# Vector bundles on degenerations of elliptic curves and Yang-Baxter equations

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# Summary

In this thesis, we study connections between vector bundles on degenerations of elliptic curves and the classical, quantum and associative Yang-Baxter equation. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and let $\mathcal{U}$ denote the universal enveloping algebra of $\mathfrak{g}$. The classical Yang-Baxter equation (CYBE) is given as follows:

$$\big[r^{12}(y_1,y_2), r^{23}(y_2,y_3)\big] + \big[r^{12}(y_1,y_2), r^{13}(y_1,y_3)\big] + \big[r^{13}(y_1,y_3), r^{23}(y_2,y_3)\big] = 0,$$

where $r : \mathbb{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ is a meromorphic function and $r^{ij}(y_i, y_j) : \mathfrak{g} \otimes \mathfrak{g} \to \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ is the embedding given by $(i,j)$. This equation plays an important role in mathematical physics, representation theory and integrable systems.

In 1982, Belavin and Drinfeld gave a classification of solutions of the CYBE. In particular, they proved that any solution of the CYBE is either *elliptic*, *trigonometric* or *rational*. Moreover, they described all elliptic and trigonometric solutions. Their work has been extended by Stolin, who gave a certain classification of rational solutions.

**Result A**. Let $E = V(wv^2 - 4u^3 - g_2uw^2 - g_3w^3) \subset \mathbb{P}^2$ be a Weierstraß cubic curve, $0 < d < n$ a pair of coprime integers and $\mathcal{A} = \mathrm{Ad}(\mathcal{P})$, where $\mathcal{P}$ is a simple vector bundle of rank $n$ and degree $d$ on $E$. Consider the map

$$\mathfrak{g} \xrightarrow{\cong} \mathcal{A}\big|_{y_1} \xrightarrow{\mathsf{res}_{y_1}^{-1}} H^0\big(\mathcal{A}(y_1)\big) \xrightarrow{\mathsf{ev}_{y_2}} \mathcal{A}\big|_{y_2} \xrightarrow{\cong} \mathfrak{g},$$

where $\mathsf{res}_{y_1}$ is the *residue map*, $\mathsf{ev}_{y_2}$ is the *evaluation map* and the first and the last maps are induced by a certain trivialization of $\mathcal{A}$. Then the tensor $r_{(E,n,d)}(y_1,y_2) \in \mathfrak{g} \otimes \mathfrak{g}$, obtained from the map above using the Killing form, is a solution of the CYBE.

This result extends an earlier construction given in works of Polishchuk and Burban-Kreußler. The core of our method is the computation of certain *triple Massey products* in the derived category $D^b\big(\mathrm{Coh}(E)\big)$.

**Result B**. Let $E$ be a cuspidal cubic curve. Then the solution $r_{(E,n,d)}$ from above is rational. We explicitly describe the *Stolin triple* $(\mathcal{L}, B, k)$ (where $\mathcal{L}$ is a Lie subalgebra of $\mathfrak{g}$, $B$ is a 2-cocycle of $\mathcal{L}$ and $k \in \mathbb{N}$) such that $r_{(E,n,d)} = r_{(\mathcal{L},B,k)}$.

**Result C**. We have found new elliptic solutions of the *associative Yang-Baxter equation* of the form

$$r(v,y) = \sum_{\substack{0\le k\le n-1\\ 0\le l\le n-1}} \nabla_{kl}\big(\sigma(v,y)\big) \sum_{\substack{1\le i\le n-l\\ 1\le j\le n-k}} e_{i,j+k} \otimes e_{j,i+l},$$

where $\sigma(v,y)$ is the Kronecker function and $\nabla_{kl}$ are certain differential operators. This leads to new identities for the higher derivatives of the Kronecker function.

**Result D**. We elaborate a relation between solutions of the associative, classical and quantum Yang-Baxter equations, generalizing results of Polishchuk.

## Contents

*To my wife and family*

# 1. Introduction

In this thesis, I incorporate and explain in greater detail the results presented in [12], [13] and [26]. Although these papers are quite different from each other with respect to the methods and tools we use - analytical, combinatorial, algebro-geometric and Lie-theoretic - there still is a common denominator for the studies we pursue. Namely, we aim for a better understanding of the Yang-Baxter equations and their solutions by application of the theory of coherent sheaves on degenerations of elliptic curves.

The Yang-Baxter equations – or to be more precise, the classical Yang-Baxter equation (CYBE), the quantum Yang-Baxter equation (QYBE) and the associative Yang-Baxter equation (AYBE) – are important objects appearing in mathematical physics, especially in integrable systems and statistical mechanics. Moreover they are studied in the context of representation theory. There are different versions for each of these equations, differing from each other with respect to the number of spectral variables involved. The version of the CYBE that we shall be mostly interested in is of the following form. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $r : (\mathbb{C}^2, 0) \to \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic function. Then $r$ is a solution of the CYBE if it satisfies the equality

$$\big[r^{12}(y_1, y_2), r^{23}(y_2, y_3)\big] + \big[r^{12}(y_1, y_2), r^{13}(y_1, y_3)\big] + \big[r^{13}(y_1, y_3), r^{23}(y_2, y_3)\big] = 0.$$

Here $r^{ij} = \tau_{ij} \circ r$ denotes $r$ followed by the obvious inclusions $\mathfrak{g}^{\otimes 2} \to \mathcal{U}(\mathfrak{g})^{\otimes 3}$ with $\mathcal{U}(\mathfrak{g})$ denoting the universal enveloping algebra of $\mathfrak{g}$, e.g. $\tau_{13}(a \otimes b) = a \otimes \mathbb{1} \otimes b$. We shall focus our studies on solutions $r$ which satisfy two additional assumptions. Firstly, we will assume that $r$ is *unitary:*

$$r^{12}(y_1, y_2) = -r^{21}(y_2, y_1).$$

Secondly, $r$ will always be *non-degenerate*, that is its image under the isomorphism

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \quad a \otimes b \mapsto \big(c \mapsto \mathrm{tr}(ac) \cdot b\big)$$

is an invertible operator for some (and hence, for a generic) value of the spectral parameters $(y_1, y_2)$. On the set of solutions of the CYBE there exists a natural action of the group of holomorphic function germs $\phi : (\mathbb{C}, 0) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ given by the rule

$$r(y_1, y_2) \mapsto \tilde{r}(y_1, y_2) = \big(\phi(y_1) \otimes \phi(y_2)\big) r(y_1, y_2).$$

It is easy to see that $\tilde{r}(y_1, y_2)$ is again a solution of the CYBE. Moreover, $\tilde{r}(y_1, y_2)$ is unitary respectively non-degenerate provided $r(y_1, y_2)$ is unitary respectively non-degenerate. The solutions $r(y_1, y_2)$ and $\tilde{r}(y_1, y_2)$ related by the equality above are called *gauge equivalent.*

It was shown by Belavin and Drinfeld [3] that any non-degenerate solution of the CYBE is either *elliptic* (two-periodic), *trigonometric* (one-periodic) or *rational.* Moreover, they classified all elliptic and trigonometric solutions completely [3, Proposition

5.1 and Theorem 6.1]. Especially, explicit formulas for these solutions can be derived, see for instance [17]. In this thesis, we are mainly interested in rational solutions. As we shall explain below, Stolin [42] derived a quite sophisticated classification of rational solutions in terms of Lie-algebraic data.

It was shown by Polishchuk [37] that solutions of the CYBE can be obtained from solutions of the AYBE. We shall study the AYBE mostly in the form

$$\begin{aligned} r^{12}(u;y_1,y_2)\,r^{23}(u+v;y_2,y_3) = \\ = r^{13}(u+v;y_1,y_3)\,r^{12}(-v;y_1,y_2) + r^{23}(v;y_2,y_3)\,r^{13}(u;y_1,y_3). \end{aligned}$$

for meromorphic function germs $r : (\mathbb{C}^3, 0) \to A \otimes A$. Here $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$ and we use the notation $r^{ij} = r \circ \rho_{ij}$ for the composition of $r$ with the canonical embedding $\rho_{ij} : A^{\otimes 2} \to A^{\otimes 3}$. In order to obtain a solution of the CYBE from $r$, consider the canonical projection $\mathrm{pr} : A \to \mathfrak{g}$ given by $X \mapsto X - \frac{\mathrm{tr} X}{n}\mathbb{1}$. Then

$$\bar{r}(y_1,y_2) = \lim_{v\to 0}(\mathrm{pr}\otimes\mathrm{pr})\,r(v;y_1,y_2)$$

-if it exists - is the corresponding solution of the CYBE. Actually Polishchuk [37], who studied only solutions of the AYBE which satisfy

$$r(v;y_1,y_2) = r(v;y_1+x,y_2+x), \tag{1.1}$$

proved that *any* elliptic solution of the CYBE can be obtained from *some* solution of the AYBE. But as was shown by Schedler [40], even for trigonometric solutions this statement can not be generalized.

Polishchuk [38] also established a connection of the AYBE with the QYBE. To this end, he imposes two further conditions on $r$. Firstly, he assumes that $r$ has the following Laurent expansion:

$$r(v;y_1,y_2) = \frac{\mathbb{1}\otimes\mathbb{1}}{v} + r_0(y_1,y_2) + v\,r_1(y_1,y_2) + v^2\,r_2(y_1,y_2) + \ldots \tag{1.2}$$

This condition is automatically satisfied in many examples, see [14, 26]. Also note that in this case, $\bar{r}_0(y_1,y_2) = (\mathrm{pr}\otimes\mathrm{pr})\,(r_0(y_1,y_2))$ is the corresponding solution of the CYBE. Secondly, Polishchuk assumes that $\bar{r}_0$ has no infinitesimal symmetries, i.e. that there is no non-trivial $a \in \mathfrak{g}$ such that

$$\Big[\bar{r}_0(y_1,y_2), a\otimes\mathbb{1} + \mathbb{1}\otimes a\Big] = 0.$$

We were able to generalize Polishchuk's results, dropping the assumption (1.1):

**Result 1** [26] *Let $r(v;y_1,y_2)$ be a non-degenerate unitary solution of the AYBE of the form (1.2). If $\bar{r}_0(y_1,y_2) = (\mathrm{pr}\otimes\mathrm{pr})\,(r_0(y_1,y_2))$ has no infinitesimal symmetries, then the following hold.*

(1) *For fixed* $v_0 \in \mathbb{C}^\times$, $\tilde{r}(y_1, y_2) = r(v_0; y_1, y_2)$ *is a solution of the QYBE:*

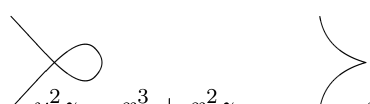
$$\tilde{r}^{12}(y_1, y_2)\, \tilde{r}^{13}(y_1, y_3)\, \tilde{r}^{23}(y_2, y_3) = \tilde{r}^{23}(y_2, y_3)\, \tilde{r}^{13}(y_1, y_3)\, \tilde{r}^{12}(y_1, y_2).$$

(2) *Let* $s(v; y_1, y_2)$ *be another non-degenerate unitary solution of the AYBE of the form (1.2) with* $(\mathrm{pr} \otimes \mathrm{pr})\,(s_0) = \bar{r}_0$. *Then there exists a* meromorphic *function* $g : \mathbb{C} \to \mathbb{C}$ *such that*

$$s(v; y_1, y_2) = \exp\Big(v\,\big(g(y_2) - g(y_1)\big)\Big) r(v; y_1, y_2). \tag{1.3}$$

Loosely speaking, statement (2) says that solutions of the AYBE with Laurent expansion as above are uniquely determined up to (a weak version of) gauge equivalence by the corresponding solutions of the CYBE, as long as the latter do not have infinitesimal symmetries. For the solutions of the AYBE constructed from stable vector bundles on Weierstraß cubic curves by Burban and Kreußler [14], see below, the map $g$ appearing in (1.3) is always holomorphic. Thus in that case, (1.3) is a proper gauge equivalence between $r$ and $s$.

The starting point of the papers [12] and [13] was a paper of Burban and Kreußler [14]. Developing ideas of Polishchuk [37], Burban and Kreußler showed that solutions of the AYBE can be obtained by the computation of certain triple Massey products in the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(E))$ for any Weierstraß cubic curve $E \subset \mathbb{P}^2$. In homogeneous coordinates such a curve is given by the equation $zy^2 = 4x^3 + g_2 xz^2 + g_3 z^3$, where $g_2, g_3$ are elements of the algebraically closed field $\Bbbk$. These curves, which are irreducible and of arithmetic genus one, fall under the following trichotomy. If $\triangle(g_2, g_3) = g_2^3 + 27g_3^2 \neq 0$, then $E$ is an elliptic curve (especially, $E$ is smooth). Otherwise $E$ is singular. Unless $g_2 = g_3 = 0$, the singularity is a node (ordinary double point), whereas in the case $g_2 = g_3 = 0$ the singularity is a cusp.



$y^2 z = x^3 + x^2 z$ $\qquad$ $y^2 z = x^3$

In particular, Burban and Kreußler used the theory of stable vector bundles on a Weierstraß cubic curve $E$ in order to make the computation of triple Massey products in $D^b(\mathrm{Coh}(E))$ accessible to explicit computations. In [12], we showed how the construction of Burban and Kreußler can be generalized for semi-stable vector bundles on an elliptic curve:

**Result 2** [12] *Fix a complex parameter* $\tau \in \mathbb{C}$ *such that* $\mathsf{Im}(\tau) > 0$ *and an invertible matrix* $B \in \mathrm{GL}_n(\mathbb{C})$. *Let* $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ *be the corresponding lattice in* $\mathbb{C}$ *and* $\mathfrak{S}(B) = \{\lambda_1, \ldots, \lambda_n\}$ *the spectrum of* $B$. *We denote by* $\Sigma = \Sigma_B$ *the lattice*

$$\big\{\lambda - \lambda' \,\big|\, \exp(2\pi i \lambda), \exp(2\pi i \lambda') \in \mathfrak{S}(B)\big\} + \Lambda \subset \mathbb{C}.$$

*Then we attach to the pair $(B, \tau)$ a meromorphic tensor-valued function*

$$r_B = r_B(v, y) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$$

*having the following properties:*

(1) *The function $r_B$ is a non-degenerate unitary solution of the AYBE.*
(2) *Moreover, $r_B$ depends* analytically *on the entries of the matrix $B$ and is holomorphic on $(\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$.*
(3) *Let $S \in \mathrm{GL}_n(\mathbb{C})$ and $A = S^{-1}BS$. Then we have:*

$$r_A(v, y) = (S^{-1} \otimes S^{-1}) r_B(v, y)(S \otimes S).$$

(4) *If $B = \mathsf{diag}\big(\exp(2\pi i\lambda_1), \ldots, \exp(2\pi i\lambda_n)\big)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then the corresponding solution $r_B$ is given by the following formula:*

$$r_B(v, y) = \sum_{k,l=1}^{n} \sigma(v - \lambda_{kl}, y)\, e_{l,k} \otimes e_{k,l},$$

*where $\lambda_{kl} = \lambda_k - \lambda_l$ and $\sigma(u, x)$ is the Kronecker function.*
(5) *If $B = J_n(1)$ is the Jordan block of size $n \times n$ with eigenvalue one then*

$$r_B(v, y) = \sum_{\substack{0\le k\le n-1 \\ 0\le l\le n-1}} \nabla_{kl}\big(\sigma(v, y)\big) \sum_{\substack{1\le i\le n-l \\ 1\le j\le n-k}} e_{i,j+k} \otimes e_{j,i+l},$$

*where $\nabla_{kl}$ are certain differential operators.*

Interestingly, the solutions we obtain from semi-stable vector bundles do not yield solutions of the CYBE. In particular, they have higher order poles and are not of the form (1.2). It is worthwhile mentioning that our results include an explicit algorithm for the construction of the solutions $r_B$, making it possible to compute the solutions presented in (4) and (5) by hand.

The final aim of our studies was to apply the theory of vector bundles on degenerations of elliptic curves to solutions of the classical Yang-Baxter equation. Before our own contributions, to be presented in [13], the state of the art was as follows. As already mentioned above, the work of Polishchuk [37] respectively Burban and Kreußler [14] yields a construction of solutions of the CYBE via solutions of the AYBE obtained from stable vector bundles works for any Weierstraß cubic curve. The input data for the construction is given by a triple $(E, n, d)$, where $E$ is a prescribed Weierstraß cubic curve and $n$ and $0 < d < n$ are coprime integers corresponding to the rank respectively degree of the stable vector bundles fixed in the construction. Polishchuk proved that if $E$ is elliptic, then the associated solution of the AYBE respectively CYBE is elliptic [37]. However, Polishchuk's approach is rather indirect and uses formulae for higher multiplications in the *Fukaya category* of a complex torus as well as *homological* *mirror symmetry* in dimension one [39]. See also [14, Section 4.3] for a detailed direct computation in the case $(n, d) = (2, 1)$. Nonetheless Polishchuk shows that in this case one precisely recovers the list of Belavin and Drinfeld of all elliptic solutions of the CYBE. In particular, if $1 \leq d \neq d' < n$ then the solutions $r_{(E,n,d)}$ and $r_{(E,n,d')}$ are *not* gauge equivalent. In the subsequent paper [38], Polishchuk describes the solutions of the AYBE corresponding to simple vector bundles on Kodaira cycles of projective lines. See also [14, Section 5.2] for a detailed computation in the case $E$ when is a Weierstraß nodal cubic curve and $(n, d) = (2, 1)$. Polishchuk's computation is based on a classification of simple vector bundles on Kodaira cycles obtained in [11, Theorem 5.3]. In this case, one obtains a certain class of trigonometric solutions of the CYBE.

The final aim of our studies was thus twofold. Firstly, we further developed ideas proposed by Polishchuk [37] concerning the *direct* construction of solutions of the CYBE via similar methods as were used for the construction of solutions of the AYBE. The first main result of [13] can be stated as follows:

**Result 3** [13] *Let $E = V(wv^2 - 4u^3 - g_2uw^2 - g_3w^3) \subset \mathbb{P}^2$ be a Weierstraß cubic curve over $\mathbb{C}$, $o \in E$ some fixed smooth point and $0 < d < n$ a pair of coprime integers. Consider the sheaf of Lie algebras $\mathcal{A} = \mathrm{Ad}(\mathcal{P})$, where $\mathcal{P}$ is a simple vector bundle of rank $n$ and degree $d$ on $E$ (note that up to automorphism, $\mathcal{A}$ does not depend on a particular choice of $\mathcal{P}$). For any pair of distinct smooth points $y_1, y_2$ of $E$, consider the map $\mathcal{A}\big|_{y_1} \longrightarrow \mathcal{A}\big|_{y_2}$ defined as follows:*

$$\mathcal{A}\big|_{y_1} \xrightarrow{\mathsf{res}_{y_1}^{-1}} H^0\big(\mathcal{A}(y_1)\big) \xrightarrow{\mathsf{ev}_{y_2}} \mathcal{A}\big|_{y_2}$$

*where $\mathsf{res}_{y_1}$ is the* residue map *and $\mathsf{ev}_{y_2}$ is the* evaluation map. *Choosing some isomorphism of Lie algebras $\xi : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ for some small neighborhood $U$ of $o$, we get the tensor $r^{\xi}_{(E,n,d)}(x, y) \in \mathfrak{g} \otimes \mathfrak{g}$. Then we have:*

(1) *The tensor $r^{\xi}_{(E,n,d)}$ is a non-degenerate unitary solution of the CYBE.*
(2) *Moreover, $r^{\xi}_{(E,n,d)}$ is* analytic *with respect to the parameters $g_2$ and $g_3$.*
(3) *A different choice of trivialization $\zeta : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ gives a gauge equivalent solution $r^{\zeta}_{(E,n,d)}$.*

In the second step, we were interested in studying rational solutions obtained by the procedure developed in Result 3. Burban and Kreußler [14] already gave some examples that indicated the validity of the natural conjecture that if $E$ is cuspidal, then the associated solution of the CYBE will be rational. We prove this conjecture in [13].

Moreover, we show that the rational solutions obtained are always of the form

$$r(y_1, y_2) = \frac{\Omega}{y_2 - y_1} + r'(y_1, y_2),\ r'(y_1, y_2) \in \mathfrak{g}[y_1] \otimes \mathfrak{g}\,[y_2]\,, \tag{1.4}$$

where $\mathfrak{g}\,[y] = \mathfrak{g} \otimes \mathbb{C}\,[y]$, $r'$ is not constant and $\Omega$ denotes the Casimir element. Thus, the solutions we obtain belong precisely to the class of rational solutions of the CYBE studied by Stolin [42]. His classification goes as follows. Stolin states that there is a bijection between rational solutions of the CYBE of the form (1.4) and triples $(\mathfrak{L}, B, k)$, where $\mathfrak{L} \subseteq \mathfrak{g}$ is a a Lie subalgebra, $B : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C}$ is a skew-symmetric 2-cocycle and where $1 \leq k \leq n-1$ encodes the following compatibility condition. Let $\mathfrak{P}_k$ denote the $k$-th parabolic subalgebra of $\mathfrak{g}$, then

(1) $\mathfrak{L} + \mathfrak{P}_k = \mathfrak{g}$.
(2) $B$ is non-degenerate on $(\mathfrak{L} \cap \mathfrak{P}_k) \times (\mathfrak{L} \cap \mathfrak{P}_k)$.

It is natural to ask which triples correspond to the solutions $r_{(E,n,d)}$ obtained from the geometric construction described in Result 3. Since the construction of a rational solution $r_{(E,n,d)}$ depends on the choice of two coprime integers $0 < d < n$ only ($E$ being the cuspidal cubic curve), it is natural to assume that one should choose $\mathcal{L} = \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. In that case, a triple $(\mathfrak{L}, B, k)$ is already determined by choosing a Frobenius functional $F$ for the Frobenius Lie algebra $\mathfrak{P}_k$, extending it by zero to $\mathfrak{g}$ and then choosing $B$ to be given by the Kirillov form $B(x, y) = F([x, y])$. Frobenius functionals were studied by Elashvili [20, 21] and also Dergachev and Kirillov [18]. For our work, the more indirect approach to this question by Elashvili is essential. In his paper [21], Elashvili gives a recursive procedure that allows to construct a Frobenius functional $F$ for a given Frobenius Lie algebra $\mathcal{F}$ from the Frobenius functional $F'$ of some smaller Frobenius Lie algebra $\mathcal{F}' \subset \mathcal{F}$ which is the stabilizer under a certain action on $\mathcal{F}$.

It turns out that in the setup we work in, the recursive procedure of Elashvili is exactly the same as the algorithm for the construction of a universal family of stable vector bundles of fixed rank and degree on the cuspidal cubic curve as established by Bodnarchuk [5]. Using the theory of matrix problems developed by Drozd and Greuel [19], Bodnarchuk showed that this universal family is basically encoded by a certain matrix $J = J(n, n-d)$, where $n$ is the rank of the vector bundles considered and $d$ denotes the degree. Interestingly, Bodnarchuk, Drozd and Greuel [5, 8] showed that these results can also be explained in terms of the representation theory of differential graded biquivers. The second main result of [13] is as follows:

**Result 4** [13] *Let $E$ be the cuspidal Weierstraß cubic curve and $0 < d < n$ be a pair of mutually prime integers. We set $B_J(a, b) = \operatorname{tr}\big(J^t \cdot [a, b]\big)$ for $a, b \in \mathfrak{g}$, where $J \in \operatorname{Mat}_{n\times n}(\mathbb{C})$ is a certain matrix uniquely determined by $n$ and $d$. Let $(\mathfrak{g}, B_J, n-d)$ be the corresponding Stolin triple and $r = r_{(\mathfrak{g}, B_J, n-d)}$ the corresponding solution of the*

*CYBE. Then the solution* $r_{(E,n,d)}$ *from Result 3 is gauge equivalent to the solution* $r_{(\mathfrak{g},B_J,n-d)}$. *Using this result we also show that the solutions* $r_{(E,n,d)}$ *and* $r_{(E,n,n-d)}$ *are* gauge equivalent.

*Remark* 1.1. Let $E$ be an elliptic curve, $E'$ the cuspidal Weierstraß cubic curve and $0 < d < n$ be a pair of mutually prime integers. Combining Results 3 and 4, we see that any elliptic solution $r_{(E,n,d)}$ has two limits of rational solutions: $r_{(E',n,n-d)}$ and $r_{(E',n,d)}$. It is not easy to give a direct proof of this statement.

To summarize, the main results of this thesis are the following:

- we study an interplay between the associative, quantum and classical Yang-Baxter equations, generalizing results of Polishchuk [37], see Theorem 6.2 and Theorem 7.1.
- we have found new elliptic solutions of the AYBE, see Theorem 18.5. This leads to new identities for (higher derivates of) the Kronecker function $\sigma$.
- we give a geometric construction of solutions of the CYBE based on the theory of vector bundles on Weierstraß cubic curves, developing ideas of Polishchuk [37] and Burban-Kreußler [14], see Theorem 9.1.
- we describe rational solutions of the CYBE arising from simple vector bundles on a cuspidal cubic curve and express them in terms of Stolin's classification, see Theorem 22.1. Via this approach, we establish new results about rational degenerations of elliptic solutions of the CYBE, see Proposition 22.10. We also give a concrete recipe to lift these rational solutions of the CYBE to solutions of the QYBE and AYBE, see Subsections 15.1 and 15.3.

1.1. **Organization of the material.** This thesis is organized as follows. In Part 1, we recall standard notions and results for the theory of the classical, quantum and associative Yang-Baxter equations. The main results of this part are Theorems 6.2 and 7.1, which yield Result 1.

Next we present the construction of solutions of the AYBE respectively the CYBE via the calculation of triple Massey products in $D^b(\mathrm{Coh}(E))$, see Part 2. After recalling the theory of (semi-) stable vector bundles on an elliptic and cuspidal cubic curve $E$ in Part 3, we then translate the constructions form Part 2 into explicit algorithms. This is contained in Part 4. In Part 5, we compute the elliptic solutions of the AYBE given in Result 2. The main results of Parts 2 to 5 are Theorems 9.1 and 18.5. Together with the results of (Sub-) Sections 14.3 and 17, these are precisely Results 2 and 3.

Part 6 is devoted to the study of rational solutions of the CYBE. First, we explain Stolin's classification [42]. Then we apply the procedure for the construction of Frobenius functionals by Elashvili [21] in order to construct a certain Stolin triple $(\mathfrak{g}, B_J, n-d)$, see Proposition 21.16 and Corollary 21.21. The main result of Part 6, presented in Theorem 22.1, establishes a concrete gauge equivalence between the rational solution corresponding to the triple $(\mathfrak{g}, B_J, n-d)$ and the rational solution $r_{(E,n,d)}$ obtained via Result 3. Finally, Proposition 22.10 gives a gauge equivalence between the rational solutions $r_{(E,n,d)}$ and $r_{(E,n,n-d)}$. Summarizing the most important statements of Part 6, we obtain Result 4.

Finally, in Part 7 we present *Mathematica* implementations for the algorithms obtained in Part 4.

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## Part 1. Yang-Baxter equations: Interplay

In this part, we recall some standard notions and results for the theory of the classical, quantum and associative Yang-Baxter equations. Moreover, we will explain our work on the interconnection between these equations as presented in [26]. All proofs contained in this part are purely analytical.

## 2. The classical Yang-Baxter equation

We start by explaining the classical Yang-Baxter equation and the classification results by Belavin and Drinfeld [3]. Although their theory is established for any finite-dimensional simple complex Lie algebra, we will concentrate on the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for reasons which will become obvious later. Let $r : \mathbb{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ be the germ of a meromorphic function in a neighbourhood of zero. Then $r$ is a solution of the *classical Yang-Baxter equation* (CYBE) if it satisfies the following equality

$$\begin{aligned}(2.1)\qquad [r^{12}(u_1,u_2), r^{23}(u_2,u_3)] + [r^{12}(u_1,u_2), r^{13}(u_1,u_3)] +\\ + [r^{13}(u_1,u_3), r^{23}(u_2,u_3)] = 0.\end{aligned}$$

Here $r^{ij} = \tau_{ij} \circ r$ denotes $r$ followed by the obvious inclusions $\mathfrak{g}^{\otimes 2} \to \mathcal{U}\big(\mathfrak{g}\big)^{\otimes 3}$ , e.g. $\tau_{13}(a \otimes b) = a \otimes \mathbb{1} \otimes b$. Throughout this thesis, we shall be interested in solutions $r$ of (2.1) which satisfy two additional conditions.

**Definition 2.1.** Let $r(u, v)$ be a solution of the CYBE.

(1) $r$ is called non-degenerate if its image under the isomorphism

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \quad a \otimes b \mapsto \big(c \mapsto \mathrm{tr}(ac) \cdot b\big)$$

is an invertible operator for some (and hence, for a generic) value of the spectral parameters $(u, v)$.

(2) $r$ is called unitary if $r^{12}(u, v) = -r^{21}(v, u)$.

On the set of solutions of (2.1) there exists an equivalence relation called *gauge equivalence*. This is induced by the action of the group of germs of holomorphic functions $\varphi : (\mathbb{C}, 0) \to \mathrm{Aut}(\mathfrak{g})$ given by

$$r(u, v) \mapsto (\varphi(u) \otimes \varphi(v))\, r(u, v).$$

The operator $\varphi(u)\otimes\varphi(v)$ is called *gauge transformation* or *gauge equivalence* and $r(u, v)$ is said to be *gauge equivalent* to $(\varphi(u) \otimes \varphi(v))\, r(u, v)$.

**Proposition 2.2.** [2] *Up to gauge equivalence, any non-degenerate unitary solution of the CYBE is equivalent to a solution $r(u, v) = r(u-v)$ depending only on the difference of the spectral parameters. Denoting this solution by $r(x)$, this means that $r$ satisfies the equation*

$$(2.2)\qquad \big[r^{12}(x), r^{23}(y)\big] + \big[r^{12}(x), r^{13}(x + y)\big] + \big[r^{13}(x + y), r^{23}(y)\big] = 0.$$

Let us illustrate the above definitions by some examples. To this end, we let

$$\{e_{i,j}\}_{1\le i\ne j\le n} \cup \{h_l\}_{1\le l\le n-1}$$

denote the standard basis of $\mathfrak{g}$. That is, for any $1 \le i, j \le n$, $e_{i,j}$ denotes the element of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ whose entry at the position $(i, j)$ is equal to one and zero everywhere else and $h_l = e_{l,l} - e_{l+1,l+1}$.

**Example 2.3.** The following functions are non-degenerate unitary solutions of (2.2) for $\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$:

1) $r(x) = \frac{1}{x}\left(\frac{1}{2}h_1 \otimes h_1 + e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2}\right)$
2) $r(x) = \frac{1}{2}\cot(x) h \otimes h + \frac{1}{\sin(x)}\left(e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2}\right) + \sin(x) e_{2,1} \otimes e_{2,1}$
3) $r(x) = \frac{\mathrm{cn}(x)}{\mathrm{sn}(x)} h \otimes h + \frac{1+\mathrm{dn}(x)}{\mathrm{sn}(x)}\left(e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2}\right) + \frac{1-\mathrm{dn}(x)}{\mathrm{sn}(x)}\left(e_{1,2} \otimes e_{1,2} + e_{2,1} \otimes e_{2,1}\right)$

The functions $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$ in example 3) are Jacobi elliptic functions on a fixed elliptic curve $E$.

*Remark* 2.4. Solution 1) of the above example was historically the first solution found for (2.2). It is due to Yang. Solution 2) was discovered by Baxter, while solution 3) was found by Baxter, Belavin and Skylanin.

The above examples illustrate the following results by Belavin and Drinfeld. In order to formulate them properly, recall that for our choice of a basis of $\mathfrak{g}$, the *Casimir element* of $\mathfrak{g} \otimes \mathfrak{g}$ is given by

$$\Omega = \sum_{1 \le i \ne j \le n} e_{i,j} \otimes e_{j,i} + \sum_{1 \le l \le n-1} \check{h}_l \otimes h_l$$

where $\check{h}_l$ is the dual of $h_l$ with respect to the trace form $(x, y) \mapsto \mathrm{tr}(x \cdot y)$ (the trace form is a scalar multiple of the non-degenerate Killing form on $\mathfrak{sl}_n(\mathbb{C})$).

**Theorem 2.5.** [3, Proposition 2.1 and 4.1] *Let $r$ be a non-constant non-degenerate solution of (2.2). Then*

- $r$ has a simple pole at zero and the residue is given by $\mathrm{Res}_x\,(r(x)) = a\Omega$, $a \in \mathbb{C}$.
- $r$ is unitary, $r^{12}(x) = -r^{21}(-x)$.

**Theorem 2.6.** [3, Theorem 1.1] *Let $r$ be a non-degenerate solution of (2.2). Then $r$ extends meromorphically to all of $\mathbb{C}$. Moreover, the poles of $r$ form a discrete subgroup $\Gamma \subset \mathbb{C}$. Write $r(x) = \sum_{1 \le k,l \le n^2-1} \alpha_{kl}(x) I_k \otimes I_l$ where $\{I_k\}_{1 \le k \le n^2-1} = \{e_{i,j}\}_{1 \le i \ne j \le n} \cup \{h_l\}_{1 \le l \le n-1}$. If*

- $\mathrm{rk}\ \Gamma = 2$, *then all $\alpha_{kl}$ are elliptic functions.*
- $\mathrm{rk}\ \Gamma = 1$, *then all $\alpha_{kl}$ are trigonometric functions.*
- $\mathrm{rk}\ \Gamma = 0$, *then all $\alpha_{kl}$ are rational functions.*

*Especially, there are three disjoint families of non-degenerate solutions* of (2.2)*: elliptic, trigonometric and rational.*

*Remark* 2.7. We will see that the definition of elliptic, trigonometric and rational solutions can be extended to solutions of the associative Yang-Baxter equation (to be introduced in the next section), see Remark 7.2.

Even though Proposition 2.2 guarantees existence of a gauge equivalence which transforms solutions of (2.1) into solutions of (2.2), we will mostly study solutions of (2.2). The reason for this is that constructing a gauge transformation which maps $r(u,v)$ to $r'(u,v) = r'(u-v)$ in practice is highly non-trivial. Moreover, as we shall see later, from a geometric point of it seems more natural to work with solutions of (2.1).
Clearly the nomenclature for solutions of (2.2) extends to solutions of (2.1). Finally, let us recall the following notion, which shall play an important role in the next sections:

**Definition 2.8.** An infinitesimal symmetry of a solution $r$ of (2.1) is given by an element $a \in \mathfrak{g}$ such that $[r(u,v), a \otimes \mathbb{1} + \mathbb{1} \otimes a] = 0$ for all $u, v$. Furthermore we say that $r$ has no infinitesimal symmetries if the only infinitesimal symmetry of $r$ is given by $a = 0$.

## 3. The associative Yang-Baxter equation

In this section, we give the definition of the associative Yang-Baxter equation. Moreover, we will recall some results by Polishchuk [37] respectively Burban and Kreußler [14] on the relationship between this equation and the classical Yang-Baxter equation. As for the classical Yang-Baxter equation, see (2.1) and (2.2), there are multiple versions of the associative Yang-Baxter equation, differing with respect to the number of variables. We start with solutions of the most general version of the *associative Yang-Baxter equation* (AYBE):

$$\begin{gathered} r^{12}(v_1, v_2; y_1, y_2)\, r^{23}(v_1, v_3; y_2, y_3) = \\ = r^{13}(v_1, v_3; y_1, y_3)\, r^{12}(v_3, v_2; y_1, y_2) + r^{23}(v_2, v_3; y_2, y_3)\, r^{13}(v_1, v_2; y_1, y_3) \end{gathered} \tag{3.1}$$

Here $r : (\mathbb{C}^4, 0) \to A \otimes A$ is the germ of a meromorphic function in a neighbourhood of zero and $A = \mathrm{Mat}_{n \times n}(\mathbb{C})$. Similar to the case of the CYBE, for $i \neq j \in \{1,2,3\}$, we use the notation $r^{ij} = r \circ \rho_{ij}$ for the composition of $r$ with the canonical embedding $\rho_{ij} : A^{\otimes 2} \to A^{\otimes 3}$. However, in this part we will mostly study solutions of (3.1) depending on the difference of the first pair of spectral parameters

$$r(v_1, v_2; y_1, y_2) = r(v_1 - v_2; y_1, y_2) = r(v; y_1, y_2).$$

That is, we study solutions of the associative Yang-Baxter equation in three spectral variables

$$\begin{gathered} r^{12}(u; y_1, y_2)\, r^{23}(u+v; y_2, y_3) = \\ = r^{13}(u+v; y_1, y_3)\, r^{12}(-v; y_1, y_2) + r^{23}(v; y_2, y_3)\, r^{13}(u; y_1, y_3). \end{gathered} \tag{3.2}$$

for meromorphic function germs $r : (\mathbb{C}^3, 0) \to A \otimes A$. Similarly to the situation of the CYBE, we are mainly interested in solutions of the AYBE with the following two additional properties:

**Definition 3.1.** Let $r(v_1, v_2; y_1, y_2)$ be a solution of (3.1). Then

(1) $r$ is called non-degenerate if $r(v_1, v_2; y_1, y_2) \in A \otimes A \cong \mathrm{End}(A)$ is invertible for generic $(v_1, v_2; y_1, y_2)$.
(2) $r$ is called unitary if $r^{12}(v_1, v_2; y_1, y_2) = -r^{21}(v_2, v_1; y_2, y_1)$.

*Remark* 3.2. For solutions of (3.2), the unitarity condition obviously translates to $r^{12}(v; y_1, y_2) = -r^{21}(-v; y_2, y_1)$.

Interestingly, solutions of the AYBE also satisfy a sort of dual version of that equation:

**Lemma 3.3.** [14, Lemma 2.7] *Let $r(v_1, v_2; y_1, y_2)$ be a unitary solution of (3.1). Then writing $r^{ij}(v_1, v_2)$ as short-hand for $r^{ij}(v_1, v_2; y_i, y_j)$, $r$ also satisfies*

$$r^{23}(v_2, v_3)\, r^{12}(v_1, v_3) = r^{12}(v_1, v_2)\, r^{13}(v_2, v_3) + r^{13}(v_1, v_3)\, r^{23}(v_2, v_1).$$

**Corollary 3.4.** *If $r(v; y_1, y_2)$ is a unitary solution of (3.2), then we also have*

$$\begin{gathered} r^{23}(u+v; y_2, y_3)\, r^{12}(u; y_1, y_2) = \\ = r^{12}(-v; y_1, y_2)\, r^{13}(u+v; y_1, y_3) + r^{13}(u; y_1, y_3)\, r^{23}(v; y_2, y_3). \end{gathered} \tag{3.3}$$

The notion of gauge equivalence can also be adapted to the situation of the AYBE:

**Definition 3.5.** Let $\phi : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ be the germ of a holomorphic function and let $r(v_1, v_2; y_1, y_2)$ be a solution of (3.1). Then the tensor valued function

$$r'(v_1, v_2; y_1, y_2) = \Big(\phi(v_1; y_1) \otimes \phi(v_2; y_2)\Big)\, r(v_1, v_2; y_1, y_2)\, \Big(\phi^{-1}(v_2; y_1) \otimes \phi^{-1}(v_1; y_2)\Big)$$

is also a solution of (3.1). The solutions $r$ and $r'$ are said to be gauge equivalent and $\phi$ is called a gauge transformation.

In order to show how this notion translates to solutions of (3.2), it is best to have a look at an example.

**Example 3.6.** Let $r(v_1, v_2; y_1, y_2) \in A \otimes A$ be a solution of (3.1), $c \in \mathbb{C}$ and $\phi(v, y) = \exp(cvy) \cdot \mathbb{1} : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ be a gauge transformation. Then

$$\exp\Big(c\,(v_2 - v_1)\,(y_2 - y_1)\Big)\, r(v_1, v_2; y_1, y_2)$$

is a solution of (3.1), gauge equivalent to $r$.

Similarly, assume that $r(v_1, v_2; y_1, y_2) = r(v; y_1, y_2)$ is a solution of (3.1) which depends only on $v = v_1 - v_2, y_1, y_2$. Thus $r(v; y_1, y_2)$ is a solution of (3.2). Consider the gauge transformation $\phi(v, y) = \exp(vg(y)) \cdot \mathbb{1} : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ for some holomorphic function $g : \mathbb{C} \to \mathbb{C}$. Then $r'(v; y_1, y_2) = \exp\big(v\,(g(y_2) - g(y_1))\big)\, r(v; y_1, y_2)$ is a solution of (3.2) as well.

