Interior stress regularity for the Prandtl Reuss and Hencky model of perfect plasticity using the Perzyna approximation

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Abstract

In this work we proof the local differentiability of the stress tensor in the Prandtl Reuss and Hencky model of perfect plasticity in dimensions $n = 2, 3, 4$. The first differentiability ressults for the Hencky model are due to Seregin [Ser90]. Bensoussan & Frehse [BF93][BF96] showed the differentiability result for the Hencky and Prandtl Reuss model by the Norton-Hoff approximation. Recently Demyanov [Dem07] was able to show the differentiability of the Prandtl Reuss model with methods similar to Seregin.
In this paper we use the Perzyna approximation to show the interior regularity of the stress tensor. For the Perzyna approximation Miersemann [Mie80] showed the local differentiability of the stresses for fixed viscosity coefficient. We obtain uniform estimates for the Perzyna model in the passage to the limit.
We also derive a certain regularity of the strain tensor in the Perzyna model needed for the differentiability proofs. From the original problems in perfect plasticity one has a priori only $\varepsilon \in BD(\Omega)$ for the strain tensor.

Keywords Prandtl Reuss model, Hencky model, perfect plasticity, Perzyna model regularity of solutions

Subject classification(2000) *primary*: 74C05 *secondary*: 49N60, 74G40, 74G65, 74B20 49J40

Contents

Chapter 1

Introduction to plasticity

We give a brief introduction into the theory of perfect plasticity and Perzyna viscoplasticity used for the approximation of the Hencky and Prandtl Reuss model.
All models introduced have in common that the behaviour of the matrial is separeted into two kinds. There is an elastic and a plastic part of the material behaviour.
We assume, that all displacements u are small, therefore we use the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(Du + Du^{\mathsf{T}})$.

1.1 Yield function and yield surface

Definition 1.1 (yield function) *Let $\mathcal{F} : \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}$ be a continuous, convex function. $\mathcal{F}$ is called a yield function.*
The set

$$\{\sigma \in \mathbb{R}^{n\times n}_{sym} | \, \mathcal{F}(\sigma) = 0\}$$

is called yield surface. The set $Z = \{\sigma \in \mathbb{R}^{n\times n}_{sym} | \, \mathcal{F}(\sigma) \le 0\}$ is called the set of all admissible stresses. (Z is closed und convex)

Sometimes the yield surface is written in the form

$$\{\sigma \in \mathbb{R}^{n\times n}_{\text{sym}} | \, F(\sigma) - \kappa = 0\}$$

with $F : \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$ continuous, convex and $\kappa > 0$. The constant κ is called the yield limit.
Remark The definition of the yield function is adopted from the book by Duvaut and Lions [DL76], other authors may define the yield function in a different way. All the proofs in this work require that the set $\{\sigma \in \mathbb{R}^{n\times n}_{\text{sym}} \,|\, \mathcal{F}(\sigma) \le 0\}$ is closed and convex.

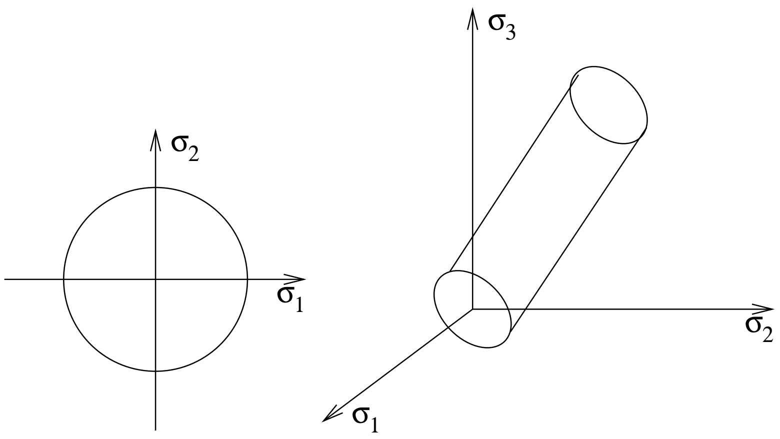


Figure 1.1: von Mises yield surface in 2 and 3D in principle stress axis

Examples for most common used yield functions:

von Mises yield criterion

For $\sigma \in \mathbb{R}^{n\times n}_{\text{sym}} \quad \sigma_D = \sigma - \frac{1}{n}\,\text{tr}(\sigma)Id$ denotes the deviator of σ.

$$\mathcal{F}(\sigma) = |\sigma_D| - \kappa = \sqrt{\sum_{i,j=1}^{n} \sigma^2_{Dij}} - \kappa$$

Tresca yield criterion

$$\mathcal{F}(\sigma) = \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} |\lambda_i - \lambda_j| - \kappa$$

where $\{\lambda_i | i = 1, \ldots, n\}$ are the eigenvalues (principal stresses) of the symmetric matrix σ.

1.2 The Prandtl Reuss law of perfect plasticity

The parameter t has the character of a memory taking the prior deformations into account. The Prandtl Reuss law is a quasi-static law.
Define $Z = \{\sigma \in \mathbb{R}^{n\times n}_{\text{sym}} | \, \mathcal{F}(\sigma) \leq 0\}$, the set of admissible stresses. Let $A \in \text{hom}(\mathbb{R}^{n\times n}_{\text{sym}}, \mathbb{R}^{n\times n}_{\text{sym}})$ be a symmetric, elliptic fourth order tensor. That is $\exists\, \alpha > 0 \;\forall \eta \in \mathbb{R}^{n\times n}_{\text{sym}} \quad \eta A \eta \geq \alpha |\eta|^2$.
The tensor A describes the elastic material behaviour and is an inverse Hookean law. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary and $\partial\Omega = \Gamma_D \,\dot{\cup}\, \Gamma_N$

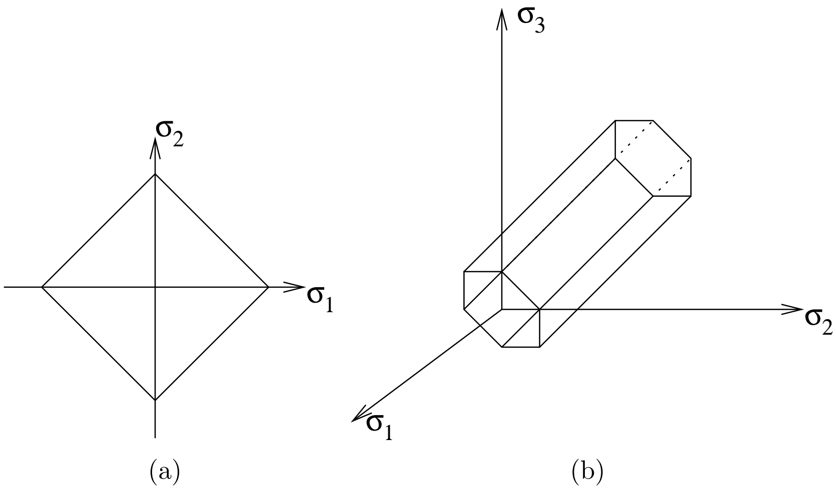


Figure 1.2: Tresca yield surface

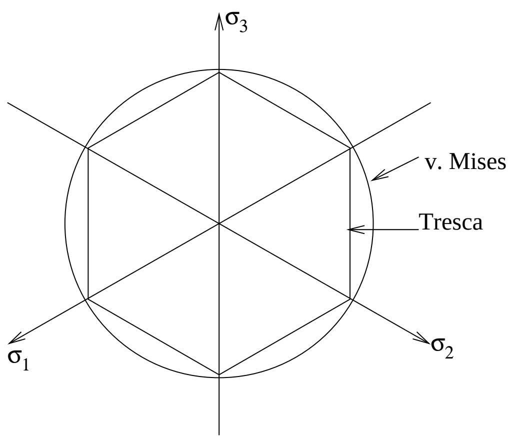


Figure 1.3: Intersection of the plane perpendicular to the axis $\sigma_1 = \sigma_2 = \sigma_3$, in principle stress axis

where Γ_D has positive $(n-1)$-dimensional Hausdorffmeasure. Let

$$f : \Omega \times [0,T] \to \mathbb{R}^n \text{ body force density in } \Omega$$
$$g : \Gamma_N \times [0,T] \to \mathbb{R}^n \text{ surface force density on } \Gamma_N$$

we abreviate $\dot{x} = \frac{\partial}{\partial t} x$

Definition 1.2 (Prandtl Reuss law) *The classic Prandtl Reuss law is:*
Find

$$\begin{aligned} \sigma : \Omega \times [0,T] &\to \mathbb{R}^{n\times n}_{sym} \\ u : \Omega \times [0,T] &\to \mathbb{R}^n \\ \sigma(x) &\in Z \\ \varepsilon(\dot{u}) &= A\dot{\sigma} + \lambda \end{aligned} \tag{1.1}$$

with λ the plastic part of the strain

$$\begin{aligned} \lambda : (\tau - \sigma) \le 0 \quad \forall \tau(x) \in Z \\ \lambda : \dot{\sigma} = 0\,. \end{aligned} \tag{1.2}$$

The equilibrium of forces holds in Ω

$$\operatorname{div} \sigma + f = 0$$

with boundary and initial values

$$\begin{aligned} \sigma \cdot \vec{n} &= g \;\; \text{on } \Gamma_N \\ \dot{u} &= \dot{U} \;\; \text{on } \Gamma_D \\ \sigma(0) &= \sigma_o \in \mathcal{K} \\ u(0) &= u_o\,. \end{aligned} \tag{1.3}$$

The inequality $\lambda : (\tau - \sigma) \le 0 \quad \forall \tau \; \mathcal{F}(\tau) \le 0$ is the principle of maximum plastic work by Hill.
If the function $t \mapsto \sigma(t)$ is differentiable in t (which we assume) the principle of maximum plastic work implies $\lambda : \dot{\sigma} = 0$.
Let $\Delta t > 0$, take $\tau = \sigma(t + \Delta t)$ respective $\tau = \sigma(t - \Delta t)$ and the principle of maximum plastic work delivers

$$\lambda : \left(\frac{\sigma(t+\Delta t) - \sigma(t)}{\Delta t} \right) \le 0 \text{ resp.}$$
$$\lambda : \left(\frac{\sigma(t-\Delta t) - \sigma(t)}{\Delta t} \right) \le 0\,.$$

thus $\lambda : \dot{\sigma} \le 0$ and $\lambda : \dot{\sigma} \ge 0$ for $\Delta t \to 0$ hence $\lambda : \dot{\sigma} = 0$.

The classic formulation of the Prandtl Reuss law is too restrictive, there need not exist any solutions.
We give now a weak formulation, there the displacement velocities are only elements of $BD(\Omega)$. For the derivation of the weak formulation we take a closer look at the plastic part λ of the strain ϵ. Write

$$\epsilon = \epsilon^e + \epsilon^p \tag{1.4}$$

with an elastic part ϵ^e and a plastic part ϵ^p. The elastic part ϵ^e is given by the linearized strain tensor $\varepsilon(\cdot)$ and the plastic part ϵ^p is just λ.

The classic Prandtl Reuss law reads

$$\varepsilon(\dot{u}) - A\dot{\sigma} = \lambda\,.$$

The principle of maximum plastic work yields

$$\big(\varepsilon(u) - A\dot{\sigma}\big) : \big(\tau - \sigma\big) \leq 0. \tag{1.5}$$

In a weak formulation the product in (1.5) between $\varepsilon(\dot{u})$ and σ is problematic because $\varepsilon(\dot{u})$ will only be a bounded measure.
Consider the product $\big(\varepsilon(\dot{u}),\sigma - \tau\big)$ and apply formal Green's formula.

$$\big(\varepsilon(\dot{u}),\tau - \sigma\big) = -\big(\dot{u}, \operatorname{div}(\tau - \sigma)\big) + \int_{\Gamma_D} \dot{U}(\tau - \sigma) \cdot \vec{n} \, \mathrm{d}\Gamma \tag{1.6}$$

$BD(\Omega)$ is continuously embeded into $L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$ (theorem A.3) so we require $\operatorname{div} \sigma \in L^n(\Omega, \mathbb{R}^n)$ for fixed t.

We abbreviate $v = \dot{u}$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega = \Gamma_N \dot{\cup} \Gamma_D$ and Γ_D has positive $(n-1)$-dimensional Hausdorffmeasure.
We assume for the body and surface forces

$$f \in L^\infty(0, T; L^n(\Omega, \mathbb{R}^n))$$
$$g \in C^0(0, T; C^0(\Gamma_N, \mathbb{R}^n))\,.$$

The stress σ is assumed $\sigma(x, t) \in L^2(0, T; L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}))$. The derivative $\dot{\sigma}$ is defined in the sense of distributions as the divergence $\operatorname{div} \sigma$.
Define

$$\mathcal{K} = \big\{\sigma \in L^2(0, T; L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}))|\, \mathcal{F}(\sigma) \leq 0 \text{ pointwise almost everywhere}\big\}$$

$$\mathcal{M} = \big\{\tau \in L^2(0, T; L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}))|\ \operatorname{div} \tau \in L^\infty(0, T; L^n(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})),\ \tau \cdot \vec{n} = g \text{ on } \Gamma_N \times [0, T]\big\}$$

Definition 1.3 (weak formulation Prandtl Reuss) *Find* $(\sigma, v) \in \big(\mathcal{M} \cap \mathcal{K}\big) \times L^1(0,T;BD(\Omega))$, *such that* $\forall \tau \in \mathcal{M} \cap \mathcal{K}$

$$\big(A\dot{\sigma}, \tau - \sigma\big) + \langle v, \operatorname{div}(\tau - \sigma)\rangle \geq \int_{\Gamma_D} V(\tau - \sigma) \cdot \vec{n} \, \mathrm{d}\Gamma \tag{1.7}$$

$$\langle \sigma, \nabla w\rangle = \langle f, w\rangle + \int_0^T \int_{\Gamma_N} g w \, \mathrm{d}\Gamma \mathrm{d}s \quad \forall w \in L^1(0,T; H^1_{\Gamma_D}(\Omega, \mathbb{R}^n)) \tag{1.8}$$

$$\begin{aligned} \sigma(0) &= \sigma_o \in \mathcal{K} \\ v(0) &= v_o \\ v &= V \;\; on\; \Gamma_D \times [0,T] \end{aligned} \tag{1.9}$$

In chapter 4.2 we will show the existence of the time derivative.
The assumption of a safe load condition (4.2) gives us the existence of solutions.

Theorem 1.1 *Under the assumption of a safe load condition* (4.2) *on page 26 in chapter 4, there exists a solution* $(\sigma, v) \in L^\infty(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym})) \times L^1(0,T; BD(\Omega))$. *The stress* σ *is unique.*

For the proof see Johnson [Joh76], Suquet [Suq81], Ionescu & Sofonea [IS93] or Anzellotti [Anz83].

1.3 Perzyna viscoplasticity

Let $Z = \{\sigma \in \mathbb{R}^{n\times n}_{\mathrm{sym}} | \, \mathcal{F}(\sigma) \leq 0\}$, $P_Z : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ denotes the projection onto Z.
Define

$$G_\mu(\tau) = \frac{1}{2\mu} |(Id - P_Z)(\tau)|^2$$

The mapping G_μ is Gâteaux differentiable[Zar71] with derivative

$$G'_\mu(\tau) = \frac{1}{\mu}(Id - P_Z)(\tau).$$

For the von Mises yield criterion we have $G'_\mu(\tau)$ explicitly (cf [DL76],[Tem85]).

$$G'_\mu(\tau) = \frac{1}{\mu} \frac{(|\tau_D| - \kappa)_+}{|\tau_D|} \tau_D$$

with

$$(a)_+ = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a \leq 0\,. \end{cases}$$

Definition 1.4 (strong formulation Perzyna viscoplasticity)

$$\varepsilon(v_\mu) = A\dot{\sigma}_\mu + G'_\mu(\sigma_\mu) \tag{1.10}$$

$$-\operatorname{div}\sigma_\mu = f \;\; in \; \Omega \times [0,T] \tag{1.11}$$

$$\begin{aligned} \sigma_\mu \cdot \vec{n} &= g \;\; on \; \Gamma_N \times [0,T] \\ v_\mu &= V \;\; on \; \Gamma_D \times [0,T] \\ \sigma_\mu(0) &= \sigma_o \in \mathcal{K} \\ v_\mu(0) &= v_o \end{aligned} \tag{1.12}$$

The the convexity and Gâteaux differentiability of G_μ yield

$$G_\mu(\tau) - G_\mu(\sigma) \geq G'_\mu(\sigma) : (\tau - \sigma) \quad \forall \tau \in \mathbb{R}^{n\times n}_{\text{sym}} \tag{1.13}$$

Inserting $\tau \in Z$ in (1.13) delivers

$$-G_\mu(\sigma) \geq G'_\mu(\sigma) : (\tau - \sigma) \; \forall \tau \in Z$$

thus

$$G'_\mu(\sigma) : (\tau - \sigma) \leq 0 \; \forall \, \tau \in Z \, .$$

This is just the principle of maximum plastic worky by Hill. The Perzyna model can be stated as follows.

$$\begin{aligned} \varepsilon(v_\mu) &= A\dot{\sigma}_\mu + G'_\mu(\sigma) \\ G_\mu(\tau) - G_\mu(\sigma_\mu) &\geq G'_\mu(\sigma_\mu) : (\tau - \sigma) \end{aligned} \tag{1.14}$$

We want to show that, by letting the viscosity coefficient μ tend to zero, the limit of the Perzyna model is the Prandtl Reuss law. The following conclusion is only formal because the solutions depend on μ.(cf [DL76])

We write $\lambda = G'_\mu(\sigma_\mu)$. For $\chi \in Z$ we have $G_\mu(\chi) = 0$ and $\lambda = 0$, for $\chi \notin Z$ we have $\lim_{\mu \searrow 0} G_\mu(\chi) = +\infty$. This gives that $G_\mu(\chi)$ tends to $ind(Z)$ the indicator function of Z.

$$ind(Z)(x) = \begin{cases} 0 & x \in Z \\ +\infty & x \notin Z \end{cases}$$

The limit of the Perzyna model is

$$\begin{aligned} \varepsilon(v) &= A\dot{\sigma} + \lambda \\ \lambda : (\tau - \sigma) &\leq 0 \; \forall \tau \in Z \end{aligned}$$

and this is just the Prandtl Reuss law.

