = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \quad e = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Lemma 18 *Let X and Y be the varieties of representations of the quiver*

$$Q: \quad \bullet \rightleftarrows \bullet$$

given by

$$X = \begin{cases} \mathbb{C} \overset{b}{\underset{0}{\rightleftarrows}} \mathbb{C}^2 \\ b \in \mathbb{C}^2 \end{cases} \qquad Y = \begin{cases} \mathbb{C} \overset{0}{\underset{a}{\rightleftarrows}} \mathbb{C}^2 \\ a \in \mathbb{C}^2 \end{cases}$$

Then $\operatorname{codim}(X \cap Y) = 2$.

PROOF. Since dim X = dim Y = 2 and $X \cap Y = \{0\}$ it follows that $\operatorname{codim}(X \cap Y) = 2$. $\square$

Lemma 19 *Let X and Y be the varieties of representations of the quiver*

$$\begin{array}{ccccc} & & \bullet & & \\ & & \uparrow\downarrow & & \\ Q: \quad \bullet & \rightleftarrows & \bullet & \rightleftarrows & \bullet \end{array}$$

given by

$$X = \begin{cases} \mathbb{C} \\ c \uparrow\downarrow d \\ \mathbb{C} \overset{b}{\underset{a}{\rightleftarrows}} \mathbb{C}^2 \overset{e}{\underset{0}{\rightleftarrows}} \mathbb{C} \\ ba, dc \in C((2)) \\ ab = cd = 0 \\ ba + dc = 0 \end{cases} \qquad Y = \begin{cases} \mathbb{C} \\ c \uparrow\downarrow d \\ \mathbb{C} \overset{0}{\underset{a}{\rightleftarrows}} \mathbb{C}^2 \overset{e}{\underset{f}{\rightleftarrows}} \mathbb{C} \\ fe, dc \in C((2)) \\ ef = cd = 0 \\ fe + dc = 0 \end{cases}$$

Then $\operatorname{codim}(X \cap Y) = 2$

PROOF. Hence $(ba)^2 = tr(ba)ba = tr(ab)ba$ the matrix ba is automatically nilpotent whenever $ab = 0$. Thus the conditions of the representation X can be written as

$$\begin{cases} b_1a_1 + d_1c_1 = 0 \\ b_1a_2 + d_1c_2 = 0 \\ b_2a_1 + d_2c_1 = 0 \\ b_2a_2 + d_2c_2 = 0 \\ b_1a_1 + b_2a_2 = 0 \\ d_1c_1 + d_2c_2 = 0. \end{cases}$$

We see that $a_1c_2 = a_2c_1$, for if not $b = d = 0$.

Similarly, we have also $\quad b_1d_2 = b_2d_1$. Thus

$$X = \{\big((a_1, a_2), (b_1, -\frac{a_1}{a_2}b_1), (c_1, \frac{a_2}{a_1}c_1), (-\frac{b_1a_1}{c_1}, \frac{a_1}{a_2}\frac{b_1a_1}{c_1}), (e_1, e_2), 0\big)\}$$

A similar argument gives

$$X = \{\big((a_1, a_2), 0, (c_1, \frac{e_2}{e_1}c_1), (-\frac{f_1e_1}{c_1}, \frac{e_1}{e_2}\frac{f_1e_1}{c_1}), (e_1, e_2), (f_1, -\frac{e_1}{e_2}f_1)\big)\}$$

So dim $X =$ dim $Y = 6$. Hence

$$X \cap Y = \{\big((a_1, a_2), (c_1, \frac{a_2}{a_1}c_1), (e_1, e_2)\big) \mid \frac{e_2}{e_1} = \frac{a_2}{a_1}\}$$

Thus dim $(X \cap Y) = 4$. This gives codim $(X \cap Y) = 2$. $\square$

Lemma 20 *Let X and Y be the varieties of representations of the quiver*

$$\bullet \rightleftarrows \bullet \rightleftarrows \bullet$$

given by

$$X = \begin{cases} \mathbb{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}^2 \underset{d}{\overset{c}{\rightleftarrows}} \mathbb{C} \\ ba, dc \in C\big((2)\big) \\ ab = cd = 0 \\ ba + dc = 0 \end{cases} \qquad Y = \quad \mathbb{C} \underset{a}{\overset{0}{\rightleftarrows}} \mathbb{C}^2 \underset{0}{\overset{c}{\rightleftarrows}} \mathbb{C}$$

Then $\operatorname{codim}(X \cap Y) = 1$

PROOF. The representation X can be written as (see the proof of Lemma 19)

$$X = \{\big((a_1, a_2), (b_1, -\frac{a_1}{a_2}b_1), (c_1, \frac{a_2}{a_1}), (-\frac{b_1a_1}{c_1}, \frac{a_1}{a_2}\frac{b_1a_1}{c_1})\big)\}$$

So dim X = dim $Y = 4$. Hence

$$X \cap Y = \{\big((a_1, a_2), (c_1, \frac{a_2}{a_1}c_1)\big)\}$$

Thus dim $(X \cap Y) = 3$. This gives codim $(X \cap Y) = 1$. $\square$

Lemma 21 *Let X and Y be the varieties of representations of the quiver*

$$Q: \begin{array}{ccccc} & & \bullet & & \\ & & \uparrow\downarrow & & \\ \bullet & \rightleftarrows & \bullet & \rightleftarrows & \bullet \end{array}$$

given by

$$X = \left\{ \begin{array}{c} \mathbb{C} \\ {}_c\uparrow\downarrow^d \\ \mathbb{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}^2 \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C} \\ ba, dc, fe \in C((2)) \\ ab = cd = ef = 0 \\ ba + dc + fe = 0 \end{array} \right. \qquad Y = \left\{ \begin{array}{c} \mathbb{C} \\ {}_c\uparrow\downarrow^0 \\ \mathbb{C} \underset{a}{\overset{0}{\rightleftarrows}} \mathbb{C}^2 \underset{0}{\overset{e}{\rightleftarrows}} \mathbb{C} \end{array} \right.$$

Then codim$(X \cap Y) = 2$.

PROOF. The condition of the representation X can be written as follows

$$\begin{cases} b_1a_1 + d_1c_1 + f_1e_1 = 0 \\ b_1a_2 + d_1c_2 + f_1e_2 = 0 \\ b_2a_1 + d_2c_1 + f_2e_1 = 0 \\ b_2a_2 + d_2c_2 + f_2e_2 = 0 \\ b_1a_1 + b_2a_2 = 0 \\ d_1c_1 + d_2c_2 = 0 \\ f_1e_1 + f_2e_2 = 0 \end{cases}$$

We going to show that $a_1c_2 = a_2c_1$. Indeed, for if in the contrary case we have

$$b_1 = \tfrac{x_1c_2 - x_2c_1}{a_1c_2 - a_2c_1} \quad b_2 = \tfrac{x_3c_2 + x_1c_1}{a_1c_2 - a_2c_1} = -\tfrac{x_1}{x_2}\tfrac{x_1c_2 - x_2c_1}{a_1c_2 - a_2c_1}$$

$$d_1 = \tfrac{a_1x_2 - a_2x_1}{a_1c_2 - a_2c_1} \quad d_2 = \tfrac{-a_1x_1 - a_2x_3}{a_1c_2 - a_2c_1} = -\tfrac{x_1}{x_2}\tfrac{a_1x_2 - a_2x_1}{a_1c_2 - a_2c_1},$$

where

$$x_1 := -f_1e_1 = f_2e_2, x_2 := -f_1e_2, x_3 = -f_2e_1.$$

Since $b_1a_1 + b_2a_2 = 0$ it follows that

$$(x_1c_2 - x_2c_1)(a_1x_2 - a_2x_1) = 0.$$

Hence $b = 0$ or $d = 0$, a contradiction.

A similar argument shows that

$$\begin{array}{llll} c_2 & = \frac{a_2}{a_1}c_1 & b_2 = -\frac{a_1}{a_2}b_1 & d_2 = -\frac{a_1}{a_2}d_1 \\ e_2 & = \frac{a_2}{a_1}e_1 & f_1 = -\frac{b_1a_1+d_1c_1}{e_1} & f_2 = \frac{a_1}{a_2}\frac{b_1a_1+d_1c_1}{e_1}. \end{array}$$

Thus dim X = dim $Y = 6$. Next,

$$X \cap Y = \{\big((a_1, a_2), (c_1, \frac{a_2}{a_1}c_1), (e_1, \frac{a_2}{a_1}e_1)\big)\}.$$

So codim$(X \cap Y) = 4$. It implies that codim$(X \cap Y) = 2$. $\square$

Lemma 22 *Let X and Y be the the varieties of representations of the quiver*

$$\begin{array}{cccccc} & & & \bullet & & \\ & & & \uparrow\downarrow & & \\ Q: & \bullet & \rightleftarrows & \bullet & \rightleftarrows & \bullet \end{array}$$

given by

$$X = \left\{ \begin{array}{c} \mathbb{C} \\ c \uparrow\downarrow d \\ \mathbb{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}^2 \underset{0}{\overset{e}{\rightleftarrows}} \mathbb{C} \\ ba, dc \in C((2)) \\ ab = cd = 0 \\ ba + dc = 0 \end{array} \right. \qquad Y = \left\{ \begin{array}{c} \mathbb{C} \\ c \uparrow\downarrow d \\ \mathbb{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}^2 \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C} \\ ba, dc, fe \in C((2)) \\ ab = cd = ef = 0 \\ ba + dc + fe = 0 \end{array} \right.$$

Then codim$(X \cap Y) = 1$.