Finally, there is also the associative Yang-Baxter equation in only two spectral variables

$$r^{12}(u; x)\, r^{23}(u+v; y) = r^{13}(u+v; x+y)\, r^{12}(-v; x) + r^{23}(v; y)\, r^{13}(u; x+y), \tag{3.4}$$

with straightforward adaptions for the definitions of non-degeneracy and unitarity. As to the definition of gauge equivalence, simply assume $g = c \in \mathbb{C}$ to be constant in example 3.6 to get a corresponding example.
The following statement, first discovered by Polishchuk for the AYBE in two variables, see [37, Lemma 1.2], gives the decisive relationship between solutions of the AYBE respectively the CYBE:

**Lemma 3.7.** [14, lemma 2.11] *Let $r(v; y_1, y_2)$ be a unitary solution of (3.2) and let* $\mathrm{pr} : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ *denote the projection* $X \mapsto X - \frac{\mathrm{tr}\,X}{n}\mathbb{1}$. *If* $(\mathrm{pr}\otimes\mathrm{pr})\, r(v; y_1, y_2)$ *exists, then*

$$\bar{r}(y_1, y_2) = \lim_{v\to 0} (\mathrm{pr}\otimes\mathrm{pr})\, r(v; y_1, y_2)$$

*is a unitary solution of (2.1).*

An interesting subclass of solutions of (3.2) is given by those solutions that have a particular kind of Laurent expansion with respect to the variable $v$:

$$r(v; y_1, y_2) = \frac{\mathbb{1}\otimes\mathbb{1}}{v} + r_0(y_1, y_2) + v\, r_1(y_1, y_2) + v^2\, r_2(y_1, y_2) + \ldots \tag{3.5}$$

Solutions with Laurent expansion of the form (3.5) behave well with respect to gauge equivalence:

**Proposition 3.8.** [14, Proposition 2.12] *Let $r$ be a solution of (3.3) of the form (3.5). For any gauge transformation $\phi$ such that*

$$r'(v_1, v_2; y_1, y_2) = \Big(\phi(v_1; y_1)\otimes\phi(v_2; y_2)\Big)\, r(v_1, v_2; y_1, y_2)\, \Big(\phi^{-1}(v_2; y_1)\otimes\phi^{-1}(v_1; y_2)\Big)$$

*is again a function of $v = v_1 - v_2$, $r'(v; y_1, y_2)$ is of the form (3.5) as well.*

*Moreover, if $r$ is unitary, then so is $r'$ and the corresponding solutions* $\bar{r}_0 = (\mathrm{pr}\otimes\mathrm{pr})r_0$ *and* $\bar{r}'_0 = (\mathrm{pr}\otimes\mathrm{pr})r'_0$ *of (2.1) are gauge equivalent as well.*

Finally, note the following result by Polishchuk which we shall generalize in Section 6:

**Theorem 3.9.** [37, Theorem 6] *Let $r(v; y)$ be a unitary solution of (3.4) with Laurent expansion analogous to (3.5) and such that* $\bar{r}_0 = (\mathrm{pr}\otimes\mathrm{pr})r_0$ *is non-degenerate. Let $s(v; y)$ be another non-degenerate unitary solution of the AYBE (3.4) with Laurent expansion analogous to (3.5) and* $(\mathrm{pr}\otimes\mathrm{pr})s_0 = \bar{r}_0$. *Then there exists $\alpha \in \mathbb{C}$ such that*

$$s(v; y) = \exp\Big(\alpha v y\Big) r(v; y).$$

## 4. Relationship with the quantum Yang-Baxter equation

The interplay between the AYBE (3.4) and the CYBE (2.2) with the quantum Yang-Baxter equation is explained by the following theorem of Polishchuk:

**Theorem 4.1.** [38, Theorem 1.4] *Let $r(v; y)$ be a non-degenerate unitary solution of (3.4) with Laurent expansion analogous to (3.5) and let $\bar{r}_0 = (\mathrm{pr} \otimes \mathrm{pr})r_0$.*

(1) $\bar{r}_0$ is a non-degenerate solution of (2.2)
(2) The following are equivalent:
   (a) For fixed $v_0 \in \mathbb{C}^\times$, $\tilde{r}(y) = r(v_0; y)$ is a solution of the quantum Yang-Baxter equation (QYBE)
$$\tilde{r}^{12}(x)\, \tilde{r}^{13}(x+y)\, \tilde{r}^{23}(y) = \tilde{r}^{23}(y)\, \tilde{r}^{13}(x+y)\, \tilde{r}^{12}(x).$$
   (b) *there exits a scalar function $\varphi(v; y)$ such that*
$$r(v; y)\, r(-v; y) = \varphi(v; y)\, (\mathbb{1} \otimes \mathbb{1})\,.$$
   (c) *there exists a scalar function $\psi(y)$ such that*
$$\frac{\partial}{\partial y}\left(r_0(y) - \overline{r}_0(y)\right) = \psi(y)\, (\mathbb{1} \otimes \mathbb{1})\,.$$
   (d) *we have*
$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\ \left[\overline{r}_0^{12}(x)\, \overline{r}_0^{13}(x+y) - \overline{r}_0^{23}(y,)\, \overline{r}_0^{12}(x) + \overline{r}_0^{13}(x+y)\, \overline{r}_0^{23}(y)\right] = 0.$$
(3) *The above conditions are satisfied if either $\overline{r}_0(y)$ has no infinitesimal symmetries or is periodic (elliptic or trigonometric).*

One of the main results of [26] is that this statement can be extended to the AYBE and CYBE in the form (3.2) respectively (2.1) and the QYBE in the form

$$r^{12}(y_1, y_2)\, r^{13}(y_1, y_3)\, r^{23}(y_2, y_3) = r^{23}(y_2, y_3)\, r^{13}(y_1, y_3)\, r^{12}(y_1, y_2). \tag{4.1}$$

The precise statement along with all proofs is presented in Section 7.

## 5. Poles of solutions of the AYBE

In this section, we study the poles of solutions of (3.2) along $y_1 = y_2$. We start with the following easy fact on $P = \sum_{1 \le i,j \le n} e_{i,j} \otimes e_{j,i} \in A \otimes A$.

**Fact 5.1.** Any tensor $\theta \in A \otimes A$ such that $\theta(x \otimes 1) = (1 \otimes x)\theta$ for all $x \in A$ is a scalar multiple of $P$. Moreover $P(1 \otimes x) = (x \otimes 1)P$ for any $x \in A$.

**Lemma 5.2.** (see [38, Lemma 1.3]) *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (3.2). Assume that $r(u; y_1, y_2)$ has a pole along $y_1 = y_2$. Then this pole is simple and $\lim_{y_2 \to y_1}(y_1 - y_2)\, r(u; y_1, y_2) = c \cdot P$ for some $c \in \mathbb{C}$.*

*Proof.* Write $r(u; y_1, y_2) = \alpha(u; y_1 - y_2) + \beta(u; y_1, y_2)$ and assume that no summand of $\beta(u; y_1, y_2)$ depends only on $u$ and $y = y_1 - y_2$. Let $\alpha(u; y) = \frac{\theta(u)}{y^k} + \frac{\eta(u)}{y^{k-1}} + \dots$ be the

Laurent expansion near $y = 0$. In order to see that $k \leq 1$, we consider the polar parts in (3.2) as $y_3 \to y_1$, which yields

$$\theta^{13}(u+v)\, r^{12}(-v; y_1, y_2) + r^{23}(v; y_2, y_1)\, \theta^{13}(u) = 0. \tag{5.1}$$

Analogously, for $y_2 \to y_1$

$$\theta^{12}(u)\, r^{23}(u+v; y_1, y_3) - r^{13}(u+v; y_1, y_3)\, \theta^{12}(-v) = 0. \tag{5.2}$$

Let $V \subseteq A$ be the minimal subspace such that $\theta(u) \in V \otimes A$ for all $u$ where $\theta(u)$ is defined. Obviously $r^{23}(u; y_1, y_2)\, \theta^{13}(u) \in V \otimes A \otimes A$ hence by (5.1) $\theta^{13}(u+v)\, r^{12}(-v; y_1, y_2) \in V \otimes A \otimes A$ as well. Thus, $r^{12}(u; y_1, y_2) \in A_1 \otimes A$, where

$$A_1 = \{\, a \in A |\, \theta(u)\, (a \otimes 1) \in V \otimes A \,\text{for all } u\}\,.$$

By non-degeneracy $A_1 = A$, thus $VA \subseteq V$. Similarly, using (5.2), we get $AV \subseteq V$, so that $V$ is a two-sided non-zero ideal in $A$. Hence $V = A$.

Let us come back to (3.2). We want to have a look at the coefficient of $(y_1 - y_2)^{1-k}$ in the expansion of (3.2) near $y_2 - y_1 = h$ equal to zero. The terms contributing to this only depend on $r^{12}(u; y_1, y_1+h)\, r^{23}(u+v; y_1+h, y_3)$ and $r^{13}(u+v; y_1, y_3)\, r^{12}(-v; y_1, y_1+h)$. Thus the coefficient of $h^{1-k}$ consists of two summands, the first one being $\eta^{12}(u)\, r^{23}(u+v; y_1, y_3) - r^{13}(u+v; y_1, y_3)\, \eta^{12}(-v)$ and second one being $\theta^{12}(u)$ times the coefficient of $h$ in $r^{23}(u+v; y_1+h, y_3)$. Note that for this last summand to be non-zero we must assume $k > 1$. Now $r^{23}(u+v; y_1+h, y_3) - r^{23}(u+v; y_1, y_3)$ equals the summand of $r^{23}(u+v; y_1+h, y_3)$ divisible by $h$, hence the coefficient of $h^{1-k}$ is exactly

$$\eta^{12}(u)\, r^{23}(u+v; y_1, y_3) - r^{13}(u+v; y_1, y_3)\, \eta^{12}(-v) + \theta^{12}(u)\, \frac{\partial r^{23}}{\partial y_1}(u+v; y_1, y_3).$$

Examining the polar parts in the above expression for $y_1 - y_3$ in a neighborhood of zero, we deduce that $\theta^{12}(u)\, \theta^{23}(u+v) = 0$. Setting $v = 0$ this amounts to saying that $\theta(u) = \{\, a \otimes b |\, ab = 0\}$. Since $V = A$ this is a contradiction. Therefore $k = 1$.

Next, we have a look at the polar parts in (3.3) near $y_3 = y_2$. We deduce $\theta^{23}(u+v)\, r^{12}(u; y_1, y_2) = r^{13}(u; y_1, y_2)\, \theta^{23}(v)$. Hence $r(u; y_1, y_2) \in A \otimes A(u)$, where

$$A(u) = \{a \in A\,| \theta(u+v)(x \otimes 1) = (1 \otimes x)\theta(v) \,\text{for all}\, v\}\,.$$

Since $A$ is non-degenerate this implies $A(u) = A$ for generic $u$, in which case $\mathbb{1} \in A(u)$ and thus $\theta(u+v) = \theta(u)$. Hence $\theta = \theta(0)$ is constant. Recalling Fact 5.1 finishes the proof. □

**Corollary 5.3.** (see [38, Lemma 1.5]) *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (3.2) of the form (3.5). Then $r(u; y_1, y_2)$ has a simple pole along $y_1 = y_2$ with residue a scalar multiple of $P$.*

*Proof.* This is essentially the same proof as that of Lemma 1.5 in [38], using Lemma 5.2 where Polishchuk refers to Lemma 1.3 of his paper. □

## 6. Uniqueness of lifts from CYBE to AYBE

In this section, we will extend Theorem 3.9 to solutions of (3.2) respectively (2.1). Let us start by generalizing one particular part of the original proof of Polishchuk's result for later usage:

**Lemma 6.1.** *Let $r(v; y_1, y_2)$ be a unitary solution of (3.2) of the form (3.5). Then $r$ is uniquely determined by $r_0$ and $r_1$. Moreover, we have*

$$\begin{aligned} &r_1^{12}(y_1,y_2) + r_1^{13}(y_1,y_3) + r_1^{23}(y_2,y_3) = \\ &= r_0^{12}(y_1,y_2)\, r_0^{13}(y_1,y_3) - r_0^{23}(y_2,y_3)\, r_0^{12}(y_1,y_2) + r_0^{13}(y_1,y_3)\, r_0^{23}(y_2,y_3). \end{aligned} \tag{6.1}$$

*Proof.* First we show that $r$ is uniquely determined by $r_0, r_1$ and $r_2$. To this end, we fix $k > 2$ and show how to construct $r_k$ from $\{r_i\}_{0 \le i \le k-1}$. Let us insert the Laurent expansion (3.5) of $r$ into (3.3) and examine the terms of total degree $k-1$ in the variables $u$ and $v$. We derive the equation

$$\begin{aligned} &r_k^{12}(y_1,y_2)\left[\frac{(-v)^k}{u+v} - \frac{u^k}{u+v}\right] + r_k^{13}(y_1,y_3)\left[\frac{u^k}{v} - \frac{(u+v)^k}{v}\right] + \\ &+ r_k^{23}(y_2,y_3)\left[\frac{v^k}{u} - \frac{(u+v)^k}{u}\right] = \ldots \end{aligned} \tag{6.2}$$

where the ride-hand side contains terms $r_i$ with $i < k$ only. The polynomials in $u$ and $v$ on the left-hand side are linearly independent for $k > 2$. Indeed, if we place everything over a common denominator and focus on the coefficients of $u^k$ in the respective terms

$$u(-v)^{k+1} - vu^{k+1},\ u^{k+1}(u+v) - u(u+v)^{k+1},\ (u+v)v^{k+1} - v(u+v)^{k+1}$$

then these are $-vu$, $u(u+v) - \binom{k+1}{2}v^2$ and $-(k+1)\,v^2$ respectively. This proves our claim that $r$ is determined by the $r_k$ with $k \le 2$.

In the next step, we show that $r_2$ is already determined by $r_0$ and $r_1$. Indeed, for $k = 2$ equation (6.2) reads

$$(v-u)\ r_2^{12}(y_1,y_2)\ - (2u+v)\ r_2^{13}(y_1,y_3) - (u+2v)\, r_2^{23}(y_2,y_3) = \ldots$$

that is

$$\begin{aligned} -u \cdot \big(r_2^{12}(y_1,y_2) + 2r_2^{13}(y_1,y_3) + r_2^{23}(y_2,y_3)\big) + \\ +v \cdot \big(r_2^{12}(y_1,y_2) - r_2^{13}(y_1,y_3) - 2r_2^{23}(y_2,y_3)\big) = \ldots \end{aligned}$$

with the the right-hand side depending on $r_0$ and $r_1$ only. Let us denote the coefficient of $-u$ on the left-hand side by $a$, that of $v$ by $b$. Since $a, b$ are determined by $r_0$ and $r_1$ only, so is $\frac{a+2b}{3} = r_2^{12}(y_1,y_2) - r_2^{23}(y_2,y_3)$. Thus, $r_2(y_1,y_2)$ is determined by $r_0$ and $r_1$. Putting $k = 1$ in (6.2), we obtain (6.1). □

Now, we can directly head to the main result of this section:

**Theorem 6.2.** *Let $r(u;y_1,y_2)$ and $s(u;y_1,y_2)$ be a unitary solutions of (3.2) of the form (3.5). Assume that the corresponding solution $\overline{r}_0(y_1,y_2) = (\mathrm{pr}\otimes\mathrm{pr})\,(r_0(y_1,y_2))$ of the CYBE (2.1) is non-degenerate, has no infinitesimal symmetries and that $\overline{s}_0(y_1y_2) = \overline{r}_0(y_1,y_2)$. Then there exists a meromorphic function $g:\mathbb{C}\to\mathbb{C}$ such that $s(u;y_1,y_2) = \exp\big(u\big(g(y_2)-g(y_1)\big)\big)r(u;y_1,y_2)$.*

*Proof.* First, we show that $r$ is uniquely determined by $r_0$. By Lemma 6.1 $r$ is uniquely determined by $r_0$ and $r_1$ and moreover $r_1$ is a solution of a certain equation in $r_0$ which is given by (6.1). If $r_1' \neq r_1$ was a solution of (6.1) with the same properties as $r_1$, then taking the difference we would obtain a meromorphic function $\alpha : (\mathbb{C}^2,0)\to A\otimes A$ with $\alpha^{21}(y_2,y_1) = \alpha(y_1,y_2)$ and

$$\alpha^{12}(y_1,y_2) + \alpha^{13}(y_1,y_3) + \alpha^{23}(y_2,y_3) = 0. \tag{6.3}$$

Using Lemma 5.2, we also know that the residue of $r(u;y_1,y_2)$ near $y_1 = y_2$ is independent of $u$. Comparing this to the Laurent expansion (3.5), we derive that $r_1(y_1,y_2)$ has no poles along $y_1 = y_2$, hence the same is true for $\alpha(y_1,y_2)$. To prove that $r_1$ is determined by $r_0$, we need only show that $\alpha$ is already zero. Choosing $y_3 = y_2$ and then applying $\mathrm{pr}\otimes\mathrm{id}\otimes\mathrm{id}$ to (6.3) we derive that $(\mathrm{pr}\otimes\mathrm{id})\,(\alpha(y_1,y_2)) = 0$. Similarly, $(\mathrm{id}\otimes\mathrm{pr})\,(\alpha(y_1,y_2)) = 0$, hence $\alpha(y_1,y_2) = f(y_1,y_2)\,\mathbb{1}\otimes\mathbb{1}$ where $f$ is a meromorphic function such that $f(y_1,y_2)+f(y_1,y_3)+f(y_2,y_3) = 0$. Since $r_1(y_1,y_2)$ has no pole along $y_1 = y_2$ by Lemma 5.2, $\alpha(y_1,y_1)$ exists. We may deduce that $2f(y_1,y_2) = -f(y_2,y_2)$, so $f$ depends only on the second variable. But then choosing $y_2 = y_1 = y_3$ we read $3f(y_1,y_1) = 0$, thus $f = 0$.We have proved that $r$ is uniquely determined by $r_0$.

It remains to prove that, provided $\overline{r}_0$ has no infinitesimal symmetries, $r$ can be uniquely recovered from $\overline{r}_0$ up to the factor $\exp\left(u\left(g(y_2)-g(y_1)\right)\right)$ for some meromorphic function $g:\mathbb{C}\to\mathbb{C}$. Note that this is equivalent to showing that $r_0(y_1,y_2)$ is uniquely determined by $\overline{r}_0(y_1,y_2) = (\mathrm{pr}\otimes\mathrm{pr})\,(r_0(y_1,y_2))$ up to a summand of the form $(g(y_2)-g(y_1))\,\mathbb{1}\otimes\mathbb{1}$. By assumption $(s_0(y_1,y_2), s_1(y_1,y_2))$ is another tuple satisfying (6.1) such that

$$s_0^{21}(y_2,y_1) = -s_0(y_1,y_2),\ s_1^{21}(y_2,y_1) = s_1(y_1,y_2).$$

We claim that

$$s_0(y_1,y_2) = r_0(y_1,y_2) + (g(y_2)-g(y_1))\,\mathbb{1}\otimes\mathbb{1}.$$

Since $\overline{s}_0(y_1,y_2) = \overline{r}_0(y_1,y_2)$, we may write

$$s_0(y_1,y_2) = r_0(y_1,y_2) + \phi^1(y_1,y_2) - \phi^2(y_2,y_1) + \psi(y_1,y_2)\,\mathbb{1}\otimes\mathbb{1}$$

for a $\mathfrak{sl}_n(\mathbb{C})$ valued function $\phi$ and a scalar function $\psi$. Denoting the left-hand side of (6.1) by $LHS(r)$, we have

$$0 = (\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\,(LHS(s) - LHS(r)) = \overline{r}_0^{12}(y_1,y_2)\,\left[\phi^3(y_3,y_2) - \phi^3(y_3,y_1)\right] +$$

$$+\overline{r}_0^{23}(y_2,y_3)\,\left[\phi^1(y_1,y_3)\,-\phi^1(y_1,y_2)\right]+\overline{r}_0^{13}(y_1,y_3)\,\left[\phi^2(y_2,y_3)-\phi^2(y_2,y_1)\right].$$

If the function $\phi$ is not constant then contracting this equation with a generic functional in the third component we derive that $\overline{r}_0$ is a sum of two decomposable tensors, that is $\bar{r}_0 = a_1 \otimes b_1 + a_2 \otimes b_2$ where all terms depend on $y_1, y_2$. But $\bar{r}_0$ is non-degenerate by assumption, so $\operatorname{span}_{\mathbb{C}}(\{a_1, a_2\}) \cong \mathfrak{g}$, which is impossible for any $n \geq 2$. Thus $\phi \in \mathfrak{g}$ is constant. Applying $(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{id})$ to $LHS(s) - LHS(r)$ yields

$$(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{id})\,(s_1^{12}(y_1,y_2)-r_1^{12}(y_1,y_2))=(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{id})\,(r_0^{12}(y_1,y_2)\phi^1- \\ -\phi^2 r_0^{12}(y_1,y_2)\Big)-\phi^1\phi^2+(\psi(y_1,y_3)-\psi(y_2,y_3))\,\overline{r}^{12}(y_1,y_2). \tag{6.4}$$

This implies that $\psi(y_1, y_3) - \psi(y_2, y_3)$ is actually independent of $y_3$, hence equal to some function $\beta(y_1, y_2)$. Also, we know by unitarity of $r$ that $\psi(y_1, y_2) = -\psi(y_2, y_1)$, thus $\beta(y_1, y_2) = \psi(y_1, y_3) + \psi(y_3, y_2)$. It follows from Lemma 5.2 that $r_0$ and $s_0$ have the same pole along $y_1 = y_2$, hence $\psi(y_1, y_1)$ exists and we may deduce that $\beta(y_1, y_2) = \psi(y_1, y_1) + \psi(y_1, y_2) = \psi(y_1, y_2)$. Thus the definition of $\beta$ reads $\psi(y_1, y_2) = \psi(y_1, y_3) - \psi(y_2, y_3)$. Therefore, defining $g(y) = \psi(y, a)$ for some fixed $a \in \mathbb{C}$, we have $\psi(y_1, y_2) = g(y_1) - g(y_2)$. Altogether

$$s_0(y_1,y_2)=r_0(y_1,y_2)+\phi^1-\phi^2+(g(y_1)-g(y_2))\;\mathbb{1}\otimes\mathbb{1} \tag{6.5}$$

Since $s_0$ and $r_0$ are both meromorphic, so is $\psi$ and thus also $g$.

Next, we replace $r(u; y_1, y_2)$ by $\exp\left(u\left(g(y_2) - g(y_1)\right)\right) r(u; y_1, y_2)$ and hence may assume that $g = 0$ in the above formula for $s_0$. Thus (6.4) yields

$$(\mathrm{pr}\otimes\mathrm{pr})\,(s_1(y_1,y_2)-r_1(y_1,y_2))=(\mathrm{pr}\otimes\mathrm{pr})\,\big(r_0(y_1,y_2)\phi^1-\phi^2 r_0(y_1,y_2)\big)-\phi^1\phi^2.$$

We exchange the first two components, make the substitutions $y_1 \leftrightarrow y_2, y_2 \leftrightarrow y_1$ and use unitarity of $r$ for both sides of the resulting equation. Comparing the result with the above equation, we derive

$$(\mathrm{pr}\otimes\mathrm{pr})\,\big(r_0(y_1,y_2)\phi^1-\phi^2 r_0(y_1,y_2)\big)=(\mathrm{pr}\otimes\mathrm{pr})\,\big(-r_0(y_1,y_2)\phi^2+\phi^1 r_0(y_1,y_2)\big)\,.$$

By Fact 7.5 a) we deduce that $[\overline{r}_0(y_1, y_1), \phi^1 + \phi^2] = 0$. But then $\phi$ is an infinitesimal symmetry of $\overline{r}_0$, so $\phi = 0$. Thus $s_0 = r_0$. □

*Remark* 6.3. As we will see in Theorem 7.1 (1), the assumption of Theorem 6.2 on the non-degeneracy of $\bar{r}_0$ is automatically satisfied if $r$ itself is non-degenerate. Moreover, we deduce from the proof of Theorem 6.2 that in that case $r$ is already uniquely determined by $r_0$.

**Corollary 6.4.** *In the notations of Theorem 6.2, assume that $r_0(y_1, y_2)$ and $s_0(y_1, y_2)$ have the same poles on $\big(\mathbb{C}^2 \setminus V\big((y_1 - y_2)\big)\big)$. Then $g$ is a holomorphic function. Thus, $r$ and $s$ are gauge equivalent.*

*Proof.* It follows from the assumption and Lemma 5.2 that the poles of $r_0$ and $s_0$ coincide. By (6.5) this implies that $g$ is holomorphic. The remaining statement follows from the discussion in example 3.6. □

## 7. Quantization of solutions of CYBE coming from solutions of AYBE

In this section, we generalize Theorem 4.1:

**Theorem 7.1.** *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (3.2) of the form (3.5) and let $\overline{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\,(r_0(y_1, y_2))$.*

(1) *$\overline{r}_0(y_1, y_2)$ is a non-degenerate unitary solution of the CYBE (2.1).*
(2) *The following conditions are equivalent:*
   (a) *for fixed $u \in \mathbb{C}^\times$, $\tilde{r}(y_1, y_2) = r(u; y_1, y_2)$ satisfies the QYBE*
   $$\tilde{r}^{12}(y_1, y_2)\,\tilde{r}^{13}(y_1, y_3)\,\tilde{r}^{23}(y_2, y_3) = \tilde{r}^{23}(y_2, y_3)\,\tilde{r}^{13}(y_1, y_3)\,\tilde{r}^{12}(y_1, y_2).$$
   (b) *there exits a scalar function $\varphi(u; y_1, y_2)$ such that*
   $$r(u; y_1, y_2)\,r(-u; y_1, y_2) = \varphi(u; y_1, y_2)\,(\mathbb{1} \otimes \mathbb{1})\,.$$
   (c) *for $i \in \{1, 2\}$ there exists a scalar function $\psi_i(y_1, y_2)$ such that*
   $$\frac{\partial}{\partial y_i}\,(r_0(y_1, y_2) - \overline{r}_0(y_1, y_2)) = \psi_i(y_1, y_2)\,(\mathbb{1} \otimes \mathbb{1})\,.$$
   (d) *we have*
   $$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\,\Big[\overline{r}_0^{12}(y_1, y_2)\,\overline{r}_0^{13}(y_1, y_3) - \\ -\overline{r}_0^{23}(y_2, y_3)\,\overline{r}_0^{12}(y_1, y_2) + \overline{r}_0^{13}(y_1, y_3)\,\overline{r}_0^{23}(y_2, y_3)\Big] = 0.$$
(3) *These conditions are satisfied if $\overline{r}_0(y_1, y_2)$ has no infinitesimal symmetries.*

*Remark* 7.2. By Theorem 7.1 1), we can extend the nomenclature of elliptic, trigonometric and rational solutions to non-degenerate unitary solutions of the AYBE. That is, a non-degenerate unitary solution $r$ of (3.2) is called elliptic, trigonometric respectively rational exactly if the non-degenerate and unitary solution $\overline{r}_0(y_1, y_2)$ of the CYBE is.

Before proving the theorem above, we first need to establish some auxiliary results. The reader might wish to postpone checking them and to go to the proof of Theorem 7.1 at the end of this section immediately.

**Lemma 7.3.** (see [38, Lemma 1.6]) *For any triple of variables $u_1, u_2, u_3$ set $u_{ij} = u_i - u_j$. Let $r(u; y_1, y_2)$ be any unitary solution of (3.2) and $s(u; y_1, y_2) = r(u; y_1, y_2)\,r(-u; y_1, y_2)$. Then*
$$r^{12}(u_{12}; y_1, y_2)\,r^{13}(u_{23}; y_1, y_3)\,r^{23}(u_{12}; y_2, y_3) - \\ -r^{23}(u_{23}; y_2, y_3)\,r^{13}(u_{12}; y_1, y_3)\,r^{12}(u_{23}; y_1, y_2) =$$

$$= s^{23}(u_{23}; y_2, y_3)\, r^{13}(u_{13}; y_1, y_3) - r^{13}(u_{13}; y_1, y_3)\, s^{23}(u_{21}; y_2, y_3) =$$

$$= r^{13}(u_{13}; y_1, y_3)\, s^{12}(u_{32}; y_1, y_2) - s^{12}(u_{12}; y_1, y_2)\, r^{13}(u_{13}; y_1, y_3).$$

*Proof.* Let us write $r^{ij}(u)$ as short-hand for $r^{ij}(u; y_i, y_j)$. Since we may assume $u = u_{12}$, $v = u_{23}$ and $u + v = u_{13}$, (3.2) may be written as

$$r^{12}(u_{12})\, r^{23}(u_{13}) = r^{13}(u_{13})\, r^{12}(u_{32}) + r^{23}(u_{23})\, r^{13}(u_{12}). \tag{7.1}$$

Analogously, putting $u = u_{13}$ and $v = u_{21}$, (3.3) reads

$$r^{23}(u_{23})\, r^{12}(u_{13}) = r^{12}(u_{12})\, r^{13}(u_{23}) + r^{13}(u_{13})\, r^{23}(u_{21}). \tag{7.2}$$

Multiplying (7.2) with $r^{23}(u_{12})$ from the right yields

$$r^{23}(u_{23})\, r^{12}(u_{13})\, r^{23}(u_{12}) = r^{12}(u_{12})\, r^{13}(u_{23})\, r^{23}(u_{12}) + r^{13}(u_{13})\, s^{23}(u_{21})$$

while switching $u_2$ and $u_3$ in (7.1) followed by multiplication with $r^{23}(u_{23})$ from the left yields

$$r^{23}(u_{23})\, r^{12}(u_{13})\, r^{23}(u_{12}) = r^{23}(u_{23})\, r^{13}(u_{12})\, r^{12}(u_{23}) + s^{23}(u_{23})\, r^{13}(u_{13}).$$

Subtracting these equations, we end up with

$$r^{12}(u_{12})\, r^{13}(u_{23})\, r^{23}(u_{12}) - r^{23}(u_{23})\, r^{13}(u_{12})\, r^{12}(u_{23}) =$$

$$= s^{23}(u_{23})\, r^{13}(u_{13}) - r^{13}(u_{13})\, s^{23}(u_{21}).$$

Switching indices 1 and 3 and using unitarity of $r$ yields the other identity. □

For the next statement we need the notion of an infinitesimal symmetry of a solution $r$ of (3.2), which is simply that of an element $a \in \mathfrak{g}$ such that $[r(u; y_1, y_2), a^1 + a^2] = 0$, where $a^1 = a \otimes \mathbb{1}$ and $a^2 = \mathbb{1} \otimes a$.

**Lemma 7.4.** (see [38, Lemma 1.7]) *Let $r(u; y_1, y_2)$ be a unitary solution of (3.2) of the form (3.5) and $s(u; y_1, y_2) = r(u; y_1, y_2)\, r(-u; y_1, y_2)$. Assuming that $r(u; y_1, y_2)$ has a simple pole along $y_1 = y_2$ with residue $cP$ for some $c \in \mathbb{C}$, we have*

$$s(u; y_1, y_2) = a \otimes 1 + 1 \otimes a + (f(u) + g(y_1, y_2))\, \mathbb{1} \otimes \mathbb{1}$$

*where $f(u) = f(-u)$, $g(y_1, y_2) = g(y_2, y_1)$ and $a \in \mathfrak{g}$ is an infinitesimal symmetry of $r(u; y_1, y_2)$. Moreover, we may write*

$$r_0(y_1, y_2) = \overline{r}_0(y_1, y_2) + \alpha(y_2) \otimes \mathbb{1} - \mathbb{1} \otimes \alpha(y_1) + h(y_1, y_2)\mathbb{1} \otimes \mathbb{1}$$

*with $\overline{r}_0(y_1, y_2)$ mapping to $\mathfrak{g} \otimes \mathfrak{g}$, $\alpha(y)$ to $\mathfrak{g}$, $h(y_1, y_2)$ a scalar function and*

$$\alpha(y) = \alpha(0) + \frac{y}{cn} a.$$

*Proof.* By assumption $r(u;y_1,y_2) = \frac{c}{y_1-y_2}P + \tilde{r}(u;y_1,y_2)$ where $\tilde{r}(u;y_1,y_2)$ does not have a pole along $y_1 = y_2$. Let us write $r(u;y_{ij})$ and $\tilde{r}(u;y_{ij})$ as short-hand for $r(u;y_i,y_j)$ and $\tilde{r}(u;y_i,y_j)$ respectively. Then starting from (3.3) we derive that for $v = -u+h$:

(7.3)
$$\begin{aligned} r^{13}(u;y_{13})\, r^{23}(-u+h;y_{23}) &= r^{23}(h;y_{23})\, r^{12}(u;y_{12}) - r^{12}(u-h;y_{12})\, r^{13}(h;y_{13}) = \\ &= [r^{23}(h;y_{23})\, r^{12}(u;y_{12}) - r^{12}(u;y_{12})\, r^{13}(h;y_{13})] + \\ &\quad + [r^{12}(u;y_{12}) - r^{12}(u-h;y_{12})]\ r^{13}(h;y_{13}). \end{aligned}$$

Let us rewrite the expression in the first bracket on the right-most side as

$$\left(r^{23}(h;y_{23})\, \frac{c}{y_1-y_2}P^{12} - \frac{c}{y_1-y_2}P^{12}\, r^{13}(h;y_{13})\right) +$$
$$+ r^{23}(h;y_{23})\, \tilde{r}^{12}(u;y_{12}) - \tilde{r}^{12}(u;y_{12})\, r^{13}(h;y_{13}).$$

Using Fact 5.1, we know that $P^{12}\, r^{13}(h;y_{13}) = r^{23}(h;y_{13})\, P^{12}$, hence the right-most side of (7.3) equals

$$\frac{r^{23}(h;y_{23}) - r^{23}(h;y_{13})}{y_1-y_2}\, cP^{12} + r^{23}(h;y_{23})\, \tilde{r}^{12}(u;y_{12}) -$$
$$-\tilde{r}^{12}(u;y_{12})\, r^{13}(h;y_{13}) + \left[\tilde{r}^{12}(u;y_{12}) - \tilde{r}^{12}(u-h;y_{12})\right]\ r^{13}(h;y_{13}).$$

Passing to the limit $y_2 \to y_1$, we see that

$$\begin{aligned} r^{13}(u;y_{13})\, r^{23}(-u+h;y_{13}) &= -\frac{\partial r^{23}}{\partial y_1}(h;y_{13})\ cP^{12} + r^{23}(h;y_{13})\, \tilde{r}^{12}(u;y_{11}) - \\ &- \tilde{r}^{12}(u;y_{11})\, r^{13}(h;y_{13}) + [\tilde{r}^{12}(u;y_{11}) - \tilde{r}^{12}(u-h;y_{11})]\ r^{13}(h;y_{13}). \end{aligned} \tag{7.4}$$

We want to apply the operator $\mu \otimes \mathrm{id} : A \otimes A \otimes A \to A \otimes A$ to this equation, where $\mu$ is the product in $A$. Observe that

$$(\mu \otimes \mathrm{id})\left(a^{13}b^{23}\right) = ab,\ (\mu \otimes \mathrm{id})\,(a^{23}b^{12} - b^{12}a^{13}) = 0$$

where $a, b \in A \otimes A$ and the notation is best explained by the example $a^{13} = a_1 \otimes \mathbb{1} \otimes a_2$ for $a = a_1 \otimes a_2$. Moreover, using that $\sum_{i,j} e_{ij} a e_{ji} = \mathrm{tr}(a)\mathbb{1}$ for any $a \in A$ clearly, we derive that for $\mathrm{tr}_1 = \mathrm{tr} \otimes \mathrm{id} : A \otimes A \to A$ we have

$$(\mu \otimes \mathrm{id})\,(a^{23}P^{12}) = \mathbb{1} \otimes \mathrm{tr}_1(a).$$

Hence applying $\mu \otimes \mathrm{id}$ to (7.4) yields

$$r(u;y_{13})\, r(-u+h;y_{13}) = -c \cdot\ \mathbb{1} \otimes \mathrm{tr}_1\left(\frac{\partial r}{\partial y_1}(h;y_{13})\right) +$$
$$+ (\mu \otimes \mathrm{id})\left(\left[\tilde{r}^{12}(u;y_{11}) - \tilde{r}^{12}(u-h;y_{11})\right]\ r^{13}(h;y_{13})\right).$$

Now, take the limit $h \to 0$. The left-hand side of yields $s(u; y_1, y_3)$. As for the right-hand side, we invoke our assumption on the existence of a certain Laurent expansion (3.5) to derive that

$$\lim_{h\to 0} \frac{\partial r}{\partial y_1}(h; y_{13}) = \frac{\partial r_0}{\partial y_1}(y_1, y_3).$$

Moreover

$$\lim_{h\to 0}\left(\left[\tilde{r}^{12}(u; y_{11}) - \tilde{r}^{12}(u-h; y_{11})\right] r^{13}(h; y_{13})\right) = \frac{\partial \tilde{r}^{12}}{\partial u}(u; y_{11})\left(\lim_{h\to 0} r^{13}(h; y_{13})\cdot h\right).$$

Again using (3.5), we see that the second factor of this last term is simply $\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$. Putting all this together, we end up with

$$s(u; y_1, y_3) = -c\cdot \mathbb{1}\otimes \mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1, y_3)\right) + \mu\left(\frac{\partial \tilde{r}}{\partial u}(u; y_1, y_1)\right)\otimes \mathbb{1}.$$

Hence we may write $s(u; y_1, y_2) = \mathbb{1}\otimes\beta(y_1, y_2) + \gamma(u, y_1)\otimes\mathbb{1}$. Note that $\beta(y_1, y_2) = \mathrm{pr}\,(\beta(y_1, y_2)) + \frac{\mathrm{tr}\,(\beta(y_1,y_2))}{n}\mathbb{1}$. Using the same trick for $\gamma(u, y_1)$, we may actually write

$$s(u; y_1, y_2) = a(u, y_1)\otimes\mathbb{1} + \mathbb{1}\otimes b(y_1, y_2) + (f(u, y_1) + g(y_1, y_2))\,\mathbb{1}\otimes\mathbb{1}$$

where now both

$$a(u, y_1) = \mathrm{pr}\,\mu\left(\frac{\partial \tilde{r}}{\partial u}(u; y_1, y_1)\right),\; b(y_1, y_2) = -c\cdot \mathrm{pr}\ \mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1, y_2)\right)$$

map to $\mathfrak{g}$. Note that unitarity of $r(u; y_1, y_2)$ implies that $s^{21}(-u; y_2, y_1) = s^{12}(u; y_1, y_2)$. Applying $\mathrm{pr}\otimes\mathbb{1}$ to this equation yields $a(u, y_1) = b(y_2, y_1)$. It follows that both $a$ and $b$ depend on the second variable only and actually coincide, hence

$$s(u; y_1, y_2) = a(y_1)\otimes\mathbb{1} + \mathbb{1}\otimes a(y_2) + (f(u, y_1) + g(y_1, y_2))\,\mathbb{1}\otimes\mathbb{1}.$$

In order to show the statement concerning the form of $s$, we have to prove that $a(y_1)$ is constant. To this end, we substitute the form of $s$ just calculated into the second equation of the equality stated in Lemma 7.3. We derive that

$$\begin{aligned} &[a^1(y_1) + a^3(y_3), r^{13}(u_{13}; y_1, y_3)] = r^{13}(u_{13}; y_1, y_3)\cdot \\ &\cdot (f(u_{32}, y_1) + f(u_{21}, y_2) - f(u_{12}, y_1) - f(u_{23}, y_2)). \end{aligned} \tag{7.5}$$

Let us focus on the left-hand side. This equals

$$\left[a^1(y_1) + a^3(y_3), \frac{cP^{13}}{y_1 - y_3}\right] + \left[a^1(y_1) + a^3(y_3), \tilde{r}^{13}(u_{13}; y_1, y_3)\right].$$

By Fact 5.1, we may rewrite the first summand as

$$\frac{a^1(y_1) - a^1(y_3)}{y_1 - y_3}cP^{13} + \frac{a^3(y_3) - a^3(y_1)}{y_1 - y_3}cP^{13}$$

thus the limit $y_3 \to y_1$ of the left-hand side of (7.5) is given by

$$\frac{d}{dy_1}\left(a^1(y_1) - a^3(y_1)\right) cP^{13} + \left[a^1(y_1) + a^3(y_1), \tilde{r}^{13}(u_{13}; y_1, y_1)\right].$$

In particular, the limit $y_3 \to y_1$ of the right-hand side of (7.5) exists as well. But $r^{13}(u_{13}; y_1, y_3)$ has a pole along $y_1 = y_3$ and the other factor on the right-hand side is independent of $y_3$. Hence we conclude that

$$f(u_{32}, y_1) + f(u_{21}, y_2) - f(u_{12}, y_1) - f(u_{23}, y_2) = 0.$$

In particular, the left-hand side of (7.5) equals zero. Focusing on the polar part yields

$$\left[a^1(y_1) + a^3(y_3), \frac{cP^{13}}{y_1 - y_3}\right] = 0$$

which, by the above, implies

$$\frac{d}{dy_1}\left(a^1(y_1) - a^3(y_1)\right) cP^{13} = 0.$$

But then $\frac{da}{dy}(y)$ must be zero, so that $a(y) = a \in \mathfrak{g}$ is constant. Therefore $s(u; y_1, y_2) = a \otimes \mathbb{1} + \mathbb{1} \otimes a + (f(u, y_1) + g(y_1, y_2))\, \mathbb{1} \otimes \mathbb{1}$. By (7.5) we also see that $a$ is an infinitesimal symmetry of $r(u; y_1, y_2)$.
Finally, we want to prove the statement concerning $r_0(y_1, y_2)$. Clearly, we may write

$$r_0(y_1, y_2) = \overline{r}_0(y_1, y_2) + \alpha(y_2, y_1) \otimes \mathbb{1} - \mathbb{1} \otimes \alpha(y_1, y_2) + h(y_1, y_2)\mathbb{1} \otimes \mathbb{1}$$

with $\alpha$ mapping to $\mathfrak{g}$. Note that in the discussion above we derived that

$$a = b(y_1, y_2) = -c \cdot \mathrm{pr}\ \mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1, y_2)\right).$$

Since both $\overline{r}_0$ and $\alpha$ map to $\mathfrak{g}$, so will their partial derivatives. This implies

$$a = -cn \cdot \mathrm{pr}\left(-\left(\frac{\partial}{\partial y_1}\right)\alpha(y_1, y_2) + \frac{\partial h}{\partial y_1}(y_1, y_2) \cdot \mathbb{1}\right).$$

Hence $\frac{\partial}{\partial y_1}\alpha(y_1, y_2) = \frac{a}{cn}$, which gives the formula for $\alpha$. In particular, $\alpha(y_1, y_2)$ does not depend on the second argument. This completes the proof of the formula for $r_0$. □

Before we finally prove Theorem 7.1, we need to state one more easy fact:

**Fact 7.5.** a) For any $x, y, \phi \in A$, $i \in \{1, 2\}$ and $\phi^1 = \phi \otimes \mathbb{1}$ respectively $\phi^2 = \mathbb{1} \otimes \phi$, we have

$$(\mathrm{pr} \otimes \mathrm{pr})\left[x \otimes y, \phi^i\right] = \left[(\mathrm{pr} \otimes \mathrm{pr})\,(x \otimes y)\,, \phi^i\right].$$

b) Let $r$ be a non-degenerate solution of (3.2), $a \in \mathfrak{g}$ and $[r, 1 \otimes a] = 0$. Then $a = 0$.

*Proof.* a) is straightforward. As to b), write $r = \sum_{i\in I} r_i' \otimes r_i''$ for some index set $I$ and let $\varphi : A \otimes A \to \mathrm{End}(A)$ denote the isomorphism given by $X \otimes Y \mapsto (Z \mapsto X\,\mathrm{tr}\,(YZ))$. Then $0 = \varphi\left([r, 1\otimes a]\right)(b) = \sum_{i\in I} r_i' \,\mathrm{tr}\left([r_i'', a]\, b\right) = \sum_{i\in I} r_i' \,\mathrm{tr}\left(r_i''\,[a,b]\right) = \varphi(r)\left([a,b]\right)$ for all $b \in A$. Now $r$ is non-degenerate, hence $\varphi(r)$ is an isomorphism. This yields $[a, b] = 0$ for all $b \in A$, especially all $b \in \mathfrak{g}$. But the Lie bracket is non-degenerate on $\mathfrak{g}$, hence we derive that $a = 0$. □

*Proof of Theorem 7.1.*

(1) By Lemma 3.7 $\bar{r}_0$ is a unitary solution of the CYBE (2.1). The rest is immediate by Lemma 5.3 and the fact that $(\mathrm{pr}\otimes\mathrm{pr})\,(P)$ is the Casimir element of $\mathfrak{g}\otimes\mathfrak{g}$ with respect to the trace form $(x, y) \mapsto \mathrm{tr}(x \cdot y)$.