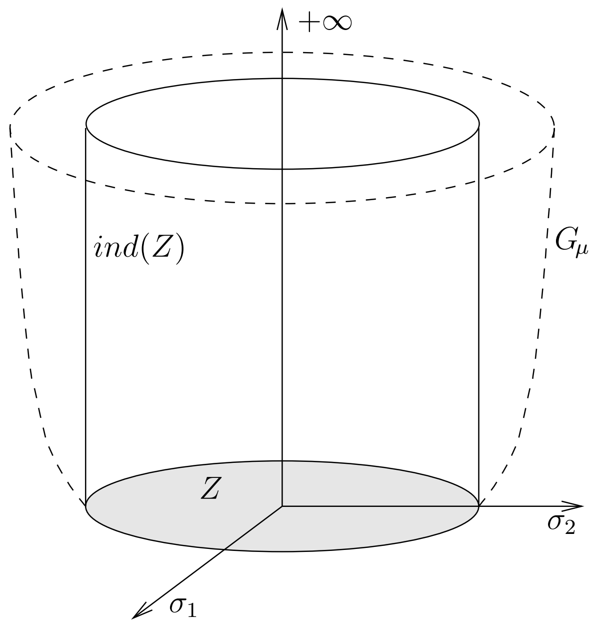


Figure 1.4: Convergence of G_μ to $ind(Z)$ with $\mu \to 0$ ([Suq81])

Definition 1.5 (weak formulation) *Find $(\sigma_\mu, v_\mu) \in \mathcal{M} \times L^1(0,T;BD(\Omega))$, such that for all $\tau \in \mathcal{M}$*

$$\big(A\dot{\sigma}_\mu, \tau - \sigma_\mu\big) + \big(G'_\mu(\sigma_\mu), \tau - \sigma_\mu\big) + \langle v, \operatorname{div}(\tau - \sigma_\mu)\rangle = \int_{\Gamma_D} V(\tau - \sigma) \cdot \vec{n}\, \mathrm{d}\Gamma \tag{1.15}$$

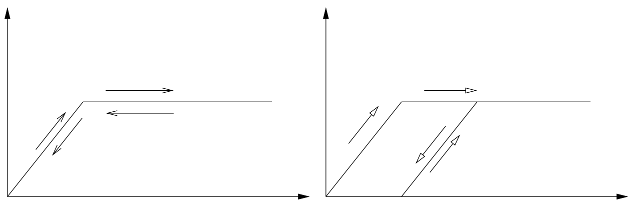
$$\langle \sigma, \nabla w\rangle = \langle f, w\rangle + \int_0^T \int_{\Gamma_N} gw\, \mathrm{d}\Gamma \mathrm{d}s \;\forall w \in L^1(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))$$

$$\begin{aligned} \sigma_\mu(0) &= \sigma_o \in \mathcal{K} \\ v_\mu(0) &= v_o \\ v &= V \;\; on\; \Gamma_D \times [0,T] \end{aligned} \tag{1.16}$$

We have now a variational equality instead of an inequality.
The existence of the time derivative of the stress tensor is demonstrated in chapter 4.2.

Theorem 1.2 *Under the assumption of a safe load condition* (4.2)*, on page 26 in chapter 4, there exists a solution $(\sigma_\mu, v_\mu) \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym})) \times L^1(0,T;BD(\Omega))$.*

For the proof see Suquet [Suq81] or Ionescu & Sofonea [IS93].



(a) Hencky model (b) Prandtl Reuss model

Figure 1.5: loading and unloading [Suq81]

1.4 Hencky model

The Hencky Model is a static model. It has no memory taking prior deformations into account ([Suq81]). Temam [Tem85] and Nečas & Hlaváček [NH91] describe the Hencky law as a special case of nonlinear elasticity .

Definition 1.6 (Hencky model) *Find* $(\sigma, u) : \Omega \to \mathbb{R}^{n\times n}_{sym} \times \mathbb{R}^n$

$$\varepsilon(u) = A\sigma + \lambda \ \textit{with} \ \mathcal{F}(\sigma) \leq 0 \tag{1.17}$$

$$\lambda : (\tau - \sigma) \leq 0 \quad \forall \tau \ \mathcal{F}(\tau) \leq 0 \tag{1.18}$$

$$\begin{aligned} -\operatorname{div}\sigma &= f \ \textit{in} \ \Omega \\ \sigma \cdot \vec{n} &= g \ \textit{on} \ \Gamma_N \\ u &= U \ \textit{on} \ \Gamma_D \end{aligned} \tag{1.19}$$

The difference to the Prandtl Reuss model can be seen in the diagramms of the tension test (figure 1.5) by relief/compression.

We assume for the body and surface forces

$$\begin{aligned} &f \in L^n(\Omega, \mathbb{R}^n) \\ &g \in C^0(\Gamma_N, \mathbb{R}^n)\,. \end{aligned}$$

Definie the sets

$$\mathcal{K} = \{\sigma \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}) \,|\, \mathcal{F}(\sigma) \leq 0 \text{ pointwise almost everywhere}\}$$

$$\mathcal{M} = \{\tau \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}) \,|\, \operatorname{div}\tau \in L^n(\Omega, \mathbb{R}^n),\ \tau \cdot \vec{n} = g \text{ on } \Gamma_N\}$$

$$\widetilde{\mathcal{M}} = \mathcal{M} \cap \{\tau| - \operatorname{div}\tau = f \text{ in } \Omega\}$$

Definition 1.7 (weak formulation of Hencky's law) *Find* $(\sigma, u) \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym}) \times BD(\Omega)$, $\sigma \in \mathcal{M} \cap \mathcal{K}$, *such that for all* $\tau \in \mathcal{M} \cap \mathcal{K}$

$$\big(A\sigma, \tau - \sigma\big) + \langle u, \operatorname{div}(\tau - \sigma)\rangle \geq \int_{\Gamma_D} U(\tau - \sigma) \cdot \vec{n} \, \mathrm{d}\Gamma \tag{1.20}$$

$$\langle \sigma, \nabla w\rangle = \langle f, w\rangle + \int_{\Gamma_N} gw \, \mathrm{d}\Gamma \quad \forall\, w \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n) \tag{1.21}$$

$$u = U \text{ on } \Gamma_D \tag{1.22}$$

The existence of a stress solution σ can be shown easily by direct methods in the calculus of variations.(cf [Lan70],[DL76]) The question of the existence for the displacements u is much more difficult to show by direct methods.
Define the energy functional

$$E(\chi) = \frac{1}{2}\big(A\chi, \chi\big) - \int_{\Gamma_D} U\chi \cdot \vec{n} \, \mathrm{d}\Gamma$$

which is the functional of complementary potential energy from linearized elasticity.

Theorem 1.3 *The Hencky problem has a unique stress solution* $\sigma \in \widetilde{\mathcal{M}} \cap \mathcal{K}$ *which is the unique minimizer of* $E(\cdot)$ *on* $\widetilde{\mathcal{M}} \cap \mathcal{K}$.

Proof "$\Rightarrow$"
Let σ be a stress solution then $\sigma \in \widetilde{\mathcal{M}} \cap \mathcal{K}$ and we have for all $\chi \in \widetilde{\mathcal{M}} \cap \mathcal{K}$

$$\big(A\sigma, \chi - \sigma\big) + \underbrace{\langle v, \operatorname{div}(\chi - \sigma)\rangle}_{=0} \geq \int_{\Gamma_D} U(\chi - \sigma) \cdot \vec{n} \, \mathrm{d}\Gamma$$

this gives

$$\big(A\sigma, \chi - \sigma\big) \geq \int_{\Gamma_D} U(\chi - \sigma) \cdot \vec{n} \, \mathrm{d}\Gamma\,.$$

But this is the Euler Lagrange equation of $E(\cdot)$.
"$\Leftarrow$" The quadratic form $\frac{1}{2}\big(A\chi, \chi\big)$ is strict convex and coercive because of the ellipticity of A. The set $\widetilde{\mathcal{M}} \cap \mathcal{K}$ is closed and convex, this implies the existence of a unique minimizer σ of $E(\cdot)$ on $\widetilde{\mathcal{M}} \cap \mathcal{K}$. This minimizer satisfies the Euler Lagrange inequality which is just the Hencky law in $\widetilde{\mathcal{M}} \cap \mathcal{K}$.

The existence of the displacement u and the connection between displacements and stresses σ was examined by Anzellotti, Giaquinta, Kohn and Temam.

Theorem 1.4 *Under the assumption of a safe load condition* (2.2), *on page 13 in chapter 2, there exists a solution* $(\sigma, u) \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym}) \times \{v \in BD(\Omega) |\, \operatorname{div} v \in L^2(\Omega)\}$. *The displacement* u *and the stress* σ *are linked together by a saddlepoint condition.*

For the proof see Anzellotti and Giaquinta [AG80, AG82] for existence of the displacement, Kohn and Temam [KT83][Tem85] for the saddlepoint condition.

1.5 The Perzyna penalized Hencky model

We define the penalty terms like in section 1.3 .

$$G_\mu(\sigma) = \frac{1}{2\mu}|(Id - P_Z)(\sigma)|^2$$
$$G'_\mu(\sigma) = \frac{1}{\mu}(Id - P_Z)(\sigma)$$

Definition 1.8 (Perzyna static) *Find* $(\sigma_\mu, u_\mu) : \Omega \to \mathbb{R}^{n\times n}_{sym} \times \mathbb{R}^n$

$$\varepsilon(u_\mu) = A\sigma_\mu + G'_\mu(\sigma_\mu) \tag{1.23}$$

$$\begin{aligned} -\operatorname{div}\sigma_\mu &= f \ \ in \ \Omega \\ \sigma_\mu \cdot \vec{n} &= g \ \ on \ \Gamma_N \\ u_\mu &= U \ \ on \ \Gamma_D \end{aligned} \tag{1.24}$$

Definition 1.9 (weak formulation) *Find* $(\sigma_\mu, u_\mu) \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym}) \times BD(\Omega) \sigma_\mu \in \mathcal{M}$*, such that for all* $\tau \in \mathcal{M}$

$$\big(A\sigma_\mu,\tau - \sigma_\mu\big) + \big(G'_\mu(\sigma_\mu),\tau - \sigma_\mu\big) + \langle u_\mu, \operatorname{div}(\tau - \sigma_\mu)\rangle = \int_{\Gamma_D} U(\tau - \sigma_\mu)\cdot \vec{n}\,\mathrm{d}\Gamma \tag{1.25}$$

$$\langle \sigma_\mu, \nabla w\rangle = \langle f, w\rangle + \int_{\Gamma_N} gw\,\mathrm{d}\Gamma \quad \forall\, w \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n)$$

$$u_\mu = U \ \ on \ \Gamma_D \tag{1.26}$$

Analogously to theorem 1.3 the existence of a stress solution can be shown by direct methods in the calculus of variations.

Theorem 1.5 *Under the assumption of the safe load condition* (2.2) *there exists a unique solution* $(\sigma_\mu, u_\mu) \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym}) \times \{v \in BD(\Omega)|\ \operatorname{div} v \in L^2(\Omega)\}$*. The solution is linked by a saddlepoint conditon.*

For the proof see Temam [Tem85].

Chapter 2

Regularity for the static Perzyna model

We now consider the Perzyna penalized Hencky model. We first show the convergence of the sequence σ_μ of stresses to the stress solution σ of the Hencky model. Then we show the local regularity of the stress tensor σ_μ for the Perzyna model.

Remark: For the estimates and convergence we can take the material tensor $A \in L^\infty(\Omega, \mathrm{hom}(\mathbb{R}^{n\times n}_{\mathrm{sym}}, \mathbb{R}^{n\times n}_{\mathrm{sym}}))$.

2.1 Estimates for the static Perzyna model

We assume for the body force density f

$$\left.\begin{aligned} f &\in L^n(\Omega, \mathbb{R}^n) \\ Df &\in L^n_{loc}(\Omega, \mathbb{R}^{n\times n}) \\ \triangle f &\in L^n_{loc}(\Omega, \mathbb{R}^n) \end{aligned}\right\} \tag{2.1}$$

We define the sets

$$\begin{aligned} Z &= \{\sigma \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \,|\, \mathcal{F}(\sigma) \le 0\} \\ \mathcal{K} &= \{\sigma \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \,|\, \sigma(x) \in Z \text{ pointwise almost everywhere in } \Omega\} \end{aligned}$$

and the set of admissible stresses

$$\mathcal{M} = \{\sigma \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \,|\, \operatorname{div}\sigma \in L^n(\Omega, \mathbb{R}^n),\ \sigma \cdot \vec{n} = g \text{ on } \Gamma_N\}$$

$$\widetilde{\mathcal{M}} = \mathcal{M} \cap \{\sigma \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \,|\, -\operatorname{div}\sigma = f \text{ in } \Omega\}$$

An important hypothesis needed for the estimates (and existence) is the safe load condition. Cf. Johnson [Joh76], Suquet [Suq81] and Temam [Tem85].

safe load condition:
There exists $\tau \in L^\infty(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ and $\delta > 0$ with

$$\left.\begin{aligned} -\operatorname{div}\tau &= f \text{ in } \Omega \\ \tau \cdot \vec{n} &= g \text{ on } \Gamma_N \\ \mathcal{F}(\tau) &\leq -\delta < 0 \text{ pointwise a.e. in } \Omega\,. \end{aligned}\right\} \tag{2.2}$$

Because we consider the mixed problem with arbitrary dirichlet boundary condition we have to assume further the existence of an admissible displacement.

existence of an admissible displacement:
There exists a displacement $\hat{u} \in H^1(\Omega, \mathbb{R}^n)$ satisfying

$$\hat{u} = U \text{ on } \Gamma_D\,. \tag{2.3}$$

Theorem 2.1 *For the solution (σ_μ, u_μ) of the static Perzyna model we have*

$$\begin{aligned} \|\sigma_\mu\|_{L^2} &\leq Const \\ \|G_\mu(\sigma_\mu)\|_{L^1} &\leq Const \\ \|G'_\mu(\sigma_\mu)\|_{L^1} &\leq Const\,. \end{aligned} \tag{2.4}$$

Proof Let τ satisfy the safe load condition
and $\hat{u}$ and be an admissible displacement. Test the weak formulation (1.25) of static Perzyna plasticity with $\sigma_\mu - \tau$.

$$\big(A\sigma_\mu, \sigma_\mu - \tau\big) + \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big) + \langle v_\mu, \operatorname{div}(\sigma_\mu - \tau)\rangle = \int_{\Gamma_D} U(\sigma_\mu - \tau)\cdot\vec{n}\,\mathrm{d}\Gamma \tag{2.5}$$

Equation (2.2) gives us $-\operatorname{div}\tau = f$ in Ω and

$$\langle v_\mu, \operatorname{div}(\sigma_\mu - \tau)\rangle = 0\,.$$

Consider now the tested penalty term. $G_\mu(\cdot)$ is convex and Gâteaux differentiable which leads to

$$\int_\Omega G_\mu(\sigma_\mu) - G_\mu(\tau)\,\mathrm{d}x \leq \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big)\,. \tag{2.6}$$

We have $G_\mu(\tau) = 0$, because τ satisfies the safe load condition. The definiteness of G_μ gives us the definiteness of the tested penalty term.

$$\big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big) \geq 0$$

This result gives the possibility to obtain further estimates of (2.5).

$$\big(A\sigma_\mu, \sigma_\mu - \tau\big) \leq \int_{\Gamma_D} U(\sigma_\mu - \tau)\cdot\vec{n}\,\mathrm{d}\Gamma \tag{2.7}$$

On the left hand side we introduce a zero addition with $\big(A\tau,\sigma_\mu - \tau\big)$ and on the right with $\big(\varepsilon(\hat{u}),\sigma_\mu - \tau\big)$. This brings a variational inequality without the boundary integral over Γ_D.

$$\big(A(\sigma_\mu - \tau),\sigma_\mu - \tau\big) \le \big(\varepsilon(\hat{u}),\sigma_\mu - \tau\big) - \big(A\tau,\sigma_\mu - \tau\big) \tag{2.8}$$

Using the ellipticity of the tensor A.

$$\alpha\|\sigma_\mu - \tau\|^2 \le \big(\varepsilon(\hat{u}) - A\tau,\sigma_\mu - \tau\big) \tag{2.9}$$

Young's inequality on the right side with $0 < \gamma < \alpha$ yields

$$(\alpha - \gamma)\|\sigma_\mu - \tau\|^2 \le \underbrace{\frac{1}{4\gamma}\|\varepsilon(\hat{u}) - A\tau\|^2}_{\le \text{ Const}} .$$

We gain $\|\sigma_\mu - \tau\|^2 \le$ Const and the boundedness of σ_μ, independent from the viscosity coefficient μ.

$$\|\sigma_\mu\|_{L^2} \le Const \tag{2.10}$$

Consider the equation (2.5), zero addition with $\big(A\tau,\sigma_\mu - \tau\big)$ and $\big(\varepsilon(\hat{u}),\sigma_\mu - \tau\big)$ leads to the uniform estimate

$$\big(G'_\mu(\sigma_\mu),\sigma_\mu - \tau\big) \le Const\,. \tag{2.11}$$

Inequality (2.6) for $G'_\mu(\sigma_\mu)$ gives the uniform bound

$$\int_\Omega G_\mu(\sigma_\mu)\,\mathrm{d}x \le Const$$

in $L^1(\Omega)$.

$$\|G_\mu(\sigma_\mu)\|_{L^1} \le Const \tag{2.12}$$

These estimates (2.2) and lemma 2 from Suquet [Suq81] yield the boundedness independent of μ

$$\|G'_\mu(\sigma_\mu)\|_{L^1} \le Const\,. \tag{2.13}$$

Because the safe load condition implies

$$\begin{aligned}
\|G'_\mu(\sigma_\mu)\|_{L^1} &= \frac{1}{\delta}\sup_{\|\chi\|_{L^\infty}\le\delta}\langle G'_\mu(\sigma_\mu),\chi\rangle \\
&\le \frac{1}{\delta}\langle G'_\mu(\sigma_\mu),\chi + \tau - \sigma_\mu\rangle + \langle G'_\mu(\sigma_\mu),\sigma_\mu - \tau\rangle \\
&\le \frac{1}{\delta}\left(Const + \int_\Omega G_\mu(\chi + \tau)\,\mathrm{d}x + \langle G'_\mu(\sigma_\mu),\sigma_\mu - \tau\rangle\right) \\
&\le \frac{1}{\delta}Const \text{ because of } \chi + \tau \in \mathcal{K} \text{ we have } G_\mu(\chi + \tau) = 0\,.
\end{aligned}$$

We now demonstrate that the displacement solutions u_μ of the static Perzyna model are better than $BD(\Omega)$.