PROOF. We have (see the proofs of Lemmas 19, 21)

$$X = \{\big((a_1, a_2), (b_1, -\frac{a_1}{a_2}b_1), (c_1, \frac{a_2}{a_1}c_1), (-\frac{b_1a_1}{c_1}, \frac{a_1}{a_2}\frac{b_1a_1}{c_1}), (e_1, e_2), 0\big)\}$$

$$Y = \{\big((a_1,a_2),\ (b_1,-\tfrac{a_1}{a_2}b_1),(c_1,\tfrac{a_2}{a_1}c_1),(d_1,-\tfrac{a_1}{a_2}d_1),(e_1,\tfrac{a_2}{a_1}e_1),$$
$$(\tfrac{b_1a_1+d_1c_1}{e_1},-\tfrac{a_1}{a_2}\tfrac{b_1a_1+d_1c_1}{e_1})\big)\}.$$

Hence

$$X\cap Y = \{\big((a_1,a_2),(b_1,-\frac{a_1}{a_2}b_1),(c_1,\frac{a_2}{a_1}c_1),(-\frac{b_1a_1}{c_1},\frac{a_1}{a_2}\frac{b_1a_1}{c_1}),(e_1,\frac{a_2}{a_1}e_1),\big)\}.$$

Thus dim $(X\cap Y)=5$. Since dim $X=$ dim $Y=6$ it follows that codimension of $(X\cap Y)$ is equal 1. $\square$

Theorem 17 *1) Two irreducible components of the variety* $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{nil}$ *of the same type do not meet.*

2) Two irreducible components of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4),\delta)_{nil}$ *of type 1 and 3 do not meet.*

3) A component C_I of type 1 meets a component C_{J_1,J_2} of type 2 if and only if

$$J_2 = I\cap(\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}\setminus J_1)$$

4) A component C_{J_1,J_2} of type 2 meets a component C_{L_1,L_2} of type 3 if and only if

$$J_1\subset L_1 \quad \textit{and} \quad L_2 = J_2\cap(\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}\setminus L_1)$$

PROOF. 1) a) Suppose C_I and C'_I are the different components of type 1. Then there exists a subquiver $Q=\bullet \rightleftarrows \bullet$ such that restriction of C_I and C'_I to this subquiver are the varieties of representations X and Y as in Lemma 18. So $\mathrm{codim}(C_I\cap C'_I)\geq 2$, and thus C_I does not meet C'_I.

b) Suppose C_{J_1,J_2} and $C_{J'_1,J'_2}$ are the different components of type 2. We use arguments similar as in a). There are 3 cases:

If $|J_1\cap J'_1|=2$ we apply Lemma 18.

If $|J_1\cap J'_1|=1$ we apply Lemma 19.

If $|J_1\cap J'_1|=0$ we apply Lemma 20.

c) Suppose C_{L_1,L_2} and $C_{L'_1,L'_2}$ are the different components of type 3. There are 2 cases:

If $|L_1 \cap L_1| = 3$ we apply Lemma 18.

If $|L_1 \cap L_1| = 2$ we apply Lemma 19.

2) It follows from Lemma 21.

3) If $J_2 \neq I \cap (\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \setminus J_1)$, then we use similar arguments as in 1). Since Lemma 18 it follows that $\operatorname{codim}(C_I \cap C_{J_1,J_2}) \geq 2$. For the converse, by applying Lemma 20, we have $\operatorname{codim}(C_I \cap C_{J_1,J_2}) = 1$.

4) If $J_1 \not\subseteq L_1$, without loss of generallity, we suppose that

$$X = \left\{ \begin{array}{c} \mathbb{C} \\ {}_c\uparrow\downarrow^d \\ \mathbb{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathbb{C}^2 \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C} \\ {}_0\uparrow\downarrow^h \\ \mathbb{C} \\ ba, dc, fe \in C((2)) \\ ab = cd = ef = 0 \\ ba + dc + fe = 0 \end{array} \right. \qquad Y = \left\{ \begin{array}{c} \mathbb{C} \\ {}_c\uparrow\downarrow 0 \\ \mathbb{C} \underset{a}{\overset{0}{\rightleftarrows}} \mathbb{C}^2 \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C} \\ {}_g\uparrow\downarrow^h \\ \mathbb{C} \\ fe, gh \in C((2)) \\ ef = hg = 0 \\ fe + gh = 0 \end{array} \right.$$

We have (see the proofs of Lemmas 22 and Lemma 19):

$$\begin{aligned} X = \ & \{\big((a_1, a_2), \ (b_1, -\tfrac{a_1}{a_2} b_1), (c_1, \tfrac{a_2}{a_1} c_1), (d_1, -\tfrac{a_1}{a_2} d_1), (e_1, \tfrac{a_2}{a_1} e_1), \\ & \qquad (\tfrac{b_1 a_1 + d_1 c_1}{e_1}, -\tfrac{a_1}{a_2} \tfrac{b_1 a_1 + d_1 c_1}{e_1}), (h_1, h_2), 0\big)\}. \\ Y = \ & \{\big((a_1, a_2), \ 0, (c_1, c_2), 0, (e_1, e_2), \\ & \qquad (f_1, \tfrac{-e_1}{e_2} f_1), (h_1, \tfrac{e_2}{e_1} h_1), (-\tfrac{f_1 e_1}{h_1}, \tfrac{e_1}{e_2} - \tfrac{f_1 e_1}{h_1})\big)\}. \end{aligned}$$

Hence
$$X \cap Y = \{\big((a_1, a_2), (c_1, \frac{a_2}{a_1} c_1), (e_1, \frac{a_2}{a_1} e_1), (h_1, \frac{e_2}{e_1} h_1), \big)\}.$$

Thus dim $(X \cap Y) = 6$. Since dim X = dim Y = 8 it follows that codim $(X \cap Y)$ $= 2$.

If $J_1 \subseteq L_1$ and $L_2 \neq J_2 \cap (\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \setminus J_1)$, then from Lemma 18 it follows codim $(C_{J_1,J_2}) \geq 2$.

For the converse we apply Lemma 22. $\square$

To describe the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$ of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4), \delta)^{\mathrm{stab}}_{\mathrm{nil}}$ we take a 4-dimemsional hypercube H^4. It has

16 vertices,
24 facets of dimension 2,
8 facets of dimension 3.

We write O_i^1, $i = 1, \ldots, 16$, for the vertices of H^4; write O_j^2, $i = j, \ldots, 24$, for the centers of the 2-facets and write O_i^3, $i = 1, \ldots, 8$, for the centers of the 3-facets. Then the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$ of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4), \delta)^{\mathrm{stab}}_{\mathrm{nil}}$ is described as follows.

Theorem 18 *The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$ has 48 vertices. They are arranged in H^4 as follows.*

The 16 vertices corresponding to the components of type 1 are the vertices O_i^1 of H^4. The 24 vertices corresponding to the components of type 2 are the centers O_j^2 of the 2-facets of H^4. The 8 vertices corresponding to the components of type 3 are the centers O_k^3 of the 3-facets of H^4.

Vertices O_i^1 and O_j^2 are connected by an edge if the vertex O_i^1 belongs to a 2-facet whose center is O_j^2.

Vertices O_j^2 and O_k^3 are connected by an edge if the 2-facet whose center is O_j^2 belongs to a 3-facet whose center is O_k^3.

PROOF. We assume that $H^4 = [-1, 1]^4$. Then the coordinates of the points O_i^1, O_j^2, O_k^3 has the forms:

$$O_i^1 : \qquad (\pm 1, \pm 1, \pm 1, \pm 1)$$

$$O_j^2 : \qquad (\pm 1, \pm 1, 0, 0)$$

$$O_k^3 : \qquad (\pm 1, 0, 0, 0)$$

We associate to the irreducible components of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_4), \delta)^{\mathrm{stab}}_{\mathrm{nil}}$ the set of points O_i^1, O_j^2, O_k^3 in H^4 by the following assignment:

A component C_I of type 1 corresponds to a point M_I^1 in H^4, where M_I^1 denotes a point in $\mathbb{R}^4$ whose i-th coordinate is 1 if $\alpha_i \in I$, and is -1 if $\alpha_i \notin I$. Thus, M_I^1 is a vertex of H^4.

A component C_{J_1,J_2} of type 2 corresponds to a point M_{J_1,J_2}^2 in H^4, where M_{J_1,J_2}^2 denotes a point in $\mathbb{R}^4$ whose i-th coordinate is 0 if $\alpha_i \in J_1$, is 1 if $\alpha_i \in J_2$, and is -1 if $\alpha_i \notin J_2$. Thus, M_{J_1,J_2}^2 is the center of a 2-facet of H^4.