(2) Setting $u_1 = u$, $u_2 = 0$ and $u_3 = -u$ in Lemma 7.3, we derive that $r(u; y_1, y_2)$ satisfies the QYBE (4.1) for $u$ fixed if and only if

$$s^{23}(u; y_2, y_3)\, r^{13}(2u; y_1, y_3) = r^{13}(2u; y_1, y_3)\, s^{23}(-u; y_2, y_3).$$

Applying Lemma 7.4 this is equivalent to

$$[r(u; y_1, y_2), 1 \otimes a] = 0 \tag{7.6}$$

which, by Fact 7.5 b) is equivalent to $a = 0$. By Lemma 7.4 this last condition holds if and only if either of conditions (b) or (c) of the Theorem are satisfied. It remains to show equivalence with condition (d). To this end, recall (6.1). Denote the right-hand side of this equation by $AYBE[r_0](y_1, y_2, y_3)$. Then (d) simply reads $(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\, AYBE[\overline{r}_0](y_1, y_2, y_3) = 0$. To show the equivalence of this with $a = 0$, express $r_0$ in terms of $\overline{r}_0$ as stated in Lemma 7.4. Then

$$-cn \cdot (\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\left(AYBE\,[r_0]\,(y_1, y_2, y_3) - AYBE\,[\overline{r}_0]\,(y_1, y_2, y_3)\right) =$$
$$= (y_1 - y_3)\,\overline{r}_0^{13}(y_1, y_3)\, a^2 + (y_1 - y_2)\,\overline{r}_0^{12}(y_1, y_2)\, a^3 + (y_3 - y_2)\,\overline{r}_0^{23}(y_2, y_3)\, a^1.$$

It is immediate by (6.1) that $(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\, AYBE\,[r_0]\,(y_1, y_2, y_3) = 0$. Hence if $a$ is zero, this implies $(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\, AYBE[\overline{r}_0] = 0$. On the other hand, assuming $(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\, AYBE[\overline{r}_0] = 0$ we deduce

$$(y_1 - y_3)\,\overline{r}_0^{13}(y_1, y_3)\, a^2 + (y_1 - y_2)\,\overline{r}_0^{12}(y_1, y_2)\, a^3 + (y_3 - y_2)\,\overline{r}_0^{23}(y_2, y_3)\, a^1 = 0.$$

We will show that this implies $a = 0$. Indeed, by Lemma 5.3 we know that $r_0(y_1, y_2) = \frac{cP}{y_1 - y_2} + \tilde{r}_0(y_1, y_2)$ with $\tilde{r}_0$ being defined along $y_1 = y_2$ and similarly for $\overline{r}_0$. Hence passing to the limit $y_1, y_2, y_3 \to y$ yields

$$(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\left[P^{13}a^2 + P^{12}a^3 + P^{23}a^1\right] = 0.$$

Let us write $a = \sum a_{ij} e_{ij}$. Looking at the coefficient of $e_{ij}\otimes e_{ji}\otimes e_{ij}$ in the above equation for $i \neq j$, we derive $a_{ij} = 0$. But then projecting the above equation to $e_{12}\otimes e_{21}\otimes \mathfrak{g}$, we may deduce that $a = 0$.

(3) As we just saw, the conditions of (2) are satisfied if $a = 0$. But $a$ is an infinitesimal symmetry of $r$ by Lemma 7.4, hence one of $r_0$. Invoking Fact 7.5 a), we deduce that $a$ is an infinitesimal symmetry of $\overline{r}_0$ and so $a = 0$. $\square$

## Part 2. Triple Massey products and the Yang-Baxter equations

Let $E$ be a reduced projective curve with trivial dualizing sheaf. Recalling the work of Polishchuk [37] respectively Burban and Kreußler [14], we show how triple Massey products in $D^b(E)$ can be used in order to construct solutions of the associative Yang-Baxter equation, see Section 8. In [13], we show how to obtain the corresponding solutions of the CYBE via similar methods directly. Our results are contained in Section 9.

We fix the following notations:

- $\Bbbk$ denotes an algebraically closed field of characteristic zero.
- Given an algebraic variety $X$, $\mathrm{Coh}(X)$ respectively $\mathrm{VB}(X)$ denotes the category of coherent sheaves respectively vector bundles on $X$.
- By $D^b_{\mathrm{Coh}}(E)$ we denote the triangulated category of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology, whereas $\mathrm{Perf}(E)$ stands for the triangulated category of perfect complexes i.e. the full subcategory of $D^b_{\mathrm{Coh}}(E)$ admitting a bounded locally free resolution.
- We always write Hom and Ext when working with coherent sheaves whereas Lin is used when we work with vector spaces. If not explicitly otherwise stated, Ext always stands for $\mathrm{Ext}^1$.
- For a vector bundle $\mathcal{F}$ on $X$ and $x \in X$ we denote by $\mathcal{F}\big|_x$ the fiber of $\mathcal{F}$ over $x$, whereas $\Bbbk_x$ denotes the skyscraper sheaf of length one supported at $x$.
- A *Weierstraß cubic curve* is a plane cubic curve given in homogeneous coordinates by an equation $zy^2 = 4x^3 + g_2xz^2 + g_3z^3$, where $g_1, g_2 \in \Bbbk$. Such a curve is always irreducible. It is singular if and only if $\Delta(g_2, g_3) = g_2^3 + 27g_3^2 = 0$. Unless $g_2 = g_3 = 0$, the singularity is a node (ordinary double point), whereas in the case $g_2 = g_3 = 0$ the singularity is a cusp.
- A *Calabi-Yau curve* is a reduced projective Gorenstein curve with trivial dualizing sheaf. Note that the complete list of such curves is actually known, see for example [41, Section 3]: $E$ is either
  1. an elliptic curve,
  2. a Kodaira cycle of $n \geq 1$ projective lines (for $n = 1$ it is a nodal Weierstraß curve), also called Kodaira fiber of type $\mathrm{I}_n$,
  3. a cuspidal plane cubic curve (Kodaira fiber II), a tachnode cubic curve (Kodaira fiber III) or a generic configuration of $n$ concurrent lines in $\mathbb{P}^{n-1}$ for $n \geq 3$.

  The irreducible Calabi-Yau curves are precisely the Weierstraß curves.
- Next, $\Omega$ will denote the sheaf of regular differential one forms on a Calabi-Yau curve $E$, which we always view as a dualizing sheaf. Taking a non-zero section $w \in H^0(\Omega)$, we get an isomorphism of $\mathcal{O}$-modules $\mathcal{O} \xrightarrow{w} \Omega$.

- Let $\bar{I}$ be an irreducible component of $E$ and $I = \bar{I} \setminus E_{\mathsf{sing}}$. We take a pair of distinct points $x, y \in I$.
- Finally, $\mathcal{P}$ is a simple vector bundle on $E$, i.e. a locally free coherent sheaf satisfying $\mathrm{End}(\mathcal{P}) = \Bbbk$. Note that we automatically have: $\mathrm{Ext}(\mathcal{P}, \mathcal{P}) \cong \Bbbk$.

## 8. Triple Massey products and the AYBE

8.1. **Algebraic Triple Massey products.** Let $E$ be a reduced projective curve over $\mathbb{C}$ with trivial dualizing sheaf $\Omega \cong \mathcal{O}$. Assume we are given two vector bundles $\mathcal{F}_1 \not\cong \mathcal{F}_2$ such that

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0 = \mathrm{Ext}(\mathcal{F}_1, \mathcal{F}_2).$$

Then for any $y_1, y_2 \in E$, $y_1 \neq y_2$, the following linear map

$$(8.1)\quad m_3 = m_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2} : \mathrm{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathrm{Ext}(\mathbb{C}_{y_1}, \mathcal{F}_2) \otimes \mathrm{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \longrightarrow \mathrm{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2}),$$

called *triple Massey product*, is defined as follows. Let

$$a \in \mathrm{Ext}(\mathbb{C}_{y_1}, \mathcal{F}_2),\ g \in \mathrm{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}),\ f \in \mathrm{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2})$$

and $0 \to \mathcal{F}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \to 0$ be an extension representing the element $a$. The vanishing of $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ and $\mathrm{Ext}(\mathcal{F}_1, \mathcal{F}_2)$ implies that we can uniquely lift the morphisms $g$ and $f$ to morphisms $\tilde{g} : \mathcal{F}_1 \to \mathcal{A}$ and $\tilde{f} : \mathcal{A} \to \mathbb{C}_{y_2}$ such that $\beta\tilde{g} = g$ and $\tilde{f}\alpha = f$. So, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & \mathcal{F}_1 & & \\
 & & & & & \swarrow^{\tilde{g}} & \downarrow^{g} & & \\
a:\ 0 & \longrightarrow & \mathcal{F}_2 & \xrightarrow{\alpha} & \mathcal{A} & \xrightarrow{\beta} & \mathbb{C}_{y_1} & \longrightarrow & 0 \\
 & & \downarrow^{f} & \swarrow_{\tilde{f}} & & & & & \\
 & & \mathbb{C}_{y_2} & & & & & &
\end{array}$$

and the triple Massey product is defined as $m_3(g \otimes a \otimes f) = \tilde{f}\tilde{g}$.

For any pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Perf}(E)$ we have the bilinear form

$$(8.2)\qquad \langle - \,,\, - \rangle = \langle - \,,\, - \rangle^{w}_{\mathcal{F},\mathcal{G}} \ :\ \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}(\mathcal{G}, \mathcal{F}) \longrightarrow \Bbbk$$

defined as the composition

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}(\mathcal{G}, \mathcal{F}) \xrightarrow{\circ} \mathrm{Ext}(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathrm{Tr}_{\mathcal{F}}} H^1(\mathcal{O}) \xrightarrow{w} H^1(\Omega) \xrightarrow{t} \Bbbk,$$

where $\circ$ denotes the composition operation, $\mathrm{Tr}_{\mathcal{F}}$ is the trace map and $t$ is the canonical morphism described in [14, Subsection 4.3]. The following result is well-known, see for example [14, Corollary 3.3] for a proof.

**Theorem 8.1.** *For any $\mathcal{F}, \mathcal{G} \in \mathrm{Perf}(E)$ the pairing $\langle - \,,\, - \rangle^{w}_{\mathcal{F},\mathcal{G}}$ is non-degenerate. In particular, we have an isomorphism of vector spaces*

$$(8.3)\qquad \mathbb{S} = \mathbb{S}_{\mathcal{F},\mathcal{G}} : \quad \mathrm{Ext}(\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G})^*,$$

*which is functorial in both arguments.*

Now let

$$(8.4)\qquad \widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2} : \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}) \longrightarrow \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_2})$$

be the image of $m^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ under the isomorphism of vector spaces

$$\mathrm{Lin}\big(\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1}) \otimes \mathrm{Ext}(\mathbb{C}_{y_1},\mathcal{F}_2) \otimes \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}), \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_2})\big) \cong$$

$$\mathrm{Lin}\big(\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}), \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_2})\big).$$

The following result is due to Polishchuk [37] respectively Burban and Kreußler [14].

**Theorem 8.2.** *Let $E$ be a Weierstrass cubic curve. The linear map $\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ satisfies the following "triangle equation"*

$$(8.5)\qquad (\widetilde{m}^{\mathcal{F}_3,\mathcal{F}_2}_{y_1,y_2})^{12}(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_3}_{y_1,y_3})^{13} - (\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_3}_{y_2,y_3})^{23}(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2})^{12} + (\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_3})^{13}(\widetilde{m}^{\mathcal{F}_2,\mathcal{F}_3}_{y_2,y_3})^{23} = 0.$$

*Here both sides of the equality (8.5) are viewed as linear maps*

$$\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}) \otimes \mathrm{Hom}(\mathcal{F}_3,\mathbb{C}_{y_3}) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_3,\mathbb{C}_{y_2}) \otimes \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_3}).$$

*Moreover, the tensor $\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ is non-degenerate and skew-symmetric:*

$$(8.6)\qquad \rho(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}) = -\widetilde{m}^{\mathcal{F}_2,\mathcal{F}_1}_{y_2,y_1},$$

*where $\rho$ is the isomorphism*

$$\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}) \longrightarrow \mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}) \otimes \mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1})$$

*given by the rule $\rho(f \otimes g) = g \otimes f$.*

*Proof.* Let us briefly sketch the main ideas of the proof. Since $E$ is Gorenstein, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Perf}(E) & \xrightarrow{\cong} & \mathrm{Hot}^b_{\mathrm{Coh}}\,(\mathrm{Inj}\,(E)) \\ \big\downarrow & & \big\downarrow \\ D^b_{\mathrm{Coh}}(E) & \xrightarrow{\cong} & \mathrm{Hot}^{+,b}_{\mathrm{Coh}}\,(\mathrm{Inj}\,(E))\,. \end{array}$$

Here $\mathrm{Hot}_{\mathrm{Coh}}\,(\mathrm{Inj}\,(E))$ denotes the subcategory of the homotopy category $\mathrm{Hot}\,(\mathrm{Inj}\,(E))$ whose objects are complexes such that all cohomologies are coherent sheaves on $E$. Note that for $B = \mathrm{Com}^{+,b}_{\mathrm{Coh}}\,(\mathrm{Inj}(E))$, we have

$$\mathrm{Hot}^{+,b}_{\mathrm{Coh}}\,(\mathrm{Inj}\,(E)) = \mathrm{H}^*(B).$$

That is, $\mathrm{Hot}^{+,b}_{\mathrm{Coh}}\,(\mathrm{Inj}\,(E))$ is the homology category of the differential graded category $B$. Hence homological perturbation theory implies that $\mathrm{Perf}(E)$ is an $A_\infty$-category [29].

As part of the $A_\infty$-structure, there exists a collection of higher products $\{m_n^\infty\}_{n\geq 2}$ on Perf$(E)$, where for any given objects $X_0$,$X_1$,...,$X_n$ of Perf$(E)$, we have

$$\begin{array}{c}\operatorname{Ext}^{i_1}(X_0, X_1) \otimes \operatorname{Ext}^{i_2}(X_1, X_2) \otimes \ldots \otimes \operatorname{Ext}^{i_n}(X_{n-1}, X_n) \\ \big\downarrow m_n^\infty \\ \operatorname{Ext}^{i_1+\ldots+i_n-(n-2)}(X_0, X_n).\end{array}$$

Moreover, these higher products satisfy the $A_\infty$-relations. That is, for each $n \geq 2$ we have

$$\sum (-1)^{r+st} m_{r+t+1}^\infty \left(\mathbb{1}^{\otimes r} \otimes m_s^\infty \otimes \mathbb{1}^{\otimes t}\right) = 0, \tag{8.7}$$

where the sum is indexed by all decompositions $n = r + s + t$ and where we adopt the sign convention of [24]. For example, taking $n = 5$, we derive

$$\begin{aligned} 0 = m_3^\infty\left(m_3^\infty \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3^\infty \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3^\infty\right) + \\ + m_4^\infty\left(m_2^\infty \otimes \mathbb{1}^{\otimes 3} - \mathbb{1} \otimes m_2^\infty \otimes \mathbb{1}^{\otimes 2} + \mathbb{1}^{\otimes 2} \otimes m_2^\infty \otimes \mathbb{1} - \mathbb{1}^{\otimes 3} \otimes m_2^\infty\right) + \\ + m_2^\infty\left(m_4^\infty \otimes \mathbb{1} - \mathbb{1} \otimes m_4^\infty\right). \end{aligned} \tag{8.8}$$

It is known that $m_2^\infty$ is just the usual composition of morphisms, but the collection $\{m_n^\infty\}_{n\geq 3}$ is not uniquely determined. However, it can be shown, see [34, 37], that $m_3 = m_3^\infty$ where $m_3$ is the triple Massey product in the sense of triangulated categories as defined in (8.1). Next, consider both sides of equation (8.8) as a linear operator mapping the tensor product

$$\mathfrak{H} = \operatorname{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \operatorname{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2) \otimes \operatorname{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \otimes \operatorname{Ext}^1(\mathbb{C}_{y_2}, \mathcal{F}_3) \otimes \operatorname{Hom}(\mathcal{F}_3, \mathbb{C}_{y_3})$$

to the vector space $\operatorname{Hom}(\mathcal{F}_1, \mathbb{C}_{y_3})$. Using the vanishing of the Hom- and Ext-spaces between the $\mathcal{F}_i$ respectively $\mathbb{C}_{y_i}$ involved, equation (8.8) implies

$$m_3 \circ (m_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3) = 0. \tag{8.9}$$

For the case where $E$ is an elliptic curve, the last crucial step of Polishchuk's proof [37, Theorem 1] goes as follows. The equality (8.6) follows from existence of an $A_\infty$-structure on Perf$(E)$ which is *cyclic* with respect to the pairing (8.2). In particular, this means that for any $a \otimes \alpha \otimes b \otimes \beta \in \mathfrak{H}$ we have

$$\langle m_3(a, \alpha, b), \beta\rangle = -\langle a, m_3(\alpha, b, \beta)\rangle = -\langle m_3(b, \beta, a), \alpha\rangle. \tag{8.10}$$

A proof of the existence of such an $A_\infty$-structure has been outlined by Polishchuk in [36, Theorem 1.1], see also [33, Theorem 10.2.2] for a different approach using non-commutative symplectic geometry. From this, both (8.5) and (8.6) can be derived.

For the case where $E$ is a cuspidal or nodal cubic curve, we refer to Burban and Kreußler [14, Section 6]. The crucial idea behind their proof is to use the continuity of Massey products. □

*Remark* 8.3. So far, there is no complete proof that one can always find an $A_\infty$-structure on $\mathrm{Perf}(E)$ for a singular Calabi-Yau curve $E$, which is cyclic with respect to the pairing (8.2). For deeper insights into the theory of dg- and $A_\infty$-algebras respectively categories, we recommend [32, 31, 30].

8.2. **Geometric Massey Products.** Our next aim is to reinterpret the map $\tilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ in another way which is more suitable for explicit computations. We have an isomorphism $\mathcal{O} \cong \Omega$ given by a nowhere vanishing differential form, e.g. by $\omega = dz$. For any $x \in E$ consider the canonical short exact sequence

$$0 \to \Omega \to \Omega(x) \xrightarrow{\mathsf{res}_x} \mathbb{C}_x \to 0. \tag{8.11}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on $E$. We identify the line bundles $\Omega$ and $\mathcal{O}$ using the differential form $\omega$, tensor the sequence (8.11) with $\mathcal{G}$ and then apply the functor $\mathrm{Hom}(\mathcal{F}, -)$. As a result, we obtain a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathrm{Hom}\big(\mathcal{F},\mathcal{G}\otimes\mathbb{C}_x\big) \longrightarrow \mathrm{Ext}(\mathcal{F},\mathcal{G}). \tag{8.12}$$

**Definition 8.4.** The linear map $\mathrm{res}^{\mathcal{F},\mathcal{G}}_x(\omega) : \mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \to \mathrm{Lin}\big(\mathcal{F}\big|_x,\ \mathcal{G}\big|_x\big)$ is the composition of the following canonical morphisms

$$\mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathrm{Hom}\big(\mathcal{F},\mathcal{G}\otimes\mathbb{C}_x\big) \longrightarrow \mathrm{Lin}\big(\mathcal{F}\big|_x,\ \mathcal{G}\big|_x\big),$$

where the first map comes from the long exact sequence (8.12) and the second one is a canonical isomorphism. The morphism $\mathrm{res}^{\mathcal{F},\mathcal{G}}_x(\omega)$ is called *residue map.*

The following lemma is a straightforward corollary of the definition and the long exact sequence (8.12).

**Lemma 8.5.** *Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on $E$ such that* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0 = \mathrm{Ext}(\mathcal{F},\mathcal{G})$*. Then for any $x \in E$ the residue map $\mathrm{res}^{\mathcal{F},\mathcal{G}}_x(\omega)$ is an isomorphism.*

**Definition 8.6.** Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on $E$ and $x, y \in E$ be a pair of *distinct* points. Then the linear map $\mathrm{ev}^{\mathcal{F},\mathcal{G}(x)}_y$ defined as the composition of the following canonical morphisms

$$\mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathrm{Hom}\big(\mathcal{F}\otimes\mathbb{C}_y,\mathcal{G}(x)\otimes\mathbb{C}_y\big) \xrightarrow{\cong} \mathrm{Lin}\big(\mathcal{F}\big|_y,\ \mathcal{G}\big|_y\big)$$

is called *evaluation map.*

**Lemma 8.7.** *Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on $E$ and $x, y \in E_{\mathrm{reg}}$ be a pair of distinct points such that* $\mathrm{Hom}\big(\mathcal{F}(y),\mathcal{G}(x)\big) = 0 = \mathrm{Ext}\big(\mathcal{F}(y),\mathcal{G}(x)\big)$*. Then the evaluation map $\mathrm{ev}^{\mathcal{F},\mathcal{G}(x)}_y$ is an isomorphism.*

*Proof.* Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \xrightarrow{\mathrm{ev}_y} \mathbb{C}_y \longrightarrow 0. \tag{8.13}$$

It induces a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{G}(x-y) \longrightarrow \mathcal{G}(x) \xrightarrow{\mathbb{1}\otimes \underline{\mathrm{ev}}_y} \mathcal{G}(x)\otimes \mathbb{C}_y \longrightarrow 0\ .$$

Using the vanishing $\mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x-y)\big) = 0 = \mathrm{Ext}\big(\mathcal{F},\mathcal{G}(x-y)\big)$, we get an isomorphism $\mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \to \mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\otimes\mathbb{C}_y\big)$. It remains to observe that $\mathrm{ev}_y^{\mathcal{F},\,\mathcal{G}(x)}$ is the composition of the following canonical isomorphisms:

$$\mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathrm{Hom}\big(\mathcal{F},\mathcal{G}(x)\otimes\mathbb{C}_y\big) \longrightarrow \mathrm{Hom}\big(\mathcal{F},\mathcal{G}\otimes\mathbb{C}_y\big) \longrightarrow \mathrm{Lin}\big(\mathcal{F}\big|_y,\,\mathcal{G}\big|_y\big).$$

□

For $\mathcal{F}_1,\mathcal{F}_2$ and $y_1,y_2$ as at the beginning of this section, consider the linear map

$$\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2} : \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_1},\mathcal{F}_2\big|_{y_1}\big) \longrightarrow \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2},\mathcal{F}_2\big|_{y_2}\big)$$

defined by the following commutative diagram of vector spaces:

(8.14)
$$\begin{array}{ccccc} & & \mathrm{Hom}\big(\mathcal{F}_1,\mathcal{F}_2(y_1)\big) & & \\ & \swarrow^{\mathrm{res}_{y_1}^{\mathcal{F}_1,\,\mathcal{F}_2}(\omega)}_{\cong} & & \searrow^{\mathrm{ev}_{y_2}^{\mathcal{F}_1,\,\mathcal{F}_2(y_1)}}_{\cong} & \\ \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_1},\mathcal{F}_2\big|_{y_1}\big) & & \xrightarrow{\quad\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}\quad} & & \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2},\mathcal{F}_2\big|_{y_2}\big). \end{array}$$

The next Theorem relates the map just defined with $\widetilde{m}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$. As we will see in the Part 4, this result is crucial for the development of concrete algorithms suitable for explicit computations.

**Theorem 8.8.** [14, Theorem 4.17] *$\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$ is the image of $\widetilde{m}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$ under the canonical isomorphism of vector spaces*

$$\mathrm{Lin}\big(\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_1})\otimes\mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_2}),\mathrm{Hom}(\mathcal{F}_2,\mathbb{C}_{y_1})\otimes\mathrm{Hom}(\mathcal{F}_1,\mathbb{C}_{y_2})\big) \cong$$
$$\mathrm{Lin}\Big(\mathrm{Lin}\big(\mathcal{F}_1\big|_{y_1},\mathcal{F}_2\big|_{y_1}\big),\ \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2},\mathcal{F}_2\big|_{y_2}\big)\Big).$$

## 9. Triple Massey products and the CYBE

In this section we prove the following theorem.

**Theorem 9.1.** *Let $E = V(wv^2 - 4u^3 - g_2uw^2 - g_3w^3) \subset \mathbb{P}^2$ be a Weierstrass cubic curve over $\mathbb{C}$, $o \in E$ some fixed smooth point and $0 < d < n$ a pair of coprime integers. Consider the sheaf of Lie algebras $\mathcal{A} = \mathrm{Ad}(\mathcal{P})$, where $\mathcal{P}$ is a simple vector bundle of rank $n$ and degree $d$ on $E$ (note that up to automorphism, $\mathcal{A}$ does not depend on a particular choice of $\mathcal{P}$). For any pair of distinct smooth points $x,y$ of $E$, consider the map $\mathcal{A}\big|_x \longrightarrow \mathcal{A}\big|_y$ defined as follows:*

(9.1)
$$\mathcal{A}\big|_x \xrightarrow{\mathsf{res}_x^{-1}} H^0\big(\mathcal{A}(x)\big) \xrightarrow{\mathsf{ev}_y} \mathcal{A}\big|_y,$$

*where* $\mathsf{res}_x$ *is the* residue map *and* $\mathsf{ev}_y$ *is the* evaluation map*. Choosing some isomorphism of Lie algebras* $\xi : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ *for some small neighborhood* $U$ *of* $o$*, we get the tensor* $r^{\xi}_{(E,n,d)}(x, y) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$*. Then we have:*

(1) *The tensor* $r^{\xi}_{(E,n,d)}$ *is a non-degenerate unitary solution of the classical Yang-Baxter equation (2.1).*
(2) *Moreover,* $r^{\xi}_{(E,n,d)}$ *is* analytic *with respect to the parameters* $g_2$ *and* $g_3$*.*
(3) *A different choice of trivialization* $\zeta : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ *gives a gauge equivalent solution* $r^{\zeta}_{(E,n,d)}$*.*

9.1. **Preliminaries from linear algebra.** In this short subsection we collect some basic results from linear algebra which will be used in what follows.

For a finite dimensional vector space $V$ over $\Bbbk$ we denote by $\mathfrak{sl}(V)$ the Lie subalgebra of $\mathrm{End}(V)$ consisting of endomorphisms with zero trace and $\mathfrak{pgl}(V) := \mathrm{End}(V)/\langle \mathbb{1}_V \rangle$. Since the proofs of all statements from this subsection are completely standard and elementary, they are left to the reader as an exercise.

**Lemma 9.2.** *The non-degenerate bilinear pairing* $\mathrm{tr} : \mathrm{End}(V) \times \mathrm{End}(V) \longrightarrow \Bbbk$, $(f, g) \mapsto \mathrm{tr}(fg)$ *induces another non-degenerate pairing* $\mathrm{tr} : \mathfrak{sl}(V) \times \mathfrak{pgl}(V) \longrightarrow \Bbbk$, $(f, \bar{g}) \mapsto \mathrm{tr}(fg)$*. In particular, for any finite dimensional vector space* $U$ *we get a canonical isomorphism of vector spaces*

$$\mathfrak{pgl}(U) \otimes \mathfrak{pgl}(V) \longrightarrow \mathrm{Lin}\big(\mathfrak{sl}(U), \mathfrak{pgl}(V)\big). \tag{9.2}$$

**Lemma 9.3.** *The* Yoneda map $Y : \mathrm{End}(V) \longrightarrow \mathrm{End}(V^*)$*, assigning to an endomorphism* $f$ *its adjoint* $f^*$*, induces an anti-isomorphisms of Lie algebras*

(1) $Y_1 : \mathfrak{sl}(V) \longrightarrow \mathfrak{sl}(V^*)$ *and*
(2) $Y_2 : \mathfrak{sl}(V) \longrightarrow \mathfrak{pgl}(V^*)$*,* $f \mapsto \bar{f}^*$*, where* $\bar{f}^*$ *is the equivalence class of* $f^*$*.*

Note that the fist part of the statement is valid for any field $\Bbbk$, whereas the second one is only true if $\mathsf{dim}_{\Bbbk}(V)$ is invertible in $\Bbbk$.

**Lemma 9.4.** *Let* $H \subseteq V$ *be a linear subspace. Then we have the canonical linear map* $r_H : \mathrm{End}(V) \longrightarrow \mathrm{Lin}(H, V/H)$ *sending an endomorphism* $f$ *to the composition* $H \longrightarrow V \xrightarrow{f} V \longrightarrow V/H$*. Moreover, the following results are true.*

(1) *We have:* $r_H(\mathbb{1}_V) = 0$*. In particular, there is an induced canonical map* $\bar{r}_H : \mathfrak{pgl}(V) \longrightarrow \mathrm{Lin}(H, V/H)$*.*
(2) *Let* $f \in \mathrm{End}(V)$ *be such that for any one-dimensional subspace* $H \subseteq V$ *we have:* $r_H(f) = 0$*. Then* $\bar{f} = 0$ *in* $\mathfrak{pgl}(V)$*.*

(3) *Let $U$ be a finite dimensional vector space and $g_1, g_2 : U \longrightarrow \mathfrak{pgl}(V)$ be two linear maps such that for any one-dimensional subspace $H \subseteq V$ we have: $\bar{r}_H \circ g_1 = \bar{r}_H \circ g_2$. Then $g_1 = g_2$.*

9.2. **Triple Massey products revisited.** Let $E$, $\mathcal{P}$ and $x, y$ be as at the beginning of this part. Consider the following vector space

$$K := \mathsf{Ker}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x) \otimes \mathrm{Ext}(\Bbbk_x, \mathcal{P}) \xrightarrow{\circ} \mathrm{Ext}(\mathcal{P}, \mathcal{P}) \cong \Bbbk\big). \tag{9.3}$$

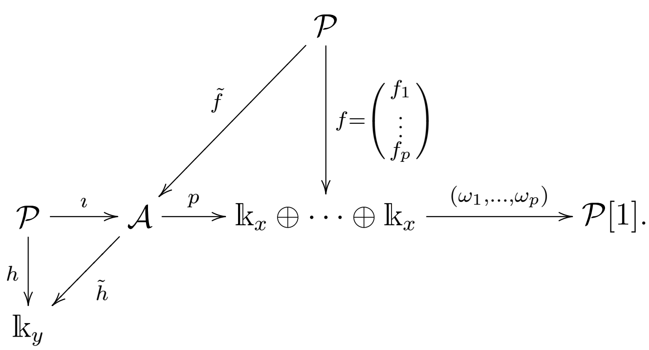
Let $H \subseteq \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$ be a one-dimensional linear subspace.

**Definition 9.5.** The triple Massey product is the linear map

$$M_H : K \longrightarrow \mathrm{Lin}\big(H, \mathrm{Hom}(\mathcal{P}, \Bbbk_y)/H\big) \tag{9.4}$$

defined as follows. Let $t = \sum_{i=1}^{p} f_i \otimes \omega_i \in K$ and $h \in H$. Consider the following commutative diagram in the triangulated category $\mathrm{Perf}(E)$:

(9.5)




The horizontal sequence is a distinguished triangle in $\mathrm{Perf}(E)$ determined by the morphism $(\omega_1, \dots, \omega_p)$. Since $\sum_{i=1}^{p} \omega_i f_i = 0$ in $\mathrm{Ext}(\mathcal{P}, \mathcal{P})$, there exists a morphism $\tilde{f} : \mathcal{P} \longrightarrow \mathcal{A}$ such that $p\tilde{f} = f$. Note that such a morphism is only defined up to a translation $\tilde{f} \mapsto \tilde{f} + \lambda\imath$ for some $\lambda \in \Bbbk$. Since $\mathrm{Hom}(\Bbbk_x, \Bbbk_y) = 0 = \mathrm{Ext}(\Bbbk_x, \Bbbk_y)$, there exists a unique morphism $\tilde{h} : \mathcal{A} \longrightarrow \Bbbk_y$ such that $\tilde{h}\imath = h$. We set:

$$\big(M_H(t)\big)(h) := \overline{\tilde{h}\tilde{f}}. \tag{9.6}$$

It is well-known that $M_H$ is well-defined, i.e. it is independent of a presentation of $t \in K$ as a sum of simple tensors and a choice of the horizontal distinguished triangle, and that $M_H$ is a linear map, see for instance [22, Exercise IV.2.3].
Next, note the following easy fact.

**Lemma 9.6.** *Let $K$ be as in (9.3). Then the linear isomorphism*

$$\overline{\mathbb{S}} : \mathrm{Hom}(\mathcal{P}, \Bbbk_x) \otimes \mathrm{Ext}(\Bbbk_x, \mathcal{P}) \xrightarrow{\mathbb{1} \otimes \mathbb{S}} \mathrm{Hom}(\mathcal{P}, \Bbbk_x) \otimes \mathrm{Hom}(\mathcal{P}, \Bbbk_x)^* \xrightarrow{\mathsf{ev}} \mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big)$$

*identifies the vector space $K$ from (9.3) with $\mathfrak{sl}\big(Hom(\mathcal{P}, \Bbbk_x)\big)$.*

Consider the linear map

$$m = m_3^{\infty} \ : \ \mathrm{Hom}(\mathcal{P}, \Bbbk_x) \otimes \mathrm{Ext}(\Bbbk_x, \mathcal{P}) \otimes \mathrm{Hom}(\mathcal{P}, \Bbbk_y) \longrightarrow \mathrm{Hom}(\mathcal{P}, \Bbbk_y).$$

It induces another linear map $K \longrightarrow \mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big)$ assigning to an element $t \in K$ the functional $g \mapsto m(t \otimes g)$. Taking the composition of this map with the canonical projection $\mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \longrightarrow \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big)$ and identifying $K$ with $\mathfrak{sl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big)$ as above, we get the linear map

$$\overline{m}_{x,y} : \ \mathfrak{sl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big) \longrightarrow \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big). \tag{9.7}$$

Finally, applying Lemma 9.2, we end up with the tensor

$$m_{x,y} \in \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big) \otimes \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big). \tag{9.8}$$

**Proposition 9.7.** *The tensor $m_{x,y}$ does not depend on a particular choice of an $A_\infty$-structure on* Perf$(E)$.

*Proof.* First note that for any choice of an $A_\infty$-structure on Perf$(E)$ and any one-dimensional linear subspace $H \subseteq \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$, the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\quad \widetilde{m}_{x,y} \quad} & \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \\ & {\scriptstyle M_H} \searrow \qquad \swarrow {\scriptstyle \bar{r}_H} & \\ & \mathrm{Lin}\big(H, \mathrm{Hom}(\mathcal{P}, \Bbbk_y)/H\big) & \end{array} \tag{9.9}$$

is commutative. Here, $M_H$ is the triple Massey product (9.4), $\widetilde{m}_{x,y} = m_{x,y} \circ \overline{\mathbb{S}}$ and $\bar{r}_H$ is the canonical linear map from Lemma 9.4. A proof of this statement can for instance be found in [34]. Let $\{m'_n\}_{n \geq 3}$ be another $A_\infty$-structure on Perf$(E)$. From the last part of Lemma 9.4 it follows that $\overline{m}_{x,y} = \overline{m'}_{x,y}$. This implies the claim. □

The following result is due to Polishchuk, see [37, Theorem 2].

**Theorem 9.8.** *Let $E$ be an elliptic curve, $\mathcal{P}$ be a simple vector bundle on $E$ and $x_1, x_2, x_3 \in E$ be pairwise distinct. Then we have the following equality*

$$\big[m^{12}_{x_1,x_2}, m^{13}_{x_1,x_3}\big] + \big[m^{12}_{x_1,x_2}, m^{23}_{x_2,x_3}\big] + \big[m^{12}_{x_1,x_2}, m^{13}_{x_1,x_3}\big] = 0, \tag{9.10}$$

*where both sides are viewed as elements of $\mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathfrak{g}_3$. Here, $\mathfrak{g}_i = \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_{x_i})\big)$ for $i = 1, 2, 3$. Moreover, the tensor $m_{x_1,x_2}$ is unitary:*

$$m_{x_2,x_1} = -\tau\big(m_{x_1,x_2}\big) \tag{9.11}$$

*where $\tau : \mathfrak{g}_1 \otimes \mathfrak{g}_2 \longrightarrow \mathfrak{g}_2 \otimes \mathfrak{g}_1$ is the map permuting both factors.*

*Idea of the proof.* The proof is very similar to that of Theorem 8.2. The identity (8.10) applied to $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{P}$ leads to the equality (9.11). The fact that $m_{x_1,x_2}$ satisfies the classical Yang-Baxter equation (9.10) follows from (9.11) and the equality

$$m \circ (m \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m) + \text{other terms} = 0$$

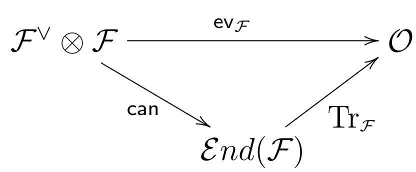
(which is one of the equalities (8.7)) viewed as a linear map

$$\mathrm{Hom}(\mathcal{P}, \Bbbk_{x_1}) \otimes \mathrm{Ext}(\Bbbk_{x_1}, \mathcal{P}) \otimes \mathrm{Hom}(\mathcal{P}, \Bbbk_{x_2}) \otimes \mathrm{Ext}(\Bbbk_{x_2}, \mathcal{P}) \otimes \mathrm{Hom}(\mathcal{P}, \Bbbk_{x_3}) \to \mathrm{Hom}(\mathcal{P}, \Bbbk_{x_3}).$$

□

*Remark* 9.9. In order to derive the identities (9.10) and (9.11) for a singular Weierstraß cubic curve $E$, we use a different approach which is similar in spirit to the work [14], see Remark 8.3. Following [37], we give another description of the tensor $m_{x,y}$ and show some kind of its continuity with respect to the degeneration of the complex structure on $E$. This approach will also allow to compute the tensor $m_{x,y}$ explicitly.

9.3. **On the sheaf of traceless endomorphism of a simple vector bundle.** Let $X$ be a projective algebraic variety over $\Bbbk$ and $\mathcal{F}$ a vector bundle on $X$. Consider the morphism of $\mathcal{O}_X$-modules $\mathrm{Tr}_{\mathcal{F}} : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}$ defined via commutativity of the following diagram of vector bundles on $X$:




**Definition 9.10.** The locally free sheaf $\mathrm{Ad}(\mathcal{F})$ of the traceless endomorphisms of $\mathcal{F}$ is the kernel of the canonical morphism $\mathrm{Tr}_{\mathcal{F}}$. In particular, we have the following short exact sequence of vector bundles on $X$:

$$0 \longrightarrow \mathrm{Ad}(\mathcal{F}) \longrightarrow \mathcal{E}nd(\mathcal{F}) \xrightarrow{\mathrm{Tr}_{\mathcal{F}}} \mathcal{O} \longrightarrow 0. \tag{9.12}$$

In the proposition below we collect some basic facts on the vector bundle $\mathrm{Ad}(\mathcal{F})$. Here we essentially use the fact that the characteristic of the base field $\Bbbk$ is zero.

**Proposition 9.11.** *In the above notation the following statements are true.*

1. *The vector bundle* $\mathrm{Ad}(\mathcal{F})$ *is a sheaf of Lie algebras on* $X$.
2. *Next, we have:* $H^0\big(\mathrm{Ad}(\mathcal{F})\big) = 0$.
3. *For any* $\mathcal{L} \in \mathrm{Pic}(X)$ *we have the natural isomorphism of sheaves of Lie algebras* $\mathrm{Ad}(\mathcal{F}) \longrightarrow \mathrm{Ad}(\mathcal{F} \otimes \mathcal{L})$ *which is induced by the natural isomorphism of sheaves of algebras* $\mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{E}nd(\mathcal{F} \otimes \mathcal{L})$.

(4) *We have a symmetric bilinear pairing*

$$\mathrm{Ad}(\mathcal{F}) \times \mathrm{Ad}(\mathcal{F}) \longrightarrow \mathcal{O}$$

*given on the level of local sections by the rule* $(f, g) \mapsto \mathrm{tr}(fg)$. *This pairing induces an isomorphism of* $\mathcal{O}$*-modules* $\mathrm{Ad}(\mathcal{F}) \longrightarrow \mathrm{Ad}(\mathcal{F})^\vee$.

As the next step, we shall describe some further properties of the sheaf $\mathrm{Ad}(\mathcal{P})$ in the case $\mathcal{P}$ is a simple vector bundle on a Calabi-Yau curve $E$.

**Definition 9.12.** Let $\{E^{(1)}, \dots, E^{(p)}\}$ be the set of the irreducible components of a Calabi-Yau curve $E$. For a vector bundle $\mathcal{F}$ on $E$ we denote by

$$\underline{\deg}(\mathcal{F}) = (d_1, \dots, d_p) \in \mathbb{Z}^p$$

its *multi-degree*, where $d_i = \deg\big(\mathcal{F}\big|_{E^{(i)}}\big)$ for $1 \le i \le p$.

For a given $\mathbb{d} \in \mathbb{Z}^p$ we denote $\mathrm{Pic}^{\mathbb{d}}(E) := \{\mathcal{L} \in \mathrm{Pic}(E) \,\big|\, \underline{\deg}(\mathcal{L}) = \mathbb{d}\}$. In particular, for $\mathbb{o} = (0, \dots, 0)$ we set: $J(E) = \mathrm{Pic}^{\mathbb{o}}(E)$. Then $J(E)$ is an algebraic group called *Jacobian* of $E$.

**Proposition 9.13.** *For* $\mathbb{k} = \mathbb{C}$ *we have the following isomorphisms of Lie groups:*

$$J(E) \cong \begin{cases} \mathbb{C}/\Lambda & \text{if } E \text{ is elliptic,} \\ \mathbb{C}^* & \text{if } E \text{ is a Kodaira cycle,} \\ \mathbb{C} & \text{in the remaining cases.} \end{cases} \tag{9.13}$$

*Moreover, for any multi-degree* $\mathbb{d}$ *we have a (non-canonical) isomorphism of algebraic varieties* $J(E) \longrightarrow \mathrm{Pic}^{\mathbb{d}}(E)$.

A proof of this result follows from [25, Exercise II.6.9] or [6, Theorem 16].

Next, recall the description of simple vector bundles on Calabi-Yau curves.