Theorem 2.2 *For fixed viscosity coefficient* μ *we have* $\varepsilon(u_\mu) \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym})$ *and* $u_\mu \in H^1(\Omega, \mathbb{R}^n)$*.*

Proof The Lipschitz continuity of $Id - P_K$ (theorem C.2) and $\|\sigma_\mu\|_{L^2} \le Const$ allows us to estimate $G'_\mu(\sigma_\mu)$.

$$\|G'_\mu(\sigma_\mu)\|_{L^2} \le Const(\mu) \tag{2.14}$$

After zero addition from $\big(\varepsilon(\hat{u}), \chi - \sigma_\mu\big)$ in (1.25) we get

$$\big(A\sigma_\mu, \chi - \sigma_\mu\big) + \langle u_\mu, \operatorname{div}(\chi - \sigma_\mu)\rangle + \big(G'_\mu(\sigma_\mu), \chi - \sigma_\mu\big) = \big(\varepsilon(\hat{u}), \chi - \sigma_\mu\big)\,. \tag{2.15}$$

Choose $\chi \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ with

$$\begin{aligned} -\operatorname{div}\chi &= f \text{ in } \Omega \\ \chi\cdot\vec{n} &= g \text{ on } \Gamma_N \\ \chi\cdot\vec{n} &= \sigma_\mu\cdot\vec{n} \text{ on } \Gamma_D\,. \end{aligned} \tag{2.16}$$

For such a χ we have

$$\begin{aligned} \operatorname{div}(\chi - \sigma_\mu) &= 0 \\ \big(\varepsilon(\hat{v}), \chi - \sigma_\mu\big) &= 0\,. \end{aligned} \tag{2.17}$$

Inserting χ into (2.15) gives

$$\big(A\sigma_\mu + G'_\mu(\sigma_\mu), \chi - \sigma_\mu\big) = 0$$

for all χ with the properties (2.16). Theorem D.4 implies

$$A\sigma_\mu + G'_\mu(\sigma_\mu) = \varepsilon(u_\mu) \tag{2.18}$$

pointwise almost everywhere in $L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ which results $u_\mu \in H^1(\Omega, \mathbb{R}^n)$ for μ fixed. Korn's inequality, (2.14) and (2.10) deliver the non uniform estimates

$$\begin{aligned} \|\varepsilon(u_\mu)\|_{L^2} &\le \frac{1}{\mu} Const \\ \|u_\mu\|_{H^1} &\le \frac{1}{\mu} Const\,. \end{aligned} \tag{2.19}$$

Theorem 2.3 *With the preceeding results we obtain*

$$\begin{aligned} \|\varepsilon(u_\mu)\|_{L^1} &\le Const \\ \|u_\mu\|_{L^{\frac{n}{n-1}}} &\le Const\,. \end{aligned} \tag{2.20}$$

Proof The pointwise Perzyna law (2.18), $\|\sigma_\mu\|_{L^2} \le Const$ and $\|G'_\mu(\sigma_\mu)\|_{L^1} \le Const$ deliver $\|\varepsilon(u_\mu)\|_{L^1} \le Const$. Korn's inequality yields for the displacements $\|u_\mu\|_{L^{\frac{n}{n-1}}} \le Const$.

2.2 Convergence of the penalized model to the Hencky law

We are now able to show the convergence of (σ_μ, u_μ) to (σ, u) solution of the Hencky model. First we will demostrate the convergence of the stress tensor and afterwards of the displacements.

Theorem 2.4 *Let $\mu \to 0$, there exists a subsequence σ_{μ_l}, converging weakly in L^2 to σ, solution of the Hencky model.*

Proof From $\|\sigma_\mu\| \leq Const$ we deduce the existence of a subsequence $\sigma_{\mu_l} \rightharpoonup \tilde{\sigma}$ for a $\tilde{\sigma} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$. We have to show that $\tilde{\sigma} = \sigma$ holds. Consider the energy functional E of σ_μ.
Claim: For all $\mu > 0 : E(\sigma_\mu) \leq E(\sigma)$.
We test the pointwise equation (2.18) with $\sigma_\mu - \sigma$ and apply Green's formula.

$$\big(A\sigma_\mu, \sigma_\mu - \sigma\big) + \underbrace{\big(G'_\mu(\sigma_\mu), \sigma_\mu - \sigma\big)}_{\geq 0} = \int_{\Gamma_D} U(\sigma_\mu - \sigma)\cdot \vec{n}\,\mathrm{d}\Gamma$$

$$\big(A\sigma_\mu, \sigma_\mu\big) \leq \big(A\sigma_\mu, \sigma\big) + \int_{\Gamma_D} U(\sigma_\mu - \sigma)\cdot \vec{n}\,\mathrm{d}\Gamma$$

using Young's inequality:

$$\frac{1}{2}\big(A\sigma_\mu, \sigma_\mu\big) \leq \frac{1}{2}\big(A\sigma, \sigma\big) + \int_{\Gamma_D} U(\sigma_\mu - \sigma)\cdot \vec{n}\,\mathrm{d}\Gamma$$

$$\Rightarrow \quad \big(A\sigma_\mu, \sigma_\mu\big) - \int_{\Gamma_D} U\sigma_\mu \cdot \vec{n}\,\mathrm{d}\Gamma \leq \big(A\sigma, \sigma\big) - \int_{\Gamma_D} U\sigma \cdot \vec{n}\,\mathrm{d}\Gamma$$

$$E(\sigma_\mu) \leq E(\sigma)\,. \tag{2.21}$$

The energy functional E_μ for the Peryzna law: $E_\mu = E + \int_\Omega G_\mu \,\mathrm{d}x$. Inserting the subsequence σ_{μ_l} in E_μ, one gets with estimate (2.13) of $G'_\mu(\sigma_\mu)$ and (2.6)

$$E(\sigma_{\mu_l}) + \mu_l \int_\Omega G_{\mu_l}(\sigma_{\mu_l})\,\mathrm{d}x \leq E(\sigma) + \mu_l \cdot Const\,.$$

With $\mu_l \to 0$, $\int_\Omega G_{\mu_l}(\sigma_{\mu_l})\,\mathrm{d}x \leq \mu_l \cdot Const \to 0$, we deduce $\mathcal{F}(\tilde{\sigma}) \leq 0$ almost everywhere, thus $\tilde{\sigma} \in \mathcal{K}$. The energy functional $E(\cdot)$ is convex which gives the lower semi continuity. One obtains for the limit $\tilde{\sigma}$

$$E(\tilde{\sigma}) \leq \liminf_{\mu_l \to 0} E(\sigma_{\mu_l}) \leq E(\sigma)\,.$$

The solution σ of the Hencky model is the unique minimizer of $E(\cdot)$ on $\mathcal{K} \cap \widetilde{\mathcal{M}}$, hence $\tilde{\sigma} = \sigma$.

Theorem 2.5 *Let $\mu \to 0$, there exists a subsequence u_{μ_l} converging weakly in $L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$ to u, displacement solution of the Hencky model.*

Proof By theorem 2.3 we know $\|u_\mu\|_{L^{\frac{n}{n-1}}} \leq Const$, extracting a suitable subsequence $(\sigma_{\mu_l}, u_{\mu_l})$, there exists a $\tilde{u} \in L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$ such that

$$(\sigma_{\mu_l}, u_{\mu_l}) \rightharpoonup (\sigma, \tilde{u}) \text{ as } \mu_l \to 0\,.$$

It remains to show that $u = \tilde{u}$. Test the pointwise almost everywhere Hencky law (2.18) with $(\sigma_{\mu_l} - \tau)$, where $\tau \in \mathcal{K} \cap \mathcal{M}$. Then

$$-\big(u_{\mu_l}, f - \operatorname{div}\tau\big) = \big(A\sigma_{\mu_l}, \sigma_{\mu_l} - \tau\big) + \big(G'_\mu(\sigma_{\mu_l}), \sigma_{\mu_l} - \tau\big)\,. \tag{2.22}$$

Using the inequality for convex differentiable functions

$$\big(G'_\mu(\sigma_{\mu_l}), \sigma_{\mu_l} - \tau\big) \geq \underbrace{G_\mu(\sigma_{\mu_l})}_{\geq 0} - \underbrace{G_\mu(\tau)}_{=0}$$

thus we obtain the variational inequality

$$\big(u_{\mu_l}, \operatorname{div}\tau - f\big) \leq \big(A\sigma_{\mu_l}, \tau - \sigma_{\mu_l}\big)\,. \tag{2.23}$$

By a lower semicontinuity argument we have $u = \tilde{u}$.
By uniqueness and a routine argument the whole sequence (σ_μ, u_μ) converges.

2.3 Local differentiability of the stress tensor

We know show the local differentiability of the stress tensor using finite differences. The estimates for the finite differences are not uniform in μ.

Theorem 2.6 *For fixed viscosity coefficient μ we have $\sigma_\mu \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{sym})$.*

Proof Let $\theta \in C_o^\infty(\Omega)$ be a cutoff function. Let $0 < h < \frac{1}{2}\operatorname{dist}(\operatorname{supp}\theta, \partial\Omega)$. We test the pointwise Perzyna law (2.18) with the difference quotient $-D_j^{-h}(\theta^2 D_j^h \sigma_\mu)$. The rule for discrete partial integration gives

$$\big(D_j^h \varepsilon(u_\mu), \theta^2 D_j^h \sigma_\mu\big) = \big(\theta A D_j^h \sigma_\mu, \theta D_j^h \sigma_\mu\big) + \big(\theta D_j^h G'_\mu(\sigma_\mu), \theta D_j^h \sigma_\mu\big)\,.$$

With theorem C.3 the term $\big(\theta D_j^h G'_\mu(\sigma_\mu), \theta D_j^h \sigma_\mu\big)$ is bounded from below.

$$\big(\theta D_j^h G'_\mu(\sigma_\mu), \theta D_j^h \sigma_\mu\big) \geq 0$$

The ellipticity of the tensor A leads to

$$\alpha \|\theta D_j^h \sigma_\mu\|^2 \leq \big(D_j^h \varepsilon(u_\mu), \theta^2 D_j^h \sigma_\mu\big)\,.$$

Apply Green's formula and the discrete product rule

$$\big(D_j^h\varepsilon(u_\mu),\theta^2 D_j^h\sigma_\mu\big) = -\big(D_j^h u_\mu, \operatorname{grad}\theta^2 D_j^h\sigma_\mu\big) - \big(D_j^h u_\mu,\theta^2 D_j^h f\big)$$

Where $E_j^{-h}\varphi(x) = \varphi(x - h\vec{e}_j)$

$$-\big(D_j^h u_\mu,\theta^2 D_j^h f\big) = \big(u_\mu,D_j^{-h}\theta^2 D_j^h f\big) + \big(u_\mu,E_j^{-h}\theta^2\triangle^h f\big)\,.$$

$\triangle^h f = D_j^{-h}(D_j^h f)$ is the finite difference approximation to the Laplace operator $\triangle f$. The assumptions (2.1) for the body force density f allow us to use the Hölder inequality.

$$\big(u_\mu,D_j^{-h}\theta^2 D_j^h f\big) + \big(u_\mu,E_j^{-h}\theta^2\triangle^h f\big) \le \|u_\mu\|_{L^{\frac{n}{n-1}}}\cdot\big(\|D_j^{-h}\theta^2 D_j^h f\|_{L^n} + \|E_j^{-h}\theta^2\triangle^h f\|_{L^n}\big)$$

With (2.1) we deduce

$$-\big(D_j^h u_\mu,\theta^2 D_j^h f\big) \le C\,.$$

The term $-\big(D_j^h u_\mu, \operatorname{grad}\theta^2 D_j^h\sigma_\mu\big)$ can be estimated by Young's inequality

$$\begin{aligned}-\big(D_j^h u_\mu, \operatorname{grad}\theta^2 D_j^h\sigma_\mu\big) &\le \frac{1}{4\gamma}\|D_j^h u_\mu\|^2 + \gamma\|2\operatorname{grad}\theta\|^2\cdot\|\theta D_j^h\sigma_\mu\|^2\\ &\le \frac{1}{4\gamma}\|D_j u_\mu\|^2 + \gamma C_\theta\|\theta D_j^h\sigma_\mu\|^2\end{aligned}$$

The norm $\|D_j u_\mu\|$ is estimated by Korn's inequality and (2.19).

$$\frac{1}{4\gamma}\|D_j u_\mu\|^2 + \gamma C_\theta\|\theta D_j^h\sigma_\mu\|^2 \le \frac{1}{4\gamma}C_{\text{Korn}}\|\varepsilon(u_\mu)\|^2 + \gamma C_\theta\|\theta D_j^h\sigma_\mu\|^2$$

Choose $0 < \gamma C_\theta < \alpha$ and it follows

$$(\alpha - \gamma C_\theta)\|\theta D_j^h\sigma_\mu\|^2 \le Const(\mu)$$

for $\theta D_j^h\sigma_\mu$.

$$\|\theta D_j^h\sigma_\mu\|^2 \le Const(\mu)$$

All together we have $\sigma_\mu \in H^1_{loc}(\Omega,\mathbb{R}^{n\times n}_{\text{sym}})$ for fixed viscosity coefficient μ.

Chapter 3

H^1_{loc} regularity for the stress tensor in the Hencky model with von Mises yield criterion

In this chapter we show that $\sigma \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ holds in the case of von Mises yield criterion.
The local differentiability of the stress tensor was first shown by Seregin [Ser90] and then by Bensoussan & Frehse [BF93] . Bensoussan and Frehse used a dual method, they penalized the Hencky model with the Norton-Hoff model and were able to show a uniform bound for the derivatives of the stress tensor in the Norton-Hoff approximation. Their proof works in arbitrary dimensions, whereas our proof only works in dimensions $n = 2, 3, 4$.
Our proof is inspired by Bensoussan & Frehse [BF93, BF02], but we use the Perzyna penalization as approximation of the Hencky model. This problem was discussed in [Pai02]. From chapter 2 we already know the local differentiability of the stress tensor σ_μ of static Perzyna model, but the estimates are not uniform in the viscosity coefficient μ.

We make the same assumptions as in chapter 2.

We assume for the body force density f:

$$\left.\begin{aligned} f &\in L^n(\Omega, \mathbb{R}^n) \\ Df &\in L^n_{loc}(\Omega, \mathbb{R}^{n\times n}) \\ \triangle f &\in L^n_{loc}(\Omega, \mathbb{R}^n) \end{aligned}\right\} \tag{3.1}$$

We already know

$$\begin{aligned} \|u_\mu\|_{L^{\frac{n}{n-1}}} &\le Const \\ \|\varepsilon(u_\mu)\|_{L^1} &\le Const \\ \left.\begin{aligned} \sigma_\mu &\in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}) \\ \varepsilon(u_\mu) &\in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}) \end{aligned}\right\} &\text{estimates dependent on } \mu \end{aligned}$$

The term $G'_\mu(\cdot)$ takes by von Mises yield criterion the form

$$G'_\mu(\sigma_\mu) = \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\,.$$

For fixed viscosity coefficient μ the pointwise penalized Hencky model holds.

$$\varepsilon(u_\mu) = A\sigma_\mu + \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\,. \tag{3.2}$$

We differentiate equation (3.2) with D_l and test with $\theta^2 D_l\sigma_\mu$, where $\theta \in C^\infty_o(\Omega)$.

$$\big(D_l\varepsilon(u_\mu),\theta^2 D_l\sigma_\mu\big) = \underbrace{\big(AD_l\sigma_\mu,\theta^2 D_l\sigma_\mu\big)}_{(*)} + \underbrace{\left(D_l\left(\frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\right), \theta^2 D_l\sigma_\mu\right)}_{(**)} \tag{3.3}$$

One can bound $(*)$ from below using the ellipticity of A.

$$\alpha\|\theta D_l\sigma_\mu\|^2 \le \big(AD_l\sigma_\mu,\theta^2 D_l\sigma_\mu\big) \tag{3.4}$$

The tested penalty term $(**)$ can also be bounded from below. First we compute the directional derivative D_l of the the penalty term.

$$\begin{aligned} D_l\left(\frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\right) &= \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}D_l\sigma_{\mu D} \\ &\quad + \frac{1}{\mu}\frac{|\sigma_{\mu D}|D_l|\sigma_{\mu D}|_\oplus - (|\sigma_{\mu D}| - \kappa)_+ D_l|\sigma_{\mu D}|}{|\sigma_{\mu D}|^2}\sigma_{\mu D} \\ &= \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}D_l\sigma_{\mu D} + \frac{1}{\mu}\frac{D_l|\sigma_{\mu D}|_\oplus}{|\sigma_{\mu D}|}\sigma_{\mu D} \\ &\quad - \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+ D_l|\sigma_{\mu D}|}{|\sigma_{\mu D}|^2}\sigma_{\mu D} \end{aligned} \tag{3.5}$$

For μ fixed the expression $(|\sigma_{\mu D}| - \kappa)_+$ is weakly differentiable.