A component C_{L_1,L_2} of type 3 corresponds to a point M_{L_1,L_2}^3 in H^4, where M_{L_1,L_2}^3 denotes a point in $\mathbb{R}^4$ whose i-th coordinate is 0 if $\alpha_i \in L_1$, is 1 if $\alpha_i \in L_2$, and is -1 if $\alpha_i \notin L_2$. Thus, M_{J_1,J_2}^3 is the center of a 3-facet of H^4

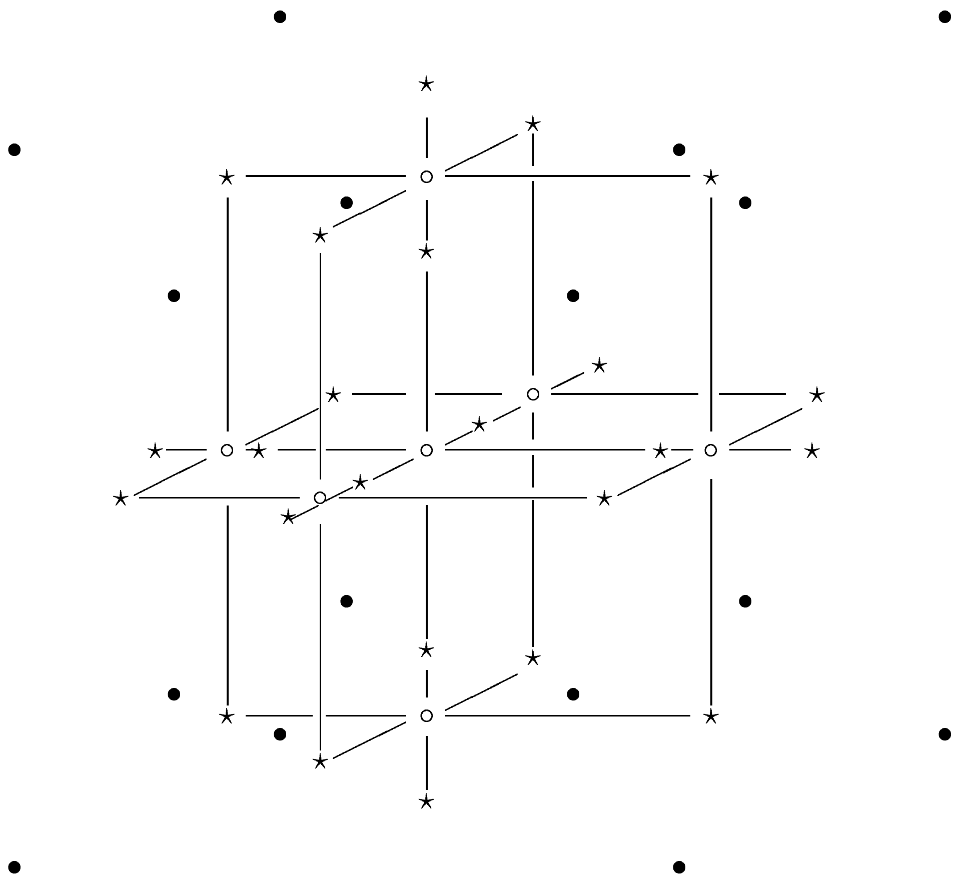
By Theorem 17 the component C_I of type 1 meets the component C_{J_1,J_2} of type 2 if and only if

$$J_2 = I \cap (\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \setminus J_1)$$

This means the i-coordinates of the corresponding points M_I^1 and M_{J_1,J_2}^2 are the same for two different values of $i \in \{1,2,3,4\}$, i.e. the points M_I^1 and M_{J_1,J_2}^2 have the distance $\sqrt{2}$. In other words M_I^1 belongs to an 2-facet whose center is M_{J_1,J_2}^2.

We use a similar argument for the other cases. $\square$

The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$ of $\operatorname{Rep}(\Pi(\mathbb{D}_4,\delta))_{\mathrm{nil}}^{\mathrm{stab}}$ is illustrated as in Figure 4 below. Here the 16 vertices of H^4 are marked by a thick dot $\bullet$, the 24 centers of the 2-facettes are marked by a star $\star$ and the 8 centers of 3-facettes with seven centers marked by $\circ$ and one center at infinity.



The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$

Figure 4

7.3 The action of the Weyl group on the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_4)$

We consider the Dynkin quiver Q of type $\tilde{\mathbb{D}}_4$:

$$\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 0 & 3 \arrow[l] \\ & 4 \arrow[u] & \end{tikzcd}$$

The reflections $r_0, r_1, r_2, r_3, r_4 : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_0}$ which generate the Weyl group are given by (see Sect. 3.6):

$$\begin{aligned} r_0(t_0,t_1,t_2,t_3,t_4) &= (-t_0+t_1+t_2+t_3+t_4, t_1, t_2, t_3, t_4), \\ r_1(t_0,t_1,t_2,t_3,t_4) &= (t_0, t_0-t_1, t_2, t_3, t_4), \\ r_2(t_0,t_1,t_2,t_3,t_4) &= (t_0, t_1, t_0-t_2, t_3, t_4), \\ r_3(t_0,t_1,t_2,t_3,t_4) &= (t_0, t_1, t_2, t_0-t_3, t_4), \\ r_4(t_0,t_1,t_2,t_3,t_4) &= (t_0, t_1, t_2, t_3, t_0-t_4). \end{aligned}$$

The reflections r_0, r_1, r_2, r_3, r_4 act as reflections on the set of dimension vectors. They induce an action on the set of irreducible components of the variety $\mathrm{Rep}(\Pi(\tilde{\mathbb{D}}_4, \delta)_{\mathrm{nil}}$:

$$r_i : \mathrm{Irr}(\mathrm{Rep}(\Pi(\tilde{\mathbb{D}}_4), \delta)_{\mathrm{nil}})) \longrightarrow \mathrm{Irr}(\mathrm{Rep}(\Pi(\tilde{\mathbb{D}}_4), \delta)_{\mathrm{nil}}))$$

We shall describe this action in the following lemma.

Lemma 23 *1) The action of the reflections r_k, for $k = 1, 2, 3, 4$.*

Let C_I be a component of type 1 and let C_{J_1,J_2} be a component of type 2 or of type 3. Then

$$r_k(C_I) = \begin{cases} C_{I \setminus \{\alpha_k\}} & \text{if } \alpha_k \in I \\ C_{I \cup \{\alpha_k\}} & \text{if } \alpha_k \notin I \end{cases}$$

$$r_k(C_{J_1,J_2}) = \begin{cases} C_{J_1,J_2} & \text{if } \alpha_k \in J_1 \\ C_{J_1,J_2 \setminus \{\alpha_k\}} & \text{if } \alpha_k \in J_2 \\ C_{J_1,J_2 \cup \{\alpha_k\}} & \text{if } \alpha_k \notin J_2. \end{cases}$$

2) The action of the reflection r_0.

Let C_I be a component of type 1, C_{J_1,J_2} be a component of type 2, and C_{L_1,L_2} be a compnent of type 3. Then

$$r_0(C_I) = \begin{cases} C_{\Omega\setminus I,I} & \text{if } \mid I \mid = 1 \\ C_I & \text{if } \mid I \mid = 2 \\ C_{I,\varnothing} & \text{if } \mid I \mid = 3 \\ C_{\varnothing} & \text{if } \mid I \mid = 4 \end{cases}$$

$$r_0(C_{J_1,J_2}) = \begin{cases} C_{J_1,J_2} & \text{if } \mid J_2 \mid = 1 \\ C_{\Omega\setminus J_1,J_1} & \text{if } \mid J_2 \mid = 0 \\ C_{\Omega\setminus J_1,\varnothing} & \text{if } \mid J_2 \mid = 2 \end{cases}$$

$$r_0(C_{L_1,L_2}) = \begin{cases} C_{L_1} & \text{if } \mid L_2 \mid = 0 \\ C_{L_2} & \text{if } \mid L_2 \mid = 1 \end{cases}$$

PROOF. It follows from Table 1 of the dimension vectors of indecomposable subrepresentations of $\mathrm{Rep}(\Pi(\tilde{\mathbb{D}}_4, \delta)_{\mathrm{nil}}$. We calculate for some cases:

$\downarrow$ $\rightarrow . \leftarrow$ $\uparrow$	0 0 1 0 0	0 1 1 0 0	1 0 1 0 0	0 0 1 1 0	0 0 1 0 1	1 1 2 1 0	1 0 2 1 1	0 1 2 1 1	1 1 2 0 1

$\downarrow r_0$

$\uparrow$ $\leftarrow . \rightarrow$ $\downarrow$	1 1 1 1 1	0 1 0 0 0	1 0 0 0 0	0 0 0 1 0	0 0 0 0 1	1 1 1 1 0	1 0 1 1 1	0 1 1 1 1	1 1 1 0 1

$\uparrow$ $\rightarrow . \rightarrow$ $\downarrow$	0 0 0 0 1	0 0 0 1 0	1 0 0 0 0	1 0 1 0 1	0 0 1 1 1	1 0 1 1 0	1 0 1 1 1	1 1 1 1 1	1 0 2 1 1

$\downarrow r_0$

$\uparrow\downarrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	0 0 1 0 1	0 0 1 1 0	1 0 1 0 0	1 0 1 0 0	0 0 1 1 1	1 0 1 1 0	1 0 2 1 1	0 0 1 0 0	1 0 1 1 1

$\downarrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	0 0 1 0 0	0 0 1 1 0	0 0 1 0 1	0 0 1 1 1	1 0 2 1 1	0 1 2 1 1

$\downarrow r_0$

$\uparrow\downarrow$ $\rightleftarrows . \rightarrow$ $\downarrow$	1 1 1 1 1	0 0 0 1 0	0 0 0 0 1	0 0 1 1 1	1 0 1 1 1	0 1 1 1 1

$\uparrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	1 0 0 0 0	1 0 1 0 0	1 0 1 1 0	1 0 1 0 1	1 0 1 1 1	1 0 2 1 1

$\downarrow r_0$

$\uparrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	1 0 1 0 0	1 0 0 0 0	1 0 1 1 0	1 0 1 0 1	1 0 2 1 1	1 0 1 1 1

$\downarrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	0 0 1 0 0	0 0 1 1 0	0 0 1 0 1	0 0 1 1 1	1 0 2 1 1	0 1 2 1 1

$\downarrow r_1$

$\downarrow$ $\leftarrow . \rightleftarrows$ $\uparrow\downarrow$	0 1 1 0 0	0 1 1 1 0	0 1 1 0 0	0 1 1 1 1	0 1 0 0 0	0 1 2 1 1