**Theorem 9.14.** *Let* $E$ *be a reduced plane cubic curve with* $p$ *irreducible components and* $\mathcal{P}$ *be a simple vector bundle on* $E$. *Then the following statements are true.*

(1) *Let* $n = \mathrm{rk}(\mathcal{P})$ *be the rank of* $\mathcal{P}$ *and* $d = d_1(\mathcal{P}) + \dots + d_p(\mathcal{P}) = \chi(\mathcal{P})$ *its degree. Then* $n$ *and* $d$ *are mutually prime.*
(2) *If* $E$ *is irreducible then* $\mathcal{P}$ *is stable.*
(3) *Let* $n \in \mathbb{N}$ *and* $\mathbb{d} = (d_1, \dots, d_p) \in \mathbb{Z}^p$ *be such that* $\gcd(n, d_1 + \dots + d_p) = 1$. *Denote by* $M_E(n, \mathbb{d})$ *the set of simple vector bundles on* $E$ *of rank* $n$ *and multi-degree* $\mathbb{d}$. *Then the map* $\det : M_E(n, \mathbb{d}) \longrightarrow Pic^{\mathbb{d}}(E)$ *is a bijection. Moreover, for any* $\mathcal{P} \not\cong \mathcal{P}' \in M_E(n, \mathbb{d})$ *we have:* $\mathrm{Hom}(\mathcal{P}, \mathcal{P}') = 0 = \mathrm{Ext}(\mathcal{P}, \mathcal{P}')$.
(4) *The group* $J(E)$ *acts transitively on* $M_E(n, \mathbb{d})$. *Moreover, given* $\mathcal{P} \in M_E(n, \mathbb{d})$ *and* $\mathcal{L} \in J(E)$, *we have:* $\mathcal{P} \cong \mathcal{P} \otimes \mathcal{L} \iff \mathcal{L}^{\otimes n} \cong \mathcal{O}$.

*Comment on the proof.* In the case of elliptic curves all these statements are due to Atiyah [1]. The case of a nodal Weierstraß cubic curve has been treated by the first-named author in [9], the corresponding result for a cuspidal cubic curve is due to Bodnarchuk and Drozd [7]. The remaining cases (Kodaira fibers of type $\mathrm{I}_2$, $\mathrm{I}_3$, III and IV) are due to Bodnarchuk, Drozd and Greuel [8]. Their method actually allows to prove this theorem for arbitrary Kodaira cycles of projective lines. In that case, one can also deduce this result from another description of simple vector bundles obtained in [11, Theorem 5.3]. On the other hand, this result is still open for $n$ concurrent lines in $\mathbb{P}^{n-1}$ if $n \geq 4$.

**Proposition 9.15.** *Let $E$ be a reduced plane cubic curve and $\mathcal{P}$ be a simple vector bundle on $E$ of rank $n$ and multi-degree $\mathbb{d}$. Then the following results are true.*

(1) *The sheaf of Lie algebras $\mathcal{A} = \mathcal{A}_{n,\mathbb{d}} := \mathrm{Ad}(\mathcal{P})$ does not depend on the choice of $\mathcal{P} \in M_E(n, \mathbb{d})$.*
(2) *We have: $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$. Moreover, this result remains true for an arbitrary Calabi-Yau curve.*
(3) *For $\mathcal{L} \in J(E)\setminus\{\mathcal{O}\}$ we have: $H^0(\mathcal{A}\otimes\mathcal{L}) \neq 0$ if and only if $\mathcal{L}^{\otimes n} \cong \mathcal{O}$. Moreover, in this case we have: $H^0(\mathcal{A} \otimes \mathcal{L}) \cong \mathbb{k} \cong H^1(\mathcal{A} \otimes \mathcal{L})$.*

*Proof.* The first part follows from the transitivity of the action of $J(E)$ on $M_E(n, \mathbb{d})$ (see Theorem 9.14) and the fact that $\mathrm{Ad}(\mathcal{P}) \cong \mathrm{Ad}(\mathcal{P} \otimes \mathcal{L})$ for any line bundle $\mathcal{L}$ (see Proposition 9.11). The second statement follows from the long exact sequence

$$0 \to H^0(\mathcal{A}) \longrightarrow \mathrm{End}(\mathcal{P}) \xrightarrow{H^0(\mathrm{Tr}_{\mathcal{P}})} H^0(\mathcal{O}) \longrightarrow H^1(\mathcal{A}) \longrightarrow \mathrm{Ext}(\mathcal{P},\mathcal{P}) \longrightarrow H^1(\mathcal{O}) \to 0,$$

the isomorphisms $\mathrm{End}(\mathcal{P}) \cong \mathbb{k} \cong \mathrm{Ext}(\mathcal{P},\mathcal{P})$, $H^0(\mathcal{O}) \cong \mathbb{k} \cong H^1(\mathcal{O})$ and the fact that $H^0(\mathrm{Tr}_{\mathcal{P}})(\mathbb{1}_{\mathcal{P}}) = \mathrm{rk}(\mathcal{P})$.

In order to show the last statement, note that we have the exact sequence

$$0 \longrightarrow H^0(\mathcal{A} \otimes \mathcal{L}) \longrightarrow \mathrm{Hom}(\mathcal{P}, \mathcal{P} \otimes \mathcal{L}) \longrightarrow H^0(\mathcal{L})$$

and $H^0(\mathcal{L}) = 0$. By Theorem 9.14 we know that $\mathrm{Hom}(\mathcal{P}, \mathcal{P} \otimes \mathcal{L}) = 0$ unless $\mathcal{L}^{\otimes n} \cong \mathcal{O}$. In the latter case, $H^0(\mathcal{A}\otimes\mathcal{L}) \cong \mathrm{End}(\mathcal{P}) \cong \mathbb{k}$. Since $\mathcal{A}\otimes\mathcal{L}$ is a vector bundle of degree zero, by the Riemann-Roch formula we obtain that $H^1(\mathcal{A} \otimes \mathcal{L}) \cong \mathbb{k}$. □

9.4. **Residues and traces.** Let $E$, $\Omega$, $w$ and $x$ be as at the beginning of the section. First recall that we have the following canonical short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \Omega(x) \xrightarrow{\mathsf{res}_x} \mathbb{k}_x \longrightarrow 0. \tag{9.14}$$

Choosing a non-zero section $w \in H^0(\Omega)$, we get the induced short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(x) \longrightarrow \mathbb{k}_x \longrightarrow 0. \tag{9.15}$$

Hence, for any vector bundle $\mathcal{F}$ we get a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\imath} \mathcal{F}(x) \xrightarrow{\mathsf{res}_x^{\mathcal{F}}} \mathcal{F} \otimes \Bbbk_x \longrightarrow 0. \tag{9.16}$$

Next, recall the following result relating categorical traces with the usual trace of an endomorphism of a finite dimensional vector space.

**Proposition 9.16.** *In the above notation, the following results are true.*

(1) *There is an isomorphism of functors* $\delta_x : \mathrm{Hom}(\Bbbk_x, - \otimes \Bbbk_x) \longrightarrow \mathrm{Ext}(\Bbbk_x, -)$ *from the category of vector bundles on $E$ to the category of vector spaces over* $\Bbbk$*, given by the boundary map induced by the short exact sequence (9.16).*

(2) *For any vector bundle $\mathcal{F}$ on $E$ and morphisms* $b : \mathcal{F} \longrightarrow \Bbbk_x, a : \Bbbk_x \longrightarrow \mathcal{F} \otimes \Bbbk_x$*, we have the equality:*

$$t^w\big(\mathrm{Tr}_{\mathcal{F}}(\delta_x(a) \circ b)\big) = \mathrm{tr}(a \circ b_x), \tag{9.17}$$

*where* $\mathrm{Tr}_{\mathcal{F}} : \mathrm{Ext}(\mathcal{F}, \mathcal{F}) \longrightarrow H^1(\mathcal{O})$ *is the trace map and $t^w$ is the composition* $H^0(\mathcal{O}) \xrightarrow{w} H^0(\Omega) \xrightarrow{t} \Bbbk$ *of the isomorphism induced by $w$ and the canonical map $t$ described in* [14, Subsection 4.3].

*Comment on the proof.* The first part of the statement is just [14, Lemma 4.18]. The content of the second part is explained by the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{R} & \longrightarrow & \Bbbk_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & \mathcal{F} & \xrightarrow{\imath} & \mathcal{F}(x) & \xrightarrow{\mathsf{res}_x^{\mathcal{F}}} & \mathcal{F} \otimes \Bbbk_x & \longrightarrow & 0.
\end{array}$$

The lowest horizontal sequence of this diagram is (9.16). The middle sequence corresponds to the element $\delta_x(a) \in \mathrm{Ext}(\Bbbk_x, \mathcal{F})$ and the top one corresponds to $\delta_x(a) \circ b \in \mathrm{Ext}(\mathcal{F}, \mathcal{F})$. The endomorphism $a \circ b_x \in \mathrm{End}(\mathcal{F}\big|_x)$ is the induced map in the fiber of $\mathcal{F}$ over $x$. The equality (9.17) follows from [14, Lemma 4.20]. □

**Proposition 9.17.** *The following diagram of vector spaces is commutative.*

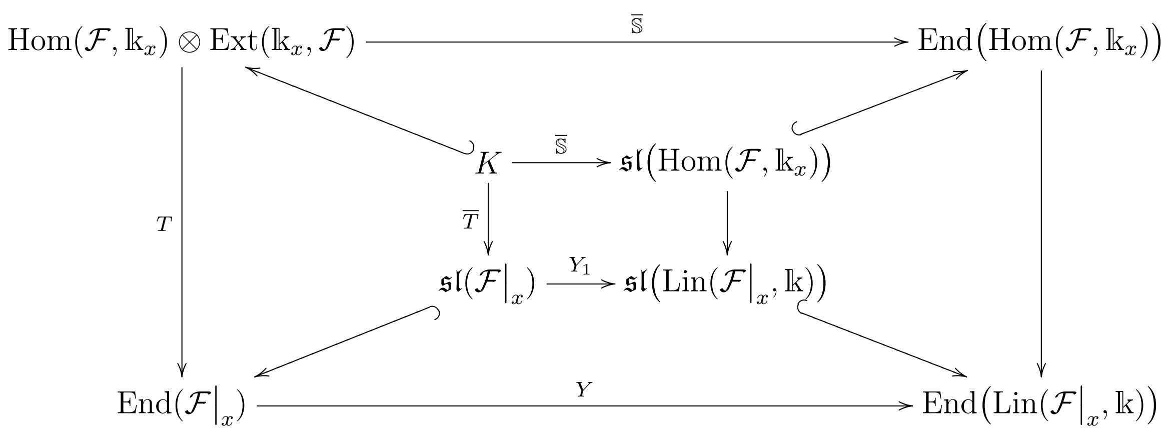
$$\begin{CD}
\mathrm{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathrm{Ext}(\Bbbk_x,\mathcal{F}) @>{\mathbb{1}\otimes\mathbb{S}}>> \mathrm{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathrm{Hom}(\mathcal{F},\Bbbk_x)^* \\
@A{\mathbb{1}\otimes\delta_x^{\mathcal{F}}}AA @AA{\mathbb{1}\otimes\mathsf{can}}A \\
\mathrm{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathrm{Hom}(\Bbbk_x,\mathcal{F}\otimes\Bbbk_x) @>{\mathbb{1}\otimes\mathrm{tr}}>> \mathrm{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathrm{Hom}(\mathcal{F}\otimes\Bbbk_x,\Bbbk_x)^* \\
@V{\circ}VV @VV{\mathsf{ev}}V \\
\mathrm{Lin}(\mathcal{F}\big|_x,\mathcal{F}\big|_x) @>{Y_1}>> \mathrm{End}\big(\mathrm{Lin}(\mathcal{F}\big|_x,\,\Bbbk)\big).
\end{CD}$$

*Here, $\mathbb{S}$ is given by (8.3), $\delta_x^{\mathcal{F}}$ is the isomorphism from Proposition 9.16, $\circ$ is the composition of morphisms composed with taking the induced map in the fiber over $x$, $Y_1$ is the canonical isomorphism of vector spaces from Lemma 9.2,* ev *and tr are canonical isomorphisms of vector spaces and* can *is the isomorphism induced by $\underline{\mathsf{res}}_x^{\mathcal{F}}$.*

*Proof.* The commutativity of the top square is given by [14, Lemma 4.21]. The commutativity of the lower square can be easily verified by diagram chasing. □

As a consequence, we get the following result.

**Lemma 9.18.** *The following diagram of vector spaces is commutative.*




*In this diagram, $\overline{\mathbb{S}}$ is the isomorphism induced by the Serre duality (8.3), $Y$ and $Y_1$ are canonical isomorphisms from Lemma 9.3, $K$ is the subspace of* $\mathrm{Hom}(\mathcal{F},\otimes\Bbbk_x)\otimes\mathrm{Ext}(\Bbbk_x,\mathcal{F})$ *defined in (9.3), $T$ is the composition of $\mathbb{1}\otimes(\delta_x^{\mathcal{F}})^{-1}$ from Proposition 9.17 and $\circ$, whereas $\overline{T}$ is the restriction of $T$. The remaining arrows are canonical morphisms of vector spaces.*

*Proof.* Commutativity of the big square is given by Proposition 9.17. For the left small square it follows from the equality (9.17) whereas the commutativity of the remaining parts of this diagram is obvious. □

9.5. **Algebraic versus geometric Massey products.** Let $E$, $\mathcal{P}$, $x$ and $y$ be as at the beginning of this section. In what follows, we shall frequently use the notation $\mathcal{A} := \mathrm{Ad}(\mathcal{P})$ and $\mathcal{E} := \mathcal{E}nd(\mathcal{P})$.

**Lemma 9.19.** *We have a canonical isomorphism of vector spaces*

$$\mathsf{res}_x := H^0(\underline{\mathsf{res}}_x^{\mathcal{A}}): \quad H^0\big(\mathcal{A}(x)\big) \longrightarrow \mathcal{A}\big|_x \tag{9.18}$$

*induced by the short exact sequence (9.16). Moreover, we have the canonical morphism*

$$\mathsf{ev}_y := H^0(\underline{\mathsf{ev}}_y^{\mathcal{A}}): \quad H^0\big(\mathcal{A}(x)\big) \longrightarrow \mathcal{A}\big|_y \tag{9.19}$$

*obtained by composing the induced map in the fibers with the canonical isomorphism $\mathcal{A}(x)\big|_y \longrightarrow \mathcal{A}\big|_y$. When $E$ is a reduced plane cubic curve, $\mathsf{ev}_y$ is an isomorphism if and only if $n \cdot \big([x]-[y]\big) \neq 0$ in $J(E)$, where $n = \mathrm{rk}(\mathcal{P})$.*

*Proof.* The short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\imath} \mathcal{A}(x) \xrightarrow{\underline{\mathsf{res}}_x^{\mathcal{A}}} \mathcal{A} \otimes \Bbbk_x \longrightarrow 0$$

yields the long exact sequence

$$0 \longrightarrow H^0(\mathcal{A}) \longrightarrow H^0\big(\mathcal{A}(x)\big) \xrightarrow{\mathsf{res}_x} \mathcal{A}\big|_x \longrightarrow H^1(\mathcal{A}).$$

Thus, the first part of the statement follows from the vanishing $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$ given by Proposition 9.15.

In order to show the second part note that we have the canonical short exact sequence

$$0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \xrightarrow{\mathsf{ev}_y} \Bbbk_y \longrightarrow 0$$

yielding the short exact sequence

$$0 \longrightarrow \mathcal{A}(x-y) \longrightarrow \mathcal{A}(x) \longrightarrow \mathcal{A}(x) \otimes \Bbbk_y \longrightarrow 0.$$

Hence, we get the long exact sequence

$$0 \longrightarrow H^0\big(\mathcal{A}(x-y)\big) \longrightarrow H^0\big(\mathcal{A}(x)\big) \xrightarrow{\mathsf{ev}_y} \mathcal{A}\big|_y \longrightarrow H^1\big(\mathcal{A}(x-y)\big).$$

Since the dimensions of $H^0\big(\mathcal{A}(x)\big)$ and $\mathcal{A}\big|_y$ are the same, $\mathsf{ev}_y$ is an isomorphism if and only if $H^0\big(\mathcal{A}(x-y)\big) = 0$. By Proposition 9.15 this vanishing is equivalent to the condition $n \cdot \big([x]-[y]\big) \neq 0$ in $J(E)$. □

Now we give a proof of the following key result, stated for the first time in [37, Theorem 4].

**Theorem 9.20.** *In the notation as at the beginning of this section, the following diagram of vector spaces is commutative:*

$$
\begin{array}{ccc}
\mathfrak{sl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big) & \xleftarrow{\;\overline{Y}_1\;} & \mathcal{A}\big|_x \\
 & & \uparrow{\scriptstyle \mathsf{res}_x} \\
\Big\downarrow{\scriptstyle \overline{m}_{x,y}} & & H^0\big(\mathcal{A}(x)\big) \\
 & & \downarrow{\scriptstyle \mathsf{ev}_y} \\
\mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) & \xleftarrow{\;\overline{Y}_2\;} & \mathcal{A}\big|_y .
\end{array}
\tag{9.20}
$$

*In this diagram, $\overline{m}_{x,y}$ is the linear map (9.7) induced by the triple product in* $\mathrm{Perf}(E)$, $\mathsf{res}_x$ *and* $\mathsf{ev}_y$ *are the linear maps (9.18) and (9.19), whereas $\overline{Y}_1$ and $\overline{Y_2}$ are obtained by composing the canonical isomorphisms $Y_1$ and $Y_2$ from Lemma 9.3 with the canonical isomorphisms induced by* $\mathrm{Hom}(\mathcal{P}, \Bbbk_z) \longrightarrow \mathrm{Lin}\big(\mathcal{P}\big|_z, \Bbbk\big)$ *for* $z \in \{x, y\}$.

We split the proof of this theorem into several logical steps.

Step 1. First note that we have a well-defined linear map

$$\imath^! : \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \longrightarrow \mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big)$$

defined as follows. Let $g \in \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)$ and $h \in \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$ be arbitrary morphisms. Then there exists a unique morphism $\tilde{h} \in \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$ such that $\imath \circ \tilde{h} = h$, where $\imath : \mathcal{P} \longrightarrow \mathcal{P}(x)$ is the canonical inclusion. Then we set: $\imath^!(g)(h) = \tilde{h} \circ g$. It follows from the definition that $\imath^!(\imath) = \mathbb{1}_{\mathrm{Hom}(\mathcal{P}, \Bbbk_y)}$. This yields the following result.

**Lemma 9.21.** *We have a well-defined linear map*

$$\overline{\imath}^! : \quad \frac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \longrightarrow \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \tag{9.21}$$

*given by the rule:* $\overline{\imath}^!(\bar{g}) = \overline{h \mapsto g \circ \tilde{h}}$.

**Lemma 9.22.** *The canonical morphism of vector spaces*

$$\jmath : H^0\big(\mathcal{A}(x)\big) \longrightarrow \frac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \tag{9.22}$$

*given by the composition*

$$H^0\big(\mathrm{Ad}(\mathcal{P})(x)\big) \hookrightarrow H^0\big(\mathcal{E}nd(\mathcal{P})(x)\big) \longrightarrow \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \longrightarrow \frac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle}$$

*is an isomorphism.*

*Proof.* The short exact sequences (9.12) and (9.15) together with the vanishing $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$ imply that we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0\big(\mathcal{O}(x)\big) & \xrightarrow{0} & \Bbbk & \longrightarrow & H^1(\mathcal{O}) \\
 & & \uparrow & & \uparrow & & \uparrow \mathrm{tr} & & \uparrow \\
0 & \longrightarrow & H^0(\mathcal{E}) & \longrightarrow & H^0\big(\mathcal{E}(x)\big) & \longrightarrow & \mathcal{E}\big|_x & \longrightarrow & H^1(\mathcal{E}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\mathsf{res}_x} & \mathcal{A}\big|_x & \longrightarrow & 0.
\end{array}$$

The fact that $\jmath$ is an isomorphism follows from a straightforward diagram chase. $\square$

**Lemma 9.23.** *The following diagram is commutative.*

$$\begin{array}{ccc}
\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) & \xrightarrow{\mathsf{ev}_y} & \mathrm{End}(\mathcal{P}\big|_y) \\
\imath^! \downarrow & & \downarrow Y \\
\mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) & \xrightarrow{\mathsf{can}} & \mathrm{End}\big(\mathrm{Lin}(\mathcal{P}\big|_y, \Bbbk)\big).
\end{array}$$

*Proof.* The result follows from a straightforward diagram chase. $\square$

**Proposition 9.24.** *The following diagram is commutative.*

$$\begin{array}{ccccc}
H^0\big(\mathcal{A}(x)\big) & & \xrightarrow{\mathsf{ev}_y} & & \mathfrak{sl}(\mathcal{P}\big|_y) \\
\jmath \downarrow & & & & \downarrow Y_2 \\
\dfrac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} & \xrightarrow{\bar{\imath}^!} & \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) & \longrightarrow & \mathfrak{pgl}\big(\mathrm{Lin}(\mathcal{P}\big|_y, \Bbbk)\big).
\end{array} \tag{9.23}$$

*In particular, if $E$ is a reduced plane cubic curve then $\bar{\imath}^!$ is an isomorphism if and only if $n \cdot \big([x] - [y]\big) \neq 0$ in $J(E)$.*

*Proof.* Note that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{sl}\big(\mathcal{P}\big|_y\big) \arrow[r, hook] & \mathrm{End}\big(\mathcal{P}\big|_y\big) \arrow[rrd, "Y"] & & \\
H^0\big(\mathcal{A}(x)\big) \arrow[u, "\mathsf{ev}_y"] \arrow[r, hook] \arrow[rd, "\jmath"'] & \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \arrow[u, "\mathsf{ev}_y"'] \arrow[r, "\imath^!"] \arrow[d] & \mathrm{End}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \arrow[r] \arrow[d] & \mathrm{End}\big(\mathrm{Lin}(\mathcal{P}\big|_y, \Bbbk)\big) \arrow[d] \\
 & \dfrac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \arrow[r, "\bar{\imath}^!"] & \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \arrow[r] & \mathfrak{pgl}\big(\mathrm{Lin}(\mathcal{P}\big|_y, \Bbbk)\big).
\end{tikzcd}$$

Indeed, the right top square is commutative by Lemma 9.23, the commutativity of the remaining parts is straightforward. This implies that the diagram (9.23) is commutative, too.

Next, observe that all maps in the diagram (9.23) but $\bar{\imath}^!$ and $\mathsf{ev}_y$ are isomorphisms. By Lemma 9.19, the map $\mathsf{ev}_y$ is an isomorphism if and only if $n \cdot \big([x] - [y]\big) \neq 0$ in $J(E)$. This proves the second part of this Proposition. ☐

Note that from the exact sequence (9.16) we get the induced map

$$R := H^0\big(\underline{\mathsf{res}}_x^{\mathcal{E}nd(\mathcal{P})}\big) : \quad \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \longrightarrow \mathrm{End}\big(\mathcal{P}\big|_x\big)$$

sending an element $g \in \mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)$ to $\big(\underline{\mathsf{res}}_x^{\mathcal{P}} \circ g\big)_x \in \mathrm{End}\big(\mathcal{P}\big|_x\big)$. Clearly, $R(\imath) = 0$, thus we obtain the induced map

$$\overline{R}: \quad \frac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \longrightarrow \mathrm{End}\big(\mathcal{P}\big|_x\big). \tag{9.24}$$

**Lemma 9.25.** *In the above notation, the following statements are true.*

(1) $\mathsf{Im}(\overline{R}) = \mathfrak{sl}\big(\mathcal{P}\big|_x\big)$.

(2) *Moreover, the map* $\overline{R}: \quad \dfrac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \longrightarrow \mathfrak{sl}\big(\mathcal{P}\big|_x\big)$ *is an isomorphism.*

*Proof.* The result follows from the commutativity of the diagram

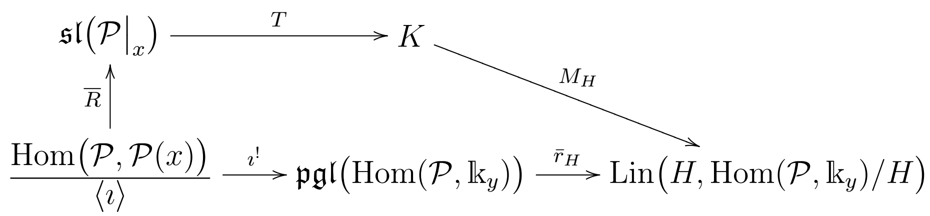
$$\begin{tikzcd}
H^0\big(\mathrm{Ad}(\mathcal{P})(x)\big) \arrow[r, "\mathsf{res}_x"] \arrow[d, "\jmath"'] & \mathfrak{sl}\big(\mathcal{P}\big|_x\big) \arrow[d, hook] \\
\dfrac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \arrow[r, "\overline{R}"] & \mathrm{End}\big(\mathcal{P}\big|_x\big)
\end{tikzcd}$$

and the facts that $\mathsf{res}_x$ and $\jmath$ are isomorphisms. ☐

Step 2. The next result is the key part of the proof of Theorem 9.20.

**Proposition 9.26.** *The following diagram is commutative.*

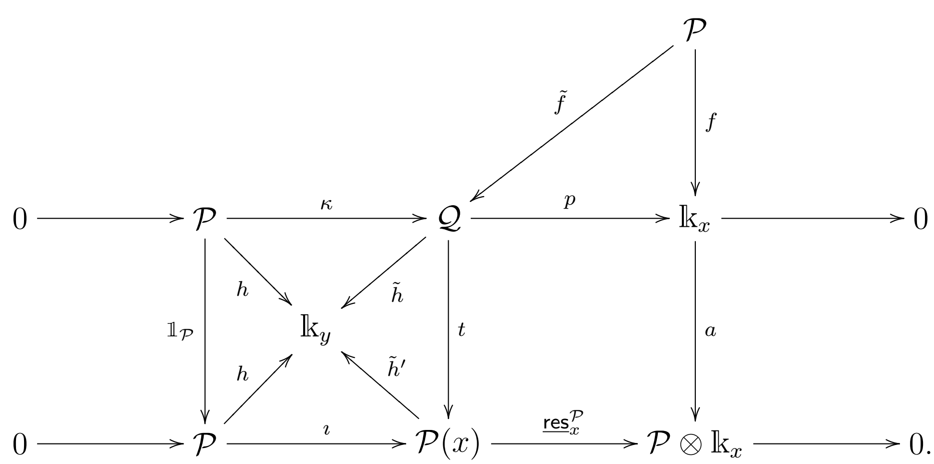
(9.25)




*Proof.* We show this result by diagram chasing. Recall that the vector space $K$ is the linear span of the simple tensors $f \otimes \omega \in \mathrm{Hom}(\mathcal{P}, \Bbbk_x) \otimes \mathrm{Ext}(\Bbbk_x, \mathcal{P})$ such that $\omega \circ f = 0$. Let $0 \longrightarrow \mathcal{P} \xrightarrow{\kappa} \mathcal{Q} \xrightarrow{p} \Bbbk_x \longrightarrow 0$ be a short exact sequence corresponding to an element $\omega \in \mathrm{Ext}(\Bbbk_x, \mathcal{P})$. Recall that by Proposition 9.16 there exists a unique $a \in \mathrm{Hom}(\Bbbk_x, \mathcal{P} \otimes \Bbbk_x)$ such that $\omega = \delta_x(a)$.

Since $\mathrm{Hom}(\Bbbk_x, \Bbbk_y) = 0 = \mathrm{Ext}(\Bbbk_x, \Bbbk_y)$, for any $h \in \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$ there exist unique elements $\tilde{h} \in \mathrm{Hom}(\mathcal{Q}, \Bbbk_y)$ and $\tilde{h}' \in \mathrm{Hom}\big(\mathcal{P}(x), \Bbbk_y\big)$ such that the following diagram is commutative:

(9.26)

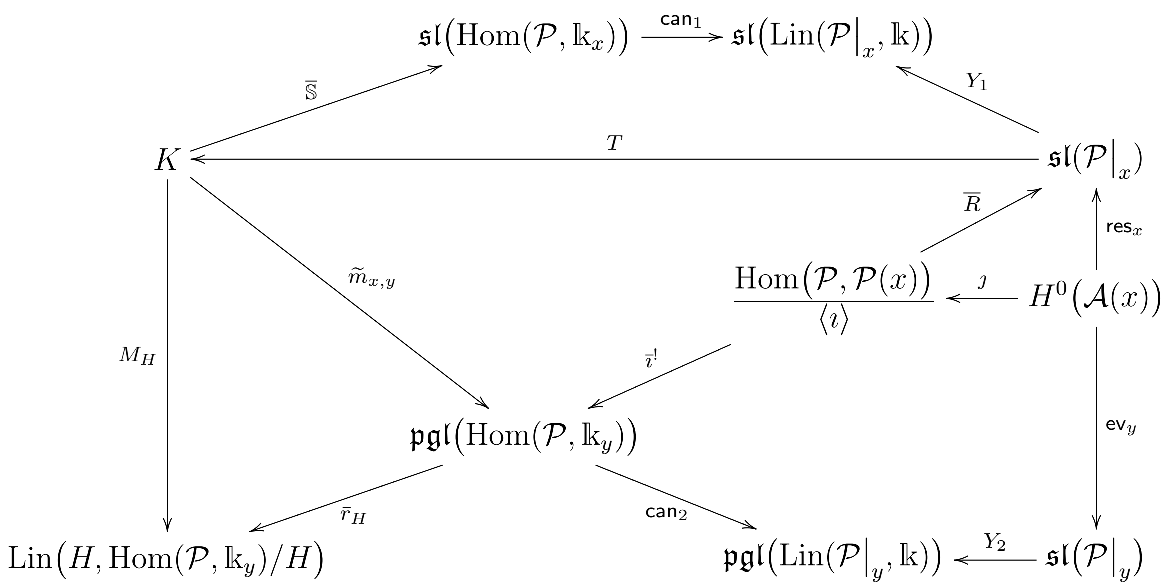



Although a lift $\tilde{f} \in \mathrm{Hom}(\mathcal{P}, \mathcal{Q})$ is only defined up to a translation $\tilde{f} \mapsto \tilde{f} + \lambda\kappa$ for some $\lambda \in \Bbbk$, we have a well-defined element $\overline{t \circ \tilde{f}} \in \dfrac{\mathrm{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle}$ such that $\overline{R}\big(\overline{t \circ \tilde{f}}\big) = a \circ f_x$. By definition, $T(a \circ f_x) = f \otimes \omega$. It remains to observe that

$$\big(\bar{r}_H \circ \bar{\imath}^!([t\tilde{f}])\big)(h) = [\tilde{h}' t \tilde{f}] = [\tilde{h} f] = \big(M_H(f \otimes \omega)\big)(h).$$

Since $\overline{R}$ and $T$ are isomorphisms and the vector space $K$ is generated by simple tensors, this concludes the proof. □

Step 3. Now we are ready to proceed with the proof of Theorem 9.20. Note that the following diagram is commutative.




Indeed, by Lemma 9.18 we have the equality $Y_1 \circ T = \mathsf{can}_1 \circ \mathbb{S}$, which gives commutativity of the top square. Next, the equality $\bar{r}_H \circ \widetilde{m}_{x,y} = M_H$ just expresses the commutativity of the diagram (9.9). The equality $\overline{R} \circ \jmath = \mathsf{res}_x$ follows from the definition of the map $\overline{R}$, see (9.24).

The equality $Y_2 \circ \mathsf{ev}_y = \mathsf{can}_2 \circ \bar{\imath}^! \circ \jmath$ is given by Proposition 9.24, yielding the commutativity of the right lower part. Finally, by Proposition 9.26 we have the equality $\bar{r}_H \circ \bar{\imath}^! = M_H \circ T \circ \overline{R}$. Since this equality is true for any one-dimensional subspace $H \subseteq \mathrm{Hom}(\mathcal{P}, \Bbbk_y)$, Lemma 9.4 implies that $\widetilde{m}_{x,y} \circ T \circ \overline{R} = \bar{\imath}^!$. This finishes the proof of commutativity of the above diagram.

It remains to conclude that the commutativity of the diagram (9.20) follows as well and Theorem 9.20 is proven. □

**Corollary 9.27.** *Let $E$ be an elliptic curve over $\Bbbk$, $\mathcal{P}$ a simple vector bundle on $E$, $\mathcal{A} = Ad(\mathcal{P})$ and $x, y \in R$ two distinct points. Let $r_{x,y} \in \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y$ be the image of the linear map $\mathsf{ev}_y \circ \mathsf{res}_x^{-1} \in \mathrm{Lin}(\mathcal{A}\big|_x, \mathcal{A}\big|_y)$ under the linear isomorphism $\mathrm{Lin}(\mathcal{A}\big|_x, \mathcal{A}\big|_y) \longrightarrow \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y$ induced by the Killing form $\mathcal{A}\big|_x \times \mathcal{A}\big|_x \longrightarrow \Bbbk, (a, b) \mapsto tr(a \circ b)$. Then $r_{x,y}$ is a solution of the classical Yang-Baxter equation: for any pairwise distinct points $x_1, x_2$ and $x_3$ of $E$ we have:*

$$\left[r^{12}_{x_1,x_2}, r^{13}_{x_1,x_3}\right] + \left[r^{12}_{x_1,x_2}, r^{23}_{x_2,x_3}\right] + \left[r^{12}_{x_1,x_2}, r^{13}_{x_1,x_3}\right] = 0, \tag{9.27}$$

*where both sides of the above identity are viewed as elements of* $\mathcal{A}\big|_{x_1} \otimes \mathcal{A}\big|_{x_2} \otimes \mathcal{A}\big|_{x_3}$. *Moreover, the tensor* $r_{x_1,x_2}$ *is unitary:*

$$r_{x_2,x_1} = -\tau\big(r_{x_1,x_2}\big), \tag{9.28}$$

*where* $\tau : \mathcal{A}\big|_{x_1} \otimes \mathcal{A}\big|_{x_2} \longrightarrow \mathcal{A}\big|_{x_2} \otimes \mathcal{A}\big|_{x_1}$ *is the map permuting both factors.*

*Proof.* By Theorem 9.20, the tensor $r_{x,y}$ is the image of the tensor $m_{x,y}$ from (9.7) under the isomorphism

$$\mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_x)\big) \otimes \mathfrak{pgl}\big(\mathrm{Hom}(\mathcal{P}, \Bbbk_y)\big) \xrightarrow{\overline{Y}_2 \otimes \overline{Y}_2} \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y.$$

Since $\overline{Y}_2$ is an anti-isomorphism of Lie algebras, the equality (9.27) is a corollary of (9.10). In the same way, the equality (9.28) is a consequence of (9.11). □

Our next goal is to generalize Corollary 9.27 to the case of singular Weierstraß cubic curves.

9.6. **Genus one fibrations and the CYBE.** We start with the following geometric data.

- Let $E \xrightarrow{p} T$ be a flat *projective* morphism of relative dimension one between algebraic varieties. We denote by $\breve{E}$ the regular locus of $p$.
- We assume there exists a section $\imath : T \to \breve{E}$ of $p$.
- Moreover, we assume that for all points $t \in T$ the fiber $E_t$ is an *irreducible* Calabi-Yau curve.
- The fibration $E \xrightarrow{p} T$ is embeddable into a smooth fibration of projective surfaces over $T$ and $\Omega_{E/T} \cong \mathcal{O}_E$.

**Example 9.28.** Let $E_T \subset \mathbb{P}^2 \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2 =: T$ be the elliptic fibration given by the equation $zy^2 = 4x^3 + g_2xz^2 + g_3z^3$ and let $\Delta(g_2, g_3) = g_2^3 + 27g_3^2$ be the discriminant of this family. This fibration has a section $(g_2, g_3) \mapsto \big((0 : 1 : 0), (g_2, g_3)\big)$ and satisfies the condition $\Omega_{E/T} \cong \mathcal{O}_E$.

The following result is well-known.

**Lemma 9.29.** *Consider* $(n, d) \in \mathbb{N} \times \mathbb{Z}$ *such that* $\gcd(n, d) = 1$. *There exists* $\mathcal{P} \in \mathrm{VB}(E)$ *such that for any* $t \in T$ *its restriction* $\mathcal{P}\big|_{E_t}$ *is simple of rank* $n$ *and degree* $d$.

*Sketch of the proof.* Let $\Sigma := \imath(T) \subset E$ and $\mathcal{I}_\Delta$ be the structure sheaf of the diagonal $\Delta \subset E \times_T E$. Let $\mathrm{FM}^{\mathcal{I}_\Delta}$ be the Fourier-Mukai transform with the kernel $\mathcal{I}_\Delta$. By [16, Theorem 2.12], $\mathrm{FM}^{\mathcal{I}_\Delta}$ is an auto-equivalence of the derived category $\mathrm{FM}^{\mathcal{I}_\Delta}$. By [15, Proposition 4.13(iv)] there exists an auto-equivalence $\mathbb{F}$ of the derived category $D^b_{\mathrm{Coh}}(E)$, which is a certain composition of the functors $\mathrm{FM}^{\mathcal{I}_\Delta}$ and $- \otimes \mathcal{O}(\Sigma)$ such that $\mathbb{F}(\mathcal{O}_\Sigma) \cong \mathcal{P}[0]$, where $\mathcal{P}$ is a vector bundle on $E$ having the required properties. □

Now we fix the following notation. Let $\mathcal{P}$ be as in Lemma 9.29 and $\mathcal{A} = \mathrm{Ad}(\mathcal{P})$. Next, we set $\overline{X} := E \times_T \breve{E} \times_T \breve{E}$ and $\overline{B} := \breve{E} \times_T \breve{E}$. Let $q : \overline{X} \longrightarrow \overline{B}$ be the canonical projection, $\Delta \subset \breve{E} \times_T \breve{E}$ the diagonal, $B := \overline{B} \setminus \Delta$ and $X := q^{-1}(B)$. The elliptic fibration $q : \overline{X} \longrightarrow \overline{B}$ has two canonical sections $h_i$, $i = 1, 2$, given by $h_i(y_1, y_2) = (y_i, y_1, y_2)$. Let $\Sigma_i := h_i(\overline{B})$ and $\overline{\mathcal{A}}$ be the pull-back of $\mathcal{A}$ on $\overline{X}$.

Note that the relative dualizing sheaf $\Omega = \Omega_{\overline{X}/\overline{B}}$ is trivial. Similarly to (9.14) one has the following canonical short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \Omega(\Sigma_1) \xrightarrow{\mathsf{res}_{\Sigma_1}} \mathcal{O}_{\Sigma_1} \longrightarrow 0, \tag{9.29}$$

see [14, Subsection 3.1.2] for a precise construction. By the assumptions from the beginning of this section, there exists an isomorphism $\mathcal{O}_{\overline{X}} \longrightarrow \Omega_{\overline{X}/\overline{B}}$ induced by a nowhere vanishing section $w \in H^0(\Omega_{E/T})$. It gives the following short exact sequence

$$0 \longrightarrow \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}(\Sigma_1) \xrightarrow{\mathsf{res}^{\mathcal{A}}_{\Sigma_1}} \overline{\mathcal{A}}\big|_{\Sigma_1} \longrightarrow 0. \tag{9.30}$$

In a similar way, we have another canonical sequence

$$0 \longrightarrow \overline{\mathcal{A}}(\Sigma_1 - \Sigma_2) \longrightarrow \overline{\mathcal{A}}(\Sigma_1) \longrightarrow \overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2} \longrightarrow 0. \tag{9.31}$$

**Proposition 9.30.** *In the above notation, the following results are true.*

1. *We have the vanishing* $q_*(\overline{\mathcal{A}}) = 0 = \mathbb{R}^1 q_*(\overline{\mathcal{A}})$.
2. *The coherent sheaf* $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big)$ *is locally free.*
3. *Moreover, we have the morphism of locally free sheaves on $B$ given by the composition* $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big) \longrightarrow q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big)$, *which is an isomorphism outside of the closed subset*

$$\Delta_n := \big\{(t, x, y) \;\big|\; n \cdot \big([x] - [y]\big) = 0 \in J(E_t)\big\} \subset B. \tag{9.32}$$

*Proof.* Let $z = (t, x, y) \in \overline{B}$ be an arbitrary point. By the base-change formula we have: $\mathbb{L}\imath_z^*\big(\mathbb{R}q_*(\overline{\mathcal{A}})\big) \cong \mathbb{R}\Gamma(\mathcal{A}\big|_{E_t}) = 0$, where the last vanishing is true by Proposition 9.15. This proves the first part of the theorem.

Thus, applying $q_*$ to the short exact sequence (9.30), we get an isomorphism

$$\mathsf{res}_1 := q_*\big(\underline{\mathsf{res}}^{\overline{\mathcal{A}}}_{\Sigma_1}\big) : \quad q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_1}\big). \tag{9.33}$$

For $i = 1, 2$, let $p_i : \overline{B} := \breve{E} \times \breve{E} \longrightarrow E$ be the composition of $i$-th canonical projection with the canonical inclusion $\breve{E} \subseteq E$. It is easy to see that we have a canonical isomorphism $\gamma : q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_i}\big) \longrightarrow p_i^*(\mathcal{A})$. This shows that the coherent sheaf $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big)$ is locally free on $\overline{B}$.

To prove the last part, consider the canonical morphism of vector bundles $q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big)$. This is an isomorphism on $B$. Moreover, by Proposition 9.15, the subset

$\Delta_n$ is precisely the support of the complex $\mathbb{R}q_*\big(\mathcal{A}(\Sigma_1 - \Sigma_2)\big)$. In particular, this shows that $\Delta_n$ is a proper closed subset of $B$. Finally, applying $q_*$ to the short exact sequence (9.31), we get a morphism of locally free sheaves

$$\mathsf{ev}_2: \quad q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow p_2^*(\mathcal{A}), \tag{9.34}$$

which is an isomorphism on the complement of $\Delta_n$. This proves the proposition. □

**Theorem 9.31.** *In the above notation, let $r \in \Gamma\big(\overline{B}, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big)$ be the meromorphic section which is the image of $\mathsf{ev}_2 \circ \mathsf{res}_1^{-1}$ under the canonical isomorphism*

$$\mathrm{Hom}\big(p_1^*(\mathcal{A}), p_2^*(\mathcal{A})\big) \longrightarrow H^0\big(p_1^*(\mathcal{A})^\vee \otimes p_2^*(\mathcal{A})\big) \longrightarrow H^0\big(p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big).$$

*The last isomorphism above is induced by the canonical isomorphism $\mathcal{A} \longrightarrow \mathcal{A}^\vee$ from Proposition 9.11. Then the following statements are true.*

(1) *The poles of $r$ lie on the divisor $\Delta$. In particular, $r$ is holomorphic on $B$.*
(2) *Moreover, $r$ is non-degenerate on the complement of the set $\Delta_n$.*
(3) *The section $r$ satisfies a version of the classical Yang-Baxter equation:*

$$\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big] = 0, \tag{9.35}$$

*where both sides are viewed as elements of $H^0\big(p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A}) \otimes p_3^*(\mathcal{A})\big)$.*

(4) *Moreover, the section $r$ is unitary. This means that*

$$\sigma^*(r) = -\tilde{r} \in H^0\big(p_2^*(\mathcal{A}) \otimes p_1^*(\mathcal{A})\big), \tag{9.36}$$

*where $\sigma$ is the canonical involution of $\overline{B} = \breve{E} \times_T \breve{E}$ and $\tilde{r}$ is the section corresponding to the morphism $\mathsf{ev}_1 \circ \mathsf{res}_2^{-1}$.*

(5) *In particular, the statement of Corollary 9.27 is also true for singular Weierstraß cubic curves.*

*Proof.* By Proposition 9.30, we have the following morphisms in $\mathrm{VB}(\overline{B})$:

$$p_1^*(\mathcal{A}) \xleftarrow{\mathsf{res}_1} q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big) \xleftarrow{\imath} q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big) \xrightarrow{\gamma} p_2^*(\mathcal{A}).$$

Moreover, $\gamma$ is an isomorphism, whereas $\mathsf{res}_1$ and $\imath$ become isomorphisms after restricting on $B$. This shows that the section $r \in \Gamma\big(\overline{B}, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big)$ is indeed *meromorphic* with poles lying on the diagonal $\Delta$. Since $\mathsf{ev}_2 \circ \mathsf{res}_1^{-1}$ is an isomorphism on $B \setminus \Delta_n$, the section $r$ is non-degenerate on $B \setminus \Delta_n$.