We have

$$D_l(|\sigma_{\mu D}| - \kappa)_+ = \begin{cases} D_l|\sigma_{\mu D}| & \text{a.e. in } \{x\in\Omega \mid |\sigma_{\mu D}| > \kappa\} \\ 0 & \text{a.e. in } \{x\in\Omega \mid |\sigma_{\mu D}| \le \kappa\}\,. \end{cases}$$

write

$$D_l|\sigma_{\mu D}|_\oplus = \begin{cases} D_l|\sigma_{\mu D}| & \text{if } |\sigma_{\mu D}| > \kappa \\ 0 & \text{if } |\sigma_{\mu D}| \leq \kappa \end{cases}$$

For the following calculations we remark[1]:

$$\begin{aligned} \sigma_{\mu D} : D_l\sigma_\mu &= \sigma_{\mu D} : D_l\sigma_{\mu D} \\ &= \frac{1}{2} D_l|\sigma_{\mu D}|^2 \\ &= |\sigma_{\mu D}| \cdot D_l|\sigma_{\mu D}| \end{aligned} \tag{3.6}$$

After taking the scalarproduct for matrices of $D_l\left(\frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\right)$ with $\theta^2 D_l\sigma_\mu$ and using

$$(D_l|\sigma_{\mu D}|_\oplus)^2 \leq (D_l|\sigma_{\mu D}|)^2 \tag{3.7}$$

we obtain

$$\begin{aligned} D_l\left(\frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\right) : \theta^2 D_l\sigma_{\mu D} &= \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}|D_l\sigma_{\mu D}|^2\theta^2 + \frac{1}{\mu}\underbrace{D_l|\sigma_{\mu D}|_\oplus D_l|\sigma_{\mu D}|}_{\geq (D_l|\sigma_{\mu D}|_\oplus)^2}\theta^2 \\ &\quad - \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+(D_l|\sigma_{\mu D}|)^2\theta^2}{|\sigma_{\mu D}|} \\ &\geq \frac{1}{\mu}\theta^2\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}|D_l\sigma_{\mu D}|^2 \\ &\quad + \underbrace{\frac{1}{\mu}\theta^2(D_l|\sigma_{\mu D}|_\oplus)^2\left(1 - \frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\right)}_{\geq 0} . \end{aligned}$$

This leads to the estimate

$$\int_\Omega \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}|\theta D_l\sigma_{\mu D}|^2 \,\mathrm{d}x \leq (**) . \tag{3.8}$$

Green's formula applied to the lefthand side of (3.3) yields

$$\big(D_l\varepsilon(u_\mu),\theta^2 D_l\sigma_\mu\big) = -\big(D_l u_\mu, \operatorname{grad}\theta^2 D_l\sigma_\mu\big) - \big(D_l u_\mu,\theta^2 D_l f\big) \tag{3.9}$$

Using partial integration and Hölders's inequality

$$\begin{aligned} -\big(D_l u_\mu,\theta^2 D_l f\big) &= \big(u_\mu, D_l(\theta^2)D_l f\big) + \big(u_\mu,\theta^2\triangle f\big) \\ &\leq \|u_\mu\|_{L^{\frac{n}{n-1}}}(C_1\|D_l f\|_{L^n} + C_2\|\triangle f\|_{L^n}) \\ &\leq C \end{aligned} \tag{3.10}$$

[1]Here we use $M_D : N = M_D : N_D$, cf appendix B

Now we consider the term $\big(D_l u_\mu, \operatorname{grad}\theta^2 D_l\sigma_\mu\big)$ and symmetrize to obtain better estimates.
recall: $\varepsilon(u) = \frac{1}{2}(Du + Du^\mathsf{T})$
We now use the summing convention.

$$-\big(D_l u_{\mu j}, D_l\sigma_{\mu ij} D_i\theta^2\big) = -2\big(\varepsilon(u_\mu)_{jl}, D_l\sigma_{\mu ij} D_i\theta^2\big) + \big(D_j u_{\mu l}, D_l\sigma_{\mu ij} D_i\theta^2\big) \tag{3.11}$$

Using the constitutive pointwise law (3.2) $\varepsilon(u_\mu) = A\sigma_\mu + \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}$ yields

$$\begin{aligned}(3.11) = -2\underbrace{\big((A\sigma)_{\mu jl}, D_l\sigma_{\mu ij} D_i\theta^2\big)}_{E_1} - \underbrace{\left(\frac{2}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu Djl}, D_l\sigma_{\mu ij} D_i\theta^2\right)}_{E_2} \\ + \underbrace{\big(D_j u_{\mu l}, D_l\sigma_{\mu ij} D_i\theta^2\big)}_{E_3}\end{aligned}$$

We estimate the term E_1.

$$\begin{aligned}-2\big(A\sigma_\mu, D_l\sigma_\mu\theta\cdot 2\operatorname{grad}\theta\big) &\le C\int_\Omega |A\sigma_\mu|\cdot|\theta||D_l\sigma_\mu|\,\mathrm{d}x \\ &\overset{\text{Young}}{\le} C\frac{1}{4\gamma}\int_\Omega |A\sigma_\mu|^2\,\mathrm{d}x + \gamma C\int_\Omega \theta^2|D_l\sigma_\mu|^2\,\mathrm{d}x \\ &\le C(\gamma) + \gamma C\|\theta D_l\sigma_\mu\|^2\end{aligned} \tag{3.12}$$

The term E_2 can be estimate as follows using $D_i\theta^2 = \theta\cdot 2D_i\theta$

$$\left(\frac{2}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu Dkl}, D_l\sigma_{\mu ij} D_i\theta^2\right) \le \int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}| - \kappa)_+|D_l\sigma_{\mu ij}|\cdot|\theta|\cdot C_\theta\,\mathrm{d}x \tag{3.13}$$

Now we have a problem because the indexpairs in this equation do not match.
We have to estimate $|D_l\sigma_\mu|$ by $|D_l\sigma_{\mu D}|$. Therefore we take a closer look at the definition of the deviator.

$$\begin{aligned}\sigma_{\mu D} &= \sigma_\mu - \frac{1}{n}\operatorname{tr}(\sigma)Id \\ \sigma_\mu &= \sigma_{\mu D} + \frac{1}{n}\operatorname{tr}(\sigma)Id \\ D_l\sigma_\mu &= D_l\sigma_{\mu D} + \frac{1}{n}D_l\operatorname{tr}(\sigma)Id\end{aligned}$$

This gives

$$|D_l\sigma_\mu| \le |D_l\sigma_{\mu D}| + |\frac{1}{n}D_l\operatorname{tr}(\sigma)Id|$$

where

$$|\frac{1}{n}D_l\,\mathrm{tr}(\sigma_\mu)Id| = \frac{1}{\sqrt{n}}|D_l\,\mathrm{tr}(\sigma)|$$

thus

$$|D_l\sigma_\mu| \leq |D_l\sigma_{\mu D}| + \frac{1}{\sqrt{n}}|D_l\,\mathrm{tr}(\sigma_\mu)|\,. \tag{3.14}$$

The following inequality is due to Bensoussan & Frehse [BF93].

Proposition:
We have the inequality

$$\int_\Omega |\theta|^2|D\,\mathrm{tr}(\sigma_\mu)|^2\,\mathrm{d}x \leq 2n^2\int_\Omega |\theta|^2|D_l\sigma_{\mu D}|^2\,\mathrm{d}x + 2n\int_\Omega |\theta|^2|f|^2\,\mathrm{d}x\,. \tag{3.15}$$

Inserting (3.14) into the righthand side of (3.13).

$$\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}|-\kappa)_+|D_l\sigma_{\mu ij}|\cdot|\theta|C_\theta\,\mathrm{d}x \leq \underbrace{\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}|-\kappa)_+|\theta|\cdot|D_l\sigma_{\mu D}|C_\theta\,\mathrm{d}x}_{T_1} + \underbrace{\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}|-\kappa)_+C_\theta\cdot\frac{1}{\sqrt{n}}|D_l\,\mathrm{tr}(\sigma_\mu)|\cdot|\theta|\,\mathrm{d}x}_{T_2} \tag{3.16}$$

For T_1 one gets with Young's inequality

$$T_1 = \int_\Omega \frac{2}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\cdot|\theta|\cdot|D_l\sigma_{\mu D}|\cdot C_\theta|\sigma_{\mu D}|\,\mathrm{d}x \leq \zeta\int_\Omega \frac{2}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}|D_l\sigma_{\mu D}|^2|\theta|^2\,\mathrm{d}x + \frac{1}{2\zeta}C_\theta^2\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}|-\kappa)_+|\sigma_{\mu D}|\,\mathrm{d}x\,. \tag{3.17}$$

Where one can show analogously to the estimates of the penalty term that

$$\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}|-\kappa)_+|\sigma_{\mu D}|\,\mathrm{d}x \leq Const\,.$$

The term T_2 is split into

$$T_2 \leq \underbrace{\frac{1}{2\varrho}\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}|-\kappa)_+C_\theta^2\,\mathrm{d}x}_{\leq Const} + \underbrace{\varrho\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}|-\kappa)_+\frac{1}{n}|\theta D_l\,\mathrm{tr}(\sigma_\mu)|^2\,\mathrm{d}x}_{T_{21}}$$

using Young's inequality. With the use of inequality (3.15) from Bensoussan & Frehse we obtain for T_{21}

$$T_{21} \leq \underbrace{\varrho\int_\Omega \frac{4}{\mu}(|\sigma_{\mu D}|-\kappa)_+n|\theta D_l\sigma_{\mu D}|^2\,\mathrm{d}x}_{T_3} + \underbrace{\varrho\int_\Omega \frac{4}{\mu}(|\sigma_{\mu D}|-\kappa)_+|\theta|^2|f|^2\,\mathrm{d}x}_{T_4}\,.$$

The assumptions (3.1) ($\triangle f \in L^n_{loc}$) yields $f \in L^\infty_{loc}$, thus

$$T_4 \le Const.$$

To obtain final estimates for E_2 we choose $\zeta = \frac{1}{4}$, $\varrho = \frac{1}{8n|\sigma_{\mu D}|}$. This yields

$$-\int_\Omega \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu Djl} D_l\sigma_{\mu ij} D_i\theta^2 \,\mathrm{d}x \le \int_\Omega \frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}|\theta D_l\sigma_{\mu D}|^2 \,\mathrm{d}x + Const\,. \quad (3.18)$$

E_3: partial integration gives

$$\begin{aligned}\big(D_j u_{\mu l}, D_l\sigma_{\mu ij} D_i\theta^2\big) &= -\int_\Omega u_{\mu l} D_i\theta^2 D_l f_i \,\mathrm{d}x - \int_\Omega u_{\mu l} D_l \sigma_{\mu ij} D_i D_j \theta^2 \,\mathrm{d}x \\ &= -\int_\Omega u_{\mu l} D_i\theta^2 D_l f_i \,\mathrm{d}x + \int_\Omega \operatorname{div} u_\mu \sigma_{\mu ij} D_i D_j\theta^2 \,\mathrm{d}x \qquad (3.19)\\ &\quad + \int_\Omega u_{\mu l}\sigma_{\mu ij} D_l D_i D_j \theta^2 \,\mathrm{d}x\,.\end{aligned}$$

We have $\operatorname{div} u_\mu = \operatorname{tr} A\sigma_\mu$ because $\operatorname{tr}\varepsilon(u_\mu) = \operatorname{div} u_\mu$ and $\operatorname{tr}\varepsilon(u_\mu) = \operatorname{tr} A\sigma_\mu + \underbrace{\operatorname{tr}\left(\frac{1}{\mu}\frac{(|\sigma_{\mu D}|-\kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\right)}_{=0}$.

By $\sigma_\mu \in L^2$ we have $A\sigma_\mu \in L^2$ and therefore $\operatorname{div} u_\mu = \operatorname{tr} A\sigma_\mu \in L^2$.

$$\begin{aligned}\int_\Omega \operatorname{div} u_\mu \sigma_{\mu ij} D_i D_j \theta^2 \,\mathrm{d}x &= \int_\Omega \operatorname{tr}(A\sigma_\mu)\sigma_{\mu ij} D_i D_j\theta^2 \,\mathrm{d}x \\ &\le C\,.\end{aligned}$$

By the assumptions (3.1) made for Df and Hölder's inequality

$$\begin{aligned}-\int_\Omega u_{\mu l} D_i\theta^2 D_l f_i \,\mathrm{d}x &\le C\|u_\mu\|_{L^{\frac{n}{n-1}}}\|D_l f_i\|_{L^n} \\ &\le C\,.\end{aligned}$$

There remains the term $\int_\Omega u_{\mu l}\sigma_{\mu ij} D_l D_i D_j\theta^2 \,\mathrm{d}x$. For dimension **n=2** we have $u_\mu \in L^{\frac{n}{n-1}}$ thus $u_\mu \in L^2$ and it follows

$$\int_\Omega u_{\mu l}\sigma_{\mu ij} D_i D_l D_j \theta^2 \,\mathrm{d}x \le C\|u_\mu\|_{L^2}\cdot\|\sigma_\mu\|_{L^2} \le C\,.$$

We now consider dimensions higher than $n=2$.
We know that $\sigma_\mu \in H^1_{loc}$ for μ fixed and $\sigma_\mu \in L^2$, the Sobolev inequalities yield:

$$\sigma_\mu \in H^1_{loc} \Rightarrow \ \sigma_\mu \in L^{\frac{2n}{n-2}}$$

For $n = 3$ we have $\sigma_\mu \in L^6,\ u_\mu \in L^{\frac{3}{2}}$
For $n = 4$ we have $\sigma_\mu \in L^4,\ u_\mu \in L^{\frac{4}{3}}$
but for $n = 5$ we have $\sigma_\mu \in L^{\frac{10}{3}},\ u_\mu \in L^{\frac{5}{4}}$ so the expression $\int_\Omega u_{\mu l}\sigma_{\mu ij} D_l D_i D_j \theta^2 \,\mathrm{d}x$ is welldefined in the case of dimension $n = 3, 4$.

In the cases $n = 3, 4$ we substitute θ by ϑ^3 where $\vartheta \in C_o^\infty(\Omega)$.

n=3:

$$\begin{aligned}\int_\Omega u_{\mu l}\sigma_{\mu ij} D_l D_i D_j \vartheta^6 \,\mathrm{d}x &\overset{\text{Hölder}}{\leq} C\|u_\mu\|_{L^{\frac{3}{2}}} \|\vartheta^3\sigma_\mu\|_{L^3} \\ &\leq C\|\vartheta^3\sigma_\mu\|_{L^6}\end{aligned}$$

with Sobolev

$$\begin{aligned}\|\vartheta^3\sigma_\mu\|_{L^6} \leq \|D_l(\vartheta^3\sigma_\mu)\|_{L^2} &\leq \|D_l\vartheta^3\sigma_\mu\|_{L^2} + \|\vartheta^3 D_l\sigma_\mu\|_{L^2} \\ &\leq C + \|\vartheta^3 D_l\sigma_\mu\|_{L^2}\end{aligned}$$

we have

$$C\|\vartheta^3 D_l\sigma_\mu\|_{L^2} \leq \frac{1}{4\rho}C^2 + \rho\|\vartheta^3 D_l\sigma_\mu\|_{L^2}^2$$

finally

$$\int_\Omega u_{\mu l}\sigma_{\mu ij} D_i D_l D_j \vartheta^3 \,\mathrm{d}x \leq C + \frac{1}{4\rho}C + \rho\|\vartheta^3 D_l\sigma_\mu\|_{L^2}^2$$

for **n=4**

$$\begin{aligned}\int_\Omega u_{\mu l}\sigma_{\mu ij} D_i D_l D_j \vartheta^3 \,\mathrm{d}x &\leq C\|\vartheta^3\sigma_\mu\|_{L^4} \\ &\leq C(\rho) + \rho\|\vartheta^3 D_l\sigma_\mu\|_{L^2}^2\end{aligned}$$

Now we choose ρ, γ, such that we can absorb terms containing $\|\theta D_l\sigma_\mu\|^2$.
This yields:

$$(\alpha - \gamma C - \rho)\|\vartheta^3 D_l\sigma_\mu\|^2 \leq Const + Const(\gamma) + Const(\rho) \tag{3.20}$$

Thus

$$\|\vartheta^3 D_l\sigma_\mu\|^2 \leq Const.$$

The sequence $(\theta D_l\sigma_\mu)_\mu$ is bounded uniformly in L^2. By the weak convergence $\sigma_\mu \rightharpoonup \sigma$ in L^2 we obtain $\theta D_l\sigma_\mu \rightharpoonup \theta D_l\sigma$ in L^2.
The stress tensor σ of the solution of the Hencky model is in $H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ for dimensions $n = 2, 3, 4$.

Chapter 4

Regularity for quasi-static Perzyna viscoplasticity

We show analogously to the static case, the regularity for the quasi-static Perzyna model. First we give estimates for the stress tensor and the penalty term. Then we show the existence of the time derivative $\dot{\sigma}_\mu$ using finite differences.

Remark: As in the case of the penalized Hencky model, for the estimates, existence and convergence we only need the material tensor to be measureable and bounded
$A \in L^\infty(\Omega, \hom(\mathbb{R}^{n\times n}_{\text{sym}}, \mathbb{R}^{n\times n}_{\text{sym}}))$.