$\square$

Components	Dimension of indecomposable subrepresentations								
$\downarrow$ $\rightarrow \cdot \leftarrow$ $\uparrow$	0 0 1 0 0	0 1 1 0 0	1 0 1 0 0	0 0 1 1 0	0 0 1 0 1	1 1 2 1 0	1 0 2 1 1	0 1 2 1 1	1 1 2 0 1
$\downarrow$ $\rightarrow \cdot \leftarrow$ $\downarrow$	0 0 0 0 1	0 0 1 0 0	0 0 1 0 1	0 0 1 1 1	0 1 1 0 1	1 0 1 0 1	1 0 2 1 1	0 1 2 1 1	1 1 2 0 1
$\downarrow$ $\leftarrow \cdot \leftarrow$ $\uparrow$	0 1 0 0 0	0 0 1 0 0	0 1 1 0 0	0 1 1 0 1	1 1 1 0 0	0 1 1 1 0	0 1 2 1 1	1 1 2 0 1	1 1 2 1 0
$\uparrow$ $\rightarrow \cdot \leftarrow$ $\uparrow$	1 0 0 0 0	0 0 1 0 0	1 0 1 0 0	1 1 1 0 0	1 0 1 1 0	1 0 1 0 1	1 1 2 0 1	1 1 2 1 0	1 0 2 1 1
$\downarrow$ $\rightarrow \cdot \rightarrow$ $\uparrow$	0 0 0 1 0	0 0 1 0 0	0 0 1 1 0	1 0 1 1 0	0 0 1 1 1	0 1 1 1 0	1 1 2 1 0	1 0 2 1 1	0 1 2 1 1
$\uparrow$ $\leftarrow \cdot \leftarrow$ $\uparrow$	1 0 0 0 0	0 1 0 0 0	1 0 1 0 0	0 1 1 0 0	1 1 1 0 0	1 1 1 0 1	1 1 1 1 0	1 1 2 1 0	1 1 2 0 1
$\uparrow$ $\rightarrow \cdot \rightarrow$ $\uparrow$	0 0 0 1 0	1 0 0 0 0	0 0 1 1 0	1 0 1 0 0	1 0 1 1 0	1 1 1 1 0	1 0 1 1 1	1 0 2 1 1	1 1 2 1 0
$\downarrow$ $\rightarrow \cdot \rightarrow$ $\downarrow$	0 0 0 1 0	0 0 0 0 1	0 0 1 0 1	0 0 1 1 0	0 0 1 1 1	1 0 1 1 1	0 1 1 1 1	1 0 2 1 1	0 1 2 1 1
$\downarrow$ $\leftarrow \cdot \leftarrow$ $\downarrow$	0 0 0 0 1	0 1 0 0 0	0 1 1 0 0	0 0 1 0 1	0 1 1 0 1	0 1 1 1 1	1 1 1 0 1	0 1 2 1 1	1 1 2 0 1
$\uparrow$ $\rightarrow \cdot \leftarrow$ $\downarrow$	1 0 0 0 0	0 0 0 0 1	1 0 1 0 0	0 0 1 0 1	1 0 1 0 1	1 0 1 1 1	1 1 1 0 1	1 0 2 1 1	1 1 2 0 1
$\downarrow$ $\leftarrow \cdot \rightarrow$ $\uparrow$	0 0 0 1 0	0 1 0 0 0	0 0 1 1 0	0 1 1 0 0	0 1 1 1 0	1 1 1 1 0	0 1 1 1 1	1 1 2 1 0	0 1 2 1 1
$\uparrow$ $\leftarrow \cdot \rightarrow$ $\uparrow$	0 0 0 1 0	1 0 0 0 0	0 1 0 0 0	0 1 1 1 0	1 0 1 1 0	1 1 1 0 0	1 1 1 1 0	1 1 1 1 1	1 1 2 1 0
$\uparrow$ $\rightarrow \cdot \rightarrow$ $\downarrow$	0 0 0 0 1	0 0 0 1 0	1 0 0 0 0	1 0 1 0 1	0 0 1 1 1	1 0 1 1 0	1 0 1 1 1	1 1 1 1 1	1 0 2 1 1
$\downarrow$ $\leftarrow \cdot \rightarrow$ $\downarrow$	0 1 0 0 0	0 0 0 0 1	0 0 0 1 0	0 1 1 1 0	0 1 1 0 1	0 0 1 1 1	0 1 1 1 1	1 1 1 1 1	0 1 2 1 1
$\uparrow$ $\leftarrow \cdot \leftarrow$ $\downarrow$	1 0 0 0 0	0 1 0 0 0	0 0 0 0 1	1 0 1 0 1	1 1 1 0 0	0 1 1 0 1	1 1 1 0 1	1 1 1 1 1	1 1 2 0 1
$\uparrow$ $\leftarrow \cdot \rightarrow$ $\downarrow$	0 0 0 1 0	1 0 0 0 0	0 1 0 0 0	0 0 0 0 1	1 1 1 1 0	1 1 1 0 1	0 1 1 1 1	1 0 1 1 1	1 1 1 1 1

Components	Dimension of indecomposable subrepresentations					
↑↓ ⇄ . → ↑	0 0 0 1 0	0 0 1 1 0	0 1 1 1 0	1 0 1 1 0	1 1 1 1 0	1 1 2 1 0
↑↓ → . ⇄ ↓	0 0 0 0 1	0 0 1 0 1	1 0 1 0 1	0 0 1 1 1	1 0 1 1 1	1 0 2 1 1
↓ ← . ⇄ ↑↓	0 1 0 0 0	0 1 1 0 0	0 1 1 1 0	0 1 1 0 1	0 1 1 1 1	0 1 2 1 1
↑ ⇄ . ← ↑↓	1 0 0 0 0	1 0 1 0 0	1 0 1 0 1	1 1 1 0 0	1 1 1 0 1	1 1 2 0 1
↑↓ ← . ← ↑↓	0 1 0 0 0	0 1 1 0 0	0 1 1 0 1	1 1 1 0 0	1 1 1 0 1	1 1 2 0 1
↑ ⇄ . ⇄ ↑	1 0 0 0 0	1 0 1 0 0	1 1 1 0 0	1 0 1 1 0	1 1 1 1 0	1 1 2 1 0
↑↓ ⇄ . ← ↓	0 0 0 0 1	0 0 1 0 1	0 1 1 0 1	1 0 1 0 1	1 1 1 0 1	1 1 2 0 1
↑↓ ← . ⇄ ↑	0 1 0 0 0	0 1 1 0 0	1 1 1 0 0	0 1 1 1 0	1 1 1 1 0	1 1 2 1 0
↑ → . ⇄ ↑↓	1 0 0 0 0	1 0 1 0 0	1 0 1 1 0	1 0 1 0 1	1 0 1 1 1	1 0 2 1 1
↓ ⇄ . → ↑↓	0 0 0 1 0	0 0 1 1 0	0 0 1 1 1	0 1 1 1 0	0 1 1 1 1	0 1 2 1 1
↑↓ → . → ↑↓	0 0 0 1 0	0 0 1 1 0	0 0 1 1 1	1 0 1 1 0	1 0 1 1 1	1 1 2 1 1
↓ ⇄ . ⇄ ↓	0 0 0 0 1	0 0 1 0 1	0 1 1 0 1	0 0 1 1 1	0 1 1 1 1	0 1 2 1 1

Components	Dimension of indecomposable subrepresentations					
$\uparrow\downarrow$ $\rightleftarrows . \rightarrow$ $\downarrow$	0 0 0 0 1	0 0 0 1 0	0 0 1 1 1	0 1 1 1 1	1 0 1 1 1	1 1 1 1 1
$\uparrow\downarrow$ $\leftarrow . \rightleftarrows$ $\downarrow$	0 1 0 0 0	0 0 0 0 1	0 1 1 0 1	1 1 1 0 1	0 1 1 1 1	1 1 1 1 1
$\uparrow$ $\leftarrow . \rightleftarrows$ $\uparrow\downarrow$	1 0 0 0 0	0 1 0 0 0	1 1 1 0 0	1 1 1 1 0	1 1 1 0 1	1 1 1 1 1
$\uparrow$ $\rightleftarrows . \rightarrow$ $\uparrow\downarrow$	0 0 0 1 0	1 0 0 0 0	1 0 1 1 0	1 0 1 1 1	1 1 1 1 0	1 1 1 1 1
$\uparrow\downarrow$ $\leftarrow . \rightarrow$ $\uparrow\downarrow$	0 1 0 0 0	0 0 0 1 0	0 1 1 1 0	1 1 1 1 0	0 1 1 1 1	1 1 1 1 1
$\uparrow$ $\rightleftarrows . \rightleftarrows$ $\downarrow$	1 0 0 0 0	0 0 0 0 1	1 0 1 0 1	1 0 1 1 1	1 1 1 0 1	1 1 1 1 1
$\uparrow\downarrow$ $\rightleftarrows . \leftarrow$ $\uparrow$	0 0 1 0 0	0 1 1 0 0	1 0 1 0 0	1 1 1 0 0	1 1 2 0 1	1 1 2 1 0
$\uparrow\downarrow$ $\rightarrow . \rightleftarrows$ $\uparrow$	0 0 1 0 0	1 0 1 0 0	0 0 1 1 0	1 0 1 1 0	1 1 2 1 0	1 0 2 1 1
$\downarrow$ $\rightarrow . \rightleftarrows$ $\uparrow\downarrow$	0 0 1 0 0	0 0 1 1 0	0 0 1 0 1	0 0 1 1 1	1 0 2 1 1	0 1 2 1 1
$\downarrow$ $\rightleftarrows . \leftarrow$ $\uparrow\downarrow$	0 0 1 0 0	0 0 1 0 1	0 1 1 0 0	0 1 1 0 1	0 1 2 1 1	1 1 2 0 1
$\uparrow\downarrow$ $\rightarrow . \leftarrow$ $\uparrow\downarrow$	0 0 1 0 0	1 0 1 0 0	0 0 1 0 1	1 0 1 0 1	1 0 2 1 1	1 1 2 0 1
$\downarrow$ $\rightleftarrows . \rightleftarrows$ $\uparrow$	0 0 1 0 0	0 0 1 1 0	0 1 1 0 0	0 1 1 1 0	1 1 2 1 1	1 1 2 1 0