To prove the last two parts of the theorem, assume first that the generic fiber of $E$ is smooth. Let $t \in T$ be such that $E_t$ is an elliptic curve. Then in the notation of Corollary 9.27, for any $z = (t, x, y) \in B$ have have:

$$\imath_z^*(r) = r_{x,y} \in \big(\mathcal{A}\big|_{E_t}\big)\Big|_x \otimes \big(\mathcal{A}\big|_{E_t}\big)\Big|_y,$$

where we use the canonical isomorphism

$$\imath_z^*\big(p_1^*(\overline{\mathcal{A}}) \otimes p_2^*(\overline{\mathcal{A}})\big) \longrightarrow \big(\mathcal{A}\big|_{E_t}\big)\Big|_x \otimes \big(\mathcal{A}\big|_{E_t}\big)\Big|_y.$$

Let $x_1$, $x_2$ and $x_3$ be three pairwise distinct points of $E_t$ and $\bar{x} = (t, x_1, x_2, x_3) \in \breve{E} \times_T \breve{E} \times_T \breve{E}$. By Corollary 9.27 we have:

$$\imath_{\bar{x}}^*\Big(\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big]\Big) = 0. \tag{9.37}$$

In a similar way, we have the equality:

$$\imath_z^*\big(\sigma^*(r) + \tilde{r}\big) = 0. \tag{9.38}$$

Since the section $r$ is continuous on $B$, the equalities (9.37) and (9.38) are true for the singular fibers of $E$ as well. In particular, the statement of Corollary 9.27 is also true for singular Weierstraß cubic curves. This implies that Theorem 9.31 is true for arbitrary genus one fibrations satisfying the conditions from the beginning of this section. □

**Corollary 9.32.** *Let $E \xrightarrow{p} T$, $\imath : T \longrightarrow E$ and $w \in H^0\big(\Omega_{E/T}\big)$ be as at the beginning of the section, $\mathcal{P}$ be a relatively stable vector bundle on $E$ of rank $n$ and degree $d$ (recall that we automatically have $\gcd(n,d) = 1$) and $\mathcal{A} = \mathrm{Ad}(\mathcal{P})$. For any closed point of the base $t \in T$ let $U$ be a small neighborhood of the point $\imath(t) \in E_{t_0}$, $V$ be a small neighborhood of $\big(t, \imath(t), \imath(t)\big) \in E \times_T E$, $O = \Gamma(U, \mathcal{O})$ and $M = \Gamma(V, \mathcal{M})$, where $\mathcal{M}$ is the sheaf of meromorphic functions on $E \times_T E$. Taking an isomorphism of Lie algebras $\xi : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n(O)$, we get the tensor-valued meromorphic function*

$$r^\xi = r^\xi_{n,d} \in \mathfrak{sl}_n(M) \otimes_M \mathfrak{sl}_n(M), \tag{9.39}$$

*which is the image of the* canonical *meromorphic section $r \in \Gamma\big(E \times_T E, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big)$ from Theorem 9.31. Then the following statements are true.*

1. *The poles of $r^\xi$ lie on the diagonal $\Delta \subset E \times_T E$.*
2. *Moreover, for a fixed $t \in T$ this function is a* unitary *solution of the classical Yang-Baxter equation (2.1) in variables $(y_1, y_2) \in \{t\} \times (U \cap E_t) \times (U \cap E_t) \subset V \subset E \times_T E$. In other words, we get a family of solutions $r^\xi_t(y_1, y_2)$ of the classical Yang-Baxter equation, which is* analytic *as the function of the parameter $t \in T$.*
3. *Let $\xi' : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n(O)$ be another isomorphism of Lie algebras and $\rho := \xi' \circ \xi^{-1}$. Then we have the following commutative diagram:*

$$\begin{array}{ccccc} & & \mathcal{A}(U) & & \\ & \swarrow^{\xi} & & \searrow^{\xi'} & \\ \mathfrak{sl}_n(O) & & \xrightarrow{\quad\rho\quad} & & \mathfrak{sl}_n(O). \end{array}$$

*Moreover, for any* $(t, y_1, y_2) \in V \setminus \Delta$ *we have:*

$$r^{\xi'}(y_1, y_2) = \big(\rho(y_1) \otimes \rho(y_2)\big) \cdot r^{\xi}(y_1, y_2) \cdot \big(\rho^{-1}(y_1) \otimes \rho^{-1}(y_2)\big). \tag{9.40}$$

*In other words, the solutions* $r^{\xi}$ *and* $r^{\xi'}$ *are* gauge equivalent.

*Remark* 9.33. One possibility to generalize Theorem 9.31 and Corollary 9.32 for an arbitrary Calabi-Yau curve $E$ can be achieved by showing that any simple vector bundle on $E$ can be obtained from the structure sheaf $\mathcal{O}$ by applying an appropriate auto-equivalence of the triangulated category $\mathrm{Perf}(E)$. Some progress in this direction has been recently achieved by Hernández Ruipérez, López Martín, Sánchez Gómez and Tejero Prieto in [27].

# Part 3. Vector bundles on degenerations of elliptic curves

In this part we study vector bundles on elliptic curves and a cuspidal cubic curve. To be more precise, Section 10 contains the theory of semi-stable vector bundles on a one-dimensional complex torus as presented in [12, Subsection 3.1]. The remaining sections of this part deal with the theory of simple vector bundles on the cuspidal cubic curve $E = V(zy^2 - x^3)$. The techniques employed for this study, i.e. the theory of the category of triples and the translation to matrix problems, were developed by Drozd and Greuel [19] and further elaborated by Bodnarchuk and Burban, see for instance [5] and [10]. As in [8], we present the study of simple vector bundles on $E$ in terms of the representation theory of differential biquivers.

## 10. Vector bundles on a one-dimensional complex torus

Let $\tau \in \mathbb{H}$, $\Lambda = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$ and $E = \mathbb{C}/\Lambda$ be the corresponding complex torus. In this section we recall the basic techniques for dealing with holomorphic vector bundles on $E$.

**Definition 10.1.** Let $A : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ be a holomorphic function satisfying the condition $A(z+1) = A(z)$ for all $z \in \mathbb{C}$. Such a function $A$, called *automorphy factor*, defines the following topological space $\mathcal{E}(A) := \mathbb{C} \times \mathbb{C}^n / \sim$, where $(z, v) \sim (z+1, v) \sim (z+\tau, A(z)v)$. Note that we have a Cartesian diagram of complex manifolds

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^n & \longrightarrow & \mathcal{E}(A) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\pi} & E \end{array}$$

and $\mathcal{E}(A)$ is a vector bundle of rank $n$ on the torus $E$.

*Remark* 10.2. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda = E$ be the quotient map. Another way to define the locally free sheaf $\mathcal{E}(A)$ is the following.

The open subsets $U \subset E$ for which all connected components of $\pi^{-1}(U)$ map isomorphically to $U$, form a basis of the topology of $E$. For such $U$, we let $U_0$ be a connected component of $\pi^{-1}(U)$ and denote $U_\gamma = \gamma + U_0$ for all $\gamma \in \Lambda$. Then $\pi_*\mathcal{O}_{\mathbb{C}}^n\big(U\big) = \prod_{\gamma\in\Lambda} \mathcal{O}_{\mathbb{C}}^n(U_\gamma)$ and we define

$$\mathcal{E}(A)\big(U\big) := \left\{ (F_\gamma(z))_{\gamma\in\Lambda} \in \pi_*(\mathcal{O}_{\mathbb{C}}^n)\big(U\big) \;\middle|\; \begin{array}{l} F_{\gamma+1}(z+1) = F_\gamma(z) \\ F_{\gamma+\tau}(z+\tau) = A(z)F_\gamma(z) \end{array} \right\}.$$

In this way we get an embedding $\mathsf{m}_A : \mathcal{E}(A) \subset \pi_*\mathcal{O}_{\mathbb{C}}^n$ as well as a trivialization $\gamma_A$ of $\pi^*\big(\mathcal{E}(A)\big)$ given by the composition $\pi^*\mathcal{E}(A) \xrightarrow{\pi^*(\mathsf{m}_A)} \pi^*\pi_*\mathcal{O}_{\mathbb{C}}^n \xrightarrow{\mathsf{can}} \mathcal{O}_{\mathbb{C}}^n$.

The following classical result is due to A. Weil.

**Theorem 10.3.** *Let $E = \mathbb{C}/\Lambda$ be a one-dimensional complex torus.*

(1) *For any holomorphic rank $n$ vector bundle $\mathcal{E}$ on the torus $E$ there exists an automorphy factor $A : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{E} \cong \mathcal{E}(A)$.*
(2) *For any automorphy factors $A : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ and $B : \mathbb{C} \to \mathrm{GL}_m(\mathbb{C})$ we have:*

$$\mathrm{Hom}\big(\mathcal{E}(A), \mathcal{E}(B)\big) \cong \mathrm{Sol}_{A,B} := \left\{ \Phi : \mathbb{C} \to \mathrm{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)A(z) = B(z)\Phi(z) \end{array} \right\}$$

*and $\mathcal{E}(A) \otimes \mathcal{E}(B) \cong \mathcal{E}(A \otimes B)$.*

*Proof.* This result is a corollary of the monoidal equivalence of the category of $\Lambda$-equivariant holomorpic vector bundles on $\mathbb{C}$ and holomorphic vector bundles on the quotient torus $E = \mathbb{C}/\Lambda$. See [4] or [28] for a detailed proof. □

**Corollary 10.4.** *For any pair of automorphy factors $A, S : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ we have an isomorphism of vector bundles $\mathcal{E}(A) \cong \mathcal{E}(B)$, where $B(z) = S(z+\tau)^{-1}A(z)S(z)$. In particular, we have an isomorphism $\mathcal{E}(A) \cong \mathcal{E}(\widehat{A})$, where $\widehat{A}(z) = \exp(2\pi i\tau)A(z)$.*

In the next step, we need an explicit description of the indecomposable semi-stable vector bundles on $E$ of degree zero.

**Theorem 10.5.** *Let $E = \mathbb{C}/\Lambda$ be a complex torus.*

(1) *The map $\mathbb{C} \to \mathsf{Pic}(E)$ assigning to $\lambda \in \mathbb{C}$ the line bundle $\mathcal{L}_\lambda := \mathcal{E}\big(\exp(2\pi i\lambda)\big)$ yields a bijection between the points of $E$ and the isomorphy classes of degree zero line bundles on $E$.*

(2) *For any $m \geq 1$ let*

$$J_m = J_m(1) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_m(\mathbb{C}).$$

*Then $\mathcal{E}(J_m)$ is isomorphic to the Atiyah bundle $\mathcal{A}_m$ defined as follows. For $m = 1$ we set $\mathcal{A}_1 = \mathcal{O}$ and for $m \geq 2$ the vector bundle $\mathcal{A}_m$ is recursively defined by the following property: it is the unique (up to an isomorphism) vector bundle occurring as the middle term of a non-split short exact sequence*

$$0 \longrightarrow \mathcal{A}_{m-1} \longrightarrow \mathcal{A}_m \longrightarrow \mathcal{O} \longrightarrow 0.$$

(3) *Let $B \in \mathrm{GL}_n(\mathbb{C})$ and $J = J_{m_1}(\mu_1) \oplus \cdots \oplus J_{m_t}(\mu_t)$ be the Jordan normal form of $B$ with $\mu_l = \exp(2\pi i\lambda_l)$ for some $\lambda_l \in \mathbb{C}$, $1 \leq l \leq t$. Then we have:*

$$\mathcal{E}(B) \cong (\mathcal{L}_{\lambda_1} \otimes \mathcal{A}_{m_1}) \oplus \cdots \oplus (\mathcal{L}_{\lambda_t} \otimes \mathcal{A}_{m_t}).$$

*In particular, $\mathcal{E}(B)$ is a semi-stable vector bundle of degree zero on the torus $E$, whose Jordan-Hölder quotients are $\mathcal{L}_{\lambda_1}, \dots, \mathcal{L}_{\lambda_t}$. Moreover, for any semi-stable vector bundle $\mathcal{E}$ of rank $n$ and degree zero on the torus $E$ there exists a matrix $B \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{E} \cong \mathcal{E}(B)$.*

*Proof.* A proof of the first two statements can for instance be found in [14, Section 8.1] or in [28]. To show the third one observe that by Corollary 10.4 we have an isomorphism $\mathcal{E}(B) \cong \mathcal{E}(J)$. Since for any $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ we have an isomorphism $\mathcal{E}\big(J_m(\lambda)\big) \cong \mathcal{L}_\lambda \otimes \mathcal{A}_m$, we have: $\mathcal{E}(J) \cong (\mathcal{L}_{\lambda_1} \otimes \mathcal{A}_{m_1}) \oplus \cdots \oplus (\mathcal{L}_{\lambda_t} \otimes \mathcal{A}_{m_t})$. Hence, the result follows from Atiyah's classification of vector bundles on $E$ [1]. □

**Corollary 10.6.** *Let* $A \in GL_n(\mathbb{C})$ *and* $\mathfrak{G}(A) = \{\exp(2\pi i\lambda_1), \dots, \exp(2\pi i\lambda_n)\}$ *be its spectrum,* $B \in \mathrm{GL}_m(\mathbb{C})$ *and* $\mathfrak{G}(B) = \{\exp(2\pi i\mu_1), \dots, \exp(2\pi i\mu_m)\}$ *be its spectrum. Assume that* $\lambda_k - \mu_l \notin \Lambda$ *for all* $1 \le k \le n$ *and* $1 \le l \le m$*. Then we have:*

$$\mathrm{Hom}\big(\mathcal{E}(A), \mathcal{E}(B)\big) = 0 = \mathrm{Ext}^1\big(\mathcal{E}(A), \mathcal{E}(B)\big).$$

*Proof.* The assumption on the eigenvalues of $A$ and $B$ implies that the degree zero semi-stable vector bundles $\mathcal{E}(A)$ and $\mathcal{E}(B)$ have no common Jordan-Hölder quotients. From this fact it follows that

$$\mathrm{Hom}\big(\mathcal{E}(A), \mathcal{E}(B)\big) = 0 = \mathrm{Hom}\big(\mathcal{E}(B), \mathcal{E}(A)\big) \cong \mathrm{Ext}^1\big(\mathcal{E}(A), \mathcal{E}(B)\big)^*,$$

where the last isomorphism is given by the Serre duality. □

**Lemma 10.7.** *Let* $\varphi(z) = \exp(-\pi i\tau - 2\pi iz)$*,* $x \in \mathbb{C}$ *and* $[x]$ *be the corresponding divisor of degree one on* $E$*. Then we have an isomorphism:*

$$\mathcal{O}_E\big([x]\big) \cong \mathcal{E}\big(\varphi(z + \frac{\tau+1}{2} - x)\big).$$

*Proof.* A proof of this result can be for instance found in [14, Section 8.1]. □

## 11. The category of triples and Matrix problems

11.1. **The category of Triples.** In this subsection, we recall a general technique to describe vector bundles on singular projective curves, see [6] and [14, Section 5.1] as well as references therein.

Let $X$ be a reduced singular (projective) curve, $\pi : \widetilde{X} \longrightarrow X$ its normalisation, $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\big) = \mathcal{A}nn_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\big)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \longrightarrow X$ the closed Artinian subspace defined by $\mathcal{I}$ (its topological support is precisely the singular locus of $X$) and by $\tilde{\eta} : \widetilde{Z} \longrightarrow \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde{\eta}} & \widetilde{X} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\eta} & X. \end{array} \tag{11.1}$$

In what follows we shall denote $\nu = \eta\tilde{\pi} = \pi\tilde{\eta}$.

In order to relate vector bundles on $X$ and $\widetilde{X}$ we need the following construction.

**Definition 11.1.** The category $\mathrm{Tri}_X$ is defined as follows.

(1) Its objects are triples $\big(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}}\big)$, where $\widetilde{\mathcal{F}} \in \mathrm{VB}_{\widetilde{X}}$, $\mathcal{V} \in \mathrm{VB}_Z$ and

$$\widetilde{\mathsf{m}} : \tilde{\pi}^*\mathcal{V} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{F}}$$

is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$-modules, called the *gluing map.*

(2) The set of morphisms $\mathrm{Hom}_{\mathrm{Tri}_X}\big((\widetilde{\mathcal{F}}_1, \mathcal{V}_1, \widetilde{\mathsf{m}}_1), (\widetilde{\mathcal{F}}_2, \mathcal{V}_2, \widetilde{\mathsf{m}}_2)\big)$ consists of all pairs $(f, g)$, where $f : \widetilde{\mathcal{F}}_1 \longrightarrow \widetilde{\mathcal{F}}_2$ and $g : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ are morphisms of vector bundles such that the following diagram is commutative

$$\begin{array}{ccc} \tilde{\pi}^*\mathcal{V}_1 & \xrightarrow{\widetilde{\mathsf{m}}_1} & \tilde{\eta}^*\widetilde{\mathcal{F}}_1 \\ {\scriptstyle \tilde{\pi}^*(g)}\downarrow & & \downarrow{\scriptstyle \tilde{\eta}^*(f)} \\ \tilde{\pi}^*\mathcal{V}_2 & \xrightarrow{\widetilde{\mathsf{m}}_2} & \tilde{\eta}^*\widetilde{\mathcal{F}}_2. \end{array}$$

The main reason to introduce the above Definition is the following result.

**Theorem 11.2.** *Let $X$ be a reduced curve. Then the following results are true.*

(1) *Let $\mathbb{F} : \mathrm{VB}_X \longrightarrow \mathrm{Tri}_X$ be the functor assigning to a vector bundle $\mathcal{F}$ the triple $(\pi^*\mathcal{F}, \eta^*\mathcal{F}, \widetilde{\mathsf{m}}_{\mathcal{F}})$, where $\widetilde{\mathsf{m}}_{\mathcal{F}} : \tilde{\pi}^*(\eta^*\mathcal{F}) \longrightarrow \tilde{\eta}^*(\pi^*\mathcal{F})$ is the canonical isomorphism. Then $\mathbb{F}$ is an equivalence of categories.*

(2) *Let $\mathbb{G} : \mathrm{Tri}_X \longrightarrow \mathrm{Coh}(X)$ be the functor assigning to a triple $(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}})$ the coherent sheaf $\mathcal{F} := \ker\big(\pi_*\widetilde{\mathcal{F}} \oplus \eta_*\mathcal{V} \xrightarrow{(\mathsf{c}\ \mathsf{m})} \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}\big)$, where $\mathsf{c} = \mathsf{c}^{\widetilde{\mathcal{F}}}$ is the canonical morphism $\pi_*\widetilde{\mathcal{F}} \longrightarrow \pi_*\tilde{\eta}_*\tilde{\eta}^*\widetilde{\mathcal{F}} = \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}$ and $\mathsf{m}$ is the composition $\eta_*\mathcal{V} \xrightarrow{\mathsf{can}} \eta_*\tilde{\pi}_*\tilde{\pi}^*\mathcal{V} \xrightarrow{=} \nu_*\tilde{\pi}^*\mathcal{V} \xrightarrow{\nu_*(\widetilde{\mathsf{m}})} \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}$. Then the coherent sheaf $\mathcal{F}$ is locally free. Moreover, the functor $\mathbb{G}$ is quasi-inverse to $\mathbb{F}$.*

For a proof of this result, see [10, Theorem 1.3]. □

**Lemma 11.3.** *Let $\mathcal{T}_i = (\widetilde{\mathcal{F}}_i, \mathcal{V}_i, \widetilde{\mathsf{m}}_i)$, $i = 1, 2$ be a pair of objects of $\mathrm{Tri}_X$ and $\mathcal{F}_i = \mathbb{G}(\mathcal{T}_i)$. Then we have:*

$$\mathbb{F}\big(\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)\big) \cong \Big(\mathcal{H}om_{\widetilde{X}}(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2), \mathcal{H}om_Z(\mathcal{V}_1, \mathcal{V}_2), h(\widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2)\Big), \tag{11.2}$$

*where $h(\widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2)$ is the morphism making the following diagram commutative:*

$$\begin{array}{ccc} \tilde{\pi}^*\mathcal{H}om_Z(\mathcal{V}_1, \mathcal{V}_2) & \xrightarrow{h(\widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2)} & \tilde{\eta}^*\mathcal{H}om_{\widetilde{X}}(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2) \\ {\scriptstyle \mathsf{can}}\downarrow & & \downarrow{\scriptstyle \mathsf{can}} \\ \mathcal{H}om_{\widetilde{Z}}(\tilde{\pi}^*\mathcal{V}_1, \tilde{\pi}^*\mathcal{V}_2) & \xrightarrow{\mathsf{cnj}(\widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2)} & \mathcal{H}om_{\widetilde{Z}}(\tilde{\eta}^*\widetilde{\mathcal{F}}_1, \tilde{\eta}^*\widetilde{\mathcal{F}}_2). \end{array}$$

*In this diagram, $\mathsf{cnj}(\widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2)(\varphi) = \widetilde{\mathsf{m}}_2 \circ \varphi \circ \widetilde{\mathsf{m}}_1^{-1}$.*

*Proof.* Since $\mathbb{F}$ and $\mathbb{G}$ are quasi-inverse equivalences of categories, $\mathcal{T}_i \cong \big(\pi^*\mathcal{F}_i, \eta^*\mathcal{F}_i, \widetilde{\mathsf{m}}_{\mathcal{F}_i}\big)$ for $i = 1, 2$. Note that we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{\pi}^*\eta^*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) & \xrightarrow{\widetilde{\mathsf{m}}_{\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2)}} & \tilde{\eta}^*\pi^*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \\
\Big\downarrow \mathsf{can} & & \Big\downarrow \mathsf{can} \\
\mathcal{H}om_{\widetilde{Z}}(\tilde{\pi}^*\eta^*\mathcal{F}_1, \tilde{\pi}^*\eta^*\mathcal{F}_2) & \xrightarrow{\mathsf{cnj}(\widetilde{\mathsf{m}}_{\mathcal{F}_1}, \widetilde{\mathsf{m}}_{\mathcal{F}_2})} & \mathcal{H}om_{\widetilde{Z}}(\tilde{\eta}^*\pi^*\mathcal{F}_1, \tilde{\eta}^*\pi^*\mathcal{F}_2).
\end{array}$$

This implies the claim. ☐

**Proposition 11.4.** *Let $\mathcal{T} = (\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}}) \in \mathrm{Tri}_X$ and $\mathcal{F} = \mathbb{G}(\mathcal{T})$. Then we have:*

$$\mathbb{F}\big(\mathrm{Ad}(\mathcal{F})\big) \cong \big(\mathrm{Ad}(\widetilde{\mathcal{F}}), \mathrm{Ad}(\mathcal{V}), \mathsf{ad}(\widetilde{\mathsf{m}})\big), \tag{11.3}$$

*where $\mathsf{ad}(\widetilde{\mathsf{m}})$ is the morphism making the following diagram commutative:*

$$\begin{array}{ccc}
\tilde{\pi}^*\mathrm{Ad}(\mathcal{F}) & \xrightarrow{\mathsf{ad}(\widetilde{\mathsf{m}})} & \eta^*\mathrm{Ad}(\widetilde{\mathcal{F}}) \\
\Big\downarrow \mathsf{can} & & \Big\downarrow \mathsf{can} \\
\mathrm{Ad}(\pi^*\mathcal{F}) & \xrightarrow{\mathsf{cnj}(\widetilde{\mathsf{m}})} & \mathrm{Ad}(\tilde{\eta}^*\widetilde{\mathcal{F}}).
\end{array}$$

*Sketch of the proof.* Let $\mathcal{F}$ be an arbitrary vector bundle on $X$. Then the following diagram of coherent sheaves is commutative:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ad}(\mathcal{F}) & \to & \pi_*\mathrm{Ad}(\pi^*\mathcal{F}) \oplus \eta_*\mathrm{Ad}(\eta^*\mathcal{F}) & \to & \nu_*\mathrm{Ad}(\nu^*\mathcal{F}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}nd(\mathcal{F}) & \to & \pi_*\mathcal{E}nd(\pi^*\mathcal{F}) \oplus \eta_*\mathcal{E}nd(\eta^*\mathcal{F}) & \to & \nu_*\mathcal{E}nd(\nu^*\mathcal{F}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X & \to & \pi_*\mathcal{O}_{\widetilde{X}} \oplus \eta_*\mathcal{O}_Z & \to & \nu_*(\mathcal{O}_{\widetilde{Z}}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The middle horizontal sequence in this diagram is exact by Lemma 11.3. It shows that the first horizontal sequence is exact, too. This proves the formula (11.3). ☐

11.2. **Reduction to Matrix problems.** In this very short subsection, we introduce the category of Matrix problems.

**Definition 11.5.** Let $\mathrm{MP}_X$ be the following Krull-Schmidt category:

- an object of $\mathrm{MP}_X$ is given by a map $\tilde{\mathrm{m}}$ for which there exists a triple $\left(\tilde{\mathcal{F}}, \mathcal{V}, \tilde{\mathrm{m}}\right) \in \mathrm{Tri}_X$.
- for two objects $\tilde{\mathrm{m}}_1, \tilde{\mathrm{m}}_2$ with corresponding triples $\left(\tilde{\mathcal{F}}_1, \mathcal{V}_1, \tilde{\mathrm{m}}_1\right)$ and $\left(\tilde{\mathcal{F}}_2, \mathcal{V}_2, \tilde{\mathrm{m}}_2\right)$ respectively, a morphism from $\tilde{\mathrm{m}}_1$ to $\tilde{\mathrm{m}}_2$ is given by a pair $(\tilde{\eta}^* F, \tilde{\pi}^* f)$ such that $\tilde{\eta}^* F \circ \tilde{\mathrm{m}}_1 = \tilde{\mathrm{m}}_2 \circ \tilde{\pi}^* f$, where $(F, f) \in \mathrm{Hom}_{\mathrm{Tri}_X}\left(\left(\tilde{\mathcal{F}}_1, \mathcal{V}_1, \tilde{\mathrm{m}}_1\right), \left(\tilde{\mathcal{F}}_2, \mathcal{V}_2, \tilde{\mathrm{m}}_2\right)\right)$.

**Corollary 11.6.** *The functor* $\mathbb{H} : \mathrm{Tri}_X \to \mathrm{MP}_X$ *which sends an object* $\left(\tilde{\mathcal{F}}, \mathcal{V}, \tilde{\mathrm{m}}\right)$ *to* $\tilde{\mathrm{m}}$ *and a morphism* $(F, f)$ *to* $(\tilde{\eta}^* F, \tilde{\pi}^* f)$ *is full and dense.*

The reasons for our interest in this category will become obvious in the following subsections.

11.3. **Matrix problem for the cuspidal cubic curve.** In this subsection, let $X = E$ be the cuspidal cubic curve given by the equation $zy^2 = x^3$ and $\Bbbk = \mathbb{C}$. Going through the construction of $\mathrm{Tri}_E$ and $\mathrm{MP}_E$ in this specific case, we will see that both objects and morphisms in $\mathrm{MP}_E$ can actually be interpreted as matrices over $\mathbb{C}$.

First note that $\tilde{X} = \tilde{E}$ equals $\mathbb{P}^1$. We choose homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$. Thus the normalization map is given by $\pi(z_0 : z_1) = (z_0^2 z_1 : z_0^3 : z_1^3)$ and the preimage of the singular point $s = (0 : 0 : 1) \in E$ is $\pi^{-1}(s) = (0 : 1) = \infty$. Let $U' = \{(z_0 : z_1) | \, z_1 \neq 0\}$ be an affine neighbourhood of $\infty \in \mathbb{P}^1$ with local coordinate $t = z_0/z_1$ and set $U = \pi(U')$. Then locally $\pi$ is given by $\mathbb{C}[U] = \mathbb{C}[t^2, t^3] \hookrightarrow \mathbb{C}[t]$. Moreover, $Z$ is the reduced point $s$ with structure sheaf $\mathcal{O}_Z = \mathbb{C}_s$ . Since the conductor equals $\mathcal{I} =< t^2, t^3 >$ on $U$, the structure sheaf $\mathcal{O}_{\tilde{Z}}$ of the non-reduced point $\tilde{Z} = \{\infty\}$ is given by $\mathbf{R} = (\mathbb{C}[\epsilon]/\epsilon^2)_\infty$. Next, recall the following result:

**Theorem 11.7.** (Birkhoff-Grothendieck). *Any vector bundle* $\mathcal{E}$ *on* $\mathbb{P}^1$ *splits into a direct sum of line bundles,* $\mathcal{E} \cong \oplus_{i\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(i)^{n_i}$. *Moreover, the degree gives an isomorphism*

$$\deg\colon \mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\cong} \mathbb{Z}.$$

Thus it follows from Theorem 11.2 that if $\mathcal{F} \in \mathrm{VB}_E$ with $\mathrm{rk}\mathcal{F} = n$ and $\mathbb{F}(\mathcal{F}) = \left(\tilde{\mathcal{F}}, \mathcal{V}, \tilde{\mathrm{m}}\right) \in \mathrm{Tri}_X$, then $\tilde{\mathcal{F}} \cong \oplus \mathcal{O}_{\mathbb{P}^1}(i)^{n_i}$ with $\sum_{i\in\mathbb{Z}} n_i = n$. Moreover, $\mathcal{V} \cong \mathcal{O}_Z^n$. Next, we want to use these results in order to show how to interpret $\tilde{\mathrm{m}}$ as a matrix.

By the construction of $\mathrm{Tri}_E$, $\tilde{\mathrm{m}} : \tilde{\pi}^* \mathcal{V} \to \tilde{\eta}^* \tilde{\mathcal{F}}$ is an isomorphism of $\mathcal{O}_{\tilde{Z}}$-modules. We have $\tilde{\pi}^* \mathcal{V} \cong \mathcal{O}_{\tilde{E}}^n$ canonically. We also have $\tilde{\eta}^* \tilde{\mathcal{F}} \cong \mathcal{O}_{\tilde{Z}}^n$, but there are some choices involved. Indeed, we need to choose trivializations $\mathcal{O}_{\tilde{E}}(i) \otimes \mathcal{O}_{\tilde{E}}/\mathcal{I} \to \mathcal{O}_{\tilde{Z}}$ for each $i$, which we will assume to be given by

$$\zeta \otimes 1 \mapsto \mathrm{pr}\left(\frac{\zeta}{z_1^i}\right) \tag{11.4}$$

for a local section $\zeta$ of $\mathcal{O}_{\tilde{E}}(i)$. Here $\mathrm{pr} : \mathbb{C}[U] \to \mathbb{C}[\epsilon]/\epsilon^2$ is the map induced by the canonical map $\mathbb{C}[t] \to \mathbb{C}[\epsilon]/\epsilon^2$, $t \mapsto \epsilon$. Thus finally, $\tilde{\mathrm{m}}$ can indeed be interpreted as an element $\mu$ of $\mathrm{GL}_n(\mathbf{R})$. In other words, $\tilde{\mathrm{m}}$ corresponds to

$$\mu = \mu_0 + \epsilon\mu_\epsilon$$

with $\mu_0, \mu_\epsilon \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. It is easy to see that $\tilde{\mathrm{m}}$ being an isomorphism is equivalent to $\mu_0 \in \mathrm{GL}_n(\mathbb{C})$.

Let us examine the morphisms in $\mathrm{MP}_E$. Since we have chosen coordinates $(z_0 : z_1)$ on $\tilde{E} = \mathbb{P}^1$, any morphism $\mathcal{O}_{\mathbb{P}^1}(i) \to \mathcal{O}_{\mathbb{P}^1}(j)$ is given by a homogeneous form $Q(z_0 : z_1)$ of degree $j - i$, that is

$$\mathrm{Hom}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(j)\right) \cong \mathbb{C}\left[z_0, z_1\right]_{j-i}.$$

Thus for two objects $\left(\tilde{\mathcal{F}}_1, \mathcal{V}_1, \tilde{\mathrm{m}}_1\right)$ and $\left(\tilde{\mathcal{F}}_2, \mathcal{V}_2, \tilde{\mathrm{m}}_2\right)$ of $\mathrm{Tri}_E$ with $\tilde{\mathcal{F}}_1 \cong \oplus\mathcal{O}_{\mathbb{P}^1}(i)^{a_i}$ and $\tilde{\mathcal{F}}_2 \cong \oplus\mathcal{O}_{\mathbb{P}^1}(j)^{b_j}$, the first part $F$ of a morphism $(F, f) : \left(\tilde{\mathcal{F}}_1, \mathcal{V}_1, \tilde{\mathrm{m}}_1\right) \to \left(\tilde{\mathcal{F}}_2, \mathcal{V}_2, \tilde{\mathrm{m}}_2\right)$ is simply given by a matrix $(F_{ij})$ where $F_{ij} \in \mathrm{Mat}_{a_i\times b_j}\left(\mathbb{C}\left[z_0, z_1\right]_{j-i}\right)$.

*Remark* 11.8. Note that if $F$ is an endomorphism, then $F_{ij} = 0$ for any $j > i$, so $(F_{ij})$ is lower triangular. Moreover, $F$ is an isomorphism if and only if each $F_{ii}$ is invertible.

Due to the chosen trivializations (11.4), any morphism $\mathcal{O}_{\mathbb{P}^1}(i) \to \mathcal{O}_{\mathbb{P}^1}(j)$ given by a homogeneous form $Q(z_0, z_1)$ of degree $j - i$ induces a map $\mathcal{O}_{\mathbb{P}^1}(i) \otimes \mathcal{O}_{\tilde{Z}} \to \mathcal{O}_{\mathbb{P}^1}(j) \otimes \mathcal{O}_{\tilde{Z}}$ given by

$$\mathrm{pr}\left(Q(z_0 : z_1)/z_1^{j-i}\right) = Q(0 : 1) + \epsilon\frac{dQ}{dz_0}(0 : 1).$$

Hence, identifying the gluing matrices $\tilde{\mathrm{m}}_1, \tilde{\mathrm{m}}_2$ with matrices $\mu_1, \mu_2$ as above, for a morphism $(\tilde{\eta}^*F, \tilde{\pi}^*f) : \mu_1 \to \mu_2$ we have

$$\tilde{\eta}^*F = F(0 : 1) + \epsilon\frac{dF}{dz_0}(0 : 1) \in \mathrm{Mat}\left(\mathbb{C}\left[\epsilon\right]/\epsilon^2\right).$$

Here we identify $F$ with the induced matrix $(F_{ij})$ from above in order for this to make sense. Finally, the second component $\tilde{\pi}^*f$ of a morphism $(\tilde{\eta}^*F, \tilde{\pi}^*f) : \mu_1 \to \mu_2$ can be interpreted as an element of $\mathrm{Mat}(\mathbb{C})$ of appropriate size as well.

11.4. **Primary reduction.** Using the results from the previous subsections, we show how to recover families of simple vector bundles of fixed rank and degree on $E$ from the study of the corresponding matrix problem. As it shall turn out, simple vector bundles correspond essentially to bricks of a given Matrix problem.

**Definition 11.9.** A vector bundle $\mathcal{F}$ on $E$ is called simple if $\mathrm{End}(\mathcal{F}) \cong \mathbb{C}$. By $\mathrm{Simp}_E$ we denote the subcategory of simple objects of $\mathrm{VB}_E$. Likewise, an object $\tilde{\mathrm{m}}$ of $\mathrm{MP}_E$ is called a brick if $\mathrm{End}(\tilde{\mathrm{m}}) \cong \mathbb{C}$ and we denote the subcategory of bricks of $\mathrm{MP}_E$ by $\mathrm{MP}^s_E$.

Since we are mostly interested in simple vector bundles on $E = V(x^3 - y^2 z)$, let us recall the following result:

**Lemma 11.10.** [14, Lemma 9.6 and 9.7] *Let $\mathcal{F}$ be a simple vector bundle of rank $n$ on $E$ and $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$. Then*

$$\tilde{\mathcal{F}} \cong \mathcal{O}_{\mathbb{P}^1}(c)^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(c+1)^{n_2} \tag{11.5}$$

*for certain $c \in \mathbb{Z}$ and $n_1 + n_2 = n$. Moreover* $\deg(\mathcal{F}) = \deg(\tilde{\mathcal{F}})$.

*Remark* 11.11. Note that if $\mathcal{F}$ is a any vector bundle of rank $n$ and degree $d$ on $E$ such that $\tilde{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$, then $n_1 = n - d$ and $n_2 = d$. Especially, $n \geq d$.

**Definition 11.12.** Let $\mathrm{VB}^0_E$ denote the full subcategory of $\mathrm{VB}_E$ satisfying (11.5) for $c = 0$. Furthermore, by $\mathrm{VB}_E(n,d)$ we denote the subcategory of $\mathrm{VB}_E$ whose objects have rank $n$ and degree $d$ and let $\mathrm{VB}^0_E(n,d)$ denote the corresponding subcategory of $\mathrm{VB}^0_E$. Moreover, by $\mathrm{Simp}^0_E$ and $\mathrm{Simp}^0_E(n,d)$ we denote the subcategory of simple vector bundles of $\mathrm{VB}^0_E$ and $\mathrm{VB}^0_E(n,d)$ respectively.

Recall that the first part of a morphism $(\tilde{\eta}^*F, \tilde{\pi}^*f)$ in $\mathrm{MP}_E$ derives from a matrix $F = (F_{ij})_{1 \leq i,j \leq 2}$. The induced block decomposition of $\tilde{\eta}^*F$ can be used to define a certain subcategory of $\mathrm{MP}_E$:

**Definition 11.13.** For any $\mathbf{r} = (r_1, r_2) \in \mathbb{N}^2$, let $\mathrm{MP}(\mathbf{r})$ denote the full subcategory of MP with objects $\mu$ having a block decomposition of the form

$$\mu = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

Here each $A_{ij}$ is a matrix over $\mathbb{C}[\epsilon]/\epsilon^2$ and we use the notation above to indicate that $A_{ij}$ is of size $r_i \times r_j$. By $\mathrm{MP}^s_E(\mathbf{r})$ we denote the subcategory of bricks of $\mathrm{MP}^s_E(\mathbf{r})$.

**Proposition 11.14.** [14, Proposition 9.11] *For any $n \in \mathbb{N}$ and $d \in \mathbb{Z}$, the restriction of the composition* $\mathrm{VB}_E \xrightarrow{\mathbb{F}} \mathrm{Tri}_E \xrightarrow{\mathbb{H}} \mathrm{MP}_E$ *yields an equivalence of categories*

$$\mathrm{VB}^0_E\,(n,d) \cong \mathrm{MP}_E(n - \bar{d}, \bar{d}),$$

*where $0 \leq \bar{d} < n$ is determined by $\bar{d} \equiv d \mod n$. Especially, we have*

$$\mathrm{Simp}^0_E(n,d) \cong \mathrm{MP}^s_E(n-\bar{d},\bar{d}).$$

It follows from the definitions that in the category $\mathrm{MP}_E(r_1,r_2)$, the isomorphism class of an object $\mu = \mu_0 + \epsilon\mu_\epsilon$ is given by all objects of the form $\tilde{\eta}^*F \cdot \mu \cdot \tilde{\pi}^* f^{-1}$. Choosing $F = \mathrm{id}$ and $f = \mu_0$, we see that $\mu$ is isomorphic to $\mathbb{1} + \epsilon\mu'_\epsilon$ where $\mu'_\epsilon = \mu_\epsilon\mu_0^{-1}$. Next, write

$$\mu_\epsilon = \left(\begin{array}{c|c} (\mu'_\epsilon)_{11} & (\mu'_\epsilon)_{12} \\ \hline (\mu'_\epsilon)_{21} & (\mu'_\epsilon)_{22} \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array} \text{ and let } F = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline (-\mu'_\epsilon)_{21} & \mathbb{1} \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

Then

$$\tilde{\eta}^*F = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & \mathbb{1} \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline (-\mu'_\epsilon)_{21} & 0 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

Choosing $f = \mathrm{id}$, we conclude that $\mu$ is isomorphic to $\mathbb{1} + \epsilon\theta$ where

$$\theta = \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

for certain matrices $X_1, X_2, Y$.

**Definition 11.15.** Let $\overline{\mathrm{MP}}_E(\mathbf{r})$ denote the subcategory of $\mathrm{MP}_E(\mathbf{r})$ with objects of the form $\mu = \mathbb{1} + \epsilon\theta$ where

$$\theta = \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

Moreover, let $\mathrm{pr}_\mathbf{r}$ denote the projection map on $\mathrm{Mat}_{(r_1+r_2)\times(r_1+r_2)}(\mathbb{C})$ given by

$$\left(\begin{array}{c|c} X_1 & Y \\ \hline M & X_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array} \mapsto \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array}.$$

It is clear that if we want to preserve the form $\mu = \mathbb{1} + \epsilon\theta$, we have to restrict to those morphisms $(\tilde{\eta}^*F, \tilde{\pi}^*f)$ starting in $\mu$ which satisfy $\tilde{\pi}^*f = (\tilde{\eta}^*F)^{-1}$.