4.1 Estimates for the stress and the penalty term

We make the following assumptions on the body force density f.

$$\left.\begin{aligned} f &\in L^\infty(0,T; L^n(\Omega, \mathbb{R}^n)) \\ Df &\in L^\infty(0,T; L^n_{loc}(\Omega, \mathbb{R}^{n\times n})) \\ \triangle f &\in L^\infty(0,T; L^n_{loc}(\Omega, \mathbb{R}^n)) \end{aligned}\right\} \tag{4.1}$$

Like in the static case assume:
safe load condition:
There exists a stress tensor $\tau \in W^{1,\infty}(0,T; L^\infty(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}))$ and $\delta > 0$

$$\left.\begin{aligned} \mathcal{F}(\tau(x,t)) \leq -\delta < 0 &\text{ for almost all } (x,t) \in \Omega \times [0,T] \\ -\operatorname{div}\tau = f &\text{ in } \Omega \times [0,T] \\ \tau \cdot \vec{n} = g &\text{ on } \Gamma_N \times [0,T] \\ \tau(x,0) = \sigma_o & \end{aligned}\right\} \tag{4.2}$$

existence of an admissible displacement:
There is a displacement $\hat{u} \in W^{1,\infty}(0,T;H^1(\Omega,\mathbb{R}^n))$ with

$$\left.\begin{aligned} \hat{u} &= U \text{ on } \Gamma_D \times [0,T] \\ \dot{\hat{u}} &= \dot{U} \text{ on } \Gamma_D \times [0,T] \\ \hat{u}(0) &= u_o \end{aligned}\right\} \tag{4.3}$$

We abbreviate: $v_\mu = \frac{\partial}{\partial t} u_\mu,\ \hat{v} = \frac{\partial}{\partial t}\hat{u},\ V = \dot{U}$

Theorem 4.1 *The sequence (σ_μ, v_μ) of solutions of quasi-static Perzyna law holds*

$$\begin{aligned} \|\sigma_\mu\|_{L^\infty(L^2)} &\le Const \\ \|G_\mu(\sigma_\mu)\|_{L^1(L^1)} &\le Const \\ \|G'_\mu(\sigma_\mu)\|_{L^1(L^1)} &\le Const\,. \end{aligned} \tag{4.4}$$

Proof Test the weak formulation (1.15) of Perzyna law with $\sigma_\mu - \tau$, where τ satisfies the safe load condition (4.2).

$$\big(A\dot{\sigma}_\mu, \sigma_\mu - \tau\big) + \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big) + \langle v_\mu, \operatorname{div}(\sigma_\mu - \tau)\rangle = \int_{\Gamma_D} V(\sigma_\mu - \tau)\cdot \vec{n}\,\mathrm{d}\Gamma \tag{4.5}$$

We have $\langle v_\mu, \operatorname{div}(\sigma_\mu - \tau)\rangle = 0$ because τ satisfies the safe load condition. Introduce zero addition with $\big(A\dot{\tau}, \sigma_\mu - \tau\big)$ on the left and with $\big(\varepsilon(\hat{v}), \sigma_\mu - \tau\big)$ on the righthand side of the equation.

$$\big(A(\dot{\sigma}_\mu - \dot{\tau}), \sigma_\mu - \tau\big) + \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big) = \big(\varepsilon(\hat{v}), \sigma_\mu - \tau\big) - \big(A\dot{\tau}, \sigma_\mu - \tau\big) \tag{4.6}$$

We write $\big(A(\dot{\sigma}_\mu - \dot{\tau}), \sigma_\mu - \tau\big)$ as time derivative.

$$\big(A(\dot{\sigma}_\mu - \dot{\tau}), \sigma_\mu - \tau\big) = \frac{1}{2}\frac{d}{dt}\big(A(\sigma_\mu - \tau), \sigma_\mu - \tau\big)$$

Integrate (4.6) from 0 to t

$$\frac{1}{2}\big(A(\sigma_\mu - \tau), \sigma_\mu - \tau\big) + \int_0^t \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big)\,\mathrm{d}s = \int_0^t \big(\varepsilon(\hat{v}) - A\dot{\tau}, \sigma_\mu - \tau\big)\,\mathrm{d}s \tag{4.7}$$

The convexity and the Gâteaux differentiability of G_μ leads like in the static case (2.6), to the definiteness of the tested penaly term.

$$\int_0^t \int_\Omega G_\mu(\sigma_\mu)\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big)\,\mathrm{d}s \tag{4.8}$$

$$\int_0^t \big(G'_\mu(\sigma_\mu), \sigma_\mu - \tau\big)\,\mathrm{d}s \ge 0$$

Using the ellipticity of A

$$\frac{\alpha}{2}\|\sigma_\mu - \tau\|^2 \leq \int_0^t \big(\varepsilon(\hat{v}) - A\dot{\tau},\sigma_\mu - \tau\big)\,\mathrm{d}s\,. \tag{4.9}$$

Young's inequality on the right hand side

$$\frac{\alpha}{2}\|\sigma_\mu - \tau\|^2 \leq \frac{1}{4\gamma}\int_0^t \|\varepsilon(\hat{v}) - A\dot{\tau}\|^2\,\mathrm{d}s + \gamma\int_0^t \|\sigma_\mu - \tau\|^2\,\mathrm{d}s \tag{4.10}$$

and

$$\frac{\alpha}{2}\|\sigma_\mu - \tau\|^2 \leq Const(\gamma) + \gamma\int_0^t \|\sigma_\mu - \tau\|^2\,\mathrm{d}s\,. \tag{4.11}$$

The Gronwall lemma finally implies

$$\|\sigma_\mu - \tau\| \leq Const$$

and $(\sigma_\mu - \tau) \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$ therefore

$$\|\sigma_\mu\|_{L^\infty(L^2)} \leq Const\,.$$

We get $\sigma_\mu \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$ independent from the viscosity coefficient μ. The estimate for σ_μ leads with (4.7) to

$$\int_0^t \big(G'_\mu(\sigma_\mu),\sigma_\mu - \tau\big)\,\mathrm{d}s \leq Const \tag{4.12}$$

and using (4.8) implies

$$\int_0^t \int_\Omega G_\mu(\sigma_\mu)\,\mathrm{d}x\,\mathrm{d}s \leq Const\,. \tag{4.13}$$

G_μ is bounded in $L^1(0,T;L^1(\Omega))$ uniformly in μ. The lemma 2 of Suquet [Suq81] delivers

$$\|G'_\mu(\sigma_\mu)\|_{L^1(L^1)} \leq Const\,. \tag{4.14}$$

4.2 Existence of the time derivative $\dot{\sigma}_\mu$ and estimates for the strain tensor

We now show the existence of the time derivative $\dot{\sigma}_\mu \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$ and uniform estimates in the viscosity coefficent.

Theorem 4.2 *The time derivative $\dot{\sigma}_\mu$ of the stress tensor exists and satisfies $\dot{\sigma}_\mu \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym}))$ with*

$$\|\dot{\sigma}_\mu\|_{L^2(L^2)} \leq Const\,.$$

Proof We discretize the weak formulation (1.15) of Perzyna viscoplasticity in time with finite backward differences.
Let $N \in \mathbb{N}^+$, $k = \frac{T}{N}$ the time stepwidth and $\eta^m = \eta(m \cdot k)$. Write

$$D_t^{-k}\eta^m = \frac{\eta^m - \eta^{m-1}}{k}$$

for finite backward differences in time. The time discretized formulation is now

$$\left(AD_t^{-k}\sigma_\mu^m, \sigma_\mu^m - \chi^m\right) + \left(G'_\mu(\sigma_\mu^m), \sigma_\mu^m - \chi^m\right) + \langle v_\mu^m, \operatorname{div}(\sigma_\mu^m - \chi^m)\rangle = \int_{\Gamma_D} V^m(\sigma_\mu^m - \chi^m) \cdot \vec{n}\, \mathrm{d}\Gamma. \tag{4.15}$$

On every time step m for μ, k fixed we have a Hencky like problem. The existence of the stress tensor of the Hencky like problem can be shown in the same way as for the Hencky Problem in theorem 1.3.
Let τ satisfy the safe load condition and $\hat{v}$ be an admissible displacement (4.3). Write $\overline{\sigma}^m = \sigma_\mu^m - \tau^m$ then $\operatorname{div}\overline{\sigma} = 0$. Test the discrete formulation (4.15) with $D_t^{-k}\overline{\sigma}^m$.

$$\left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\overline{\sigma}^m\right) + \left(G'_\mu(\sigma_\mu^m), D_t^{-k}\overline{\sigma}^m\right) = \int_{\Gamma_D} V^m D_t^{-k}\overline{\sigma}^m \cdot \vec{n}\, \mathrm{d}\Gamma \tag{4.16}$$

After sorting terms

$$\left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\sigma_\mu^m\right) + \left(G'_\mu(\sigma_\mu^m), D_t^{-k}\overline{\sigma}^m\right) = \int_{\Gamma_D} V^m D_t^{-k}\overline{\sigma}^m \cdot \vec{n}\, \mathrm{d}\Gamma + \left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\tau^m\right) \tag{4.17}$$

Introduce zero additions $\left(\varepsilon(\hat{v}^m), D_t^{-k}\overline{\sigma}^m\right)$ and using the ellipticity of A.

$$\alpha\|D_t^{-k}\sigma_\mu^m\|^2 + \left(G'_\mu(\sigma_\mu^m), D_t^{-k}\overline{\sigma}^m\right) = \left(\varepsilon(\hat{v}^m), D_t^{-k}\overline{\sigma}^m\right) + \left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\tau^m\right) \tag{4.18}$$

Consider $\left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\tau^m\right)$ on the right hand side. The tensor A is symmetric $A^* = A$ we can write

$$\left(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\tau^m\right) = \left(D_t^{-k}\sigma_\mu^m, AD_t^{-k}\tau^m\right).$$

Apply the Schwarz und Young inequality on $\left(\varepsilon(\hat{v}^m), D_t^{-k}\overline{\sigma}^m\right)$. We exploit, that for k sufficiently small $\|D_t^{-k}\tau^m\| \le \|\dot{\tau}^m\|$ holds.

$$\left(\varepsilon(\hat{v}^m), D_t^{-k}\sigma_\mu^m\right) - \left(\varepsilon(\hat{v}^m), D_t^{-k}\tau^m\right) \le \frac{1}{4\rho}\|\varepsilon(\hat{v}^m)\|^2 + \rho\|D_t^{-k}\sigma_\mu^m\|^2 + \|\varepsilon(\hat{v}^m)\| \cdot \|\dot{\tau}^m\| \tag{4.19}$$

We have $\|\varepsilon(\hat{v}^m)\| \le Const$ und $\|\dot{\tau}^m\| \le Const$. Using Young's inequality for $\left(D_t^{-k}\sigma_\mu^m, AD_t^{-k}\tau^m\right)$

$$\left(D_t^{-k}\sigma_\mu^m, AD_t^{-k}\tau^m\right) \le \gamma\|D_t^{-k}\sigma_\mu^m\|^2 + \frac{1}{4\gamma}\|AD_t^{-k}\tau^m\|^2 \tag{4.20}$$

With

$$\begin{aligned}\frac{1}{4\gamma}\|AD_t^{-k}\tau^m\|^2 &\le \frac{1}{4\gamma}\|A\|_{L^\infty}^2 \cdot \|D_t^{-k}\tau^m\|^2 \\ &\le C\|\dot{\tau}^m\|^2 \\ &\le Const\end{aligned} \tag{4.21}$$

These estimates and the choice $0 < \gamma + \rho < \alpha$ yield

$$(\alpha - \gamma - \rho)\|D_t^{-k}\sigma_\mu^m\|^2 + \big(G'_\mu(\sigma_\mu^m), D_t^{-k}\sigma_\mu^m\big) \le \big(G'_\mu(\sigma_\mu^m), D_t^{-k}\tau^m\big) + Const. \tag{4.22}$$

Futhermore

$$\begin{aligned}\big(G'_\mu(\sigma_\mu^m), D_t^{-k}\tau^m\big) &= \int_\Omega G'_\mu(\sigma_\mu^m) : (D_t^{-k}\tau^m)\,\mathrm{d}x \\ &\le \int_\Omega |G'_\mu(\sigma_\mu^m)| \cdot |D_t^{-k}\tau^m|\,\mathrm{d}x \\ &\le \int_\Omega |G'_\mu(\sigma_\mu^m)| \cdot |\dot{\tau}^m|\,\mathrm{d}x \\ &\le C\int_\Omega |G'_\mu(\sigma_\mu^m)|\,\mathrm{d}x\,.\end{aligned} \tag{4.23}$$

We get

$$(\alpha - \gamma - \rho)\|D_t^{-k}\sigma_\mu^m\|^2 + \big(G'_\mu(\sigma_\mu^m), D_t^{-k}\sigma_\mu^m\big) \le C\int_\Omega |G'_\mu(\sigma_\mu^m)|\,\mathrm{d}x + Const\,. \tag{4.24}$$

We multiply this equation by k and sum over m from 1 to N and remember $k \cdot D_t^{-k}\eta^m = \eta^m - \eta^{m-1}$.

$$(\alpha-\gamma-\rho)\sum_{m=1}^N k\cdot\|D_t^{-k}\sigma_\mu^m\|^2 + \underbrace{\sum_{m=1}^N \big(G'_\mu(\sigma_\mu^m), \sigma_\mu^m - \sigma_\mu^{m-1}\big)}_{(*)} \le \sum_{m=1}^N k\cdot Const + C\sum_{m=1}^N k\cdot\int_\Omega |G'_\mu(\sigma_\mu^m)|\,\mathrm{d}x \tag{4.25}$$

The term $(*)$ is definite, using the inequality for convex differentiable functions we get

$$\int_\Omega G_\mu(\sigma_\mu^m) - G_\mu(\sigma_\mu^{m-1})\,\mathrm{d}x \le \big(G'_\mu(\sigma_\mu^m), \sigma_\mu^m - \sigma_\mu^{m-1}\big)\,.$$

Summing from $m = 1, \dots, N$ gives a telescope sum.

$$\begin{aligned}(*) = \sum_{m=1}^N \big(G'_\mu(\sigma_\mu^m), \sigma_\mu^m - \sigma_\mu^{m-1}\big) &\ge \sum_{m=1}^N \int_\Omega G_\mu(\sigma_\mu^m) - G_\mu(\sigma_\mu^{m-1})\,\mathrm{d}x \\ &= \int_\Omega G_\mu(\sigma_\mu^N) - G_\mu(\sigma_\mu^0)\,\mathrm{d}x \ge 0\end{aligned} \tag{4.26}$$

By assumption $\sigma_\mu^0 = \sigma_o \in \mathcal{K}$ holds and $\sum_{m=1}^N k \cdot Const = N \cdot \frac{T}{N} \cdot Const$. We know that $G'_\mu(\sigma_\mu) \in L^1(0,T; L^1(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}))$ is uniformly bounded in μ. For k sufficiently small enough

$$C\sum_{m=1}^N k \cdot \int_\Omega |G'_\mu(\sigma_\mu^m)|\,\mathrm{d}x \le C\|G'_\mu(I_p^k\sigma_\mu)\|_{L^1(L^1)} \le Const\,. \tag{4.27}$$

Where I_p^k denotes the piece wise constant interpolation in time.

$$(\alpha - \gamma - \rho) \sum_{m=1}^{N} k \cdot \|D_t^{-k} \sigma_\mu^m\|^2 \leq Const + Const \cdot T\,. \tag{4.28}$$

The sequence $(D_t^{-k}(I_p^k \sigma_\mu))_k$ is bounded in $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ for k sufficiently small. The uniqueness of the solution follows from a monotonicity argument, the existence of the derivative $\dot{\sigma}_\mu \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ follows with the estimate

$$\|\dot{\sigma}_\mu\|_{L^2(L^2)} \leq Const(T)\,. \tag{4.29}$$

Like in the static case the solutions v_μ of quasi-static Perzyna model are more regular than $BD(\Omega)$.

Theorem 4.3 *For fixed viscosity coefficient μ we have $\varepsilon(v_\mu) \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym}))$ and $v_\mu \in L^2(0,T;H^1(\Omega,\mathbb{R}^n))$.*

Proof In the weak formulation (1.15) we get after a zero addition of $\big(\varepsilon(\hat{v}),\chi - \sigma_\mu\big)$ the equation

$$\big(A\dot{\sigma}_\mu,\chi - \sigma_\mu\big) + \langle v_\mu, \operatorname{div}(\chi - \sigma_\mu)\rangle + \big(G'_\mu(\sigma_\mu),\chi - \sigma_\mu\big) = \big(\varepsilon(\hat{v}),\chi - \sigma_\mu\big)\,. \tag{4.30}$$

Choose $\chi \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ with

$$\begin{aligned} -\operatorname{div}\chi &= f \text{ in } \Omega \times [0,T] \\ \chi \cdot \vec{n} &= g \text{ on } \Gamma_N \times [0,T] \\ \chi \cdot \vec{n} &= \sigma_\mu \cdot \vec{n} \text{ on } \Gamma_D \times [0,T]\,. \end{aligned} \tag{4.31}$$

For such a χ and all $t \in [0,T]$

$$\begin{aligned} \operatorname{div}(\chi - \sigma_\mu) &= 0 \\ \big(\varepsilon(\hat{v}),\chi - \sigma_\mu\big) &= 0\,. \end{aligned} \tag{4.32}$$

We get

$$\big(A\dot{\sigma}_\mu - G'_\mu(\sigma_\mu),\chi - \sigma_\mu\big) = 0\,.$$

The application of theorem D.4 yields

$$A\dot{\sigma}_\mu + G'_\mu(\sigma_\mu) = \varepsilon(v_\mu) \tag{4.33}$$

for almost every $x \in \Omega$ and it follows that $\varepsilon(v_\mu) \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$.
From the Lipschitz continuity of $Id - P_K$ and $\|\sigma_\mu\|_{L^\infty(L^2)} \leq Const$ we deduce

$$\|G'_\mu(\sigma_\mu)\|_{L^2(L^2)} \leq Const(\mu)$$

like in theorem 2.2 we obtain

$$\begin{aligned} \|\varepsilon(v_\mu)\|_{L^2(L^2)} &\leq \frac{1}{\mu} Const \\ \|v_\mu\|_{L^2(H^1)} &\leq \frac{1}{\mu} Const\,. \end{aligned} \tag{4.34}$$

Theorem 4.4 *We have*

$$\begin{aligned} \|\varepsilon(v_\mu)\|_{L^2(L^1)} &\leq Const \\ \|v_\mu\|_{L^2(L^{\frac{n}{n-1}})} &\leq Const\,. \end{aligned} \tag{4.35}$$

Proof By $\|\sigma_\mu\|_{L^2(L^2)} \leq Const$ and $\|\dot{\sigma}_\mu\|_{L^2(L^2)} \leq Const$ we obtain form equation (4.7) that $\int_0^T \big(G'_\mu(\sigma_\mu),\sigma_\mu - \tau\big)\, ds$ is bounded in $L^2(0,T;\mathbb{R})$. Using again lemma 2 from Suquet [Suq81] we obtain $\|G'_\mu(\sigma_\mu)\|_{L^2(L^1)} \leq Const$. With these estimates we deduce from the pointwise Perzyna law (4.34) $\|\varepsilon(v_\mu)\|_{L^2(L^1)} \leq Const$. Korn's inequality gives $\|v_\mu\|_{L^2(L^{\frac{n}{n-1}})} \leq Const$.