Components	Dimension of indecomposable subrepresentations								
↑↓ ⇄.⇄ ↑	0 0 1 0 0	0 0 1 1 0	1 0 1 0 0	0 1 1 0 0	1 0 1 1 0	1 1 1 0 0	0 1 1 1 0	1 1 1 1 0	1 1 2 1 0
↑↓ →.⇄ ↑↓	0 0 1 0 0	0 0 1 0 1	0 0 1 1 0	1 0 1 0 0	0 0 1 1 1	1 0 1 1 0	1 0 1 0 1	1 0 1 1 1	1 0 2 1 1
↓ ⇄.⇄ ↑↓	0 0 1 0 0	0 1 1 0 0	0 0 1 0 1	0 0 1 1 0	0 1 1 0 1	0 0 1 1 1	0 1 1 1 0	0 1 1 1 1	0 1 2 1 1
↑↓ ⇄.← ↑↓	0 0 1 0 0	1 0 1 0 0	0 1 1 0 0	0 0 1 0 1	1 1 1 0 0	0 1 1 0 1	1 0 1 0 1	1 1 1 0 1	1 1 2 0 1
↑↓ ⇄.⇄ ↓	0 0 0 0 1	0 0 1 0 1	1 0 1 0 1	0 0 1 1 1	0 1 1 0 1	1 0 1 1 1	0 1 1 1 1	1 1 1 0 1	1 1 1 1 1
↑↓ ←.⇄ ↑↓	0 1 0 0 0	0 1 1 0 0	0 1 1 1 0	0 1 1 0 1	1 1 1 0 0	0 1 1 1 1	1 1 1 0 1	1 1 1 1 0	1 1 1 1 1
↑ ⇄.⇄ ↑↓	1 0 0 0 0	1 0 1 0 0	1 0 1 0 1	1 1 1 0 0	1 0 1 1 0	1 1 1 0 1	1 1 1 1 0	1 0 1 1 1	1 1 1 1 1
↑↓ ⇄.→ ↑↓	0 0 0 1 0	0 0 1 1 0	0 1 1 1 0	1 0 1 1 0	0 0 1 1 1	1 1 1 1 0	1 0 1 1 1	0 1 1 1 1	1 1 1 1 1
↑↓ ⇄.⇄ ↑↓	0 0 1 0 0	1 0 1 0 0	0 0 1 1 0	0 0 1 0 1	0 1 1 0 0	1 1 1 0 0	1 0 1 1 0	0 0 1 1 1	0 1 1 0 1
	1 0 1 0 1	0 1 1 1 0	1 1 1 0 1	1 1 1 1 0	1 0 1 1 1	0 1 1 1 1	1 1 1 1 1		

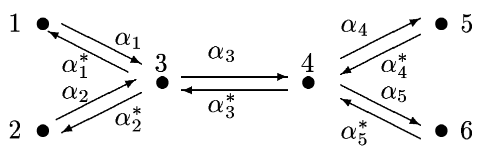
The dimension vectors of indecomposable subpresentations of
the irreducible components of $\mathrm{Rep}(\Pi(\tilde{\mathbb{D}}_4), \delta)_{\mathrm{nil}}$

Table 1

7.4 The nilpotent variety $\mathbf{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta)_{\mathbf{nil}}$

Let $\overline{Q}(\Gamma) = (\overline{Q}_0, \overline{Q}_1, s, t)$ be the McKay quiver of type $\widetilde{\mathbb{D}_5}$. We assume that $\overline{Q}_0 = \{1, 2, 3, 4, 6\}$ and that $\overline{Q}_1$ consists of the arrows

$$\begin{array}{llll} \alpha_1: & 1 \to 3 & \alpha_1^*: & 3 \to 1 \\ \alpha_2: & 2 \to 3 & \alpha_2^*: & 3 \to 2 \\ \alpha_3: & 3 \to 4 & \alpha_3^*: & 4 \to 3 \\ \alpha_4: & 4 \to 5 & \alpha_4^*: & 5 \to 4 \\ \alpha_5: & 4 \to 6 & \alpha_5^*: & 6 \to 4. \end{array}$$



Since $\delta = (1, 1, 2, 2, 1, 1)$ we have

$$\begin{array}{rll} \operatorname{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) & = \{(a_i, a_i^*)_i \mid & (a_i, a_i^*) \in \mathbb{C}^2 \oplus \mathbb{C}^2, i = 1, 2, 4, 5 \\ & & (a_3, a_3^*) \in \mathbb{C}^4 \oplus \mathbb{C}^4 \\ & & a_1^* a_1 = 0, a_2^* a_2 = 0, a_4 a_4^* = 0, a_5 a_5^* = 0 \\ & & a_3^* a_3 - a_1 a_1^* - a_2 a_2^* = 0 \\ & & a_4^* a_4 + a_5^* a_5 - a_3 a_3^* = 0\}. \end{array}$$

Let Ω be the subset of $\overline{Q}_1$ consisting of all arrows $\alpha_i, i = 1, 2, 3, 4, 5$. The nilpotent variety $\operatorname{Rep}(\Pi(\tilde{\mathbb{D}})_5, \delta)_{\mathrm{nil}}$ has 174 irreducible components. We shall describe these components in detail.

1) **The components of type 1.**

Let $J_1 = \{\alpha_3, \alpha_k, \alpha_l \mid k < 3 < l\}$ be a subset consisting of 3 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{array}{rll} C^{\overrightarrow{<}}_{J_1, J_2} & := \operatorname{Cl}(\{(a_i, a_i^*)_i \in \operatorname{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ & & \operatorname{rank} a_3 = 2, \operatorname{rank} a_3^* = 1, \\ & & a_3 a_3^*,\ a_3^* a_3 \in C((2))\}) \end{array}$$

$$C^{\overleftarrow{>}}_{J_1,J_2} := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ \text{rank } a_3 = 1, \text{ rank } a_3^* = 2, \\ a_3 a_3^*,\ a_3^* a_3 \in C((2))\}). \end{array}$$

Then $C^{\overrightarrow{<}}_{J_1,J_2}$ and $C^{\overleftarrow{>}}_{J_1,J_2}$ are the irreducible components of the variety $\mathrm{Rep}(\Pi(\tilde{\mathbb{D}})_5,\delta)_{\mathrm{nil}}$. They are called the components of type 1. There are 32 components of type 1.

2) **The components of type 2.**

2a) Let I be a subset of Ω. We define

$$C_I := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in I, \\ a_j = 0 \text{ for } \alpha_j \notin I\} \end{array}$$

2b) Let $J_1 = \{\alpha_3, \alpha_k\}$ be a subset consisting of 2 elements of Ω, and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$C^{>}_{\{\alpha_3,\alpha_k,3>k\},J_2} := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ a_3 a_3^* = 0,\ a_3^* a_3 \in C((2))\}), \end{array}$$

$$C^{<}_{\{\alpha_3,\alpha_k,3<k\},J_2} := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ a_3 a_3^* \in C((2)),\ a_3^* a_3 = 0\}). \end{array}$$

2c) Let $J_1 = \{\alpha_3, \alpha_k, \alpha_l, \alpha_m\}$ be a subset consisting of 4 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$C^{\overrightarrow{<}}_{J_1,J_2} := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ \text{rank } a_3 = 2, \text{ rank } a_3^* = 1, \\ a_3 a_3^*,\ a_3^* a_3 \in C((2))\}), \end{array}$$

$$C^{\overleftarrow{>}}_{J_1,J_2} := \mathrm{Cl}(\{(a_i,a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}),\delta) \mid \begin{array}{l} a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ \text{rank } a_3 = 1, \text{ rank } a_3^* = 2, \\ a_3 a_3^*,\ a_3^* a_3 \in C((2))\}). \end{array}$$

There are 32 components in the case 2a), 32 components in the case 2b), and 16 components in the case 2c). We call them the components of type 2.

3) **The components of type 3.**

3a) Let $J_1 = \{\alpha_k, \alpha_l; k, l \neq 3; \mid k-l \mid = 1\}$ be a subset consisting of 2 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C_{J_1,J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2\}). \end{aligned}$$

3b) Let $J_1 = \{\alpha_3\}$ be a subset of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C_{\{\alpha_3\},J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin \{\alpha_3\} \cup J_2\}). \end{aligned}$$

3c) Let $J_1 = \{\alpha_3, \alpha_k, \alpha_l, \alpha_m\}$ be a subset consisting of 4 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C^{>}_{\{\alpha_3,\alpha_k,\alpha_l,\alpha_m;k,l>3\},J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \quad & \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \\ & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ & a_3 a_3^* = 0,\ a_3^* a_3 \in C((2))\}), \end{aligned}$$

$$\begin{aligned} C^{<}_{\{\alpha_3,\alpha_k,\alpha_l,\alpha_m;k,l<3\},J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \quad & \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \\ & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ & a_3 a_3^* \in C((2)),\ a_3^* a_3 = 0\}). \end{aligned}$$

There are 16 components in the case 3a), 16 components in the case 3b), and 8 components in the case 3c). We call them the components of type 3.