**Corollary 11.16.** [14, Lemma 9.31] *The following hold:*

(1) *In the category $\overline{\mathrm{MP}}_E(\mathbf{r})$, the set of morphisms between objects $\mu = \mathbb{1} + \epsilon\theta$ and $\mu' = \mathbb{1} + \epsilon\theta'$ is given by*

$$\mathrm{Hom}_{\overline{\mathrm{MP}}_E(\mathbf{r})}(\mu,\mu') = \{S \in \mathbb{S} \,|\, \mathrm{pr}_\mathbf{r}(S\theta) = \mathrm{pr}_\mathbf{r}(\theta' S)\},$$

*where*

$$\mathbb{S} = \left\{ \left(\begin{array}{c|c} S_1 & 0 \\ \hline A & S_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array} \right\} \subset \mathrm{GL}_{r_1+r_2}(\mathbb{C}).$$

(2) *The natural inclusion $\overline{\mathrm{MP}}_E(\mathbf{r}) \hookrightarrow \mathrm{MP}_E(\mathbf{r})$ is an equivalence of categories.*

Combining the results obtained so far yields:

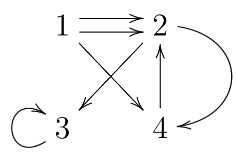
**Corollary 11.17.** *There is an equivalence of categories* $\mathrm{VB}^0_E(n,d) \cong \overline{\mathrm{MP}}_E(n-\bar{d},\bar{d})$. *Especially, we have* $\mathrm{Simp}^0_E(n,d) \cong \overline{\mathrm{MP}}^s_E(n-\bar{d},\bar{d})$, *where the latter category's objects consist of all bricks in* $\overline{\mathrm{MP}}_E(n-\bar{d},\bar{d})$.

## 12. Differential Biquivers

In this and the next section we explain how to classify objects in $\mathrm{VB}^s_E(n,d)$. The first step in solving this problem is the translation to the category of bricks of the corresponding matrix problem as stated in Corollary 11.17. The next step is to express the classification of isomorphism classes of objects in $\mathrm{MP}^s_E(\mathbf{r})$ in terms of the representation theory of a certain differential biquiver. Before going into details, we recall some general notions and results from the theory of differential biquivers.

12.1. **Differential biquivers.** Recall that a quiver $Q$ is given by a set of vertices $I$ and for each $i,j \in I$ a (possibly empty) set of arrows $Q(i,j)$.

**Example 12.1.** Let $I = \{1,2,3,4\}$. An example for a quiver with vertex set $I$ is given by




**Definition 12.2.** The category of representations $\mathrm{Rep}_Q$ of a quiver $Q$ with vertex set $I$ over a field $\Bbbk$ is defined as follows:

- an object $M$ consists of a collection of $\Bbbk$-vector spaces $\{M_i\}_{i\in I}$ together with $k$-linear maps $M_a : M(i) \to M(j)$ for each arrow $a \in Q(i,j)$.
- a morphism $f$ between a representation $M$ and a representation $N$ is given by a collection of $\Bbbk$-linear maps $\{f_i\}_{i\in I}$ such that for each arrow $a : i \to j$ the following diagram is commutative

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & N_i \\ {\scriptstyle M(a)}\downarrow & & \downarrow{\scriptstyle N(a)} \\ M_j & \xrightarrow{f_j} & N_j. \end{array}$$

To any quiver $Q$ we may associate the path algebra $\Bbbk Q$ over the field $\Bbbk$ as follows. A path is a concatenation of composable arrows, where two arrows $a \in Q(i,j)$ and $b \in Q(l,m)$ are called composable if $j = l$, in which case the concatenation is written as $ba$. As a vector space, $\Bbbk Q$ has a basis given by the set of paths in $Q$ plus the set of so called trivial paths $\{e_i\}_{i\in I}$, where $e_i$ behaves as a loop at the vertex $i$ with respect to composition. Multiplication of two basis elements of $\Bbbk Q$ is given by concatenation, the product being zero if the paths are not composable. It is a standard fact from the representation theory of quivers that $\mathrm{Mod}_{\Bbbk Q}$, the category of modules over $\Bbbk Q$, is equivalent to the category $\mathrm{Rep}_Q$.

**Definition 12.3.** Recall the notion of a differential biquiver:

- A biquiver $Q = (I, Q_0, Q_1)$ is given by a vertex set $I$ and two sets of arrows, $Q_0$ and $Q_1$ such that both $(I, Q_0)$ and $(I, Q_1)$ are quivers. The elements of $Q_0$ and $Q_1$ are called solid and dotted arrows respectively.
- Given a biquiver $Q = (I, Q_0, Q_1)$, we equip the path algebra $\Bbbk Q = k(Q_0 \cup Q_1)$ with a grading as follows:
  - to any solid arrow and any trivial path $e_i$ we assign the degree zero.
  - to any dotted arrow we assign the degree one.
  - to any path we assign the degree given by the number of dotted arrows it contains.
- Given a biquiver $Q = (I, Q_0, Q_1)$, then a $k$-linear map $\partial : \Bbbk Q \to \Bbbk Q$ is called differential for $Q$ if it satisfies:
  - $\partial$ raises degree by one and $\partial^2 = 0$
  - $\partial(e_i) = 0$ for each trivial path $e_i$
  - for any two paths $a, b \in \Bbbk Q$, the Leibniz rule holds

$$\partial(xy) = \partial(x)\, y + (-1)^{\deg x}\, x\, \partial(y).$$

- A differential biquiver is a tuple $(Q, \partial)$ where $Q$ is a biquiver and $\partial$ is a differential for $Q$.

Next, we will explain the category of representations of a differential biquiver.

**Definition 12.4.** The category of representations $\mathrm{Rep}_{(Q,\partial)}$ of a differential biquiver $Q = (I, Q_0, Q_1)$ over the field $\Bbbk$ is defined as follows:

- an object $M$ is given by a representation of the quiver $Q_0$.
- in order to explain morphisms in $\mathrm{Rep}_{(Q,\partial)}$, we need to introduce another quiver $\Gamma$. The quiver $\Gamma$ consists of two copies of $Q_0$, denoted $Q_0$ and $Q_0'$ plus some additional arrows from vertices of $Q_0$ to vertices of $Q_0'$. Namely, for each $i \in I$, there is an arrow $w_i : i \mapsto i'$ and for each dotted arrow $v : i \dashrightarrow j$ in $Q_1$, there is a corresponding arrow $v : i \to j$ in $\Gamma$. Now a morphism $S$ between representations $M$ and $N$ is a representation of the quiver $\Gamma$ such that the restriction of $S$ to $Q_0$ and $Q_0'$ are exactly $M$ and $N$ respectively and such that for any solid arrow $a : i \to j$ the following relation holds:

$$S\left(\partial(a)\right) = N(a')S(w_i) - S(w_j)M(a).$$

  The composition $T \circ S$ of two morphisms $S : L \to M$ and $T : M \to N$ is defined as follows:

– for $w_i : i \to i'$: $(T \circ S)(w_i) = T(w_i) \cdot S(w_i)$.
– for a dotted arrow $v : j \dashrightarrow i$, let us write

$$\partial(v) = \sum \alpha\, p_1 u p_2 u' p_3,$$

where $p_i$ are paths of solid arrows and $u$,$u'$ are dotted arrows such that $p_1 u p_2 u' p_3$ is a path from $i$ to $j$ and $\alpha \in k$. Then

$$(T \circ S)(v) = T(w_i) \cdot S(v) + T(v) \cdot S(w_j) + \sum \alpha\, N(p_1) \cdot T(u) \cdot M(p_2) \cdot S(u') \cdot L(p_3).$$

Finally, let us give the following definitions:

**Definition 12.5.** Let $(Q, \partial)$ be a differential biquiver and $M \in \mathrm{Rep}_{(Q,\partial)}$. Then the dimension vector $\dim(M)$ is defined to be the vector $(\dim_{\Bbbk} M_i)_{i\in I}$. Moreover, $M$ is called a brick if $\mathrm{End}(M) \cong \Bbbk$.

12.2. **Small reduction.**

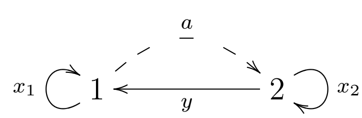
**Definition 12.6.** Let $(Q, \partial)$ be a differential biquiver. Then $(Q, \partial)$ is said to be of BT-type if there exists

- a set of distinguished loops $\mathfrak{x} = \{x_i \in Q_0(i, i) | \, i \in I\}$.
- an injective map $v : Q_0 \backslash \mathfrak{x} \hookrightarrow Q_1$ mapping a solid arrow $a : i \to j$ to a dotted arrow $v_a : j \dashrightarrow i$.

such that for each distinguished loop $x_i \in \mathfrak{x}$ the following condition holds:

$$\partial(x_i) = \sum_{c:\cdot \to i} c \cdot v_c - \sum_{d: i \to \cdot} v_d \cdot d.$$

**Example 12.7.** The following differential biquiver $(Q, \partial)$ is of BT-type for the obvious choices of $\mathfrak{x}$ and $v$:

$(Q, \partial)$ 


with differential given by

$$\partial(x_1) = ya \quad \partial(x_2) = -ay \quad \partial(a) = 0 \ .$$

The reason why we are interested in differential biquivers of BT-type is the following property:

**Proposition 12.8.** [8, Proposition 7.2] *Let $(Q, \partial)$ be a differential biquiver of* BT*-type, $b \in Q_0$ and let $M \in \mathrm{Rep}_{(Q,\partial)}$ be a brick. Then $M(b)$ has maximal rank.*

As we shall see below, this property of BT-type differential biquivers is essential for the inductive construction of bricks of a given dimension vector. Moreover, the procedure will translate to an algorithm for constructing simple vector bundles on a cuspidal cubic curve.

12.3. **Differential biquiver for the cuspidal cubic curve.** Let $\mu = \mathbb{1} + \epsilon\theta \in \overline{\mathrm{MP}}_E(\mathbf{r})$,

$$\theta = \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \end{array} .$$

Then $\mu$ is a representation of the differential biquiver given in example 12.7. A straightforward comparison of the definitions involved yields the following result:

**Lemma 12.9.** *The procedure described above extends to an equivalence of categories* $\overline{\mathrm{MP}}_E \cong \mathrm{Rep}_{(Q,\partial)}$.

Combining this result with Corollary 11.16 yields:

**Corollary 12.10.** $\mathrm{MP}^s_E(r_1, r_2) \cong \mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2)$.

Hence by Proposition 11.14, we can study simple vector bundles on $E$ via studying the categories $\mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2)$.

**Lemma 12.11.** [5, Section 3.2] *If* $r_1, r_2 \in \mathbb{N}$ *are not coprime, then* $\mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2) = \emptyset$.

By Proposition 12.8, the following definitions make sense:

**Definition 12.12.** Let $r_1, r_2 \in \mathbb{N}^2$ be coprime.

- If $r_1 > r_2$, let $\mathcal{R}_{(12)} : \mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2) \to \mathrm{Rep}_{(Q,\partial)}(r_1 - r_2, r_2)$ be given by

$$\left(\begin{array}{cc|c} X_1 & Y & 0 \\ 0 & X_2 & \mathbb{1} \\ \hline 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right).$$

- If $r_1 < r_2$, let $\mathcal{R}_{(21)} : \mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2) \to \mathrm{Rep}_{(Q,\partial)}(r_1, r_2 - r_1)$ be given by

$$\left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & X_1 & Y \\ 0 & 0 & X_2 \end{array}\right) \mapsto \left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right).$$

**Proposition 12.13.** [5, Section 7.1] *The following hold true for any coprime integers* $r_1, r_2 \in \mathbb{N}^2$*:*

- *the assignment* $\mathcal{R}_{(12)}$ *is an equivalence of categories*

$$\mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2) \xrightarrow{\cong} \mathrm{Rep}^s_{(Q,\partial)}(r_1 - r_2, r_2) .$$

- *the assignment* $\mathcal{R}_{(21)}$ *is an equivalence of categories*

$$\mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2) \xrightarrow{\cong} \mathrm{Rep}^s_{(Q,\partial)}(r_1, r_2 - r_1) .$$

Clearly, this result indicates a certain reduction procedure. In order to make this precise, note that any pair of coprime integers $a, b \in \mathbb{N}$ induces a sequence of pairs of coprime integers $\{(a_i, b_i)\}$ ending with the pair $(1,1)$ as follows. Assuming that $(a,b) \neq (1,1)$, we set

$$\epsilon(a,b) = \begin{cases} (a-b,b), & a > b \\ (a, b-a), & a < b. \end{cases}$$

We put $(a_0, b_0) = (a,b)$ and, as long as $(a_i, b_i) \neq (1,1)$, we set $(a_{i+1}, b_{i+1}) = \epsilon(a_i, b_i)$. By Proposition 12.13, the sequence $\{(a_i, b_i)\}$ (and hence the tuple $(a,b)$) induces a sequence of equivalences

$$\mathcal{R}(a,b): \ \mathrm{Rep}^s_{(Q,\partial)}(a,b) \xrightarrow{\cong} \mathrm{Rep}^s_{(Q,\partial)}(a_1,b_1) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathrm{Rep}^s_{(Q,\partial)}(1,1)$$

*Remark* 12.14. The functor $\mathcal{R}(a,b)$ can be interpreted as a path on the graph

$$(21) \circlearrowright \circ \circlearrowleft (12) \ ,$$

where the arrow $(ij)$ corresponds to the functor $\mathcal{R}_{(ij)}$.

## 13. Vector bundles on the cuspidal cubic curve

Let $E \subset \mathbb{P}^2$ be the cuspidal cubic curve given by $zy^2 = x^3$. In this section we summarize the results from the previous sections in order to obtain the following special case of [8, Theorem 1.2]:

**Theorem 13.1.** *For $(n,d) \in \mathbb{N} \times \mathbb{Z}$, let $\overline{\mathrm{Simp}}_E(n,d)$ denote the set of isomorphism classes of $\mathrm{Simp}_E(n,d)$. If $n, d$ are not coprime, then $\overline{\mathrm{Simp}}_E(n,d) = \emptyset$. Otherwise, the determinant yields a bijection $\det\colon \mathrm{Spl}_E(n,d) \to \mathsf{Pic}(E)$.*

Moreover, we will give a concrete algorithm that, starting only with two coprime integers $0 < d < n$, constructs the family $\mathrm{Simp}^0_E(n,d)$ explicitly. Finally, we shall give an important property regarding homomorphism and extension spaces in $\mathrm{Simp}^0_E(n,d)$.

### 13.1. Classification.

We sketch the proof of Theorem 13.1. Combining Proposition 11.14 and Corollary 12.10, there exists an equivalence of categories

$$\mathrm{Simp}^0_E(n,d) \cong \mathrm{Rep}^s_{(Q,\partial)}(n-\bar{d}, \bar{d}),$$

where $0 \leq \bar{d} < n$ is determined by $\bar{d} \equiv d \bmod n$. If $n$ and $d$ are not coprime, then Lemma 12.11 yields that $\mathrm{Simp}^0_E(n,d) = \emptyset$. Otherwise, the functor $\mathcal{R}(n-\bar{d}, \bar{d}) : \mathrm{Rep}^s_{(Q,\partial)}(n-\bar{d}, \bar{d}) \to \mathrm{Rep}^s_{(Q,\partial)}(1,1)$ is an equivalence. It can be shown that the induced equivalence $\mathrm{Simp}^0_E(n,d) \to \mathrm{Simp}^0_E(1,1)$ leaves the determinant unchanged. Finally, $\mathrm{Rep}^s_{(Q,\partial)}(1,1)$ is given by matrices of the form

$$\theta_\lambda = \left(\begin{array}{c|c} \lambda & 1 \\ \hline 0 & 0 \end{array}\right), \lambda \in \mathbb{C},$$

and the determinant induces a map $\theta_\lambda \mapsto \lambda$. Since the Picard group $\mathsf{Pic}(E)$ is isomorphic to $\mathbb{C}$ as well, this finishes the proof.

13.2. **Algorithm for construction of simple vector bundles.** Next we present an algorithm which constructs $\mathrm{Simp}^0_E(n,d)$ for two coprime integers $0 \leq d < n$ explicitly:

**Step 1: construction of the matrix $\mathcal{J}_\lambda(n-d,d)$.**
We introduce the following map defined on all tuples of coprime integers $(a,b) \neq (1,1)$:

$$\epsilon(a,b) = \begin{cases} (a-b,b), & a > b \\ (a,b-a), & a < b \end{cases}$$

By assumption $(n,d)$ is a tuple of coprime integers. Hence it induces a finite sequence of tuples ending with $(1,1)$, defined as follows. We put $(a_0,b_0) = (n-d,d)$ and, as long as $(a_i,b_i) \neq (1,1)$, we set $(a_{i+1},b_{i+1}) = \epsilon(a_i,b_i)$. Next, for fixed $\lambda \in \mathbb{C}$, let

$$\mathcal{J}_\lambda(1,1) = \left(\begin{array}{c|c} \lambda & 1 \\ \hline 0 & 0 \end{array}\right) \in \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

Assuming

$$\mathcal{J}_\lambda(a,b) = \left(\begin{array}{c|c} \mathcal{J}_1 & \mathcal{J}_2 \\ \hline 0 & \mathcal{J}_3 \end{array}\right)$$

with $\mathcal{J}_1 \in \mathrm{Mat}_{a\times a}(\mathbb{C})$ and $\mathcal{J}_3 \in \mathrm{Mat}_{b\times b}(\mathbb{C})$ has already been defined and that $(a,b) = \epsilon(p,q)$, we set

$$\mathcal{J}_\lambda(p,q) = \begin{cases} \left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & \mathcal{J}_1 & \mathcal{J}_2 \\ 0 & 0 & \mathcal{J}_3 \end{array}\right), & p = a \\ \\ \left(\begin{array}{cc|c} \mathcal{J}_1 & \mathcal{J}_2 & 0 \\ 0 & \mathcal{J}_3 & \mathbb{1} \\ \hline 0 & 0 & 0 \end{array}\right), & q = b. \end{cases}$$

Hence, to $(n,d)$ we may associate the $n \times n$ matrix $\mathcal{J}_\lambda(n-d,d)$ that is obtained from the matrix $\mathcal{J}_\lambda(1,1)$ and the sequence $\{(n-d,d),...,(1,1)\}$ by applying the recursive procedure described above.

**Example 13.2.** Let $(n,d)=(5,2)$. We obtain an induced sequence $\{(3,2),(1,2),(1,1)\}$ and $\mathcal{J}_\lambda(3,2)$ is constructed as follows

$$\left(\begin{array}{c|c}\lambda & 1\\ \hline 0 & 0\end{array}\right)\to\left(\begin{array}{c|cc}0 & 1 & 0\\ \hline 0 & \lambda & 1\\ 0 & 0 & 0\end{array}\right)\to\left(\begin{array}{ccc|cc}0 & 1 & 0 & 0 & 0\\ 0 & \lambda & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right).$$

**Step 2: Translation to** $\mathrm{Tri}_E$.
Let $T(\lambda)=\left(\tilde{\mathcal{F}},\mathcal{V},\tilde{\mathrm{m}}(\lambda)\right)\in\mathrm{Tri}_E$ be given by $\tilde{\mathcal{F}}=\mathcal{O}_{\mathbb{P}^1}^{n-d}\oplus\mathcal{O}_{\mathbb{P}^1}(1)^d$, $\mathcal{V}=\mathbb{C}_s^n$ and $\tilde{\mathrm{m}}(\lambda)=\mu_0+\epsilon\mu_\epsilon(\lambda)$, where $\mu_0=\mathbb{1}$ and $\mu_\epsilon(\lambda)=\mathcal{J}_\lambda(n-d,d)$. Setting $\mathcal{H}=\left\{\left(\tilde{\mathcal{F}},\mathcal{V},\tilde{\mathrm{m}}(\lambda)\right)\right\}_{\lambda\in\mathbb{C}}$, we obtain $\mathrm{Simp}_E^0(n,d)$ by applying $\mathbb{G}:\mathrm{Tri}_E\to\mathrm{VB}_E$ to $\mathcal{H}$.

Finally, let us note the following result:

**Lemma 13.3.** [8, Remark 11.2] *Let* $\mathcal{J}'_\lambda(a,b)=\mathcal{J}_0(a,b)+\lambda\cdot\mathbb{1}$. *Then* $\tilde{\mathrm{m}}(\lambda)$ *is isomorphic to* $\tilde{\mathrm{m}}'(\lambda)=\mu_0+\epsilon\mu_\epsilon(\lambda)$ *given by* $\mu_0=\mathbb{1}$ *and* $\mu_\epsilon(\lambda)=\mathcal{J}'_\lambda(a,b)$ *in* $\mathrm{MP}_E$. *Especially,*

$$\mathrm{Simp}_E^0(n,d)=\left\{\mathbb{G}\left(\mathcal{O}_{\mathbb{P}^1}^{n-d}\oplus\mathcal{O}_{\mathbb{P}^1}(1)^d,\mathbb{C}_s^n,\tilde{\mathrm{m}}'(\lambda)\right)\right\}_{\lambda\in\mathbb{C}}.$$

13.3. **Hom and Ext vanishing.** Motivated by Lemma 13.3, we introduce the following objects:

**Definition 13.4.** For any $\lambda\in\mathbb{C}$, let $\mathcal{E}(\lambda)=\mathbb{G}\left(\mathcal{O}_{\mathbb{P}^1}^{n-d}\oplus\mathcal{O}_{\mathbb{P}^1}(1)^d,\mathbb{C}_s^n,\tilde{\mathrm{m}}'(\lambda)\right)$ where $\tilde{\mathrm{m}}'(\lambda)$ is defined as in Lemma 13.3.

The following result shall prove essential for the computation of triple Massey products later on:

**Proposition 13.5.** [8, Proposition 12.3] *Let* $\mathcal{E}(\lambda_1),\mathcal{E}(\lambda_2)\in\mathrm{Simp}_E^0(n,d)$ *and* $\lambda_1\neq\lambda_2$. *Then* $\mathrm{Hom}\left(\mathcal{E}(\lambda_1),\mathcal{E}(\lambda_2)\right)=0$.

# Part 4. From vector bundles on Weierstraß cubic curves to solutions of the Yang-Baxter equations

We use the theory of vector bundles on Weierstraß cubic curves as developed in Part 3 in order to obtain concrete algorithms for the computation of solutions of the Yang-Baxter equations from the abstract procedures described in Part 2. This will allow us to prove further results on the solutions obtained.

In Section 14, we present the algorithm for elliptic solutions of the AYBE as developed in [12, 14], while in Section 15 we discuss the procedure for rational solutions of the AYBE which was already established by Burban and Kreußler [14]. As we discussed earlier, such solutions yield solutions of the CYBE under certain conditions. As we proved in Section 9, there is a direct way to compute these. The corresponding algorithm is presented in Section 16. The procedure is discussed in detail in [13], while some of the ideas were already developed by Polishchuk [37].

## 14. From vector bundles on the elliptic curve to solutions of the AYBE

In this section we show how the abstract results from Section 8 can be translated to a concrete algorithm for the computation of elliptic solutions $r : (\mathbb{C}^2, 0) \to A \otimes A$ of

$$r^{12}(u,x)r^{23}(u+v,y) = r^{13}(u+v,x+y)r^{12}(-v,x) + r^{23}(v,y)r^{13}(u,x+y), \tag{14.1}$$

where $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$. First, we present the algorithm, see Subsection 14.1. The proofs for all statements involved are contained in subsections 14.2 and 14.3. In particular, we demonstrate how the procedure of Subsection 14.1 is connected to the concepts of Section 8, especially (8.14).

### 14.1. Construction of the elliptic solutions $r_B$ of the AYBE.

In this subsection we present an algorithm attaching to a pair $(B,\tau) \in \mathrm{GL}_n(\mathbb{C}) \times \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane, a non-degenerate unitary solution of the associative Yang-Baxter equation (14.1) with values in $\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$. The explanation of this method as well as proofs will be given in the next subsection.
In what follows, we denote $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Let $\mathfrak{S}(B) = \{\lambda_1, \dots, \lambda_n\}$ be the spectrum of $B$ and $\Sigma = \Sigma_B \subset \mathbb{C}$ be the lattice $\{\lambda - \lambda' \mid \exp(2\pi i\lambda), \exp(2\pi i\lambda') \in \mathfrak{S}(B)\} + \Lambda$. We construct the tensor-valued function

$$r_B : (\mathbb{C}\setminus\Sigma) \times (\mathbb{C}\setminus\Lambda) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$$

in the following way.

- For any $v \in \mathbb{C}$ consider the function
$$e(z) = e(z,v,\tau) := -\exp\big(-2\pi i(z+v+\tau)\big).$$
- Let $\mathsf{Sol} = \mathsf{Sol}_{B,\,v,\,\tau}$ be the following complex vector space:
$$\mathsf{Sol} = \left\{ \Phi : \mathbb{C} \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)B = e(z)B\Phi(z) \end{array} \right\}.$$
- For any $y \in \mathbb{C}\setminus\Lambda$ consider the *evaluation map* $\mathrm{ev}_y : \mathsf{Sol} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ given by the formula $\mathrm{ev}_y(\Phi) = \frac{1}{\bar\theta(y+\frac{\tau+1}{2})}\Phi(y)$, where
$$\bar\theta(y) = \theta_3(y|\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos\big(2\pi n y\big)$$
is the *third* Jacobian theta-function with $q = \exp(\pi i\tau)$. Next, consider the *residue map* $\mathrm{res}_0 : \mathsf{Sol} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ given by the formula $\mathrm{res}_0(\Phi) = \Phi(0)$.

**Proposition 14.1.** *For any $v \in \mathbb{C}$ and $B \in \mathrm{GL}_n(\mathbb{C})$ the vector space $\mathrm{Sol}_{B,\,v,\,\tau}$ has dimension $n^2$. Moreover, if $v \notin \Sigma$ then the linear map $\mathrm{res}_0 : \mathrm{Sol}_{B,\,v,\,\tau} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ is an isomorphism.*

For a proof of this Proposition, see Corollary 14.4, Theorem 14.5 and Remark 14.6.

Next, we continue the construction of the tensor valued function $r_B$.

- For any pair $(v, y) \in (\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$ consider the linear map $\tilde{r}_B(v, y)$ given by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Mat}_{n\times n}(\mathbb{C}) & \xrightarrow{\;\tilde{r}_B(v,y)\;} & \mathrm{Mat}_{n\times n}(\mathbb{C}) \\ {\scriptstyle \mathrm{res}_0}\nwarrow & & \nearrow{\scriptstyle \mathrm{ev}_y} \\ & \mathrm{Sol}_{B,\,v,\,\tau} & \end{array} \tag{14.2}$$

  In other words, $\tilde{r}_B(v, y) := \mathrm{ev}_y \circ \mathrm{res}_0^{-1}$.

- Let $r_B(v, y) \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the tensor corresponding to the linear map $\tilde{r}_B(v, y)$ via the canonical map of vector spaces

$$\mathsf{can} : \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})\big)$$

  sending a simple tensor $X \otimes Y$ to the linear map $Z \mapsto \mathrm{Tr}(XZ)Y$.

**Theorem 14.2.** *Let $(B, \tau) \in \mathrm{GL}_n(\mathbb{C}) \times \mathbb{H}$.*

(1) *The function $(\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda) \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$, assigning to a pair $(v, y)$ the tensor $r_B(v, y)$ constructed above, is a non-degenerate holomorphic unitary solution of the associative Yang-Baxter equation (14.1). Moreover, this function is meromorphic on $\mathbb{C} \times \mathbb{C}$.*

(2) *Let $S \in \mathrm{GL}_n(\mathbb{C})$ and $A = S^{-1}BS$. Then for any $(v, y) \in (\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$ we have the following equality:*

$$r_A(v, y) = (S^{-1} \otimes S^{-1})r_B(v, y)(S \otimes S).$$

*In particular, the solutions $r_A$ and $r_B$ are gauge-equivalent in the sense of* [14, Definition 2.5].

The proof of this Theorem is given in several steps, see Theorem 14.7, Proposition 14.8 and Proposition 14.10.

14.2. **Identification of the geometric method and Algorithm 14.1.** In this subsection we will prove Theorem 14.2 and Proposition 14.1. The main idea is to connect the algorithm of the previous subsection with (8.14).

**Lemma 14.3.** *For $B \in \mathrm{GL}_n(\mathbb{C})$ and $v \in \mathbb{C}$ we set $\mathcal{F}_v = \mathcal{E}\big(\exp(2\pi i v)B\big) \cong \mathcal{E}(B) \otimes \mathcal{L}_v$. Then for any $v_1, v_2 \in \mathbb{C}$ and $y \in E$ we have:*

$$\dim_{\mathbb{C}}\Big(\mathrm{Hom}\big(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y)\big)\Big) = n^2.$$

*Proof.* The vector bundle $\mathcal{F}_{v_2}(y)$ is semi-stable of slope one. Hence, we have:

$$\mathrm{Ext}\big(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y)\big) \cong \mathrm{Hom}\big(\mathcal{F}_{v_2}(y), \mathcal{F}_{v_1}\big)^* = 0.$$

Thus, the statement of Lemma is a consequence of the Riemann-Roch formula. □

**Corollary 14.4.** *For any $v, y \in \mathbb{C}$ the dimension of the complex vector space*

$$\mathrm{Sol} = \mathsf{Sol}_{B,v,y,\tau} := \left\{ \Phi : \mathbb{C} \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)B = e(z)B\Phi(z) \end{array} \right\} \tag{14.3}$$

*is $n^2$, where $e(z) = e(z, v, y, \tau) = -\exp\big(-2\pi i(z + v - y + \tau)\big)$.*

*Proof.* By Theorem 10.3 and Lemma 10.7, we have an isomorphism of vector spaces $\mathrm{Sol} \cong \mathrm{Hom}\big(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y)\big)$, where $v = v_1 - v_2$. Hence, by Lemma 14.3, the dimension of Sol is $n^2$. Taking $y = 0 \in E$, we also recover the first part of Proposition 14.1. □

**Theorem 14.5.** *Let $B \in \mathrm{GL}_n(\mathbb{C})$ and $\omega = \bar{\theta}'(\frac{1+\tau}{2})dz \in H^0(\Omega_E)$. Let $U \subset \mathbb{C}$ be a small neighborhood of $0$. Using the projection map $\pi : \mathbb{C} \to E$, we identify $U$ with a small neighborhood of $\pi(0) \in E$. Then for all $v_1, v_2; y_1, y_2 \in U$ such that $y_1 \neq y_2$ the following diagram of vector spaces is commutative:*

$$\begin{array}{ccccc}
\mathrm{Lin}\big(\mathcal{F}_{v_1}\big|_{y_1}, \mathcal{F}_{v_2}\big|_{y_1}\big) & \xleftarrow{\mathrm{res}_{y_1}^{\mathcal{F}_{v_1},\mathcal{F}_{v_2}}(\omega)} & \mathrm{Hom}\big(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y_1)\big) & \xrightarrow{\mathrm{ev}_{y_2}^{\mathcal{F}_{v_1},\mathcal{F}_{v_2}(y_1)}} & \mathrm{Lin}\big(\mathcal{F}_{v_1}\big|_{y_2}, \mathcal{F}_{v_2}\big|_{y_2}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Mat}_{n\times n}(\mathbb{C}) & \xleftarrow{\quad \mathrm{res}_{y_1} \quad} & \mathrm{Sol}_{B,v,y_1,\tau} & \xrightarrow{\quad \mathrm{ev}_{y_2} \quad} & \mathrm{Mat}_{n\times n}(\mathbb{C}),
\end{array}$$

*where $v = v_1 - v_2$, the middle vertical arrow is the isomorphism from Theorem 10.3, whereas the first and the last vertical arrows are isomorphisms induced by trivializations $\gamma$ from Remark 10.2. The maps $\mathrm{res}_{y_1}$ and $\mathrm{ev}_{y_2}$ are given by the formulae:*

$$\mathrm{res}_{y_1}\big(\Phi(z)\big) = \Phi(y_1) \quad \text{and } \mathrm{ev}_{y_2}\big(\Phi(z)\big) = \frac{1}{\bar{\theta}(y_2 - y_1 + \frac{\tau+1}{2})}\Phi(y_2),$$

*where $\bar{\theta}(y)$ is the third Jacobian theta-function.*

*Proof.* The proof of this Theorem is literally the same as the one given in [14, Section 8.2], see in particular [14, Corollary 8.10]. □

*Remark* 14.6. Let $v_1, v_2 \in \mathbb{C}$ be such that $v_1 - v_2$ does not belong to the lattice $\Sigma$. By Corollary 10.6 we get the vanishing $\operatorname{Hom}(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}) = 0 = \operatorname{Ext}(\mathcal{F}_{v_1}, \mathcal{F}_{v_2})$. Next, Lemma 8.5 implies that the morphism $\operatorname{res}_{y_1}^{\mathcal{F}_{v_1}, \mathcal{F}_{v_2}}(\omega)$ is an isomorphism. The commutativity of the left square of the diagram from Theorem 14.5 implies that the linear map $\operatorname{res}_{y_1}$ is an isomorphism, too. Setting $y_1 = 0$, we obtain a proof of the second part of Proposition 14.1.

Hence we can finally prove Theorem 14.2:

**Theorem 14.7.** *Let $B \in \operatorname{GL}_n(\mathbb{C})$, $v_1, v_2 \in \mathbb{C}$ such that $v = v_1 - v_2 \notin \Sigma$ and $y_1, y_2 \in \mathbb{C}$ such that $y_2 - y_1 \notin \Lambda$. Consider the linear map $\tilde{r}_B(v_1, v_2; y_1, y_2) : \operatorname{Mat}_{n\times n}(\mathbb{C}) \to \operatorname{Mat}_{n\times n}(\mathbb{C})$ defined via the commutative diagram*

$$\begin{array}{ccc} \operatorname{Mat}_{n\times n}(\mathbb{C}) & \xrightarrow{\tilde{r}_B(v_1,v_2;y_1,y_2)} & \operatorname{Mat}_{n\times n}(\mathbb{C}) \\ & {}_{\operatorname{res}_{y_1}}\nwarrow \quad \nearrow_{\operatorname{ev}_{y_2}} & \\ & \operatorname{Sol}_{B,v,y_1,\tau} & \end{array} \tag{14.4}$$

*where $\operatorname{res}_{y_1}$ and $\operatorname{ev}_{y_2}$ are as in Theorem 14.5. Let $r_B(v_1, v_2; y_1, y_2) \in \operatorname{Mat}_{n\times n}(\mathbb{C}) \otimes \operatorname{Mat}_{n\times n}(\mathbb{C})$ be the tensor corresponding to the linear map $\tilde{r}_B(v_1, v_2; y_1, y_2)$ via the canonical isomorphism of vector spaces*

$$\operatorname{Mat}_{n\times n}(\mathbb{C}) \otimes \operatorname{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \operatorname{Lin}\big(\operatorname{Mat}_{n\times n}(\mathbb{C}),\ \operatorname{Mat}_{n\times n}(\mathbb{C})\big),$$

*which sends a simple tensor $X \otimes Y$ to the linear map $Z \mapsto \operatorname{Tr}(XZ)Y$. Then the obtained function of four variables*

$$r_B : \mathbb{C}^4_{(v_1,v_2;\,y_1,y_2)} \longrightarrow \operatorname{Mat}_{n\times n}(\mathbb{C}) \otimes \operatorname{Mat}_{n\times n}(\mathbb{C})$$

*satisfies the following version of the associative Yang-Baxter equation:*

$$r_B(v_1, v_2; y_1, y_2)^{12} r_B(v_1, v_3; y_2, y_3)^{23} = r_B(v_1, v_3; y_1, y_3)^{13} r_B(v_3, v_2; y_1, y_2)^{12} + \tag{14.5}$$
$$+ r_B(v_2, v_3; y_2, y_3)^{23} r_B(v_1, v_2; y_1, y_3)^{13}.$$

*Moreover, the tensor-valued function $r$ is unitary, i.e. it satisfies the condition*

$$r_B(v_1, v_2; y_1, y_2)^{12} = -r_B(v_2, v_1; y_2, y_1)^{21}. \tag{14.6}$$

*Proof.* For $v \in \mathbb{C}$ we set $\mathcal{F}_v = \mathcal{E}\big(\exp(2\pi i v)B\big) \cong \mathcal{E}(B) \otimes \mathcal{L}_v$ and $\mathcal{F}_i = \mathcal{F}_{v_i}$ for both $i = 1, 2$. By Theorems 8.2 we obtain $\widetilde{m}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$, which satisfies equation 8.5. Via the canonical isomorphism described in Theorem 8.8, $\widetilde{m}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$ maps to $\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$, which was defined by the diagram 8.14. Theorem 14.5 implies that $\tilde{r}_B(v_1, v_2; y_1, y_2)$ coincides with $\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$. The main ideas for the rest of the proof are as follows.

• Let $A$ be an arbitrary automorphy factor, $\mathcal{F} = \mathcal{E}(A)$ and $y \in E$. Then we have an isomorphism of vector spaces

$$\gamma(A, y) : \mathrm{Hom}(\mathcal{F}, \mathbb{C}_y) \longrightarrow \mathrm{Hom}(\mathcal{F} \otimes \mathbb{C}_y, \mathbb{C}_y) \longrightarrow \mathcal{F}\big|_y^* \longrightarrow \mathbb{C}^n,$$

induced by the trivialization $\gamma_A$ from Remark 10.2. For any $v \in \mathbb{C}$ we denote by $\gamma(v, y)$ the isomorphism $\mathrm{Hom}(\mathcal{F}_v, \mathbb{C}_y) \to \mathbb{C}^n$. We obtain a linear map $\bar{r}_B(v_1, v_2; y_1, y_2)$, defined by the following commutative diagram of vector spaces:

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}_{v_1}, \mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_{v_2}, \mathbb{C}_{y_2}) & \xrightarrow{\widetilde{m}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}} & \mathrm{Hom}(\mathcal{F}_{v_2}, \mathbb{C}_{y_1}) \otimes \mathrm{Hom}(\mathcal{F}_{v_1}, \mathbb{C}_{y_2}) \\ \Big\downarrow {\scriptstyle \gamma(v_1,y_1)\otimes\gamma(v_2,y_2)} & & \Big\downarrow {\scriptstyle \gamma(v_2,y_1)\otimes\gamma(v_1,y_2)} \\ \mathbb{C}^n \otimes \mathbb{C}^n & \xrightarrow{\bar{r}_B(v_1,v_2;y_1,y_2)} & \mathbb{C}^n \otimes \mathbb{C}^n. \end{array}$$

• Using the canonical isomorphism $\mathrm{Lin}(\mathbb{C}^n \otimes \mathbb{C}^n,\ \mathbb{C}^n \otimes \mathbb{C}^n) \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$, we end up with a tensor-valued meromorphic function

$$\mathbb{C}^2_{(v_1,v_2)} \times \mathbb{C}^2_{(y_1,y_2)} \xrightarrow{r_B} \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}),$$

satisfying the Yang-Baxter equation (14.5) and the unitarity condition (14.6). Moreover, for any $v_1, v_2; y_1, y_2 \in \mathbb{C}$ such that $v_1 - v_2 \notin \Sigma$ and $y_1 - y_2 \notin \Lambda$ the tensor $r_B(v_1, v_2; y_1, y_2)$ coincides with the image of $\tilde{r}_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2}$ under the composition of the canonical isomorphism of vector spaces

$$\mathrm{Lin}\Big(\mathrm{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big),\ \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big) \longrightarrow \mathrm{Lin}\big(\mathcal{F}_2\big|_{y_1}, \mathcal{F}_1\big|_{y_1}\big) \otimes \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big)$$

with the isomorphism $\mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ induced by trivializations $\gamma$ from Remark 10.2. □

14.3. **Remarks on the solutions $r_B$.** In the previous subsections we have seen how one can attach a unitary solution

$$\mathbb{C}^2 \times \mathbb{C}^2 \xrightarrow{r_B} \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$$

of the associative Yang-Baxter equation (14.5) to a matrix $B \in \mathrm{GL}_n(\mathbb{C})$, see diagram (14.4) from Theorem 14.7.

**Proposition 14.8.** *For general $v_1, v_2, u; y_1, y_2, x \in \mathbb{C}$ we have the equality*

$$r_B(v_1 + u, v_2 + u;\ y_1 + x, y_2 + x) = r_B(v_1, v_2;\ y_1, y_2).$$

*In other words, the function $r_B(v_1, v_2; y_1, y_2)$ depends only on the differences $v = v_1 - v_2$ and $y = y_2 - y_1$. In particular, the function $r_B(v, y) = r_B(v_1, v_2; y_1, y_2)$ satisfies the associative Yang-Baxter equation (14.1).*

*Proof.* Since the vector space $\mathrm{Sol}_{B,\,v,\,y_1,\,\tau}$ from Theorem 14.5 only depends on the difference $v = v_2 - v_1$, whereas $\mathrm{res}_{y_1}$ and $\mathrm{ev}_{y_2}$ depend only on $y_1$ and $y_2$, we have the equality $r_B(v_1 + u, v_2 + u; y_1, y_2) = r_B(v_1, v_2; y_1, y_2)$. To show the translation invariance of the function $r_B$ with respect to the second pair of spectral variables note that we have the following commutative diagram:

$$
\begin{array}{ccccc}
 & & \mathrm{Sol}_{B,\,v,\,y_1,\,\tau} & & \\
 & \stackrel{\mathrm{res}_{y_1}}{\swarrow} & & \stackrel{\mathrm{ev}_{y_2}}{\searrow} & \\
\mathrm{Mat}_{n\times n} & & \downarrow{\scriptstyle t_x} & & \mathrm{Mat}_{n\times n} \\
 & \stackrel{\mathrm{res}_{y_1+x}}{\nwarrow} & & \stackrel{\mathrm{ev}_{y_2+x}}{\nearrow} & \\
 & & \mathrm{Sol}_{B,\,v,\,y_1+x,\,\tau} & &
\end{array}
$$

where $t_x\big(\Phi(z)\big) = \Phi(z-x)$. It proves that $r_B(v_1, v_2; y_1+x, y_2+x) = r_B(v_1, v_2; y_1, y_2)$. □

*Remark* 14.9. Proposition 14.8 implies that in order to compute the linear map $r_B(v, y)$ we can take $y_1 = 0$ and $y_2 = y$ in the commutative diagram (14.4). In particular, the solution $r_B(v, y)$ can be computed using the diagram (14.2).