4.3 Convergence of the penalized model to the Prandtl Reuss law

With the estimates of the preceeding section we are able to show the convergence of the Perzyna penalized model to the Prandtl Reuss model. Frist we show the convergence of the stress tensor.

Theorem 4.5 *There exists a subsequence σ_{μ_l} such that*

$$\begin{aligned} \sigma_{\mu_l} &\rightharpoonup \sigma \ \ in\ L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym})) \\ \dot{\sigma}_{\mu_l} &\rightharpoonup \dot{\sigma} \ \ in\ L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym})) \end{aligned}$$

Where σ denotes the weak stress solution of the Prandtl Reuss model.

Proof By the boundedness of $\sigma_\mu, \dot{\sigma}_\mu$ in $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$ we can extract a suitable subsequence σ_{μ_l} such that

$$\begin{aligned} \sigma_{\mu_l} &\rightharpoonup \tilde{\sigma} \\ \dot{\sigma}_{\mu_l} &\rightharpoonup \dot{\tilde{\sigma}} \end{aligned}$$

for $\tilde{\sigma}, \dot{\tilde{\sigma}} \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$. The weak limit satisfies $\tilde{\sigma} \in \mathcal{M} \cap \{\sigma | -\operatorname{div}\sigma = f\}$ since it is closed and convex. Test the pointwise almost everywhere penalized Prandtl Reuss law (4.33) with $\sigma_{\mu_l} - \tau$ where $\tau \in \mathcal{K} \cap \mathcal{M}$ and $-\operatorname{div}\tau = f$. We have a.e. in [0,T]

$$0 = \big(A\dot{\sigma}_{\mu_l},\sigma_{\mu_l} - \tau\big) + \big(G'_\mu(\sigma_{\mu_l}),\sigma_{\mu_l} - \tau\big)\,. \tag{4.36}$$

The tested penalty term is positive semi definite using the inequality for convex differentiable functions.

$$\big(A\dot{\sigma}_{\mu_l},\sigma_{\mu_l}-\tau\big) \le 0 \tag{4.37}$$

We have $\tilde{\sigma} \in \mathcal{K}$ because for a.e. $t \in [0,T]$ we have

$$\int_0^T \int_\Omega |(Id - P_Z)(\sigma_{\mu_l})| \,\mathrm{d}x\,\mathrm{d}s \le \mu_l \cdot Const.$$

If we insert $\tau = \tilde{\sigma}$ into equation (4.37), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big) - \big(A\dot{\sigma}_{\mu_l},\tilde{\sigma}\big) \le 0\,. \tag{4.38}$$

Integrating in time from 0 to t and bearing in mind, that $\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big)(0) = \big(A\sigma_o,\sigma_o\big) = Const > 0$.

$$\frac{1}{2}\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big)(t) \le \int_0^t \big(A\dot{\sigma}_{\mu_l},\tilde{\sigma}\big)\,\mathrm{d}s \tag{4.39}$$

Letting the the penalty parameter $\mu \to 0$ and using the weak covergence of σ_{μ_l}

$$\begin{aligned}\lim_{\mu_l\to 0}\frac{1}{2}\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big)(t) &\le \int_0^t \big(A\dot{\tilde{\sigma}},\tilde{\sigma}\big)\,\mathrm{d}s \\ &= \frac{1}{2}\big(A\tilde{\sigma},\tilde{\sigma}\big)\Big|_{s=0}^{s=t} \le \frac{1}{2}\big(A\tilde{\sigma},\tilde{\sigma}\big)(t)\,.\end{aligned}$$

Note that $\tilde{\sigma}(0)$ is defined, since $\dot{\tilde{\sigma}} \in L^2$ implies $\sigma \in C(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$.
Thus $\sigma_{\mu_l} \to \tilde{\sigma}$ strongly in L^2 and therefore we can pass to the limit in the variational inequality (4.37).

$$\big(A\dot{\tilde{\sigma}},\tilde{\sigma}-\tau\big) \le 0$$

Hence $\tilde{\sigma}$ is a solution of the Prandtl Reuss law.

Theorem 4.6 *There exists a subsequence v_{μ_l} converging weakly in $L^2(0,T;L^{\frac{n}{n-1}}(\Omega,\mathbb{R}^n))$ to v displacement velocity solution of the Prandtl Reuss model.*

Proof The know that $\|v_\mu\|_{L^2(L^{\frac{n}{n-1}})} \le Const$ so we can extract a suitable subsequence $(\sigma_{\mu_l}, v_{\mu_l})$ converging weakly to

$$(\sigma_{\mu_l}, v_{\mu_l}) \rightharpoonup (\sigma, \tilde{v})$$

with $\tilde{v} \in L^2(0,T;L^{\frac{n}{n-1}}(\Omega,\mathbb{R}^n))$. Test the pointwise almost everywhere penalized Prandtl Reuss law with $\sigma_{\mu_l} - \tau$, where $\tau \in \mathcal{M} \cap \mathcal{K}$.

$$-\big(v_{\mu_l},f - \operatorname{div}\tau\big) = \big(A\dot{\sigma_{\mu_l}},\sigma_{\mu_l}-\tau\big) + \big(G'_\mu(\sigma_{\mu_l}),\sigma_{\mu_l}-\tau\big)\,. \tag{4.40}$$

Again the tested Penalty term is definite and we have

$$0 \le \big(A\dot{\sigma}_{\mu_l},\tau-\sigma_{\mu_l}\big) + \big(v_{\mu_l},\operatorname{div}\tau - f\big)\,. \tag{4.41}$$

A lower semicontinuity argument gives us the desired result.
By uniqueness and a routine argument the whole sequence converges.

4.4 Local differentiability of the stress tensor

With the estimates of the preceeding sections we are now able to show the local differentiability of the stress tensor.

Assumption $\sigma_\mu(0) = \sigma_o \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$

Theorem 4.7 *For fixed viscosity coefficient μ we have $\sigma_\mu \in H^1(0,T; H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{sym}))$.*

Proof Let $\theta \in C_o^\infty(\Omega)$ be a cutoff function and $0 < h < \frac{1}{2}\operatorname{dist}(\operatorname{supp}\theta, \partial\Omega)$. Test the pointwise Perzyna law (4.34) with the difference quotient $-D_j^{-h}(\theta^2 D_j^h \sigma_\mu)$. Using discrete partial integration

$$\big(D_j^h \varepsilon(v_\mu),\theta^2 D_j^h \sigma_\mu\big) = \big(\theta A D_j^h \dot{\sigma}_\mu,\theta D_j^h \sigma_\mu\big) + \big(\theta D_j^h G'_\mu(\sigma_\mu),\theta D_j^h \sigma_\mu\big) \tag{4.42}$$

Theorem C.3 gives the definiteness of the term $\big(\theta D_j^h G'_\mu(\sigma_\mu),\theta D_j^h \sigma_\mu\big)$.

$$\big(\theta D_j^h G'_\mu(\sigma_\mu),\theta D_j^h \sigma_\mu\big) \geq 0.$$

We have the inequality

$$\big(\theta A D_j^h \dot{\sigma}_\mu,\theta D_j^h \sigma_\mu\big) \leq \big(D_j^h \varepsilon(v_\mu),\theta^2 D_j^h \sigma_\mu\big)\,. \tag{4.43}$$

Write $\big(\theta A D_j^h \dot{\sigma}_\mu,\theta D_j^h \sigma_\mu\big)$ as time derivative

$$\big(\theta A D_j^h \dot{\sigma}_\mu,\theta D_j^h \sigma_\mu\big) = \frac{1}{2}\frac{d}{d\,t}\big(\theta A D_j^h \sigma_\mu,\theta D_j^h \sigma_\mu\big)\,.$$

Integrate equation (4.43) from 0 to t.

$$\big(\theta A D_j^h \sigma_\mu,\theta D_j^h \sigma_\mu\big) + \big(\theta A D_j^h \sigma_\mu(0),\theta \sigma_\mu(0)\big) \leq \int_0^t \big(D_j^h \varepsilon(v_\mu),\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s \tag{4.44}$$

The ellipticity and the assumption $\sigma_o \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ lead to

$$\alpha\|\theta D_j^h \sigma_\mu\|^2 \leq \int_0^t \big(D_j^h \varepsilon(v_\mu),\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s + Const\,. \tag{4.45}$$

Consider the right hand side and apply Green's formula.

$$\int_0^t \big(D_j^h \varepsilon(v_\mu),\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s = -\int_0^t \big(D_j^h v_\mu, \operatorname{grad}\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s - \int_0^t \big(D_j^h v_\mu,\theta^2 D_j^h f\big)\,\mathrm{d}s \tag{4.46}$$

Discrete partial integration gives

$$-\int_0^t \big(D_j^h v_\mu,\theta^2 D_j^h f\big)\,\mathrm{d}s = \int_0^t \big(v_\mu,D_j^{-h}\theta^2 D_j^h f\big)\,\mathrm{d}s + \int_0^t \big(v_\mu,E_j^{-h}\theta^2 \triangle^h f\big)\,\mathrm{d}s$$

Using the Hölder inequality and (4.1)

$$\int_0^t \big(v_\mu, D_j^{-h}\theta^2 D_j^h f\big) + \big(v_\mu, E_j^{-h}\theta^2 \triangle^h f\big) \mathrm{d}s \leq \int_0^t \|v_\mu\|_{L^{\frac{n}{n-1}}} \cdot \big(\|D_j^{-h}\theta^2 D_j^h f\|_{L^n} + \|E_j^{-h}\theta^2 \triangle^h f\|_{L^n}\big) \mathrm{d}s$$
$$\leq Const$$

$$-\int_0^t \big(D_j^{-h} v_\mu, \theta^2 D_j^h f\big)\,\mathrm{d}s \leq Const$$

We estimate $-\int_0^t \big(D_j^h v_\mu, \operatorname{grad}\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s$ with Young's and Korn's inequality.

$$\begin{aligned}
-\int_0^t \big(D_j^h v_\mu, \operatorname{grad}\theta^2 D_j^h \sigma_\mu\big)\,\mathrm{d}s &\leq \frac{1}{4\gamma}\int_0^t \|D_j^h v_\mu\|^2\,\mathrm{d}s + \gamma \int_0^t \|2\operatorname{grad}\theta\|^2 \cdot \|\theta D_j^h \sigma_\mu\|^2\,\mathrm{d}s \\
&\leq \frac{1}{4\gamma}\int_0^t \|D_j^h v_\mu\|^2\,\mathrm{d}s + \gamma C_\theta \int_0^t \|\theta D_j^h \sigma_\mu\|^2\,\mathrm{d}s \\
&\leq \frac{1}{4\gamma}\int_0^t C_{Korn}\|\varepsilon(v_\mu)\|^2\,\mathrm{d}s + \gamma C_\theta \int_0^t \|\theta D_j^h \sigma_\mu\|^2\,\mathrm{d}s
\end{aligned}$$

Altogether

$$\alpha\|\theta D_j^h \sigma_\mu\|^2 \leq Const + t\cdot Const(\mu) + \gamma C_\theta \int_0^t \|\theta D_j^h \sigma_\mu\|^2\,\mathrm{d}s\,.$$

The Gronwall lemma implies

$$\|\theta D_j^h \sigma_\mu\|^2 \leq Const(\mu)\,.$$

We have for μ fixed $\sigma_\mu \in H^1(0,T; H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}))$.

Chapter 5

H^1_{loc} regularity for the stress tensor in the Prandtl Reuss model with von Mises yield criterion

We show Analogously to the Hencky model the local differentiability of the stress tensor in the Prandtl Reuss model. The first differentiability results are due to Bensoussan & Frehse [BF94, BF96].
They used the Norton-Hoff model as approximation. We use the Perzyna model as approximation.
The quasi-static Perzyna model will be discretized in time and we obtain a system of Hencky like problems.

The assumptions for the bodyforce density f are the same as in chapter 4.

$$\left.\begin{aligned} f &\in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n)) \\ Df &\in L^\infty(0,T;L^n_{loc}(\Omega,\mathbb{R}^{n\times n})) \\ \triangle f &\in L^\infty(0,T;L^n_{loc}(\Omega,\mathbb{R}^n)) \end{aligned}\right\} \tag{5.1}$$

Initialvalue of σ_μ: $\sigma_\mu(0) = \sigma_o \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$

Pointwise penalized Prandtl Reuss model for μ fixed.

$$\varepsilon(v_\mu) = A\dot{\sigma}_\mu + \frac{1}{\mu}\frac{(|\sigma_{\mu D}| - \kappa)_+}{|\sigma_{\mu D}|}\sigma_{\mu D}\,.$$

We already know

$$\begin{aligned} \|\varepsilon(v_\mu)\|_{L^1(L^1)} &\le Const \\ \|v_\mu\|_{L^1(L^{\frac{n}{n-1}})} &\le Const \\ \|\dot{\sigma}_\mu\|_{L^2(L^2)} &\le Const \end{aligned}$$

5.1 Discretisation in time

Let $N \in \mathbb{N}^+$ and $k = \frac{T}{N}$ the time stepwidth. We discretize $\dot{\sigma}_\mu$ by finite backward differences in time.

$$\dot{\sigma}_\mu^m \approx D_t^{-k}\sigma_\mu^m = \frac{\sigma_\mu^m - \sigma_\mu^{m-1}}{k}$$

We have a system of N equations.

$$\left.\begin{aligned}
A\left(\frac{\sigma_\mu^1 - \sigma_\mu^0}{k}\right) + \frac{1}{\mu}\frac{(|\sigma_{\mu D}^1| - \kappa)_+}{|\sigma_{\mu D}^1|}\sigma_{\mu D}^1 &= \varepsilon(v_\mu^1) \\
A\left(\frac{\sigma_\mu^2 - \sigma_\mu^1}{k}\right) + \frac{1}{\mu}\frac{(|\sigma_{\mu D}^2| - \kappa)_+}{|\sigma_{\mu D}^2|}\sigma_{\mu D}^2 &= \varepsilon(v_\mu^2) \\
&\vdots \\
A\left(\frac{\sigma_\mu^m - \sigma_\mu^{m-1}}{k}\right) + \frac{1}{\mu}\frac{(|\sigma_{\mu D}^m| - \kappa)_+}{|\sigma_{\mu D}^m|}\sigma_{\mu D}^m &= \varepsilon(v_\mu^m) \\
&\vdots \\
A\left(\frac{\sigma_\mu^N - \sigma_\mu^{N-1}}{k}\right) + \frac{1}{\mu}\frac{(|\sigma_{\mu D}^N| - \kappa)_+}{|\sigma_{\mu D}^N|}\sigma_{\mu D}^N &= \varepsilon(v_\mu^N)
\end{aligned}\right\} \tag{5.2}$$

5.2 H^1_{loc} for μ, k fixed

On every timestep m we have $\sigma_\mu^m \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ for fixed $k, \mu > 0$.
Proof By induction over the timesteps
Let $m = 1$. Test the equation with $-D_j^{-h}(\theta^2 D_j^h \sigma_\mu^1)$ where $\theta \in C_o^\infty(\Omega)$.

$$\big(D_j^h\varepsilon(v_\mu^1), \theta^2 D_j^h\sigma_\mu^1\big) = \frac{1}{k}\big(AD_j^h(\sigma_\mu^1 - \sigma_\mu^0), \theta^2 D_j^h\sigma_\mu^1\big) + \underbrace{\left(D_j^h\Big(\frac{1}{\mu}\frac{(|\sigma_{\mu D}^1| - \kappa)_+}{|\sigma_{\mu D}^1|}\sigma_{\mu D}^1\Big), \theta^2 D_j^h\sigma_\mu^1\right)}_{\geq 0 \text{ by monotonicity}} \tag{5.3}$$

Where

$$\frac{1}{k}\big(AD_j^h(\sigma_\mu^1 - \sigma_\mu^0), \theta^2 D_j^h\sigma_\mu^1\big) = \underbrace{\frac{1}{k}\big(A\theta D_j^h\sigma_\mu^1, \theta D_j^h\sigma_\mu^1\big)}_{\geq \frac{\alpha}{k}\|\theta D_j^h\sigma_\mu^1\|^2} - \frac{1}{k}\big(\theta AD_j^h\sigma_\mu^0, \theta D_j^h\sigma_\mu^0\big) \tag{5.4}$$

and

$$\big(D_j^h\varepsilon(v_\mu^1), \theta^2 D_j^h\sigma_\mu^1\big) + \frac{1}{k}\big(\theta AD_j^h\sigma_\mu^0, \theta D_j^h\sigma_\mu^1\big) \geq \frac{\alpha}{k}\|\theta D_j^h\sigma_\mu^1\|^2 \tag{5.5}$$

Using Young's inequality for $\frac{1}{k}\big(AD_j^h\sigma_\mu^0, \theta D_j^h\sigma_\mu^1\big)$ we get

$$\frac{1}{k}\big(\theta AD_j^h\sigma_\mu^0, \theta D_j^h\sigma_\mu^1\big) \leq \frac{\gamma_1}{k}\|\theta D_j^h\sigma_\mu^1\|^2 + \frac{1}{4\gamma_1 k}\|A\theta D_j^h\sigma_\mu^0\|^2 \tag{5.6}$$