4) **The components of type 4.**

4a) Let $J_1 = \{\alpha_3, \alpha_k, \alpha_l; \mid k-l \mid = 1\}$ be a subset consisting of 3 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C_{J_1,J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2\}). \end{aligned}$$

4b) Let $J_1 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$ be a subset consisting of 4 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C_{J_1,J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2\}). \end{aligned}$$

There are 8 components in the case 4a) and 2 components in the case 4b). We call them the components of type 4.

5) **The components of type G.**

$G1$) Let $J_1 = \{\alpha_3, \alpha_k, \alpha_l, \}$ be a subset consisting of 3 elements of Ω and let J_2 be a subset of $\Omega \setminus J_1$. We define

$$\begin{aligned} C^{>}_{\{\alpha_3,\alpha_k,\alpha_l,;k,l<3\},J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \quad & \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \\ & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ & a_3a_3^* = 0,\ a_3^*a_3 \in C((2))\}), \end{aligned}$$

$$\begin{aligned} C^{<}_{\{\alpha_3,\alpha_k,\alpha_l,\alpha_m;k,l>3\},J_2} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \quad & \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \\ & a_i^* = 0 \text{ for } \alpha_i \in J_2, \\ & a_j = 0 \text{ for } \alpha_j \notin J_1 \cup J_2 \\ & a_3a_3^* \in C((2)),\ a_3^*a_3 = 0\}). \end{aligned}$$

$G2$) We define

$$\begin{aligned} C^{\vec{<}}_{\Omega} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & \mathrm{rank}\ a_3 = 2,\ \mathrm{rank}\ a_3^* = 1, \\ & a_3a_3^*,\ a_3^*a_3 \in C((2))\}) \end{aligned}$$

$$\begin{aligned} C^{\overleftarrow{>}}_{\Omega} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad & \mathrm{rank}\ a_3 = 1,\ \mathrm{rank}\ a_3^* = 2, \\ & a_3a_3^*,\ a_3^*a_3 \in C((2))\}). \end{aligned}$$

$G3$) We define

$$C^{<}_{\Omega} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad a_3a_3^* \in C((2)),\ a_3^*a_3 = 0\ \})$$

$$C^{>}_{\Omega} \quad := \mathrm{Cl}(\{(a_i, a_i^*)_i \in \mathrm{Rep}(\Pi(\widetilde{\mathbb{D}_5}), \delta) \mid \quad a_3a_3^* = 0\ a_3^*a_3 \in C((2))\}).$$

There are 8 components in the case $G1$), 2 components in the case $G2$), and 2 components in the case $G3$).

Lemma 24 *The components of type G are the ghost component, i.e. they are not stable.*

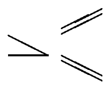
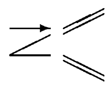
PROOF. The components $C^{<}_{\{\alpha_3,\alpha_k,\alpha_l,\alpha_m;k,l>3\},J_2}$, $C^{\lessgtr}_{\Omega}$ and $C^{\overleftarrow{\gtrless}}_{\Omega}$ of type G have two subrepresentations whose dimension vectors are:

$$\delta'_1 = (0,0,0,1,0,0) \text{ and } \delta'_2 = (1,1,1,2,1,1).$$

Thus, they are not stable because $\theta(\delta'_1)\theta(\delta'_2) < 0$ for all $\theta \in \mathbb{H}(\delta)$. $\square$

Lemma 25 *The local intersection of the irreducible components of the variety $Rep(\Pi(\tilde{\mathbb{D}})_5, \delta)_{nil}$ is given in Table 3.*

PROOF. We calculate, for example, the codimension of $X \cap Y$ with

X =  Y = 

We have

$$X = \left\{ \begin{array}{c} \mathbb{C} \\ a \uparrow\downarrow b \\ \mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2 \underset{d}{\overset{c}{\rightleftarrows}} \mathbb{C} \\ BA \in C((2)) \\ AB = 0 \\ ba, dc \in C((2)) \\ ab = cd = 0 \\ ba + dc = 0 \end{array} \right. \qquad Y = \left\{ \begin{array}{c} \mathbb{C} \\ a \uparrow\downarrow b \\ \mathbb{C}^2 \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^2 \underset{d}{\overset{c}{\rightleftarrows}} \mathbb{C} \\ BA, AB \in C((2)) \\ \operatorname{rank}(A) = 2; \operatorname{rank}(B) = 1 \\ ba, dc \in C((2)) \\ ab = cd = 0 \\ ba + dc = AB \end{array} \right.$$

Let

$$a = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad c = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Since AB $= 0$ and BA $\in$ C((2)) we have

$$A = \begin{pmatrix} A_1 & A_1\mu \\ A_2 & A_2\mu \end{pmatrix} \quad B = \begin{pmatrix} B_1 & B_1\lambda \\ B_2 & B_2\lambda \end{pmatrix} \quad \text{with} \quad B_1 + B_2\mu = 0.$$

Since ab $= 0$; ba + dc $= 0$; ba, dc $\in$ C((2)) we have

$$\begin{cases} b_1a_1 + d_1c_1 = 0 \\ b_1a_2 + d_1c_2 = 0 \\ b_2a_1 + d_2c_1 = 0 \\ b_2a_2 + d_2c_2 = 0 \\ b_1a_1 + b_2a_2 = 0 \\ d_1c_1 + d_2c_2 = 0. \end{cases}$$

The equalities above give (see the proof in Lemma 19, Sect. 7.2)

$$b_1 = -\frac{a_1}{a_2}b_1 \qquad c_2 = \frac{a_2}{a_1}c_1 \qquad d_1 = -\frac{b_1a_1}{c_1} \qquad d_2 = -\frac{a_1}{a_2}\frac{b_1a_1}{c_1}.$$

Thus X is the set of complex matrices:

$$\begin{pmatrix} A_1 & A_1\mu \\ A_2 & A_2\mu \end{pmatrix} \quad \begin{pmatrix} B_1 & B_1\lambda \\ B_2 & B_2\lambda \end{pmatrix} \quad \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad \begin{pmatrix} b_1 & -\frac{a_1}{a_2}b_1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & \frac{a_2}{a_1}c_1 \end{pmatrix} \quad \begin{pmatrix} -\frac{b_1a_1}{c_1} & \frac{a_1}{a_2}\frac{b_1a_1}{c_1} \end{pmatrix} \quad \text{with} \quad B_1 + B_2\mu = 0.$$

So dim X = 9.

Since AB, BA $\in$ C((2)), rank(A) = 2 and rank(B) = 1 we have

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \quad B = \begin{pmatrix} B_1 & B_1\lambda \\ B_2 & B_2\lambda \end{pmatrix} \quad \text{with} \quad W_1 + W_2\lambda = 0.$$

where $W_1 = A_1B_1 + A_3B_2$ and $W_2 = A_2B_1 + A_4B_2$.

Since ab = 0; ba, dc $\in$ C((2)) and ba + dc = AB we have

$$\begin{cases} b_1a_1 + d_1c_1 = W_1 \\ b_1a_2 + d_1c_2 = W_1\lambda \\ b_2a_1 + d_2c_1 = W_2 \\ b_2a_2 + d_2c_2 = W_2\lambda \\ b_1a_1 + b_2a_2 = 0 \\ d_1c_1 + d_2c_2 = 0. \end{cases}$$

It follows that (see the proof in Lemma 21, Sect. 7.2)

$$a_1c_2 = a_2c_1 \qquad b_1d_2 = b_2d_1$$

The equalities above give

$$a_2 = \lambda a_1 \qquad b_2 = -\frac{1}{\lambda}b_1 \qquad c_2 = \lambda c_1,$$

$$d_1 = \frac{W_1 - b_1 a_1}{c_1} \qquad d_2 = -\frac{1}{\lambda}\frac{W_1 - b_1 a_1}{c_1}.$$

Thus Y is the set of complex matrices:

$$\begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \quad \begin{pmatrix} B_1 & B_1\lambda \\ B_2 & B_2\lambda \end{pmatrix} \quad \begin{pmatrix} a_1 & \lambda a_1 \end{pmatrix} \quad \begin{pmatrix} b_1 & -\frac{1}{\lambda} b_1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & \lambda c_1 \end{pmatrix} \quad \begin{pmatrix} -\frac{W_1 - b_1 a_1}{c_1} & -\frac{1}{\lambda}\frac{W_1 - b_1 a_1}{c_1} \end{pmatrix} \quad \text{with} \quad W_1 + W_2\lambda = 0.$$

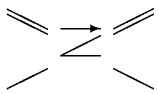
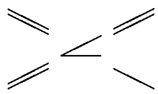
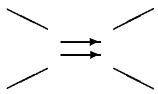
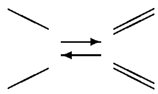
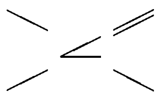
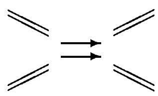
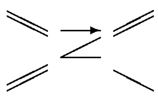
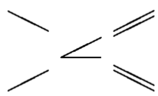
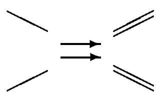
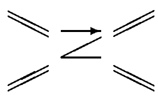
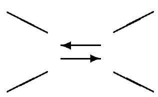
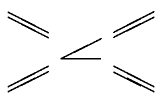
So $\dim Y = 9$.