**Proposition 14.10.** *Let $B, S \in \mathrm{GL}_n(\mathbb{C})$ and $A := S^{-1}BS$. Then we have:*

$$r_A(v, y) = \big(S^{-1} \otimes S^{-1}\big)\; r_B(v, y)\; \big(S \otimes S\big).$$

*Proof.* For simplicity of notation we denote $\mathrm{Sol}_B = \mathrm{Sol}_{B,\,v,\,y,\,\tau}$ and $r_B = r_B(v, y)$. Observe that we have an isomorphism of vector spaces $\varphi_S : \mathrm{Sol}_B \to \mathrm{Sol}_A$ mapping a function $\Phi \in \mathrm{Sol}_B$ to $S^{-1}\,\Phi\,S \in \mathrm{Sol}_A$. We have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Mat}_{n\times n}(\mathbb{C}) & \xleftarrow{\ \mathrm{res}_0\ } & \mathrm{Sol}_B & \xrightarrow{\ \mathrm{ev}_y\ } & \mathrm{Mat}_{n\times n}(\mathbb{C}) \\
{\scriptstyle c_S}\downarrow & & \downarrow{\scriptstyle \varphi_S} & & \downarrow{\scriptstyle c_S} \\
\mathrm{Mat}_{n\times n}(\mathbb{C}) & \xleftarrow{\ \mathrm{res}_0\ } & \mathrm{Sol}_A & \xrightarrow{\ \mathrm{ev}_y\ } & \mathrm{Mat}_{n\times n}(\mathbb{C}),
\end{array}
\tag{14.7}
$$

where $c_S(X) = S^{-1}XS$. This implies that for any $X \in \mathrm{Mat}_{n\times n}(C)$ we have:

$$\tilde{r}_A(S^{-1}XS) = S^{-1}\tilde{r}_B(X)S.$$

The matrix $S$ defines the following linear automorphism

$$\psi_S : \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}),\ \mathrm{Mat}_{n\times n}(\mathbb{C})\big) \longrightarrow \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}),\ \mathrm{Mat}_{n\times n}(\mathbb{C})\big)$$

sending $l \in \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})\big)$ to the linear map $X \xrightarrow{\psi_S(l)} S^{-1}\,l(SXS^{-1})\,S$. Then we have: $\psi_S(\tilde{r}_B) = \tilde{r}_A$.

Finally, let $\mathsf{can} : \mathrm{Mat}_{n\times n}(\mathbb{C})\otimes\mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}),\, \mathrm{Mat}_{n\times n}(\mathbb{C})\big)$ be the canonical isomorphism of vector spaces mapping a simple tensor $X \otimes Y$ to the linear map $Z \mapsto \mathrm{Tr}(XZ)Y$. Then the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) & \xrightarrow{\ \mathsf{can}\ } & \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}),\, \mathrm{Mat}_{n\times n}(\mathbb{C})\big) \\
\Big\downarrow{\scriptstyle c_S\otimes c_S} & & \Big\downarrow{\scriptstyle \psi_S} \\
\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}) & \xrightarrow{\ \mathsf{can}\ } & \mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}),\, \mathrm{Mat}_{n\times n}(\mathbb{C})\big).
\end{array}$$

But this implies that $r_A(v, y) = \big(S^{-1} \otimes S^{-1}\big)\ r_B(v, y)\ \big(S \otimes S\big)$. □

It still remains to be shown that $r_B$ is a meromorphic function in $v$ and $y$, holomorphic on $(\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$ and with an analytic dependence on the matrix $B$. Although this fact can be verified by a direct computation, we prefer to give an abstract proof based on the technique of semi-universal families of semi-stable sheaves.

• Let $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathcal{P} \in \mathrm{VB}(E \times G)$ be defined as follows

$$\mathcal{P} := \mathbb{C} \times G \times \mathbb{C}^n / \sim, \quad \text{where} \quad (z, g, v) \sim (z + 1, g, v) \sim (z + \tau, g, g \cdot v)$$

for all $(z, g, v) \in \mathbb{C} \times G \times \mathbb{C}^n$. Note that we have a Cartesian diagram

$$\begin{array}{ccc}
(\mathbb{C} \times G) \times \mathbb{C}^n & \longrightarrow & \mathcal{P} \\
\Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow \\
\mathbb{C} \times G & \xrightarrow{\ \pi\times\mathbb{1}\ } & E \times G,
\end{array}$$

where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda = E$ is the quotient map. Note that for any $g \in G$ we have an isomorphism $\mathcal{P}\big|_{E\times\{g\}} \cong \mathcal{E}(g)$, where $\mathcal{E}(g)$ is the semi-stable degree zero vector bundle on $E$ determined by the automorphy factor $g \in \mathrm{GL}_n(\mathbb{C})$. Thus, the constructed vector bundle $\mathcal{P}$ is a *semi-universal* family of degree zero semi-stable vector bundle on the torus $E$.

• Let $I = \mathsf{Pic}^0(E)$ be the Jacobian of $E$. One can identify $I$ with the torus $E$ using the following construction. Consider the line bundle $\mathcal{L}$ on $E \times E = \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ defined as the quotient space $\mathcal{L} := (\mathbb{C} \times \mathbb{C}) \times \mathbb{C}/ \sim$, where

$$(z, w, v) \sim (z + 1, w, v) \sim (z, w + 1, v) \sim (z, w + \tau, v) \sim (z + \tau, w, \exp(2\pi i w)v)$$

for all $(z, w, v) \in (\mathbb{C} \times \mathbb{C}) \times \mathbb{C}$. The constructed line bundle $\mathcal{L}$ is a universal family of degree zero vector bundles on $E$.

• We denote $X = E\times I\times I\times E\times E\times G$, $T = I\times I\times E\times E\times G$ and set $q : X \to T$ and $p : X \to E\times G$ to be the canonical projection maps. Similarly, for $i = 1, 2$ we define $p_i : X \to E\times I$ and $h_i : T \to X$ to be given by the formulae $p_i(x, v_1, v_2, y_1, y_2, g) = (x, v_i)$ and $h_i(v_1, v_2, y_1, y_2, g) = (y_i, v_1, v_2, y_1, y_2, g)$. Note that $h_1$ and $h_2$ are sections of the canonical projection $q$.

• For $i = 1, 2$ we define $\mathcal{F}_i := p^*\mathcal{P}\otimes p_i^*\mathcal{L}$. Obviously, for any point $t = (v_1, v_2, y_1, y_2, g) \in T$ we have: $\mathcal{F}_i\big|_{q^{-1}(t)} \cong \mathcal{P}\big|_{E\times\{g\}}\otimes\mathcal{L}\big|_{E\times\{v_i\}} \cong \mathcal{E}\big(\exp(2\pi v_i)\cdot g\big)$.

**Lemma 14.11.** *The coherent sheaf $q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is supported on a proper closed analytic subset of $T$.*

*Proof.* By Grauert's direct image Theorem, the sheaf $q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is coherent, hence it is supported on a closed analytic subset $\Delta$ of the base $T$. Since $\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is a vector bundle on $X$, it is flat over $T$ and for any point $t = (v_1,\, v_2,\, y_1,\, y_2,\, g) \in T$ we have a base-change isomorphism

$$q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)\otimes\mathbb{C}_t \cong \mathrm{Hom}_E\big(\mathcal{E}(g)\otimes\mathcal{L}_{v_1}, \mathcal{E}(g)\otimes\mathcal{L}_{v_2}\big).$$

By Corollary 10.6, we have the vanishing $\mathrm{Hom}_E\big(\mathcal{E}(g)\otimes\mathcal{L}_{v_1}, \mathcal{E}(g)\otimes\mathcal{L}_{v_2}\big) = 0$ for generically chosen $v_1, v_2 \in J$ and $g \in G$. Hence, $\Delta$ is a proper subset of $T$. □

The following result can be proved along the same lines as Lemma 14.11.

**Lemma 14.12.** *Let $D_i := \mathsf{Im}(h_i) \subseteq X$. Then the sheaf $q_*\mathcal{H}om_X\big(\mathcal{F}_1(D_2), \mathcal{F}_2(D_1)\big)$ is supported on a proper closed analytic subset $\Delta'$ of $T$ and $q_*\mathcal{H}om_X\big(\mathcal{F}_1, \mathcal{F}_2(D_1)\big)$ is a vector vector bundle of rank $n^2$.*

• Let $\breve{T} := T\setminus(\Delta\cup\Delta')$ and $\breve{X} := q^{-1}(\breve{T})$. For the sake of simplicity we denote the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\breve{X}$ by the same symbols. Let $\omega \in H^0\big(\Omega_{\breve{X}/\breve{T}}\big)$ be the pull-back of the differential form $dz \in H^0\big(\Omega_E\big)$. Note that we are in the situation of [14, Section 5.3]. In particular, we have the following commutative diagram in $\mathrm{Coh}(\breve{T})$, where all arrows are isomorphisms of vector bundles on $\breve{T}$:

$$\begin{array}{ccc} & q_*\mathcal{H}om_{\breve{X}}\big(\mathcal{F}_1, \mathcal{F}_2(D_1)\big) & \\ {}^{\mathrm{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)}\swarrow & & \searrow{}^{\mathrm{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)}} \\ \mathcal{H}om_{\breve{T}}\big(h_1^*\mathcal{F}_1, h_1^*\mathcal{F}_2\big) & \xrightarrow{\tilde{r}_{h_1,h_2}^{\mathcal{F}_1,\mathcal{F}_2}} & \mathcal{H}om_{\breve{T}}\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big). \end{array} \tag{14.8}$$

The morphisms $\mathrm{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)$ and $\mathrm{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)}$ are induced by the short exact sequences

$$0\to\Omega_{\breve{X}/\breve{T}}\to\Omega_{\breve{X}/\breve{T}}(D_1)\xrightarrow{\mathrm{res}_{D_1}}\mathcal{O}_{D_1}\to 0,\quad 0\to\mathcal{O}_{\breve{X}}(-D_2)\to\mathcal{O}_{\breve{X}}\to\mathcal{O}_{D_1}\longrightarrow 0,$$

see [14, Section 5.3]. By [14, Theorem 5.17], after tensoring the diagram (14.8) with $\mathbb{C}_t$, where $t = (v_1, v_2, y_1, y_2, g) \in \breve{T}$, and applying base change isomorphisms, we get the commutative diagram (8.14). In particular, the function $\tilde{r}_B(v_1, v_2; y_1, y_2)$ from Theorem 14.7 is just the isomorphism of vector bundles $\tilde{r}^{\mathcal{F}_1, \mathcal{F}_2}_{h_1, h_2}$ written with respect of the trivialization $\gamma$, described in Remark 10.2. This implies that the tensor $r_B(v, y)$ is non-degenerate.
In a similar way, the isomorphism $\tilde{r}_B(v_1, v_2; y_1, y_2)$ determines a holomorphic section $r^{\mathcal{F}_1, \mathcal{F}_2}_{h_1, h_2} \in H^0\big(\breve{T}, \mathcal{H}om_{\breve{T}}\big(h_1^*\mathcal{F}_2, h_1^*\mathcal{F}_1\big) \otimes \mathcal{H}om_{\breve{T}}\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big)\big)$. Trivializing $\mathcal{F}_1$ and $\mathcal{F}_2$ as in Remark 10.2, the section $r^{\mathcal{F}_1, \mathcal{F}_2}_{h_1, h_2}$ becomes the tensor-valued function $r_B(v, y)$ from Theorem 14.7. This proves that $r_B(v, y)$ is holomorphic on $(\mathbb{C} \setminus \Sigma_B) \times (\mathbb{C} \setminus \Lambda)$ and as a function of the input matrix $B$. To show that $r_B(v, y)$ is meromorphic on $\mathbb{C} \times \mathbb{C}$ note that $\mathrm{res}^{\mathcal{F}_1, \mathcal{F}_2}_{h_1}(\omega)$ and $\mathrm{ev}^{\mathcal{F}_1, \mathcal{F}_2(D_1)}_{h_2}$ are morphisms of vector bundles of rank $n^2$ on the whole base $T$ and $\tilde{r}^{\mathcal{F}_1, \mathcal{F}_2}_{h_1, h_2} = \mathrm{ev}^{\mathcal{F}_1, \mathcal{F}_2(D_1)}_{h_2} \circ \big(\mathrm{res}^{\mathcal{F}_1, \mathcal{F}_2}_{h_1}(\omega)\big)^{-1}$ is a meromorphic isomorphism of $\mathcal{H}om_T\big(h_1^*\mathcal{F}_1, h_1^*\mathcal{F}_2\big)$ and $\mathcal{H}om_T\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big)$.

## 15. From vector bundles on the cuspidal cubic curve to solutions of the AYBE

Let $E$ be the cuspidal cubic curve given by $V(y^2z - x^3) \subset \mathbb{P}^2$. In this section we present an algorithm which takes as input a pair of coprime numbers $(n, d)$ with $0 \le d < n$ and produces a rational solution $r_{(n,d)}$ of the AYBE

$$(15.1) \qquad \begin{aligned} & r^{12}(u; y_1, y_2)\, r^{23}(u + v; y_2, y_3) = \\ &= r^{13}(u + v; y_1, y_3)\, r^{12}(-v; y_1, y_2) + r^{23}(v; y_2, y_3)\, r^{13}(u; y_1, y_3). \end{aligned}$$

Here $r : (\mathbb{C}^3, 0) \to A \otimes A$ is the germ of a meromorphic function for $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$ with values in $A \otimes A$ where $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$. We will first present the algorithm, see Subsection 15.1. Then we verify the algorithm and all statements involved, see Subsection 15.2. The main idea is to demonstrate how the procedure of Subsection 15.1 is connected to the concepts of Section 8. Finally, in Subsection 15.3 we give some examples of solutions of the CYBE (2.1) obtained from solutions of (15.1). We see that these solutions satisfy the assumptions of Theorem 7.1 and hence yield solutions of the QYBE (4.1) as well.

15.1. **Construction of the rational solutions $r_{(n,d)}$ of the AYBE.** Let $(n, d)$ be a pair of coprime integers such that $0 < d < n$. We give an algorithm that constructs a rational solution $r_{(n,d)}$ of the AYBE (15.1).

1. Construct the matrix $J = \mathcal{J}_0(n - d, d)$ from Algorithm 13.2.

2. For the partition of $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$ induced by the partition of $J$ as in step 1, we introduce the following subspace of the polynomial ring $A[z]$:

$$W_{n,d} = \left\{ F(z) = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right) + \left(\begin{array}{c|c} W' & 0 \\ \hline Y' & Z' \end{array}\right) z + \left(\begin{array}{c|c} 0 & 0 \\ \hline Y'' & 0 \end{array}\right) z^2 \right\}.$$

Next, for $F(z) \in W_{n,d}$ we denote

$$F_0 = \left(\begin{array}{c|c} W' & X \\ \hline Y'' & Z' \end{array}\right) \text{ and } F_\epsilon = \left(\begin{array}{c|c} W & 0 \\ \hline Y' & Z \end{array}\right).$$

Then for $v, y_1 \in \mathbb{C}$, we define the following subspace of $W_{n,d}$:

$$\mathrm{Sol}_{n,d}^{v,y_1} = \left\{ F(z) \in W_{n,d} \,\middle|\, [F_0, J] + (y_1 - v)F_0 + F_\epsilon = 0 \right\}.$$

**Proposition 15.1.** *The vector space* $\mathrm{Sol}_{n,d}^{v,y_1}$ *has dimension* $n^2$ *and for* $y_1 \neq y_2 \in \mathbb{C}$, $\mathrm{ev}_{y_2} : \mathrm{Sol}_{n,d}^{v,y_1} \to A$ *defined by* $\mathrm{ev}_{y_2}\big(F(z)\big) = \frac{1}{y_2 - y_1} F(y_2)$ *is an isomorphism. For* $v \neq 0$, $\mathrm{res}_{y_1} : \mathrm{Sol}_{n,d}^{v,y_1} \to A$ *given by* $\mathrm{res}_{y_1}\big(F(z)\big) = F(y_1)$ *is an isomorphism as well.*

*Remark* 15.2. The proof of this and the next statement can be found in the following subsection.

We will assume that $v \neq 0$ and $y_1 \neq y_2$ for the rest of this section. Then we get a linear automorphism $\tilde{r}_{(n,d)}$ of the matrix algebra $A$ given by the formula $\tilde{r}_{(n,d)}(v; y_1, y_2) = \mathrm{ev}_{y_2} \circ \mathrm{res}_{y_1}^{-1}$.

3. Note that we have a canonical isomorphism of vector spaces

$$\mathrm{can} : A \otimes A \to \mathrm{End}_{\mathbb{C}}(A),\ X \otimes Y \mapsto \big(Z \mapsto \mathrm{tr}(XZ)Y\big).$$

For fixed $v, y_1, y_2$, we set $r_{(n,d)}(v; y_1, y_2) = \mathrm{can}^{-1}\big(\tilde{r}_{(n,d)}(v; y_1, y_2)\big)$.

**Theorem 15.3.** *The tensor-valued function* $r_{(n,d)} : \left(\mathbb{C}^3_{(v;y_1,y_2)}, 0\right) \to A \otimes A$ *is a rational solution of (15.1). Moreover* $r_{(n,d)}(v; y_1, y_2)$ *is holomorphic on* $\big(\mathbb{C}^3 \setminus V\big(v(y_1 - y_2)\big)\big)$.

**Example 15.4.** For any $n \in \mathbb{N}$, let $P = \sum_{1 \le i,j \le n} e_{ij} \otimes e_{ji} \in A \otimes A$.

i) Let $(n, d) = (2, 1)$. Then we have

$$r_{(2,1)}(v; y_1, y_2) = \frac{1}{2v} \mathbb{1} \otimes \mathbb{1} + \frac{1}{y_2 - y_1} P +$$
$$+ (v - y_1)\, e_{21} \otimes \check{h} + (v + y_2)\, \check{h} \otimes e_{21} - \frac{v(v - y_1)(v + y_2)}{2} e_{21} \otimes e_{21},$$

where $\check{h} = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2})$.

ii) Let $(n,d)=(3,1)$. Then we have

$$r_{(3,1)}(v;y_1,y_2)=\frac{1}{3v}\mathbb{1}\otimes\mathbb{1}+\frac{1}{y_2-y_1}P-$$

$$-e_{21}\otimes\check{h}_1+\check{h}_1\otimes e_{21}+e_{32}\otimes e_{12}-e_{12}\otimes e_{32}-y_1e_{32}\otimes\check{h}_2+y_2\check{h}_2\otimes e_{32}+$$

$$+(v-y_1)e_{31}\otimes e_{12}+(v+y_2)e_{12}\otimes e_{31}+ve_{32}\otimes(e_{11}-e_{33})+v(e_{11}-e_{33})\otimes e_{32}+$$

$$+\frac{1}{3}v(y_1-3v)e_{32}\otimes e_{21}+\frac{1}{3}v(y_2+3v)e_{21}\otimes e_{32}+v(v-y_1)e_{31}\otimes\check{h}_1-$$

$$-v(v+y_2)\check{h}_1\otimes e_{31}+\frac{2}{3}v^2(y_1-v)e_{31}\otimes e_{21}-\frac{2}{3}v^2(y_2+v)e_{21}\otimes e_{31}+$$

$$+\frac{1}{3}v^2(y_2+v)(3v-y_1)e_{32}\otimes e_{31}+\frac{1}{3}v^2(y_1+v)(3v+y_2)e_{31}\otimes e_{32}+$$

$$+\frac{2}{3}ve_{21}\otimes e_{21}+\frac{2}{3}v^3(v-y_1)(v+y_2)e_{31}\otimes e_{31}$$

$$+\frac{1}{3}v(-6v^2+3v(y_1-y_2)+2y_1y_2)e_{32}\otimes e_{32},$$

where $\check{h}_1=\operatorname{diag}(\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$ and $\check{h}_2=\operatorname{diag}(\frac{1}{3},\frac{1}{3},-\frac{2}{3})$.

15.2. **Identification of the geometric method and Algorithm 15.1.** Similarly to the case of the elliptic curve, the key observation of this subsection is that (8.14) can be connected to the algorithm presented in the previous subsection. First recall the description of the simple vector bundles on the cuspidal Weierstraß cubic curve as in Section 13.

Let $\pi:\mathbb{P}^1\longrightarrow E$ be the normalization of $E$. We choose homogeneous coordinates $(z_0:z_1)$ on $\mathbb{P}^1$ in such a way that $\pi\big((0:1)\big)$ is the singular point of $E$. In what follows, we denote $\infty=(0:1)$ and $0=(1:0)$. Abusing the notation, for any $x\in\mathbb{k}$ we also denote by $x\in\breve{E}$ the image of the point $\tilde{x}=(1:x)\in\mathbb{P}^1$, identifying in such a way $\breve{E}$ with $\mathbb{A}^1=\mathbb{P}^1\setminus\{\infty\}=:U_\infty$. Denote $u=\frac{z_0}{z_1}$ then $\mathbb{k}[U_\infty]=\mathbb{k}[u]$. Let $R=\mathbb{k}[\varepsilon]/\varepsilon^2$ and $\mathbb{k}[u]\longrightarrow R$, $u\mapsto\varepsilon$ be the canonical projection. Then in the notation of the previous subsection we have: $Z\cong\operatorname{Spec}(\mathbb{k})$ and $\widetilde{Z}\cong\operatorname{Spec}(R)$.

1. By the theorem of Birkhoff-Grothendieck, for any $\mathcal{F}\in\operatorname{VB}(\mathbb{P}^1)$ we have:

$$\pi^*\mathcal{F}\cong\bigoplus_{c\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}. \tag{15.2}$$

A choice of homogeneous coordinates on $\mathbb{P}^1$ yields two distinguished sections $z_0,z_1\in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$. In such a way, for any $e\in\mathbb{N}$ we get a distinguished basis of the space

$\mathrm{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(e)\big)$ given by the monomials $z_0^e, z_0^{e-1}z_1, \dots, z_1^e$. Next, for any $c \in \mathbb{Z}$ we fix the following isomorphism

$$\mathcal{O}_{\mathbb{P}^1}(c)\big|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}$$

sending a local section $p$ to $\frac{p}{z_1^c\big|_{\widetilde{Z}}}$. Thus, for any vector bundle $\widetilde{\mathcal{F}} = \bigoplus_{c\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}$ of rank $n$ on $\mathbb{P}^1$ we have the induced isomorphism $\zeta^{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}}\big|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}^{\oplus n}$.

2. Given $0 < d < n$ mutually prime and $\lambda \in \Bbbk$, we take the matrix $J = \mathcal{J}_0(n-d, d)$ constructed in Algorithm 13.2 and let

$$M = M_{n,d,\lambda} = \mathbb{1} + \varepsilon\big(\lambda\mathbb{1} + J\big) \in \mathrm{GL}_n(R). \tag{15.3}$$

Any such matrix defines the morphism $\mathsf{m} : \eta_*\mathcal{O}_Z \longrightarrow \nu_*\mathcal{O}_{\widetilde{Z}}$. Let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_{n,d} = \mathcal{O}_{\mathbb{P}^1}^{\oplus e} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$. Then we consider the following vector bundle $\mathcal{P} = \mathcal{P}_{n,d,\lambda}$ on $E$:

$$0 \longrightarrow \mathcal{P} \xrightarrow{\binom{\imath}{q}} \pi_*\widetilde{\mathcal{P}} \oplus \eta_*\mathcal{O}_Z^{\oplus n} \xrightarrow{(\zeta^{\widetilde{\mathcal{P}}}\ \mathsf{m})} \nu_*\mathcal{O}_{\widetilde{Z}}^{\oplus n} \longrightarrow 0. \tag{15.4}$$

Then $\mathcal{P}$ is a simple vector bundle of rank $n$ and degree $d$ on the cuspidal Weierstraß cubic curve $E$. Moreover, in an appropriate sense (15.4) describes a universal family of stable vector bundles of rank $n$ and degree $d$ on $E$, see [14, Theorem 5.1.40]. The following result shall prove useful later:

**Corollary 15.5.** *Let $0 < d < n$ be a pair of coprime integers, and $J = \mathcal{J}_0(n-d, d) \in \mathrm{Mat}_{n\times n}(\Bbbk)$ be the matrix constructed in Algorithm 13.2. Consider the vector bundle $\mathcal{A}$ given by the following short exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{\binom{\jmath}{r}} \pi_*\widetilde{\mathcal{A}} \oplus \eta_*\big(\mathrm{Ad}(\mathcal{O}_Z^{\oplus n})\big) \xrightarrow{\left(\zeta^{\mathrm{Ad}(\widetilde{\mathcal{P}})}\ \mathsf{cnj}(\mathsf{m})\right)} \eta_*\big(\mathrm{Ad}(\mathcal{O}_{\widetilde{Z}}^{\oplus n})\big) \longrightarrow 0, \tag{15.5}$$

*where $\widetilde{\mathcal{P}} = \mathcal{O}_{\mathbb{P}^1}^{\oplus e} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$ and $\widetilde{\mathcal{A}} = \mathrm{Ad}(\widetilde{\mathcal{P}})$. Then $\mathcal{A} \cong \mathrm{Ad}(\mathcal{P})$, where $\mathcal{P}$ is a simple vector bundle on $E$ of rank $n$ and degree $d$. Moreover, for any trivialization $\xi : \widetilde{\mathcal{P}}\big|_{U_\infty} \longrightarrow \mathcal{O}_{U_\infty}^{\oplus n}$ we get the following isomorphisms of sheaves of Lie algebras*

$$\mathcal{A}\big|_{\breve{E}} \xrightarrow{\jmath} \pi_*\big(\mathrm{Ad}(\widetilde{\mathcal{P}})\big)\big|_{\breve{E}} \longrightarrow \pi_*\mathrm{Ad}\big(\mathcal{O}_{U_\infty}^{\oplus n}\big) \xrightarrow{\mathsf{can}} \mathrm{Ad}\big(\mathcal{O}_{\breve{E}}^{\oplus n}\big), \tag{15.6}$$

*where the second morphism is induced by $\xi$.*

3. In the above notation, for any $x \in \breve{E} \cong \mathbb{A}^1$ the corresponding line bundle $\mathcal{O}_E\big([x]\big)$ is given by the triple $\big(\mathcal{O}_{\mathbb{P}^1}(1), \Bbbk, 1 - x\cdot\varepsilon\big)$, see [14, Lemma 5.1.27].

The following theorem finishes the proof of correctness of Algorithm 15.1:

**Theorem 15.6.** [14, Section 10, in particular Algorithm 10.7] *Let*

$$\chi: \ \mathcal{P}\big|_{Ereg} \xrightarrow{\cong} \pi_*\widetilde{\mathcal{P}}\big|_{Ereg} \xrightarrow{\cong} \mathcal{O}^n_{Ereg}$$

*denote the trivialization induced by* $\zeta^{\widetilde{\mathcal{P}}}$*. For* $\lambda_1 \neq \lambda_2 \in \mathbb{C}$*, set* $\mathcal{F}_i = \mathcal{P}_{n,d,\lambda_i}$ *for both* $i = 1, 2$*. Then for any* $y_1 \neq y_2 \in E_{\mathrm{reg}}$ *and* $\lambda = \lambda_1 - \lambda_2 \in \mathbb{C}^\times$*, the following diagram is commutative*

$$\begin{array}{ccccc} \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big) & \xleftarrow{\mathrm{res}_{y_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)} & \mathrm{Hom}\big(\mathcal{F}_1, \mathcal{F}_2(y_1)\big) & \xrightarrow{\mathrm{ev}_{y_2}^{\mathcal{F}_1,\mathcal{F}_2(y_1)}} & \mathrm{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big) \\ \downarrow & & \downarrow \phi & & \downarrow \\ \mathrm{Mat}_{n\times n}(\mathbb{C}) & \xleftarrow[\mathrm{res}_{y_1}]{} & W_{n,d} & \xrightarrow[\mathrm{ev}_{y_2}]{} & \mathrm{Mat}_{n\times n}(\mathbb{C}), \end{array}$$

*where all vertical arrows are induced by the trivialization* $\chi$*. Moreover,* $\mathrm{im}\,\phi = \mathrm{Sol}^{\lambda,y_1}_{n,d}$*.*

15.3. **Obtaining rational solutions of the CYBE from solutions of the AYBE.** Combining Theorem 7.1 1) with example 15.4, we derive that $r_{(2,1)}$ and $r_{(3,1)}$ induce solutions of the CYBE (2.1). Let us denote these by $c_{(2,1)}$ and $c_{(3,1)}$ respectively. Also, let $\Omega$ be the Casimir element of $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$ with respect to the trace form $(x, y) \mapsto \mathrm{tr}(x \cdot y)$

$$\Omega = \sum_{1\le i\neq j\le n} e_{i,j} \otimes e_{j,i} + \sum_{1\le l\le n-1} h_l \otimes \check{h}_l.$$

Observing that $(\mathrm{pr} \otimes \mathrm{pr})(P) = \Omega$, we derive the following formulae:

$$c_{(2,1)}(y_1, y_2) = \frac{\Omega}{y_2 - y_1} + y_2 \check{h} \otimes e_{21} - y_1 e_{21} \otimes \check{h} \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$$

and

$$c_{(3,1)}(y_1, y_2) = \frac{\Omega}{y_2 - y_1} + y_2 \check{h}_2 \otimes e_{32} - y_1 e_{32} \otimes \check{h}_2 + y_2 e_{12} \otimes e_{31} - y_1 e_{31} \otimes e_{12} -$$

$$-e_{21} \otimes \check{h}_1 + \check{h}_1 \otimes e_{21} + e_{32} \otimes e_{12} - e_{12} \otimes e_{32} \in \mathfrak{sl}_3(\mathbb{C}) \otimes \mathfrak{sl}_3(\mathbb{C}).$$

Moreover, it can be verified that neither $c_{(2,1)}$ nor $c_{(3,1)}$ has any infinitesimal symmetries. Thus Theorem 7.1 1) yields that for fixed $v_0 \in \mathbb{C}^\times$, both $r_{(2,1)}(v_0; y_1, y_2)$ and $r_{(3,1)}(v_0; y_1, y_2)$ satisfy the QYBE (4.1). In the next section, we give a more direct way to compute the solutions $c_{(n,d)}$

## 16. From vector bundles on the cuspidal cubic curve to solutions of the CYBE

In this section we describe the recipe to compute the solution $c_{(n,d)}$ of the CYBE corresponding to the triple $(E, n, d)$, where $E$ is the cuspidal Weierstraß cubic curve and $0 < d < n$ a pair of coprime integers ($c_{(n,d)}$ was denoted $r_{(E,n,d)}$ in the introduction). As before, we first present the algorithm, see Subsection 16.1. Then we develop a general formula for the solutions $c_{(n,d)}$, see Subsection 16.2. Only then will we verify the algorithm , see Subsection 16.3. The main idea is identify the algorithm of Subsection 16.1 with the procedure described in Section 9.

16.1. **Construction of the rational solutions $c_{(n,d)}$ of the CYBE.** Let $0 < d < n$ such that $\gcd(n, d) = 1$ be given.

(1) We first compute the matrix $J = \mathcal{J}_0(n - d, d)$ given by Algorithm 13.2.
(2) Next, for the block decomposition of $\mathrm{Mat}_{n\times n}(\Bbbk)$ induced by $J$, consider the following subspace of $\mathfrak{sl}_n \otimes \Bbbk[z]$:

$$V_{n,d} = \left\{ F(z) = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right) + \left(\begin{array}{c|c} W' & 0 \\ \hline Y' & Z' \end{array}\right) z + \left(\begin{array}{c|c} 0 & 0 \\ \hline Y'' & 0 \end{array}\right) z^2 \right\}.$$

Given $F(z) \in V_{n,d}$, we denote

$$F_0 = \left(\begin{array}{c|c} W' & X \\ \hline Y'' & Z' \end{array}\right) \text{ and } F_\epsilon = \left(\begin{array}{c|c} W & 0 \\ \hline Y' & Z \end{array}\right).$$

For $u \in \mathbb{C}$, we determine a basis of the following subspace of $V_{n,d}$:

$$\mathrm{Sol}^u_{n,d} = \left\{ F(z) \in V_{n,d} \,\middle|\, [F_0, J] + uF_0 + F_\epsilon = 0 \right\}.$$

(3) We choose a basis for $\mathfrak{sl}_n(\Bbbk)$ and compute the images of the basis vectors under the map

$$\mathfrak{sl}_n(\Bbbk) \xrightarrow{\mathrm{res}_u^{-1}} \mathrm{Sol}^u_{n,d} \xrightarrow{\mathrm{ev}_v} \mathfrak{sl}_n(\Bbbk).$$

Here $\mathrm{res}_u(F(z)) = F(u)$ and $\mathrm{ev}_v(F(z)) = \frac{1}{v-u} F(v)$. For a proof that $\mathrm{res}_u$ is an isomorphism, we refer to Subsection 16.3.
(4) Using the trace form, we obtain a canonical isomorphism of vector spaces

$$\mathrm{can} : \mathfrak{sl}_n(\Bbbk) \otimes \mathfrak{sl}_n(\Bbbk) \to \mathrm{End}_\Bbbk(\mathfrak{sl}_n(\Bbbk)),\ X \otimes Y \mapsto \big(Z \mapsto \mathrm{tr}(XZ)Y\big).$$

For fixed $u, v$, we set $r_{(E,n,d)}(u, v) = \mathrm{can}^{-1}\big(\mathrm{ev}_v \circ \mathrm{res}_u^{-1}\big) \in \mathfrak{sl}_n(\Bbbk) \otimes \mathfrak{sl}_n(\Bbbk)$.

**Theorem 16.1.** *The map $c_{(n,d)}$ is a non-degenerate unitary solution of the classical Yang-Baxter equation (2.1).*

*Remark* 16.2. In the introduction, the solution $c_{(n,d)}$ was denoted $r_{(E,n,d)}$ , where $E$ is the cuspidal cubic curve. The proof of the statement above can be found in Subsection 16.3.

16.2. **The solution $c_{(n,d)}$ for a particular choice of basis.** It is convenient to develop a more concrete formula for $c_{(n,d)}$. To this end, we fix the standard basis $\{e_{i,j}\}_{1\leq i\neq j\leq n}\cup\{h_l\}_{1\leq l\leq n-1}$ of $\mathfrak{sl}_n(\Bbbk)$. Since $\mathrm{res}_u:\mathrm{Sol}^u_{n,d}\to\mathfrak{sl}_n(\Bbbk)$ given by $F(z)\mapsto F(u)$ is an isomorphism, we have

$$\begin{array}{rl} \mathrm{res}_u^{-1}(e_{i,j}) = e_{i,j} + G^u_{i,j}(z) & 1\leq i\neq j\leq n \\ \mathrm{res}_u^{-1}(h_l) = h_l + G^u_l(z) & 1\leq l\leq n-1, \end{array}$$

where $G^u_{i,j}(z), G^u_l(z)\in V_{n,d}$ are uniquely determined by the properties

$$\begin{gathered} e_{i,j} + G^u_{i,j}(z), h_l + G^u_l(z) \in \mathrm{Sol}^u_{n,d} \\ G^u_{i,j}(u) = 0 = G^u_l(u). \end{gathered} \tag{16.1}$$

**Proposition 16.3.** *In the notations as above, we have*

$$c_{(n,d)}(u,v) = \frac{1}{v-u}\left[\Omega + \left(\sum_{1\leq i\neq j\leq n} e_{j,i}\otimes G^u_{i,j}(v)\right) + \left(\sum_{1\leq l\leq n-1} \check{h}_l\otimes G^u_l(v)\right)\right],$$

*where $\check{h}_l$ denotes the dual of $h_l$ with respect to the trace form and $\Omega$ denotes the Casimir element. Especially, $c_{(n,d)}$ is rational.*

*Proof.* It follows directly from the definitions that

$$\begin{array}{rl} \mathrm{ev}_v\circ\mathrm{res}_u^{-1}(e_{i,j}) = \frac{1}{v-u}\left(e_{i,j} + G^u_{i,j}(v)\right) & 1\leq i\neq j\leq n \\ \mathrm{ev}_v\circ\mathrm{res}_u^{-1}(h_l) = \frac{1}{v-u}\left(h_l + G^u_l(v)\right) & 1\leq l\leq n-1, \end{array}$$

Since $e_{j,i}$ respectively $\check{h}_l$ is the dual of $e_{i,j}$ respectively $h_l$ with respect to the trace form, applying $\mathrm{can}^{-1}$ yields

$$\begin{array}{rl} \left(e_{i,j}\mapsto \frac{1}{v-u}\left(e_{i,j}+G^u_{i,j}(v)\right)\right)\mapsto e_{j,i}\otimes\frac{1}{v-u}\left(e_{i,j}+G^u_{i,j}(v)\right) & 1\leq i\neq j\leq n \\ \left(h_l\mapsto \frac{1}{v-u}\left(h_l+G^u_l(v)\right)\right)\mapsto \check{h}_l\otimes\frac{1}{v-u}\left(h_l+G^u_l(v)\right) & 1\leq l\leq n-1. \end{array}$$

But

$$\Omega = \sum_{1\leq i\neq j\leq n} e_{i,j}\otimes e_{j,i} + \sum_{1\leq l\leq n-1}\check{h}_l\otimes h_l,$$

hence the statement easily follows. □

16.3. **Identification of the geometric method and Algorithm 16.1.** In this subsection we prove Theorem 16.1. We fix the following notation.

1. As before, $E = V(zy^2 - x^3)$ is the cuspidal Weierstraß cubic curve, $\pi : \mathbb{P}^1 \longrightarrow E$ the normalization map, $(z_0 : z_1)$ are homogeneous coordinates on $\mathbb{P}^1$, $\infty = (0 : 1)$ and $0 = (1 : 0)$. As in the previous subsection, we assume that $\pi(\infty)$ is the singular point of $E$. For any $x \in \Bbbk$ we set $\tilde{x} = (1 : x) \in \mathbb{P}^1$. Let $U_0 = \mathbb{P}^1 \setminus \{\infty\}$. Using the normalization map $\pi$, we identify $U_0$ and $\breve{E}$. Moreover, for any $x \in \Bbbk$ we also denote by $x = \pi(\tilde{x})$ the corresponding smooth point of $E$. Let $z = \frac{z_1}{z_0}$, then $w := dz$ is the meromorphic differential form on $\mathbb{P}^1$ which descends to a regular differential form on $E$ yielding an isomorphism $\Omega_E \longrightarrow \mathcal{O}_E$.
2. For any $c \in \mathbb{Z}$ we fix the trivialization $\xi : \mathcal{O}_{\mathbb{P}^1}(c)\big|_{U_0} \longrightarrow \mathcal{O}_{U_0}$ given on the level of local sections by the rule $p \mapsto \dfrac{p}{z_0^c\big|_{U_0}}$. Thus, for any vector bundle $\widetilde{\mathcal{F}} = \oplus \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}$ we get the induced trivialization $\xi^{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}}\big|_{U_0} \longrightarrow \mathcal{O}_{U_0}^{\oplus n}$, where $n = \mathrm{rk}(\widetilde{\mathcal{F}})$.
3. For a pair of coprime integers $0 < d < n$, let $\mathcal{A}$ be the sheaf of Lie algebras on $E$ given by the short exact sequence (15.5). Then we automatically have the trivialization $\jmath^{\mathcal{A}} : \mathcal{A}\big|_{\breve{E}} \longrightarrow \mathcal{O}_{\breve{E}}^{\oplus n}$, given by the formula (15.6) and the trivialization $\xi^{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}}\big|_{U_0} \longrightarrow \mathcal{O}_{U_0}^{\oplus n}$.

The main result of this subsection is the following theorem, which yields Theorem 16.1.

**Theorem 16.4.** *In the above notation, the following results are true.*

1. *Let $x \neq y \in \breve{E}$. Then the following diagram is commutative:*

$$
\begin{array}{ccccc}
\mathcal{A}\big|_x & \xleftarrow{\ \mathsf{res}_x(w)\ } & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\ \mathsf{ev}_y\ } & \mathcal{A}\big|_y \\
\downarrow{\scriptstyle \jmath_x^{\mathcal{A}}} & & & & \downarrow{\scriptstyle \jmath_y^{\mathcal{A}}} \\
\mathfrak{sl}_n(\Bbbk) & & \downarrow{\scriptstyle \bar{\pi}^*} & & \mathfrak{sl}_n(\Bbbk) \\
\downarrow & & & & \downarrow \\
\mathrm{Mat}_n(\Bbbk) & \xleftarrow{\ \overline{\mathsf{res}}_x\ } & H^0\big(\widetilde{\mathcal{E}}(1)\big) & \xrightarrow{\ \overline{\mathsf{ev}}_y\ } & \mathrm{Mat}_n(\Bbbk)
\end{array}
$$

*where the following notation is used.*

(1) $\mathsf{res}_x(w) = \mathsf{res}_x^{\mathcal{A}}(w)$ *is the residue map (9.18) at the point $x$ given by the differential form $w = dz$ and $\mathsf{ev}_y = \mathsf{ev}_y^{\mathcal{A}}$ is the evaluation map (9.19).*

(2) $\xi_x^{\mathcal{A}}$ *and* $\xi_y^{\mathcal{A}}$ *are isomorphisms given by the trivialization $\jmath^{\mathcal{A}}$.*

(3) $\widetilde{\mathcal{E}} = \mathcal{E}nd(\widetilde{\mathcal{P}})$, *where* $\widetilde{\mathcal{P}} = \mathcal{O}_{\mathbb{P}^1}^{\oplus e} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$. *Note that we have:*

$$\widetilde{\mathcal{E}} = \left(\begin{array}{ccc||ccc} \mathcal{O} & \dots & \mathcal{O} & \mathcal{O}(-1) & \dots & \mathcal{O}(-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \dots & \mathcal{O} & \mathcal{O}(-1) & \dots & \mathcal{O}(-1) \\ \hline\hline \mathcal{O}(1) & \dots & \mathcal{O}(1) & \mathcal{O} & \dots & \mathcal{O} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{O}(1) & \dots & \mathcal{O}(1) & \mathcal{O} & \dots & \mathcal{O} \end{array}\right) \tag{16.2}$$

*where* $\mathcal{O}(c) := \mathcal{O}_{\mathbb{P}^1}(c)$ *for* $c \in \{-1, 0, 1\}$. *The sizes of the diagonal blocks in the presentation (16.2) are* $e \times e$ *and* $d \times d$ *respectively.*

(4) *In the above notation we have:*

$$H^0\big(\widetilde{\mathcal{E}}(1)\big) = \left\{ F = \left( \begin{array}{c|c} z_0 W + z_1 W' & X \\ \hline z_0^2 Y + z_0 z_1 Y' + z_1^2 Y'' & z_0 Z + z_1 Z' \end{array} \right) \right\} \tag{16.3}$$

*where* $W, W' \in \mathrm{Mat}_{e\times e}(\Bbbk)$, $Z, Z' \in \mathrm{Mat}_{d\times d}(\Bbbk)$, $Y, Y', Y'' \in \mathrm{Mat}_{d\times e}(\Bbbk)$ *and* $X \in \mathrm{Mat}_{e\times d}(\Bbbk)$.