We have $\sigma^0_\mu \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ by assumption and h small enough that $\|\theta A D^h_j \sigma^0_\mu\| \le \|\theta A D_j \sigma^0_\mu\| \le Const$ holds and we obtain

$$\frac{1}{k}\big(\theta A D^h_j \sigma^0_\mu, \theta D^h_j \sigma^1_\mu\big) \le \frac{\gamma_1}{k}\|\theta D^h_j \sigma^1_\mu\|^2 + \frac{1}{4\gamma_1 k} Const \tag{5.7}$$

For $\mu > 0$ fixed $\varepsilon(v^1_\mu) \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ and $v^1_\mu \in H^1(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$.

$$\big(D^h_j \varepsilon(v^1_\mu), \theta^2 D^h_j \sigma^1_\mu\big) = -\big(D^h_j v^1_\mu, \operatorname{grad}\theta^2 D^h_j \sigma^1_\mu\big) - \big(D^h_j v^1_\mu, \theta^2 D^h_j f^1\big) \tag{5.8}$$

where

$$\begin{aligned} -\big(D^h_j v^1_\mu, \theta^2 D^h_j f^1\big) &= \big(v^1_\mu, D^{-h}_j \theta^2 D^h_j f^1\big) + \big(v^1_\mu, E^{-h}_j \theta^2 \triangle^h f^1\big) \\ &\le \|v^1_\mu\|_{L^{\frac{n}{n-1}}} \big(\|D^{-h}_j \theta^2 D^h_j f^1\|_{L^n} + \|E^{-h}_j \theta^2 \triangle^h f^1\|_{L^n}\big) \\ &\le C\big(\mu, (m,k)\big) \end{aligned}$$

Futhermore we have

$$\begin{aligned} -\big(D^h_j v^1_\mu, \operatorname{grad}\theta^2 D^h_j \sigma^1_\mu\big) &\le \frac{1}{4\gamma_2}\|D^h_j v^1_\mu\|^2 + \gamma_2 \|2 \operatorname{grad}\theta\|^2 \cdot \|\theta D^h_j \sigma^1_\mu\|^2 \\ &\le \frac{1}{4\gamma_2}\|D^h_j v^1_\mu\|^2 + \gamma_2 C \|\theta D^h_j \sigma^1_\mu\|^2 \\ &\le \frac{1}{4\gamma_2} C_{Korn} \|\varepsilon(v^1_\mu)\|^2 + \gamma_2 C \|\theta D^h_j \sigma^1_\mu\|^2 \\ &\le \frac{1}{4\gamma_2} C(\mu, k) + \gamma_2 C \|\theta D^h_j \sigma^1_\mu\|^2 \end{aligned}$$

By a suitable choice of $\gamma_1, \gamma_2 > 0$ we can absorb terms containing $\|\theta D^h_j \sigma^1_\mu\|^2$.

$$\Big(\frac{\alpha}{k} - \frac{\gamma_1}{k} - \gamma_2 C\Big)\|\theta D^h_j \sigma^1_\mu\|^2 \le Const$$

$$\Rightarrow \quad \|\theta D^h_j \sigma^1_\mu\|^2 \le C(\mu, k)$$

These estimates yield $\sigma^1_\mu \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ for μ, k fixed. By induction over m we obtain for μ, k fixed $\sigma^m_\mu \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$.

5.3 H^1_{loc} uniform estimates

We are now able to show that $\sigma^m_\mu \in H^1_{loc}(\Omega, \mathbb{R}^{n\times n}_{\text{sym}})$ independent of the choice of μ and k. We will proceed like in the case of the penalized Hencky model.
Every row j of the discretized system (5.2) is differentiated by D_r and then tested with $\theta^2 D_r \sigma^j_\mu$.
On timestep m we have the equation

$$\big(\theta A D_r D^{-k}_t \sigma^m_\mu, \theta D_r \sigma^m_\mu\big) + \big(D_r G'_\mu(\sigma^m_\mu), \theta^2 D_r \sigma^m_\mu\big) = \big(D_r \varepsilon(v^m_\mu), \theta^2 D_r \sigma^m_\mu\big).$$

The term $\big(\theta A D_r D_t^{-k}\sigma_\mu^m, \theta D_r \sigma_\mu^m\big)$ can be bounded from below by the ellipticity of A.

$$\begin{aligned}\frac{1}{k}\big(\theta A D_r(\sigma_\mu^m - \sigma_\mu^{m-1}), \theta D_r\sigma_\mu^m\big) &= \frac{1}{k}\big(\theta A D_r\sigma_\mu^m, \theta D_r\sigma_\mu^m\big) - \frac{1}{k}\overbrace{\big(\theta A D_r\sigma_\mu^{m-1}, \theta D_r\sigma_\mu^m\big)}^{\text{estimate by Young}} \\ &\geq \frac{1}{2k}\big(\theta A D_r\sigma_\mu^m, \theta D_r\sigma_\mu^m\big) - \frac{1}{2k}\big(\theta A D_r\sigma_\mu^{m-1}, \theta D_r\sigma_\mu^{m-1}\big) \\ &\geq \frac{1}{2k}\alpha\|\theta D_r\sigma_\mu^m\|^2 - \frac{1}{2k}\alpha\|\theta D_r\sigma_\mu^{m-1}\|^2 \end{aligned} \tag{5.9}$$

We have now

$$\frac{1}{2k}\alpha\|\theta D_r\sigma_\mu^m\|^2 - \frac{1}{2k}\alpha\|\theta D_r\sigma_\mu^{m-1}\|^2 + \big(D_r G'_\mu(\sigma_\mu^m), \theta^2 D_r\sigma_\mu^m\big) \leq \big(\theta D_r\varepsilon(v_\mu^m), \theta D_r\sigma_\mu^m\big) \tag{5.10}$$

Like in the case of the penalized Hencky model we can bound the differentiated and tested penalty term from below.

$$\int_\Omega \frac{1}{\mu}\frac{(|\sigma_{\mu D}^m| - \kappa)_+}{|\sigma_{\mu D}^m|}|\theta D_r\sigma_{\mu D}^m|^2 \leq \big(D_r G'_\mu(\sigma_\mu^m), \theta^2 D_r\sigma_\mu^m\big) \tag{5.11}$$

Consider $\big(D_r\varepsilon(v_\mu^m), \theta^2 D_r\sigma_\mu^m\big)$, Green's formula yields

$$\big(D_r\varepsilon(v_\mu^m), \theta^2 D_r\sigma_\mu^m\big) = -\big(D_r v_\mu^m, \operatorname{grad}\theta^2 D_r\sigma_\mu^m\big) - \big(D_r v_\mu^m, \theta^2 D_r f^m\big)\,. \tag{5.12}$$

Where

$$\begin{aligned}-\big(D_r v_\mu, \theta^2 D_r f^m\big) &\leq \|v_\mu\|_{L^{\frac{n}{n-1}}}\big(C\|D_r f^m\|_{L^n} + C\|\triangle f^m\|_{L^n}\big) \\ &\leq C\|v_\mu\|_{L^{\frac{n}{n-1}}} \qquad \text{because } Df, \triangle f \in L^\infty(L^n_{loc})\end{aligned}$$

Analogously to the penalized Hencky model we symmetrize the term $\big(D_r v_\mu, \operatorname{grad}\theta^2 D_r\sigma_\mu^m\big)$. From now on we use the summing convention

$$-\big(D_r v_{\mu j}^m, D_r\sigma_{\mu ij}^m D_i\theta^2\big) = -2\big(\varepsilon(v_\mu^m)_{jr}, D_r\sigma_{\mu ij}^m D_i\theta^2\big) + \big(D_j v_{\mu j}^m, D_r\sigma_{\mu ij}^m D_i\theta^2\big) \tag{5.13}$$

Using the constitutive law: $\varepsilon(v_\mu^m) = A\frac{\sigma_\mu^m - \sigma_\mu^{m-1}}{k} + \frac{1}{\mu}\frac{(|\sigma_{\mu D}^m| - \kappa)_+}{|\sigma_{\mu D}^m|}\sigma_{\mu D}^m$

$$\begin{aligned}= &\underbrace{-2\left(\left(A\frac{\sigma_\mu^m - \sigma_\mu^{m-1}}{k}\right)_{jr}, D_r\sigma_{\mu ij}^m D_i\theta^2\right)}_{E_1} - \underbrace{\left(\frac{2}{\mu}\frac{(|\sigma_{\mu D}^m| - \kappa)_+}{|\sigma_{\mu D}^m|}\sigma_{\mu Djr}^m, D_r\sigma_{\mu ij}^m D_i\theta^2\right)}_{E_2} \\ &+ \underbrace{\big(D_j v_{\mu r}^m, D_r\sigma_{\mu ij}^m D_i\theta^2\big)}_{E_3}\end{aligned}$$

We estimate E_1 by Young's inequality.

$$-2\left(A\frac{\sigma_\mu^m-\sigma_\mu^{m-1}}{k}, D_r\sigma_\mu^m 2\theta\,\mathrm{grad}\,\theta\right) \le \frac{1}{4\gamma}C\int_\Omega \left|A\frac{\sigma_\mu^m-\sigma_\mu^{m-1}}{k}\right|^2 \mathrm{d}x + \gamma C\int_\Omega \theta^2|D_r\sigma_\mu^m|^2\,\mathrm{d}x$$
$$\le \frac{1}{4\gamma}\|A\dot\sigma_\mu^m\|^2 + \gamma C\|\theta D_r\sigma_\mu^m\|^2 \tag{5.14}$$

We now estimate E_2 and proceed like in the case of the penalized Hencky model.

$$E_2 \le \int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+|D_r\sigma_{\mu ij}^m|\cdot|\theta|C\,\mathrm{d}x$$
$$\le \underbrace{\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+)|\theta|\cdot|D_r\sigma_{\mu D}^m|C\,\mathrm{d}x}_{T_1} + \underbrace{\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+ C\frac{1}{\sqrt{n}}|D\,\mathrm{tr}(\sigma_\mu^m)|\cdot|\theta|\,\mathrm{d}x}_{T_2} \tag{5.15}$$

With Young's inequality we have

$$T_1 \le \zeta\int_\Omega \frac{2}{\mu}\frac{(|\sigma_{\mu D}^m|-\kappa)_+}{|\sigma_{\mu D}^m|}|D_r\sigma_{\mu D}^m|^2|\theta|^2\,\mathrm{d}x + \frac{1}{2\zeta}C^2\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+|\sigma_{\mu D}^m|\,\mathrm{d}x \tag{5.16}$$

Where $\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+|\sigma_{\mu D}^m|\,\mathrm{d}x \le Const$ and

$$T_1 \le \zeta\int_\Omega \frac{2}{\mu}\frac{(|\sigma_{\mu D}^m-\kappa|)_+}{|\sigma_{\mu D}^m|}|D_r\sigma_{\mu D}^m|^2\,\mathrm{d}x + \frac{1}{2\zeta}Const \tag{5.17}$$

We now split T_2 into

$$T_2 \le \underbrace{\frac{1}{2\varrho}\int_\Omega \frac{1}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+ C^2\,\mathrm{d}x}_{\le Const} + \underbrace{\varrho\int_\Omega \frac{2}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+\frac{1}{n}|\theta D_l\,\mathrm{tr}(\sigma_\mu^m)|^2\,\mathrm{d}x}_{T_{21}}$$

using Young's inequality. Using the inequality (3.15) from Bensoussan & Frehse we obtain for T_{21}

$$T_{21} \le \underbrace{\varrho\int_\Omega \frac{4}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+ n|\theta D_r\sigma_{\mu D}^m|^2\,\mathrm{d}x}_{T_3} + \underbrace{\varrho\int_\Omega \frac{4}{\mu}(|\sigma_{\mu D}^m|-\kappa)_+|\theta|^2|f^m|^2\,\mathrm{d}x}_{T_4}\,. \tag{5.18}$$

By assumption (5.1) ($\triangle f \in L^\infty(L^n_{loc})$) we obtain $f \in L^\infty(L^\infty_{loc})$, thus

$$T_4 \le Const\,.$$

For a final estimate of E_2 choose the parameter $\zeta = \frac{1}{4}$, $\varrho = \frac{1}{8n|\sigma_{\mu D}^m|}$. We have

$$E_2 \le \int_\Omega \frac{1}{\mu}\frac{(|\sigma_{\mu D}^m|-\kappa)_+}{|\sigma_{\mu D}^m|}|\theta D_r\sigma_{\mu D}^m|^2\,\mathrm{d}x + Const\,. \tag{5.19}$$

Partial integration of E_3 delivers:

$$\begin{aligned}\big(D_j v^m_{\mu r}, D_r\sigma^m_{\mu ij} D_i\theta^2\big) = &-\int_\Omega v^m_{\mu r} D_i\theta^2 D_r f^m_i \,\mathrm{d}x + \int_\Omega \operatorname{div} v^m_\mu \sigma^m_{\mu ij} D_i D_j\theta^2 \,\mathrm{d}x \\ &+ \int_\Omega v^m_{\mu r}\sigma^m_{\mu ij} D_r D_i D_j \theta^2 \,\mathrm{d}x\end{aligned}$$

We have $\operatorname{div} v^m_\mu = \operatorname{tr}(A\frac{\sigma^m_\mu - \sigma^{m-1}_\mu}{k})$ and $\sigma^m_\mu \in L^2$ thus

$$\begin{aligned}\int_\Omega \operatorname{div} v^m_\mu \sigma^m_{\mu ij} D_i D_j\theta^2 \,\mathrm{d}x &= \int_\Omega \operatorname{tr}\left(A\frac{\sigma^m_\mu - \sigma^{m-1}_\mu}{k}\right)\sigma^m_{\mu ij} D_i D_j\theta^2 \,\mathrm{d}x \\ &\le C\Big\|A\frac{\sigma^m_\mu - \sigma^{m-1}_\mu}{k}\Big\|^2 + C\|\sigma^m_\mu\|^2 \\ &\le C\|A\dot\sigma^m_\mu\|^2 + C\end{aligned} \tag{5.20}$$

By assumption $Df \in L^\infty(0,T; L^n_{loc}(\Omega, \mathbb{R}^{n\times n}))$, this yields

$$\begin{aligned}-\int_\Omega v^m_{\mu r} D_i\theta^2 D_r f^m_i \,\mathrm{d}x &\le C\|v^m_\mu\|_{L^{\frac{n}{n-1}}}\cdot\|D_r f^m_i\|_{L^n} \\ &\le C\|v^m_\mu\|_{L^{\frac{n}{n-1}}}\end{aligned} \tag{5.21}$$

The term $\int_\Omega v^m_{\mu r}\sigma^m_{\mu ij} D_r D_i D_j\theta^2 \,\mathrm{d}x$ remains. Like in the case of the penalized Hencky model we use the Sobolev inequalities.
For space dimension **n=2** we have $v^m_\mu \in L^{\frac{n}{n-1}}$ thus $v^m_\mu \in L^2$ and it follows

$$\int_\Omega v^m_{\mu r}\sigma^m_{\mu ij} D_i D_r D_j\theta^2 \,\mathrm{d}x \le C\|v^m_\mu\|_{L^2}\|\sigma^m_\mu\|_{L^2} \le C\,.$$

We now consider dimensions higher than $n = 2$.
We already know that $\sigma^m_\mu \in H^1_{loc}$ for μ fixed and $\sigma^m_\mu \in L^2$ by Sobolev we obtain:

$$\sigma^m_\mu \in H^1_{loc} \Rightarrow \ \sigma^m_\mu \in L^{\frac{2n}{n-2}}$$

For $n = 3$ we have $\sigma^m_\mu \in L^6,\ v^m_\mu \in L^{\frac{3}{2}}$
For $n = 4$ we have $\sigma^m_\mu \in L^4,\ v^m_\mu \in L^{\frac{4}{3}}$
but in the case $n = 5$ we have $\sigma^m_\mu \in L^{\frac{10}{3}},\ v^m_\mu \in L^{\frac{5}{4}}$ thus the term $\int_\Omega v^m_{\mu r}\sigma^m_{\mu ij} D_r D_i D_j\theta^2 \,\mathrm{d}x$ is welldefined for space dimensions $n = 3, 4$.