Hence $X \cap Y$ is the set of matrices

$$\begin{pmatrix} A_1 & A_1\mu \\ A_2 & A_2\mu \end{pmatrix} \quad \begin{pmatrix} B_1 & B_1\lambda \\ B_2 & B_2\lambda \end{pmatrix} \quad \begin{pmatrix} a_1 & \lambda a_1 \end{pmatrix} \quad \begin{pmatrix} b_1 & -\frac{1}{\lambda} b_1 \end{pmatrix}$$

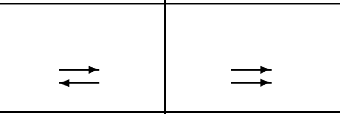
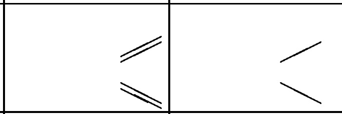
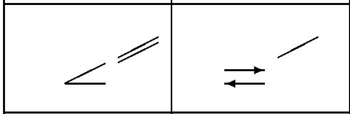
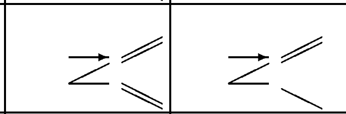
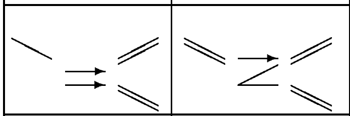
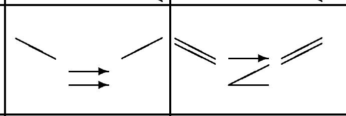
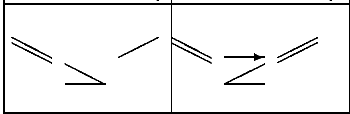
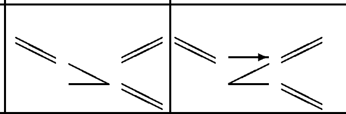
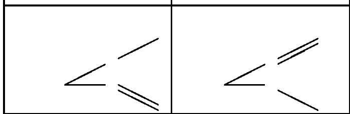
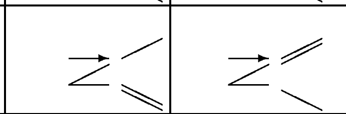
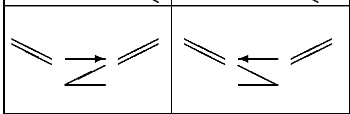
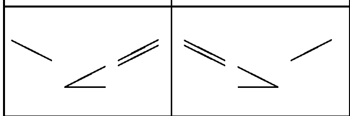
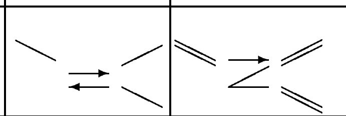
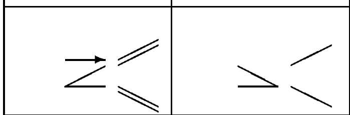
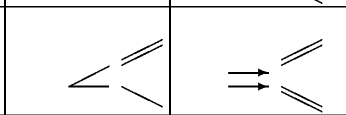
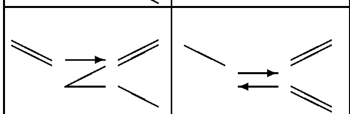
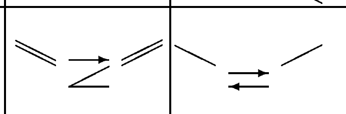
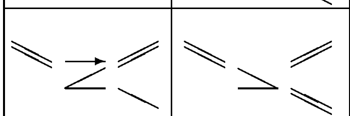
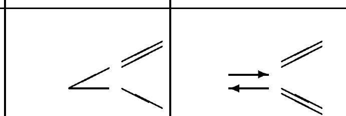
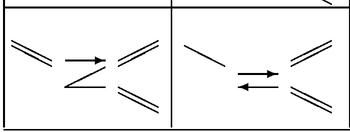
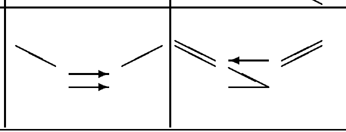
$$\begin{pmatrix} c_1 & \lambda c_1 \end{pmatrix} \quad \begin{pmatrix} -\frac{b_1 a_1}{c_1} & \frac{1}{\lambda}\frac{b_1 a_1}{c_1} \end{pmatrix} \quad \text{with} \quad B_1 + B_2\mu = 0.$$

So $\dim(X \cap Y) = 8$. This gives codim $(X \cap Y) = 1$. $\square$

Components	♯	Types	Components	♯	Types
	32	1		8	3c
	32	2a		8	4a
	32	2b		2	4b
	16	2c		8	$G1$
	16	3a		2	$G2$
	16	3b		2	$G3$

The 174 irreducible components of $\mathrm{Rep}(\Pi(\tilde{\mathbb{D}})_5, \delta)_{\mathrm{nil}}$

Table 2

X	Y	codim (X∩Y)	X	Y	codim (X∩Y)
		1			1
		1			1
		1			1
		1			1
		2			2
		2			2
		2			2
		2			3
		2			2
		2			2
		2			2

The local intersection

Table 3

Theorem 19 *1) Two irreducible components of the variety* $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_5), \delta)_{nil}^{\mathrm{stab}}$ *of the same type do not meet.*

2) Two irreducible components of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_5), \delta)_{nil}^{\mathrm{stab}}$ *of type* i *and* j *with* $\mid i-j \mid\neq 1$ *do not meet.*

3) The intersection of the components of type 1 and 2 is as follows.

3.1) A component $C^{\vec{<}}_{J_1,J_2}$ *of type 1 meets a component* C_I *of type 2a if and only if*

$$\alpha_3 \in I \text{ and } J_2 = I \cap (\Omega \setminus J_1).$$

A component $C^{\overleftarrow{>}}_{J_1,J_2}$ *of type 1 meets a component* C_I *of type 2a if and only if*

$$\alpha_3 \notin I \text{ and } J_2 = I \cap (\Omega \setminus J_1).$$

3.2) A component $C^{\vec{<}}_{J_1,J_2}$ *or* $C^{\overleftarrow{>}}_{J_1,J_2}$ *of type 1 meets a component* $C^{<}_{L_1,L_2}$ *of type 2a if and only if*

$$L_1 \subset J_1 \text{ and } J_2 = L_2 \cap (\Omega \setminus J_1).$$

3.3) A component $C^{\vec{<}}_{J_1,J_2}$ *of type 1 meets a component* $C^{\vec{<}}_{L_1,L_2}$ *of type 2b if and only if*

$$J_1 \subset L_1 \text{ and } L_2 = J_2 \cap (\Omega \setminus L_1).$$

A component $C^{\vec{<}}_{J_1,J_2}$ *of type 1 does not meet any component* $C^{\overleftarrow{>}}_{L_1,L_2}$ *of type 2c.*

4) The intersection of the components of type 2 and 3 is as follows.

4.1) The components of type 2a and 3c do not meet. This is also true for the components of type 2b and 3a, and for the components of type 2c and 3b.

4.2) A component C_I *of type 2a meets a component* C_{J_1,J_2} *of type 3a if and only if*

$$J_2 = I \cap (\Omega \setminus J_1).$$

4.3) A component C_I *of type 2a meets a component* $C_{\{\alpha_3\},J_2}$ *of type 3b if and only if*

$$J_2 = I \cap (\Omega \setminus \{\alpha_3\}).$$

4.4) A component $C^{<}_{J_1,J_2}$ or $C^{>}_{J_1,J_2}$ of type 2b meets a component $C_{\{\alpha_3\},L_2}$ of type 3b if and only if

$$J_2 = L_2 \cap (\Omega \setminus J_1).$$

4.5) A component $C^{<}_{J_1,J_2}$ of type 2b meets a component $C^{<}_{L_1,L_2}$ of type 3c if and only if

$$J_1 \subset L_1 \text{ and } L_2 = J_2 \cap (\Omega \setminus L_1).$$

A component $C^{<}_{J_1,J_2}$ of type 2b does not meet any component $C^{>}_{L_1,L_2}$ of type 3c.

4.6) A component $C^{\vec{<}}_{J_1,J_2}$ (resp. $C^{\overleftarrow{>}}_{J_1,J_2}$) of type 2c meets a component C_{L_1,L_2} of type 3a if and only if

$$\alpha_3 \in L_2 \text{ (resp. } \alpha_3 \notin L_2) \text{ and } J_2 = L_2 \cap (\Omega \setminus J_1).$$

4.6) A component $C^{\vec{<}}_{J_1,J_2}$ or $C^{\overleftarrow{>}}_{J_1,J_2}$ of type 2c meets a component $C^{<}_{L_1,L_2}$ of type 3c if and only if

$$J_1 = L_1 \text{ and } J_2 = L_2.$$

5) The intersection of the components of type 3 and 4 is as follows.

5.1) A component C_{J_1,J_2} of type 3a meets a component C_{L_1,L_2} of type 4a if and only if

$$J_1 \subset L_1 \text{ and } L_2 = J_2 \cap (\Omega \setminus L_1).$$

5.2) A component $C_{\{\alpha\},J_2}$ of type 3b meets a component C_{L_1,L_2} of type 4a if and only if

$$L_2 = J_2 \cap (\Omega \setminus L_1).$$

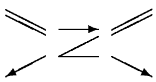
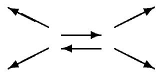
5.3) A component $C^{<}_{J_1,J_2}$ or $C^{>}_{J_1,J_2}$ of type 3c meets a component C_{L_1,L_2} of type 4a if and only if

$$L_1 \subset J_1 \text{ and } J_2 = L_2 \cap (\Omega \setminus J_1).$$

A component of type 4b does not meet any component of type 3b and 3c.