Replace in the case $n = 3, 4 \quad \theta$ by ϑ^3 where $\vartheta \in C^\infty_o(\Omega)$.

n=3:

$$\begin{aligned}\int_\Omega v^m_{\mu r}\sigma^m_{\mu ij} D_r D_i D_j\vartheta^6 \,\mathrm{d}x &\overset{\text{Hölder}}{\le} C\|v^m_\mu\|_{L^{\frac{3}{2}}}\|\vartheta^3\sigma^m_\mu\|_{L^3} \\ &\le C\|\vartheta^3\sigma^m_\mu\|_{L^6}\end{aligned}$$

by Sobolev

$$\begin{aligned}\|\vartheta^3\sigma_\mu^m\|_{L^6} \le \|D_r(\vartheta^3\sigma_\mu^m)\|_{L^2} &\le \|D_r\vartheta^3\sigma_\mu^m\|_{L^2} + \|\vartheta^3 D_r\sigma_\mu^m\|_{L^2}\\ &\le C + \|\vartheta^3 D_r\sigma_\mu^m\|_{L^2}\end{aligned}$$

and

$$C\|\vartheta^3 D_r\sigma_\mu^m\|_{L^2} \le \frac{1}{4\rho}C^2 + \rho\|\vartheta^3 D_r\sigma_\mu^m\|_{L^2}^2$$

finally

$$\int_\Omega v_{\mu r}^m \sigma_{\mu ij}^m D_i D_r D_j \vartheta^3 \,\mathrm{d}x \le C + \frac{1}{4\rho}C^2 + \rho\|\vartheta^3 D_r\sigma_\mu^m\|_{L^2}^2$$

for **n=4** we have

$$\begin{aligned}\int_\Omega v_{\mu r}^m \sigma_{\mu ij}^m D_i D_r D_j \vartheta^3 \,\mathrm{d}x &\le C\|\vartheta^3\sigma_\mu^m\|_{L^4}\\ &\le C + \rho\|\vartheta^3 D_r\sigma_\mu^m\|_{L^2}^2\end{aligned}$$

Combining now these estimates we have

$$\begin{aligned}&\frac{\alpha}{2k}\|\theta D_r\sigma_\mu^m\|^2 - \frac{\alpha}{2k}\|\theta D_r\sigma_\mu^{m-1}\|^2\\ &\quad\le C\|v_\mu^m\|_{L^{\frac{n}{n-1}}} + \frac{1}{4\gamma}\|A\dot\sigma_\mu^m\|^2 + C\|\operatorname{tr}(A\dot\sigma_\mu^m)\|^2 + \gamma C\|\theta D_r\sigma_\mu^m\|^2 + \rho\|\theta D_r\sigma_\mu^m\|^2\end{aligned} \tag{5.22}$$

Choose $0 < \rho, \gamma$, such that $\gamma C + \rho = \frac{1}{2}\alpha$ and multiply the equation by k.

$$\begin{aligned}\frac{\alpha}{2}\|\theta D_r\sigma_\mu^m\|^2 - \frac{\alpha}{2}\|\theta D_r\sigma_\mu^{m-1}\|^2 &\le nCk\|v_\mu^m\|_{L^{\frac{n}{n-1}}} + nk\|A\dot\sigma_\mu^m\|^2 + nCk\|\operatorname{tr}(A\dot\sigma_\mu^m)\|^2\\ &\quad + k\frac{\alpha}{2}\|\theta D_r\sigma_\mu^m\|^2 + kn\cdot Const\end{aligned}$$

We have now a system of N inequalities. Summing these inequalities from timestep 1 to m we obtain a telescope sum on the righthand side.

$$\begin{aligned}\frac{\alpha}{2}\|\theta D_r\sigma_\mu^m\|^2 - \frac{\alpha}{2}\|\theta D_r\sigma_\mu^0\|^2 &\le nC\sum_{p=1}^m k\|v_\mu^p\|_{L^{\frac{n}{n-1}}} + n\sum_{p=1}^m k\|A\dot\sigma_\mu^p\|^2 + nC\sum_{p=1}^m k\|\operatorname{tr}(A\dot\sigma_\mu^p)\|^2\\ &\quad + \frac{\alpha}{2}\sum_{p=1}^m k\|\theta D_r\sigma_\mu^p\|^2 + mkn\cdot Const\end{aligned}$$

We know

$$\begin{aligned}\|v_\mu\|_{L^1(L^{\frac{n}{n-1}})} &\le Const\\ \|\dot\sigma_\mu\|_{L^2(L^2)} &\le Const\end{aligned}$$

thus

$$nC\sum_{p=1}^{m} k\|v_\mu^p\|_{L^{\frac{n}{n-1}}} \leq nC\|v_\mu\|_{L^1(L^{\frac{n}{n-1}})} \leq \; n\cdot Const$$
$$n\sum_{p=1}^{m} k\|A\dot{\sigma}_\mu^p\|^2 \leq n\|A\dot{\sigma}_\mu\|^2_{L^2(L^2)} \leq \; n\cdot Const$$
$$nC\sum_{p=1}^{m} k\|\operatorname{tr}(A\dot{\sigma}_\mu^p)\|^2 \leq \; nC\|tr(A\dot{\sigma}_\mu)\|^2_{L^2(L^2)} \leq n\cdot Const$$

and finally

$$\|\theta D_r\sigma_\mu^m\|^2 \leq n\cdot Const + \sum_{p=1}^{m} k\|\theta D_r\sigma_\mu^p\|^2\,. \tag{5.23}$$

By a discrete version of the Gronwall lemma we have

$$\|\theta D_r\sigma_\mu^m\|^2 \leq Const\,. \tag{5.24}$$

These estimates are independent of μ and k.

Appendix A

The space $BD(\Omega)$

In the case of perfect plasticity the right functionspace for the displacements is the space $BD(\Omega)$ of fuctions with bounded deformation.
The linearized strain tensor ε is in this case only a bounded Radon measure.
While the process of plastic deformation slip lines can occur, these are zones in which the deformation gradient contains discontinuities in its tangential component. An adequate formulation in the setting of sobolev spaces cannot take account of the mecanical qualities of the material.

The literature for this appendix can be found in Suquet [Suq78b], Temam and Strang [TS78],[TS80] and the book [Tem85].

Let $\Omega \subset \mathbb{R}^n$ be an open connected and bounded subset of $\mathbb{R}^n$.

Definition A.1 *$M_1(\Omega)$ denotes the space of all bounded Radon measures on Ω.*

This is a space of distributions μ on Ω, such that

$$\sup_{\substack{\phi \in C_o^\infty(\Omega) \\ \|\phi\|_\infty = 1}} \langle \mu, \phi \rangle < \infty$$

The pairing $\langle \cdot, \cdot \rangle$ is defined as the integral with respect to the measure μ.

$$\langle \mu, \phi \rangle = \int_\Omega \phi \, d\mu$$

The space $M_1(\Omega)$ is isomorphic to the dualspace $(C_o(\Omega))^*$ of the continuous functions with compact support in Ω.

Definition A.2 *By M_{sym} we denote the space of all second order symmetric tensor with values in the space of bounded measures.*

$$\begin{aligned} m \in M_{sym} :\Leftrightarrow \; & m \in \mathbb{R}^{n \times n}_{sym} \\ & m_{ij} \in M_1(\Omega) \quad 1 \le i, j \le n \end{aligned}$$

Definition A.3 *We define the space $BD(\Omega)$ of functions with bounded deformation as follows*

$$BD(\Omega) = \{u \in L^1(\Omega, \mathbb{R}^n) \,|\, \varepsilon(u) \in M_{sym}\}\,. \tag{A.1}$$

This space endowed with the natural norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|\varepsilon(u)\|_{M_{sym}} \tag{A.2}$$

is a nonreflexive Banachspace. The smooth fuctions are not dense in $BD(\Omega)$ with respect to the topology generated by this norm.

If the boundary of Ω is Lipschitz continuous we have the following trace theorem.

Theorem A.1 (trace theorem) *Let $\partial\Omega$ be Lipschitz continuous. There exists a continuous surjective linear operator $\gamma : BD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^n)$, such that for all $u \in BD(\Omega) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$*

$$\gamma(u) = u_{|\partial\Omega} \tag{A.3}$$

holds.

Theorem A.2 (generalized Green's formula) *If the boundary $\partial\Omega$ is Lipschitz continuous we have for all $\phi \in C^1(\overline{\Omega})$*

$$\int_\Omega \left(u_j \frac{\partial\phi}{\partial x_i} + u_i \frac{\partial\phi}{\partial x_j} \right) \mathrm{d}x + 2\int_\Omega \phi\varepsilon(u)\,\mathrm{d}x = \int_{\partial\Omega} \phi \cdot \big(\gamma(u_i)\cdot\vec{n}_j + \gamma(u_j)\cdot\vec{n}_i\big)\,\mathrm{d}\Gamma \tag{A.4}$$

With $\vec{n} = (\vec{n}_1, \dots, \vec{n}_n)$ the unit outward normal on $\partial\Omega$.

Theorem A.3 (Embedding) *Let Ω be a bounded domain with Lipschitz boundary. Then*

- *The space $BD(\Omega)$ is continuously embedded into $L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$.*
- *For $1 \le p < \frac{n}{n-1}$ the injection $BD(\Omega) \hookrightarrow L^p(\Omega, \mathbb{R}^n)$ is compact.*

We have the following regularity theorem for distributions.

Theorem A.4 (Regularity theorem) *If $u \in \mathscr{D}'(\Omega, \mathbb{R}^n)$ and $\varepsilon(u) \in M_{sym}$ then $u \in BD(\Omega)$.*

Together with the embedding theorem A.3 the distribution u lies in $L^{\frac{n}{n-1}}(\Omega, \mathbb{R}^n)$.

A Korn type inequality exists in $BD(\Omega)$. It is defined for the quotientspace $BD(\Omega)$ modulo the rigid dispacements, these are the kernel of $\varepsilon(\cdot)$.
Let $\mathscr{R} = \ker(\varepsilon)$

Theorem A.5 (Norm equivalence) *On $BD(\Omega)_{/\mathscr{R}}$ we have by $\|\varepsilon(u)\|_{M_{sym}}$ an equivalent norm to the norm of $BD(\Omega)$.*

Appendix B

The deviator of a matrix

Definition B.1 (Deviator) *The deviator A_D of a matrix $A \in \mathbb{R}^{n\times n}_{sym}$ is defined as*

$$A_D = A - \frac{1}{n}\,\mathrm{tr}(A) Id.$$

The mapping $A \mapsto A_D$ is linear with kernel $\ker(\cdot_D) = \{\lambda \cdot Id \,|\, \lambda \in \mathbb{R}\}$. The image of the deviator mapping is the subspace of all matrices with trace zero.

$$\begin{aligned} \mathrm{tr}(A_D) &= \mathrm{tr}\left(A - \frac{1}{n}\,\mathrm{tr}(A) Id\right) \\ &= \mathrm{tr}(A) - \frac{1}{n}\,\mathrm{tr}(A)\cdot \mathrm{tr}(Id) \\ &= \mathrm{tr}(A) - \mathrm{tr}(A) = 0. \end{aligned}$$

Theorem B.1 *For $A, B \in \mathbb{R}^{n\times n}_{sym}$ we have $A_D : B = A_D : B_D$.*

Proof Consider the identities $tr(Id) = n \qquad Id : A = \mathrm{tr}(A)$

$$\begin{aligned} A_D : B &= \left(A - \frac{1}{n}\,\mathrm{tr}(A) Id\right) : B \\ &= A : B - \frac{1}{n}\,\mathrm{tr}(A)\,\mathrm{tr}(B) \\ A_D : B_D &= \left(A - \frac{1}{n}\,\mathrm{tr}(A) Id\right) : \left(B - \frac{1}{n}\,\mathrm{tr}(B) Id\right) \\ &= A : B - \frac{2}{n}\,\mathrm{tr}(A)\,\mathrm{tr}(B) - \frac{1}{n^2}\,\mathrm{tr}(A)\,\mathrm{tr}(B)\cdot n \\ &= A : B - \frac{1}{n}\,\mathrm{tr}(A)\,\mathrm{tr}(B) \end{aligned}$$

It follows $A_D : A = |A_D|^2$ and the proof shows that $|A_D|^2 \leq |A|^2$.

Appendix C

Projections onto closed convex sets in Hilbertspaces

The proofs of the theorems given in this appendix can be found in the article [Zar71] by Zarantonello.

Let X be a real Hilbertspace and $A \subset X$ a nonempty, closed, convex subset. We denote by $\big(\cdot\,,\cdot\big)$ the scalarproduct and by $\|\cdot\|$ the induced norm.

Theorem C.1 *Given a nonempty, closed, convex subset A, then there exists a unique mapping $P : X \to A$ with*

$$\|x - P(x)\| = \operatorname{dist}(x, A) = \inf_{y \in A} \|x - y\| \; \forall x \in X.$$

An equivalent characterization of $P(\cdot)$ is the variational inequality

$$\big(x - P(x), a - P(x)\big) \leq 0 \; \forall \, x \in A.$$

In the case then A is a closed subspace P is the orthogonal projection $x - P(x) \in A^{\perp}$.

Theorem C.2 *Let $P_A : X \to A$ the projection onto A and $(Id - P_A)$ the complement of P_A. The projection P_A and the complement $(Id - P_A)$ are Lipschitz continuous.*

Theorem C.3 *We have for the projection P_A and the complement $Id - P_A$*

- $\big(P_A x - P_A y, x - y\big) \geq \|P_A x - P_A y\|^2 \quad \forall \, x, y \in X$
- $\big((Id - P_A)x - (Id - P_A)y, x - y\big) \geq \|(Id - P_A)x - (Id - P_A)y\|^2 \quad \forall \, x, y \in X$

This statement shows that P_A and $Id - P_A$ are monotone operators.

The next theorem characterizes the projection by a differential equation.

Theorem C.4 *A Lipschitz continuous mapping $\Pi : X \to X$ is a projection onto a closed convex subset iff the following differential equation holds.*

$$(Id - \Pi)x = \frac{1}{2}\nabla\|(Id - \Pi)x\|^2 \quad \forall x \in X \tag{C.1}$$

∇ denotes the gradient taken with respect to the Gâteaux differential.

This result gives the identity for P_A

$$\frac{1}{2}\nabla\|x\|^2 - \frac{1}{2}\nabla\|(Id - P_A)x\|^2 = x - (Id - P_A)x = P_Ax \tag{C.2}$$

and

Theorem C.5 *The projection P_A onto the closed convex subset A and their complement $Id - P_A$ are gradient mappings.*

From P_A and $Id - P_A$ being monotone operators and gradient mappings we get:

Theorem C.6 *$\|(Id - P_A)x\|^2$ und $\|x\|^2 - \|(Id - P_K)x\|^2$ are convex functions.*

Appendix D

Properties of the linearized strain tensor ε

We now assume for the displacements $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ or a least that the derivative Du exists in the sense of distributions.

Theorem D.1 *The kernel of the linearized strain tensor $\varepsilon(\cdot)$ consists of the so called rigid displacements.*

$$\begin{aligned} u(x) \in \ker(\varepsilon) \Leftrightarrow u(x) = Ax + b \text{ with } A \in \mathbb{R}^{n\times n}_{sym} \\ A^\top = -A \text{ and } b \in \mathbb{R}^n \end{aligned} \tag{D.1}$$

The proof can be found in [Tem85].

Theorem D.2 (Korn's inequality) *Let $\Omega \subset \mathbb{R}^n$ be an open connected subset with Lipschitz continuous boundary.*

- *For $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ $1 < p < \infty$ with a constant $c_o > 0$ dependent of Ω*

$$\int_\Omega |\varepsilon(u)|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x \geq c_o \|u\|^p_{W^{1,p}} \tag{D.2}$$

- *Let $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n)$ $1 < p < \infty$ with $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n) = \{u \in W^{1,p}(\Omega, \mathbb{R}^n) | \, u = 0$ on Γ_0 in the trace sense$\}$ and $\Gamma_0 \subset \partial\Omega$ with positive $(n-1)$-dimensional Hausdorffmeasure. Then there exists constans $C_1, C_2 > 0$ dependent of Ω and Γ_0 with*

$$C_1 \|u\|_{W^{1,p}} \leq \|\varepsilon(u)\|_{L^p} \leq C_2 \|u\|_{W^{1,p}} \tag{D.3}$$

- *In the case $p = 1$, $\varepsilon(u) \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$ and $\gamma(u) = 0$ on Γ_0, there exists a positive constant C_3 dependent of $\Omega$and Γ_0 such that*

$$\|u\|_{L^{\frac{n}{n-1}}} \leq C_3 \|\varepsilon(u)\|_{L^1} \tag{D.4}$$

The proofs can be found in the book of Temam [Tem85].

Theorem D.3 (generalized Green's formula) *Let $\Omega \subset \mathbb{R}^n$ be a open connected subset with Lipschitz continuous boundary. Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ $1 < p < \infty$, p^* dual exponent to p $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $\chi \in L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{sym})$ with $\operatorname{div}\chi \in L^{p^*}(\Omega, \mathbb{R}^n)$. The divergence $\operatorname{div}\chi$ has to be taken in the distributional sense. Then the generalized Green's formula holds*

$$\int_\Omega \varepsilon(u) : \chi \,\mathrm{d}x + \int_\Omega u \operatorname{div}\chi \,\mathrm{d}x = \int_{\partial\Omega} u\chi \cdot \vec{n} \,\mathrm{d}\Gamma \tag{D.5}$$

For the proof see [Tem85].

The mapping $\varepsilon : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n) \to L^p(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ is continuous and linear. The Image $R(\varepsilon)$ of ε is closed in $L^p(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}})$. The closed range theorem yields

$$R(\varepsilon) = \ker(\varepsilon^*)^\perp.$$

How does the adjoint operator ε^* look like? We have: $\varepsilon^* : L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \to \left(W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n)\right)^*$.

$$\langle \varepsilon(u), \tau \rangle_{L^p \times L^{p^*}} = \langle u, \varepsilon^*(u) \rangle_{W^{1,p}_{\Gamma_0} \times \left(W^{1,p}_{\Gamma_0}\right)^*}$$

Using the generalized Green's formula (D.5) we can compute the adjoint operator.

$$\int_\Omega \varepsilon(u) : \tau \,\mathrm{d}x = -\int_\Omega u \operatorname{div}\tau \,\mathrm{d}x + \int_{\partial\Omega} u\tau \cdot \vec{n} \,\mathrm{d}\Gamma$$

Where $\tau \in L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ with $\operatorname{div}\tau \in L^{p^*}(\Omega, \mathbb{R}^n)$ ($\operatorname{div}\tau$ in the distributional sense)

Theorem D.4 *If for $\chi \in L^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ $1 < p < \infty$*

$$\int_\Omega \chi : \tau \,\mathrm{d}x = 0 \quad \forall \tau \in V \tag{D.6}$$

holds, with $V := \{\tau \in L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{sym}) | \operatorname{div}\tau = 0 \text{ in } \Omega,\ \tau \cdot \vec{n} = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}$ then there exists a unique $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n)$ such that

$$\chi = \varepsilon(u).$$

This means

$$\{\varepsilon(u) | \, u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^n)\} = V^\perp$$

The strains are the annihilator of the divergence free tensor fields in $L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{sym})$.

Proof Let $\int_\Omega \chi : \tau \,\mathrm{d}x = 0$ $\forall \tau \in V$. The closed range theorem implies $\chi \in R(\varepsilon)$ if $\chi \in \ker(\varepsilon^*)^\perp$. The kernel of ε^* is

$$\ker(\varepsilon^*) = \{\tau \in L^{p^*}(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) | \operatorname{div}\tau = 0 \text{ and } \tau \cdot \vec{n} = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}$$

We have $\chi \in \ker(\varepsilon^*)$ and this gives $\chi \in R(\varepsilon)$. From Korn's inequality we obtain the injectivity of $\varepsilon(\cdot)$. The injectivity of ε yields the uniqueness of the displacement u.

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