PROOF. All statements follow from Table 3 of local intersection in Lemma 25. We check a few cases. For example :

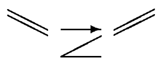
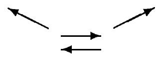
1) The two components

$C^{\overset{\rightarrow}{<}}_{\{\alpha_1,\alpha_3,\alpha_4\},\{\alpha_5\}} =$  $\qquad C_{\{\alpha_3\},\{\alpha_4,\alpha_5\}} =$ 

of type 1 and 3b do not meet because there exists a subquiver

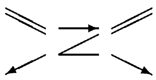
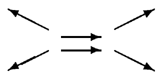
$$Q = \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet$$

such that restriction of these components to the quiver Q are the varieties of representations

$X =$  $\qquad Y =$  .

From Table 3 we have $\operatorname{codim}(X \cap Y) = 2$.

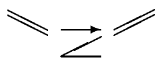
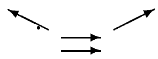
2) Let

$C^{\overset{\rightarrow}{<}}_{\{\alpha_1,\alpha_3,\alpha_4\},\{\alpha_5\}} =$  $\qquad C_{\{\alpha_3,\alpha_4,\alpha_5\}} =$ 

be the two components of type 1 and 2a. We have

$$\operatorname{codim}\ (C^{\overset{\rightarrow}{<}}_{\{\alpha_1,\alpha_3,\alpha_4\},\{\alpha_5\}} \cap C_{\{\alpha_3,\alpha_4,\alpha_5\}}) = \operatorname{codim}\ (X \cap Y)$$

where X and Y are the varieties of representations

$X =$  $\qquad Y =$  .

From Table 3 we have $\operatorname{codim}(X \cap Y) = 1$. So the component $C^{\overset{\rightarrow}{<}}_{\{\alpha_1,\alpha_3,\alpha_4\},\{\alpha_5\}}$ meets the component $C_{\{\alpha_3,\alpha_4,\alpha_5\}}$. $\square$

REMARK. Each component of type 2 meets 4 components of type 1 (Figure 5), each component of type 3 meets 6 components of type 2 (Figure 6), and each component of type 4 meets 8 components of type 3 (Figure 7). Hence, to describe the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$ of the irreducible components of the variety $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_5),\delta)^{\mathrm{stab}}_{\mathrm{nil}}$ we take a 5-dimemsional hypercube H^5. It has

32 vertices
80 edges
80 facets of dimension 2
40 facets of dimension 3
10 facets of dimension 4.

Then the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$ is described as follows.

Theorem 20 . *The intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$ has 162 vertices. They are arranged in the 5-dimensional cube H^5 as follows.*

The 32 vertices corresponding to the components of type 1 are the vertices O^1_i, $i=1,\dots,32$, of H^5. The 80 vertices corresponding to the components of type 2 are the centers O^2_j, $j=1,\dots,80$, of the 2-facets of H^5. The 40 vertices corresponding to the components of type 3 are the centers O^3_k, $k=1,\dots,40$, of the 3-facets of H^5. The 10 vertices corresponding to the components of type 4 are the centers O^4_h, $h=1,\dots,10$, of the 4-facets of H^5.

Vertices O^1_i and O^2_j are connected by an edge if the vertex O^1_i belongs to a 2-facet whose center is O^2_j.

Vertices O^3_k and O^2_j are connected by an edge if the 2-facet whose center is O^2_j belongs to a 3-facet whose center is O^3_k.

Vertices O^4_h and O^3_k are connected by an edge if the 3-facet whose center is O^3_k belongs to a 4-facet whose center is O^4_h.

CONJECTURE. One may hope that the following statements might be true:

1) The number of stable components of $\mathrm{Rep}(\Pi(\widetilde{\mathbb{D}}_n),\delta)_{\mathrm{nil}}$ is

$$\sum_{\substack{k=0 \\ k\neq 1}}^{n-1} \binom{k}{n} 2^{n-k}$$

where $\binom{k}{n} 2^{n-k}$ is the number of k-facettes in a n-dimensional cube H^n.

2) One can describe the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_n)$ by induction on n the intersection diagram $\Gamma(\widetilde{\mathbb{D}}_5)$.

A component of type 2 meets

4 components of type 1

Figure 5

A component of type 3 meets

6 components of type 2

Figure 6

A component of type 4 meets 8 components of type 3

Figure 7

REFERENCES

[1] W. Crawley-Boevey and M. P. Holland; Noncommutative deformation of Kleinian singularities, Duke Math.J 92(1998), 605-635.

[2] E. Brieskorn, Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen, Math. Annalen 166(1966), 76-102.

[3] E. Brieskorn, Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, Math. Annalen 178(1968), 255-270.

[4] Le Bruyn and C. Procesi, Semisimple representations of quivers, Trans. Amer. Math. Soc. 317 (1990), 585-599.

[5] H. Cassens; Lineare Modification algebraische Quotienten Darstellung des McKay - Köchers und Kleinsche Singularitäten, Dissertation, Universität Hamburg, 1994.

[6] V. Dlab and C. M. Ringel, Idecomposable Representation of Graphs and Algebra, Mem. Amer. Math. Soc., Providence, 1976, no.173.

[7] Cassens, Heiko; Slodowy, Peter, On Kleinian Singularities and quivers, Arnold, V.I.(ed) et al., Singularities, the Brieskorn anniversary volume. Proseeding of the conference dedicated to E.Brieskorn on his 60 th birthday, Germany, July 1996.

[8] L. Hille, Aktionen algebraischer Gruppen, geometrische Quotienten and Köcher, Habilitationsschcrift, Hamburg 2002.

[9] L. Hille, Jose Antonio de la Pena, Stable representations of quivers, Journal of Pure and Applied Algebra, 172 (2002) 205-224.

[10] A. Kas, on the resolution of certain holomorphic mappings; in "Global Analysis", papier in honour of K. Kodaira; Ed. P.C. Spencer, S. Iyanaga, Princeton Univ. Press, Princeton, 1969, 289 - 294.

[11] A. D. King, Moduli of representations of finite dimensional algrbras, Quart. J. Math. Oxford 45(1994), 515-530.

[12] P. B. Kronheimer, The construction of ALE spaces as hyper- Kaehler quotients, J. Diff. Geometry 29(1989), 665-683.

[13] G. Lusztig, Canonical bases arising from quantized enveloping algebras II, Progr. Theor. Phys. 102(1990).

[14] Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), 365-421.

[15] H. Nakajima, Quiver varities and Kac-Moody algebra, Duke Math.J 91(1998), 515-560.

[16] O. Riemenschneider, Cyclic quotient surface singularities: Constructing the Artin component via the McKay - Quiver, Singularities and Complex Analytic Geometry, Sunrikaiseki Kenkyuosho Kokyuroku(RIMS Symposium Report), Nr. 1033(1998), 163-171.

[17] P. Slodowy; algebraic groups and resolutions of Kleinian singularities; RIMS Kyoto University; Japan (1998).

[18] P. Slodowy; simple singularities and algebraic group; Lecture Notes in Math.,815, Springer (1980).

[19] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory. Third enlarged edition, Springer-Verlag, 1994.

[20] P. Du Val, On isolated singularities which do not affect the condition of adjunction I, II, III; Proc. Cambridge Phil. Soc. 30, 453-459, 483-491(1934).

[21] P. Du Val, On absolute and non-absolute singularities of algebraic surface; Istanbul Uni. Fe. Fak. Mecm. A9(1944), 159-215.

[22] V.G.Kac Infinite root systems, representations of graphs and invariant theory; Inventiones Math. 56(1980), 57-92.

[22] V.G. Kac Infinite root systems, representations of graphs and invariant theory II; Journal of Algebra. 78(1982), 141-162.

[23] F. Klein, Volesungenüber das Icosaeder und die Auflösung der Gleichungen vom fünften Grade; Teubner, Leipzig, 1884, and Birkhäuser, Basel, 1993 Englisch translation: Lectures on the icosahedron and the solution of equations of the fifth degree, Trübner and Co., London,1888, Dover, New York, 1956.

[24] G.N. Tjurina, Locally semiuniversal flat deformations of isolated singularities of complex spaces; Math. USSR Izvesta 3(1969) 967-999.

[25] G.N. Tjurina, Resolution of singularities of flat deformations of rational double points, Functional Anal. Appl.(1970), 68-73.

[26] J. McKay, Graphs, Singularities and finite groups; Proc. Sympos. Pure Math. Vol. 37, 183-186(1980).

[27] R. Hartshorne, Algebraic Geometry; Springer(1977).

[28] W. Crawley-Boevey, On the exceptional fibres of Kleinian singularities, Am. J. Math. 122(5)(2000),1027-1037.

[29] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, Compos. Math. 126, 257-293,(2001).

[30] D. Bättig, H. Knörrer, Singularitäten, Birkhäuser Verlag, 1987.

[31] K.S. Brown, Buildings, Graduate Texts in Math., Springer Verlag, 1989.

[32] K. Lamotke, Regular Solids and Isolated Singularities, Vieweg Verlag, Wiesbaden,1986.

[33] P. Slodowy, Platonic Solids, Kleinian Singularities and Lie groups; in "Algebraic Geometry", Proc. Ann Arbor 1981, Ed. I. Dolgachev, Springer Lecture Notes in Math. 1008, 102-138(1983).

[34] H. Kraft, Geometrische Methoden in der Invariantentheorie, Vieweg Verlag, Wiesbaden,1985.

[35] T. A. Springer, Invariant theory, Lecture Notes in Math. 585, Springer 1977.

[36] P. Gabriel, Unzerlegbare Darstellungen, Manusscripta Math. 6(1972), 71-103.

[37] F. C. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Math. Notes 31, Princeton Univ. (1984).

[38] W. Barth, C. Peters, A. Van de Van, Compact complex surfaces, Erg. d. Math. u.i Grenzg. 3. Folge Bd.10, Springer 1987.

[39] H. Nakajima, Varieties associated with quivers, Canadian Math. Soc. Conf. Proc., Vol 19 1996.