(5) *For any* $F \in H^0\big(\widetilde{\mathcal{E}}(1)\big)$ *as in (16.3) we have:*

$$\overline{\mathsf{res}}_x(F) = F(1, x) \quad \textit{and} \quad \overline{\mathsf{ev}}_y(F) = \frac{1}{y - x} F(1, y). \tag{16.4}$$

(6) *The morphism* $\bar{\pi}^*$ *is defined as follows. We compose the canonical map* $\pi^* : H^0\big(\mathcal{A}(x)\big) \longrightarrow H^0\big(\pi^*\mathcal{A}(\tilde{x})\big)$ *with the morphism induced by the following morphism of sheaves*

$$\pi^*\mathcal{A}(\tilde{x}) \longrightarrow \pi^*\pi_*\widetilde{\mathcal{A}}(\tilde{x}) \longrightarrow \widetilde{\mathcal{A}}(\tilde{x}) \hookrightarrow \widetilde{\mathcal{E}}(\tilde{x}) \xrightarrow{t_\sigma} \widetilde{\mathcal{E}}(1),$$

*where* $t_\sigma$ *is the isomorphism induced by* $\sigma = z_1 - x z_0 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$.

*2. Let* $\mathrm{Sol}_{n,d} := \mathsf{Im}(\bar{\pi}^*) \subset H^0\big(\widetilde{\mathcal{E}}(1)\big)$. *Then we have:*

$$\mathrm{Sol}_{n,d} = \Big\{ F \,\Big|\, \mathrm{tr}(W + Z) = 0 = \mathrm{tr}(W' + Z'),\ [F_0, J] + x F_0 + F_\varepsilon = 0 \Big\}, \tag{16.5}$$

*where* $F$ *is a matrix whose entries are homogeneous forms as in (16.3), whereas*

$$F_0 := \left( \begin{array}{c|c} W' & X \\ \hline Y'' & Z' \end{array} \right) \ \textit{and}\ F_\epsilon := \left( \begin{array}{c|c} W & 0 \\ \hline Y' & Z \end{array} \right). \tag{16.6}$$

*Proof.* First note that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
 & & \mathcal{A}\big|_x & \xleftarrow{\mathsf{res}_x^{\mathcal{A}}(w)} & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\mathsf{ev}_y^{\mathcal{A}}} & \mathcal{A}\big|_y & & \\
 & & \downarrow \hat{\pi}_x^* & & \downarrow \hat{\pi}^* & & \downarrow \hat{\pi}_y^* & & \\
 & & \widetilde{\mathcal{A}}\big|_{\tilde{x}} & \xleftarrow{\mathsf{res}_{\tilde{x}}^{\widetilde{\mathcal{A}}}(w)} & H^0\big(\widetilde{\mathcal{A}}(\tilde{x})\big) & \xrightarrow{\mathsf{ev}_{\tilde{y}}^{\widetilde{\mathcal{A}}}} & \widetilde{\mathcal{A}}\big|_{\tilde{y}} & & \\
 & \jmath_x^{\mathcal{A}} & \big\downarrow & & \big\downarrow & & \big\downarrow & \jmath_y^{\mathcal{A}} & \\
 & & \widetilde{\mathcal{E}}\big|_{\tilde{x}} & \xleftarrow{\mathsf{res}_{\tilde{x}}^{\widetilde{\mathcal{E}}}(w)} & H^0\big(\widetilde{\mathcal{E}}(\tilde{x})\big) & \xrightarrow{\mathsf{ev}_{\tilde{y}}^{\widetilde{\mathcal{E}}}} & \widetilde{\mathcal{E}}\big|_{\tilde{y}} & & \\
 & & \downarrow \xi_{\tilde{x}}^{\widetilde{\mathcal{E}}} & & \downarrow (t_\sigma)_* & & \downarrow \xi_{\tilde{y}}^{\widetilde{\mathcal{E}}} & & \\
\mathfrak{sl}_n(\Bbbk) & \hookrightarrow & \mathrm{Mat}_n(\Bbbk) & \xleftarrow{\overline{\mathsf{res}}_x} & H^0\big(\widetilde{\mathcal{E}}(1)\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathrm{Mat}_n(\Bbbk) & \hookleftarrow & \mathfrak{sl}_n(\Bbbk).
\end{array}
$$

Let us explain our notation. In the notation used in (15.5), the composition

$$\gamma_{\mathcal{A}}: \quad \pi^*\mathcal{A} \xrightarrow{\pi^*(\imath)} \pi^*\pi_*\widetilde{\mathcal{A}} \xrightarrow{\mathsf{can}} \widetilde{\mathcal{A}}$$

is an isomorphism of vector bundles on $\mathbb{P}^1$. The morphisms $\hat{\pi}_x^*$ and and $\hat{\pi}_y^*$ are the induced maps in the fibers, obtained by composing $\pi^*$ and $\gamma_{\mathcal{A}}$. Similarly, $\hat{\pi}^*$ is the induced map of global sections. The commutativity of the first two top squares follows from the "locality" of the morphisms $\underline{\mathsf{res}}_x(w)$ and $\underline{\mathsf{ev}}_y$, see [14, Proposition 2.2.8 and Proposition 2.2.12] as well as [14, Section 5.2] for a detailed proof.

Next, recall that $\widetilde{\mathcal{E}} = \mathcal{E}nd_{\mathbb{P}^1}(\widetilde{\mathcal{P}})$ and $\widetilde{\mathcal{A}} = \mathrm{Ad}(\widetilde{\mathcal{P}})$, so we have the obvious inclusion $\widetilde{\mathcal{A}} \hookrightarrow \widetilde{\mathcal{E}}$. This morphism induces inclusion of the fibers of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{E}}$ over $\tilde{x}$ and $\tilde{y}$ as well as the map of global sections. The commutativity of two middle squares is obvious.
The commutativity of two lower squares is given by [14, Corollary 5.2.1] and [14, Corollary 5.2.2] respectively. In particular, the explicit formulae (16.4) for the maps $\overline{\mathsf{res}}_x$ and $\overline{\mathsf{ev}}_y$ are given there. Finally, see [14, Subsection 5.2.2] for the proof of commutativity of two side diagrams. This proves the first part of Theorem 16.4.
To prove the second part of the theorem note that $\mathrm{Sol}_{n,d} = \overline{\mathrm{Sol}}_{n,d} \cap \ker(T)$, where we use the following notation.

$$\overline{\mathrm{Sol}}_{n,d} := \mathsf{Im}\Big(H^0\big(\mathcal{E}(x)\big) \xrightarrow{\bar{\pi}^*} H^0\big(\widetilde{\mathcal{E}}(1)\big)\Big) \subset H^0\big(\widetilde{\mathcal{E}}(1)\big), \tag{16.7}$$

where $\bar{\pi}^* = (t_\sigma)_* \circ \mathsf{cnj}(\gamma_{\mathcal{P}}) \circ \pi^*$. Next, the map $T$ is such that the following diagram

$$(16.8)\qquad \begin{array}{ccc} H^0\big(\mathcal{E}nd(\widetilde{\mathcal{P}})(\tilde{x})\big) & \xrightarrow{(t_\sigma)_*} & H^0\big(\widetilde{\mathcal{E}}(1)\big) \\ \Big\downarrow{\scriptstyle H^0(\mathrm{Tr}_{\widetilde{\mathcal{P}}}(\tilde{x}))} & & \Big\downarrow{\scriptstyle T} \\ H^0\big(\mathcal{O}_{\mathbb{P}^1}(\tilde{x})\big) & \xrightarrow{t_\sigma} & H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \end{array}$$

is commutative. It is not difficult to show that for $F \in H^0\big(\widetilde{\mathcal{E}}(1)\big)$ written in the matrix form (16.3) we actually have:

$$(16.9)\qquad T(F) = \mathrm{tr}(W+Z)z_0 + \mathrm{tr}(W'+Z')z_1.$$

Let $\mathcal{P}$ be a simple vector bundle of rank $n$ and degree $d$ on $E$ given by (15.4). Then $\mathcal{E} = \mathcal{E}nd_E(\mathcal{P})$ is given by the triple $\big(\mathcal{E}nd_{\mathbb{P}^1}(\widetilde{\mathcal{P}}), \mathrm{Mat}_{n\times n}(\mathcal{O}_Z), \mathsf{cnj}(M)\big)$, see Lemma 11.3. The isomorphism $\mathsf{cnj}(M) : \mathrm{Mat}_{n\times n}(R) \longrightarrow \mathrm{Mat}_{n\times n}(R)$ is given by the formula $X \mapsto M \circ X \circ M^{-1}$, where $M$ is the matrix given by (15.3) for $\lambda = 0$. Note that $M^{-1} = \mathbb{1} - \varepsilon J(e,d)$.

As it was already mentioned above, the line bundle $\mathcal{O}_E\big([x]\big)$ corresponds to the triple $\big(\mathcal{O}_{\mathbb{P}^1}(1), \Bbbk, \mathbb{1} - x\cdot\varepsilon\big)$. Since the tensor product of vector bundles on $E$ corresponds to the tensor product in $\mathrm{Tri}(E)$, we have the following statement for $F \in H^0\big(\widetilde{\mathcal{E}}(1)\big)$:

$$(16.10)\qquad F \in \overline{\mathrm{Sol}}_{n,d} \iff F\big|_{\widetilde{Z}} = (1 - x\cdot\varepsilon)\cdot M \cdot A \cdot M^{-1} \in \mathrm{Mat}_{n\times n}(\Bbbk)[\varepsilon]$$

for some $A \in \mathrm{Mat}_n(\Bbbk)$, where $F\big|_{\widetilde{Z}} = F_0 + \varepsilon F_\epsilon$ and $F_0, F_\varepsilon$ are given by (16.6). See also [14, Subsection 5.2.5] or a computation in a similar situation. Now, the fact that $\mathrm{Sol}_{n,d} = \overline{\mathrm{Sol}}_{n,d} \cap \ker(T)$, combined with the formulae (16.9) and (16.10), proves the formula (16.5). The theorem is proved. $\square$

## Part 5. Computations of elliptic solutions of the AYBE

In this part we compute the solutions of the associative Yang-Baxter equation (14.1) attached to a diagonal matrix and to a Jordan block, see sections 17 and 18 respectively. The general procedure for these computations was described in Subsection 14.1. Since the computation of solutions attached to a Jordan block involves some rather lengthy combinatorics, we postpone the most tiresome proofs (those of Proposition 18.3 and Lemma 18.20) to Section 19. Those are precisely the proofs omitted in [12, Section 4.2].

## 17. Solution obtained from a diagonal matrix

All our computations are based on the following standard fact.

**Lemma 17.1.** *Let $\varphi(z) = \exp(-\pi i\tau - 2\pi i z)$. Then the vector space*

$$\left\{ f : \mathbb{C} \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f \quad \text{is holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = \varphi(z) f(z) \end{array} \right\} \tag{17.1}$$

*is one-dimensional and generated by the third Jacobian theta-function*

$$\bar{\theta}(z) = \theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$$

*Proof.* A proof of this result can for instance be found in [35, Chapter 1]. □

**Theorem 17.2.** *Let $B = \mathsf{diag}\big(\exp(2\pi i\lambda_1), \dots, \exp(2\pi i\lambda_n)\big)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then the corresponding solution of the associative Yang-Baxter equation described in Theorem 14.7 is given by the following formula:*

$$r_B(v, y) = \sum_{k,\,l=1}^{n} \sigma(v - \lambda_{kl}, y)\, e_{l,k} \otimes e_{k,l}, \tag{17.2}$$

*where $\lambda_{kl} = \lambda_k - \lambda_l$ for all $1 \le k, l \le n$ and $\sigma(u, x)$ is the Kronecker function.*

*Proof.* Let $\Phi(z) = \big(a_{kl}(z)\big)$ be an element of $\mathrm{Sol} = \mathrm{Sol}_{B,\,v,\,0,\,\tau}$, where $v = v_1 - v_2$. Then for all $1 \le k, l \le n$ we have:

$$\left\{ \begin{array}{lcl} a_{kl}(z+1) & = & a_{kl}(z) \\ a_{kl}(z+\tau) & = & \exp\big(-\pi i\tau - 2\pi i(z + v + \frac{\tau+1}{2} - \lambda_{kl})\big) a_{kl}(z). \end{array} \right.$$

Hence, there exist $\beta_{kl} \in \mathbb{C}$ such that $a_{kl}(z) = \beta_{kl}\bar{\theta}(z + v + \frac{\tau+1}{2} - \lambda_{kl})$.

If $A = (\alpha_{kl}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is such that $\mathrm{res}_0\big(\Phi(z)\big) = A$ then $\beta_{kl} = \frac{1}{\bar{\theta}(v + \frac{\tau+1}{2} - \lambda_{kl})}\alpha_{kl}$. If $C = (\gamma_{kl}) := \mathrm{ev}_y\big(\Phi(z)\big)$ then for all $1 \le k, l \le n$ we have

$$\gamma_{kl} = \frac{\bar{\theta}(v + y - \lambda_{kl} + \frac{\tau+1}{2})}{\bar{\theta}(v - \lambda_{kl} + \frac{\tau+1}{2})\,\bar{\theta}(y + \frac{\tau+1}{2})}\alpha_{kl} = \frac{1}{i\exp(-\pi i\frac{\tau}{4})}\frac{\theta(v - \lambda_{kl} + y)}{\theta(v - \lambda_{kl})\theta(y)}\alpha_{kl},$$

where we have used the well-known relation between the first and the third Jacobian theta functions $\bar{\theta}(z + \frac{\tau+1}{2}) = i\exp(-\pi i(z + \frac{\tau}{4}))\theta(z)$. Hence, the linear map $\tilde{r}_B(v, y) : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ sends the basis vector $e_{k,l}$ to $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}\,\sigma(v - \lambda_{kl}, y)e_{k,l}$. Neglecting the constant $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}$, we end up with the solution $r_B(v, y)$ given by (17.2). □

## 18. Solution attached to a Jordan block

In this section we compute the solution of the associative Yang-Baxter equation (14.1) attached to a Jordan block of size $n \times n$. Fist note the following easy fact.

**Lemma 18.1.** *For any $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$ the solution $r_{J_n(\lambda)}(v, y)$ constructed in Theorem 14.7, is gauge equivalent to $r_J(v, y)$, where $J_n(\lambda)$ is the Jordan block of size $n \times n$ with eigenvalue $\lambda$ and $J = J_n(1)$.*

*Proof.* Since the matrices $J_n(\lambda)$ and $\lambda \cdot J$ are conjugate, Proposition 14.10 implies that the corresponding solutions are gauge equivalent. From the algorithm of the construction of solutions of (14.1) presented in Theorem 14.7 it is clear that the matrices $\lambda \cdot J$ and $J$ give the same solutions. □

Hence, it suffices to describe the solution of the associative Yang-Baxter equation (14.1) attached to the Jordan block $J$.

**Definition 18.2.** Let $n \in \mathbb{N}$ be fixed. For all $1 \leq k \leq n-1$ we set $a_k = \frac{(-1)^k}{k}$,

$$A_0 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & a_1 & 0 \end{pmatrix} \quad \text{and} \quad A_k = -a_k \cdot \mathbb{1}_{n \times n}.$$

Next, consider the following matrix $N$ from $\mathrm{Mat}_{n^2 \times n^2}(\mathbb{C})$:

$$N = \begin{pmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_1 \\ 0 & \cdots & 0 & A_0 \end{pmatrix}. \tag{18.1}$$

Next, we list all statements concerning $N$ necessary for the following discussion. We recommend that the reader should take only a passing look at them now and postpone studying each statement and its proof to a later moment when it becomes relevant in the discussion. The proofs are contained in Section 19.1.

**Proposition 18.3.** *The following hold:*

(1) For $r \in \mathbb{N}$,

$$N^r = \begin{pmatrix} N_0^{(r)} & N_1^{(r)} & \cdots & N_{n-1}^{(r)} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & N_1^{(r)} \\ 0 & \cdots & 0 & N_0^{(r)} \end{pmatrix}$$

where each $N_i^{(r)}$, $0 \leq i \leq n-1$ is a $n \times n$ matrix given by

$$N_i^{(r)} = \sum_{k=0}^{r} \binom{r}{k} A_0^{r-k} \sum_{s \in S_i^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}}$$

where for $k, i \in \mathbb{N}$

$$S_i^k = \left\{ (s_j)_{0 \leq j \leq k} \in \mathbb{N}^{k+1} \;\middle|\; \substack{s_0 = i \\ s_k = 0 \\ s_{j+1} < s_j,\, 0 \leq j \leq k} \right\}.$$

(2) For all $i, j, r \in \mathbb{N}$ we have

$$e_j^t \, N_i^{(r)} \, e_1 = \sum_{k=0}^{r} \binom{r}{k} \left( \sum_{t \in S_{j-1}^{r-k}} \frac{(-1)^{j-1}}{\prod_{l=0}^{r-k-1} (t_l - t_{l+1})} \right) \left( \sum_{s \in S_i^k} \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1} (s_l - s_{l+1})} \right).$$

(3) $N$ is nilpotent. More precisely $N^{2n-1} = 0$.
(4) For all $1 \leq i, j \leq n$ and any $r \in \mathbb{N}$, $e_j^t \, N_i^{(i+j+r)} \, e_1 = 0$.
(5) We have $\exp(N) = J \otimes \tilde{J}$.
(6) For any $i, j, \chi \in \mathbb{N}$, $(u_p)_{1 \leq p \leq \chi} \in [0, 2n-1]^\chi$ and $u = \sum_{p=1}^{\chi} u_p$:

$$\sum_{(\alpha_p, \beta_p)_{1 \leq p \leq \chi} \in \mathcal{W}_{(i,j)}^{\chi}} \prod_{p=1}^{\chi} e_{\beta_p + 1}^t \, N_{\alpha_p}^{(u_p)} \, e_1 = e_{j+1}^t \, N_i^{(u)} \, e_1$$

where

$$\mathcal{W}_{(i,j)}^{\chi} = \left\{ (\alpha_p, \beta_p)_{1 \leq p \leq \chi} \in (\mathbb{N} \times \mathbb{N})^\chi \;\middle|\; (\alpha_p, \beta_p) \neq (0,0);\ \left( \sum_{p=1}^{\chi} \alpha_p, \sum_{p=1}^{\chi} \beta_p \right) = (i, j) \right\}.$$

By Proposition 18.3 (3), the following definition makes sense:

**Definition 18.4.** Consider the differential operator $\nabla = -\frac{1}{2\pi i} \cdot \frac{d}{dz}$ acting on the vector space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$. For all $0 \leq k, l \leq n-1$ we define the linear operator $\nabla_{k,l} : \mathcal{M} \to \mathcal{M}$ given by the following formula:

$$\nabla_{k,l} = e_{n(n-k-1)+l+1}^t \, \exp(\nabla\, N) \, e_{n(n-1)+1}. \tag{18.2}$$

Since the matrix $N$ is nilpotent, the operators $\nabla_{k,l}$ are polynomials in $\nabla$. Note that $\nabla_{0,0} = e_{n(n-1)+1}^t \, \exp(\nabla\, N) \, e_{n(n-1)+1}$ is the identity operator.

Now we can state the main result of this section.

**Theorem 18.5.** *Let $J$ be the Jordan block of size $n \times n$ with eigenvalue one. Then the corresponding solution of the associative Yang-Baxter equation, described in Theorem 14.7, is given by the following formula:*

$$r_J(v,y) = \left( \sum_{\substack{0\le k\le n-1\\0\le l\le n-1}} \nabla_{k,l}\big(\sigma\,(v,y)\big) \sum_{\substack{1\le i\le n-l\\1\le j\le n-k}} e_{i,j+k}\otimes e_{j,i+l} \right), \tag{18.3}$$

*where $\sigma(v,y)$ is the Kronecker function and $\nabla_{k,l}$ acts on the first spectral variable.*

*Remark* 18.6. Let $1 \le a,b,c,d \le n$. Then the coefficient of the tensor $e_{a,b}\otimes e_{c,d}$ in the expression for $r_J(v,y)$ from Equation (18.3) is zero unless $d \ge a$ and $b \ge c$. Moreover, this coefficient depends only on the differences $d-a$ and $b-c$.

**Example 18.7.** Let $n = 2$ and $J = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Note that

$$N = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 \end{array}\right), \quad N^2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

and that all higher powers of $N$ are zero. Hence,

$$\exp(\nabla N) = 1 + \nabla N + \frac{\nabla^2 N^2}{2} = \left(\begin{array}{cc|cc} \mathbb{1} & 0 & \nabla & 0\\ -\nabla & \mathbb{1} & -\nabla^2 & \nabla\\ \hline 0 & 0 & \mathbb{1} & 0\\ 0 & 0 & -\nabla & \mathbb{1} \end{array}\right)$$

and we derive that

$$\begin{aligned} r_J(v,y) &= \sigma(v,y)\big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\big) + \\ &\quad \nabla\sigma(v,y)\big(e_{12}\otimes h - h\otimes e_{12}\big) - \nabla^2\sigma(v,y)e_{12}\otimes e_{12}, \end{aligned}$$

where $h = e_{11} - e_{22}$.

*Remark* 18.8. From the fact that the function $r_J(v,y)$ from example 18.7 satisfies the associative Yang-Baxter equation (14.1) we obtain the following identity for derivatives of the Kronecker function with respect to the first spectral variable:

$$\begin{aligned} \sigma'(u,x+y)\sigma'(v,y) - \sigma'(u,x)\sigma'(u+v,y) - \sigma'(-v,x)\sigma'(u+v,x+y) \\ = \sigma(u,x)\sigma''(u+v,y) - \sigma(-v,x)\sigma''(u+v,x+y). \end{aligned}$$

**Example 18.9.** For $n=3$ and $J=\left(\begin{smallmatrix}1&1&0\\0&1&1\\0&0&1\end{smallmatrix}\right)$ we have

$$N=\left(\begin{array}{ccc|ccc|ccc}
0&0&0&1&0&0&-\frac{1}{2}&0&0\\
-1&0&0&0&1&0&0&-\frac{1}{2}&0\\
\frac{1}{2}&-1&0&0&0&1&0&0&-\frac{1}{2}\\
\hline
0&0&0&0&0&0&1&0&0\\
0&0&0&-1&0&0&0&1&0\\
0&0&0&\frac{1}{2}&-1&0&0&0&1\\
\hline
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&\frac{1}{2}&-1&0
\end{array}\right).$$

Note that

$$\exp(\nabla N)\,e_7=\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}-\frac{1}{2}\\0\\0\\1\\0\\0\\0\\-1\\\frac{1}{2}\end{pmatrix}\nabla+\begin{pmatrix}1\\1\\-\frac{1}{2}\\0\\-2\\1\\0\\0\\1\end{pmatrix}\frac{\nabla^2}{2}+\begin{pmatrix}0\\-3\\0\\0\\0\\3\\0\\0\\0\end{pmatrix}\frac{\nabla^3}{6}+\begin{pmatrix}0\\0\\6\\0\\0\\0\\0\\0\\0\end{pmatrix}\frac{\nabla^4}{24}$$

and that $\nabla_{k,l}=e^t_{3(2-k)+l+1}\big(\exp(\nabla N)\,e_7\big)$. Carrying out computations, we end up with the following solution of the associative Yang-Baxter equation:

$$\begin{aligned}
r_J(v,y)=&\sigma\sum_{1\le i,j\le 3}e_{i,j}\otimes e_{j,i}+\nabla\sigma\sum_{\substack{1\le i\le 3\\1\le j\le 2}}\big(e_{i,j+1}\otimes e_{j,i}-e_{j,i}\otimes e_{i,j+1}\big)+\\
&\Big(-\frac{1}{2}\nabla+\frac{1}{2}\nabla^2\Big)\sigma\sum_{1\le i\le 3}e_{i,3}\otimes e_{1,i}+\Big(\frac{1}{2}\nabla+\frac{1}{2}\nabla^2\Big)\sigma\sum_{1\le i\le 3}e_{1,i}\otimes e_{i,3}\\
&\Big(\frac{1}{2}\nabla^2-\frac{1}{2}\nabla^3\Big)\sigma\sum_{1\le i\le 2}e_{i,3}\otimes e_{1,i+1}+\Big(\frac{1}{2}\nabla^2+\frac{1}{2}\nabla^3\Big)\sigma\sum_{1\le i\le 2}e_{1,i+1}\otimes e_{i,3}+\\
&-\nabla^2\sigma\sum_{1\le i,j\le 2}e_{i,j+1}\otimes e_{j,i+1}+\Big(-\frac{1}{4}\nabla^2+\frac{1}{4}\nabla^4\Big)\sigma e_{1,3}\otimes e_{1,3},
\end{aligned}$$

where $\sigma=\sigma(v,y)$ is the Kronecker function.

*Proof of Theorem 18.5.* We divide the proof into several steps.

**Computation of a basis of Sol**. First, we compute a basis of the vector space

$$
\mathrm{Sol}_{J,\,v,\,0,\,\tau} := \left\{ \Phi : \mathbb{C} \to \mathrm{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)J = e(z)J\Phi(z) \end{array} \right\}, \tag{18.4}
$$

where $e(z) = e(z, v, \tau) = -\exp\bigl(-2\pi i(z + v + \tau)\bigr)$. The proof of the following result is straightforward.

**Lemma 18.10.** *Let $C : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ and $D :\to \mathrm{GL}_m(\mathbb{C})$ be a pair of automorphy factors. Let $\widetilde{D} := \bigl(D^{-1}\bigr)^t$ be the transpose of the inverse matrix of $D$. Next, we set*

$$
C \otimes \widetilde{D} = \begin{pmatrix} c_{11}\widetilde{D} & \dots & c_{1n}\widetilde{D} \\ \vdots & \ddots & \vdots \\ c_{n1}\widetilde{D} & \dots & c_{nn}\widetilde{D} \end{pmatrix}.
$$

(1) *In the notations of Theorem 10.3, we have isomorphisms*

$$
\mathrm{Hom}\bigl(\mathcal{E}(C),\, \mathcal{E}(D)\bigr) \xrightarrow{\cong} \mathrm{Sol}_{C,\,D} \xrightarrow{\alpha} \mathrm{Sol}_{(\mathbb{1}),\, C\otimes\widetilde{D}} \xrightarrow{\cong} H^0\bigl(\mathcal{E}(C \otimes \widetilde{D})\bigr),
$$

*where for $\Phi = \bigl(f_{ij}\bigr)_{1\le i,j\le n} \in \mathrm{Sol}_{C,\,D}$ we set $\alpha\bigl(\Phi\bigr) = \bigl(f_{n(i-1)+j}\bigr)_{1\le i,j\le n}$.*

(2) *We have: $J \otimes \widetilde{J} = \exp(N)$, where $N$ is the matrix from Definition 18.2.*

The following result is due to Polishchuk and Zaslow [39, Proposition 2].

**Proposition 18.11.** *As above, let $\nabla = -\frac{1}{2\pi i} \cdot \frac{d}{dz}$ and $e(z) = -\exp\bigl(-2\pi i(z + v + \tau)\bigr)$. Then we have an isomorphism of vector spaces:*

$$
\delta : H^0\Bigl(\mathcal{E}\bigl(e(z)\bigr)\Bigr) \otimes \mathbb{C}^{n^2} \longrightarrow H^0\Bigl(\mathcal{E}\bigl(e(z)\cdot \exp(N)\bigr)\Bigr)
$$

*given by the rule $\delta(f\otimes u) = \bigl(\exp(\nabla N)f\bigr)u = \sum_{m=0}^{\infty} \frac{\nabla^m(f)}{m!} N^m(u)$ for any $f \in H^0\Bigl(\mathcal{E}\bigl(e(z)\bigr)\Bigr)$ and $u \in \mathbb{C}^{n^2}$.*

*Let $\bar{\theta}_v(z) = \bar{\theta}(z+v+\frac{\tau+1}{2})$. Then we have an isomorphism of vector spaces $\Delta : \mathbb{C}^{n^2} \to \mathrm{Sol}_{J,\,v,\,0,\,\tau}$ mapping a vector $u \in \mathbb{C}^{n^2}$ to the matrix-valued function $\Delta(u)$, where for any $1 \le k, l \le n$ we have:*

$$
\bigl(\Delta(u)\bigr)_{k,\,l}(z) = e^t_{n(k-1)+l}\bigl(\exp(\nabla N)\bar{\theta}_v(z)\bigr)u.
$$

*Proof.* By Lemma 17.1, the vector space $H^0\bigl(\mathcal{E}(e(z))\bigr)$ is one-dimensional and $\bar{\theta}_v(z) = \bar{\theta}(z+v+\frac{\tau+1}{2})$ is its basis element. Hence, Proposition 18.11 is a consequence of Lemma 18.10 and Proposition 18.11. □

**Definition 18.12.** In the notations of Proposition 18.11, let $U$ be the element of $\mathrm{Sol} = \mathrm{Sol}_{J,\,v,\,0,\,\tau}$ corresponding to $u = e_{n(n-1)+1} \in \mathbb{C}^{n^2}$. Note that $\bigl(U(z)\bigr)_{n,1} = \bar{\theta}_v(z)$.

**Proposition 18.13.** *Let $K = J_n(0)$ be the Jordan block of size $n \times n$ with eigenvalue zero. For all $1 \leq i, j \leq n$ we set $F_{ij} = K^{n-i} U K^{j-1}$. Then we have:*

1. *All matrix-valued functions $F_{ij} : \mathbb{C} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ belong to* Sol.
2. *If $1 \leq p$, $q \leq n$ are such that $i < p \leq n$ or $1 \leq q < j$ then we have: $\big(F_{ij}\big)_{p,\,q} = 0$. Moreover, $\big(F_{ij}\big)_{i,\,j} = \bar{\theta}_v$. In other words, all non-zero entries of $F_{ij}$ are located in the rectangle whose lower left corner is $(i, j)$.*
3. *Moreover, $\big\{F_{ij}\big\}_{1\leq i,j\leq n}$ is a basis of the vector space Sol.*

*Proof.* The statement that $F_{ij}$ belongs to Sol is equivalent to the equality

$$K^{n-i}\, U(z+\tau)\, K^{j-1}\, J = e(z) J\, K^{n-i}\, U(z)\, K^{j-1}. \tag{18.5}$$

Since the matrices $K$ and $J$ commute, Equality (18.5) is equivalent to

$$K^{n-i}\big(U(z+\tau)J - e(z)JU(z)\big)K^{j-1} = 0,$$

which is true since $U$ belongs to Sol. The second part of the Proposition follows from the definition of the functions $F_{ij}$. From this part also follows that all elements of the set $\big\{F_{ij}\big\}_{1\leq i,j\leq n}$ are linearly independent. By Corollary 14.4, the dimension of Sol is $n^2$. Thus, $\big\{F_{ij}\big\}_{1\leq i,j\leq n}$ is a basis of Sol. $\square$

**Example 18.14.** Let $n = 2$. Similarly to example 18.7, we obtain:

$$F_{2,1} = U = \begin{pmatrix} \nabla\bar{\theta}_v & -\nabla^2\bar{\theta}_v \\ \bar{\theta}_v & -\nabla\bar{\theta}_v \end{pmatrix}.$$

Moreover, we have:

$$F_{1,1} = \begin{pmatrix} \bar{\theta}_v & -\nabla\bar{\theta}_v \\ 0 & 0 \end{pmatrix},\ F_{2,2} = \begin{pmatrix} 0 & \nabla\bar{\theta}_v \\ 0 & \bar{\theta}_v \end{pmatrix} \text{ and } F_{1,2} = \begin{pmatrix} 0 & \bar{\theta}_v \\ 0 & 0 \end{pmatrix}.$$

**Computation of $\mathrm{res}_0^{-1}$.** As the next step, we compute the preimages of the elementary matrices $\big\{e_{a,b}\big\}_{1\leq a,b\leq n}$ under the isomorphism $\mathrm{res}_0 : \mathrm{Sol}_{\to}\mathrm{Mat}_{n\times n}(\mathbb{C})$.

Let $X = (x_{p,q})_{1\leq p,q\leq n} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a given matrix and $A \in \mathrm{Sol}$ be such that

$\mathrm{res}_0(A) = X$. By Proposition 18.13, we have an expansion $A = \sum_{1\leq i,\,j\leq n} \eta_{i,j} F_{i,j}$ for certain uniquely determined $\eta_{i,j} \in \mathbb{C}$. It is clear that for all $1 \leq p, q \leq n$ we get:

$$x_{p,q} = \sum_{\substack{p\leq i\leq n \\ 1\leq j\leq q}} \eta_{i,j}\big(F_{i,j}(0)\big)_{p,q}. \tag{18.6}$$

Next, for all $1 \leq p, q \leq n$ we have $\left(F_{p,q}(z)\right)_{p,q} = \bar{\theta}_v(z)$. This implies that

$$\eta_{p,q} = \frac{1}{\bar{\theta}_v(0)} \left[ x_{p,q} - \sum_{\substack{p\leq i\leq n \\ 1\leq j\leq q \\ (i,j)\neq(p,q)}} \eta_{i,j} \left(F_{i,j}(0)\right)_{p,q} \right]. \tag{18.7}$$

Hence $\eta_{p,q}$ can be expressed as a linear combination of those $x_{i,j}$ for which $p \leq i \leq n$ and $1 \leq i \leq q$. Moreover, due to the recursive structure of the Equality (18.7), it is clear that $\eta_{p,\,q}$ can be written as a certain linear combination of $x_{i,j}$, whose structure is controlled by the set of paths starting at $(p, q)$ and ending at $(i, j)$. In order to make this more precise, let us make the following definition.

**Definition 18.15.** For any $\chi \in \mathbb{N}$ and $(i, j), (p, q) \in \mathbb{N} \times \mathbb{N}$ such that $i \geq p$ and $j \leq q$ we denote

$$\mathcal{W}^{\chi}_{(p,q),(i,j)} = \left\{ (\alpha_s, \beta_s)_{0\leq s\leq\chi} \in (\mathbb{N} \times \mathbb{N})^{\chi+1} \,\middle|\, \substack{\alpha_s\leq\alpha_{s+1},\, \beta_s\geq\beta_{s+1} \\ (\alpha_s,\beta_s)\neq(\alpha_{s+1},\beta_{s+1}) \\ (\alpha_0,\beta_0)=(p,q) \\ (\alpha_\chi,\beta_\chi)=(i,j)} \right\}.$$

In other words, $\mathcal{W}^{\chi}_{(p,q),(i,j)}$ is the set of all paths of length $\chi$ on the square lattice $\mathbb{N} \times \mathbb{N}$ starting at $(p, q)$, ending at $(i, j)$ and going in the "south-west" direction.

Applying the recursive formula (18.7), we end up with the following result.

**Lemma 18.16.** *Let* $X = \left(x_{i,j}\right)_{1\leq i,j\leq n} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ *and* $\left\{\eta_{p,q}(X)\right\}_{1\leq p,q\leq n}$ *be such that the equality (18.6) is true. Then we have:*

$$\eta_{p,q}(X) = \sum_{\substack{p\leq i\leq n \\ 1\leq j\leq q}} x_{i,j} \left[ \sum_{\chi=0}^{i-p+q-j} \sum_{\mathcal{W}^{\chi}_{(p,q),(i,j)}} \frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=0}^{\chi-1} \left(F_{\alpha_{s+1},\beta_{s+1}}(0)\right)_{\alpha_s,\beta_s} \right], \tag{18.8}$$

*where the third sum runs over all elements* $\left(\alpha_s, \beta_s\right)_{0\leq s\leq\chi}$ *of* $\mathcal{W}^{\chi}_{(p,q),(i,j)}$.

**Corollary 18.17.** *Let* $\left\{e_{a,b}\right\}_{1\leq a,b\leq n}$ *be the standard basis of* $\mathrm{Mat}_{n\times n}(\mathbb{C})$. *If* $a \geq p$ *and* $b \leq q$ *then we have:*

$$\eta_{p,q}\left(e_{a,b}\right) = \sum_{\chi=0}^{a-p+q-b} \sum_{(\alpha_s,\beta_s)_{0\leq s\leq\chi}\in\mathcal{W}^{\chi}_{(p,q),(a,b)}} \frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=0}^{\chi-1} \left(F_{\alpha_{s+1},\beta_{s+1}}(0)\right)_{\alpha_s,\beta_s},$$

*whereas in the remaining cases* $\eta_{p,q}(e_{a,b}) = 0$. *Hence,* $res_0^{-1}(e_{a,b}) = \gamma^{a,b}$, *where*

$$\gamma^{a,b}(z) = \sum_{\substack{1\le p\le a\\ b\le q\le n}} F_{p,q}(z) \left[ \sum_{\chi=0}^{a-p+q-b} \sum_{(\alpha_s,\beta_s)_{0\le s\le\chi}\in\mathcal{W}^{\chi}_{(p,q),(a,b)}} \frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=0}^{\chi-1} \big(F_{\alpha_{s+1},\beta_{s+1}}(0)\big)_{\alpha_s,\beta_s} \right].$$

Denote the matrix entries of $\big(\gamma^{a,b}(z)\big)_{c,d}$ by $\gamma^{a,b}_{c,d}(z)$. If $c < a$ or $d < b$ then $\gamma^{a,b}_{c,d}(z) = 0$. On the other hand, if $a \ge c$ and $d \ge b$ then we get:

$$\gamma^{a,b}_{c,d}(z) = \sum_{\substack{c\le p\le a\\ b\le q\le d}} \Big[\big(F_{p,q}(z)\big)_{c,d} \sum_{\chi=0}^{a-p+q-b} \sum_{\mathcal{W}^{\chi}_{(p,q),(a,b)}} \frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=0}^{\chi-1} \big(F_{\alpha_{s+1},\beta_{s+1}}(0)\big)_{\alpha_s,\beta_s}\Big]. \tag{18.9}$$

*Remark* 18.18. In the formula (18.9) the function $\gamma^{a,b}_{c,d}$ depends on one variable $z$. However, from its definition it is clear that it also depends on the parameter $v \in \mathbb{C} \setminus \Lambda$. Hence, in what follows, we shall consider it as a function of two variables $z$ and $v$.

**Computation of $\tilde{r}_J(v, y) : \mathbf{Mat}_{n\times n} \to \mathbf{Mat}_{n\times n}$.** Recall that $\tilde{r}_B(v, y) : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ is the composition $\mathrm{ev}_y \circ \mathrm{res}_0^{-1}$. The tensor $r_B(v, y) \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ is the image of $\tilde{r}_B(v, y)$ under the canonical isomorphism

$$\mathrm{Lin}\big(\mathrm{Mat}_{n\times n}(\mathbb{C}), \mathrm{Mat}_{n\times n}(\mathbb{C})\big) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C}).$$

In the standard basis $\{e_{a,b}\}_{1\le a,b\le n}$ of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ this map is given as follows:

$$\Big(e_{a,b} \mapsto \sum_{1\le c,\, d\le n} \alpha^{a,b}_{c,d}\, e_{c,d}\Big) \mapsto \sum_{1\le c,\, d\le n} \alpha^{a,b}_{c,d}\, e_{b,a} \otimes e_{c,d}.$$

Hence, the solution of the associative Yang-Baxter equation (14.1) attached to the Jordan block $J$ is the following:

$$r_J(v, y) = \frac{1}{\bar{\theta}\big(\frac{1+\tau}{2} + y\big)} \sum_{\substack{0\le k\le n-1\\ 0\le l\le n-1}} \sum_{\substack{1\le i\le n-l\\ 1\le j\le n-k}} \gamma^{j+k,i}_{j,i+l}(v, y)\, e_{i,j+k} \otimes e_{j,i+l}, \tag{18.10}$$

where $\gamma^{j+k,i}_{j,i+l}(v, y)$ are given by (18.9). Our next goal is to simplify this expression.

**Definition 18.19.** For any $i, j, \chi$ in $\mathbb{N}$, we set

$$\mathcal{W}^{\chi}_{(i,j)} = \left\{ (\alpha_s, \beta_s)_{1\le s\le\chi} \in (\mathbb{N}\times\mathbb{N})^{\chi} \;\middle|\; (\alpha_s, \beta_s) \ne (0,0);\; \left(\sum_{s=1}^{\chi} \alpha_s, \sum_{s=1}^{\chi} \beta_s\right) = (i, j) \right\}.$$

We may think of $\mathcal{W}^{\chi}_{(i,j)}$ as the set paths of length $\chi$ in $\mathbb{N}\times\mathbb{N}$ starting at the point $(0,0)$, ending in $(i, j)$ and going in the "north-east" direction. Note that any path $\{(\alpha_s, \beta_s)\}_{0\le s\le\chi}$ from $\mathcal{W}^{\chi}_{(j+a,i+b),(j+k,i)}$ corresponds to the element $\{(\alpha_{s+1} - \alpha_s, \beta_s -$

$\beta_{s+1})_s\big\}_{1\le s\le\chi}$ of $\mathcal{W}^{\chi}_{(k-a,b)}$. For the sake of simplicity, we shall use the same notation $\big\{(\alpha_s,\beta_s)\big\}_{0\le s\le\chi}$ for both elements.

In these notations, the formula (18.9) can be rewritten as follows:

$$\gamma^{j+k,i}_{j,i+l}(y)=\sum_{\substack{0\le a\le k\\0\le b\le l}}\left[\big(F_{j+a,i+b}(y)\big)_{j,i+l}\cdot\sum_{\chi=0}^{k-a+b}\sum_{\mathcal{W}^{\chi}_{(k-a,b)}}\frac{(-1)^{\chi}}{\bar\theta_v(0)^{\chi+1}}\prod_{s=1}^{\chi}\big(F_{\alpha_{s+1},\beta_{s+1}}(0)\big)_{\alpha_s,\beta_s}\right],$$

where the third sum is taken over all elements $\big(\alpha_s,\beta_s\big)_{1\le s\le\chi}$ of $\mathcal{W}^{\chi}_{(k-a,b)}$. Recall that $\big(F_{\alpha,\beta}\big)_{\gamma,\delta}=0$ if $\gamma>\alpha$ or $\beta>\delta$. For $\alpha\ge\gamma$ and $\delta\ge\beta$ we have: $\big(F_{\alpha,\beta}\big)_{\gamma,\delta}=U_{(n-(\alpha-\gamma),\delta-\beta+1)}$, where $\big(U(v,z)\big)_{\alpha,\beta}=e^t_{n(\alpha-1)+\beta}\exp(\nabla N)\bar\theta_v(z)\,e_{n(n-1)+1}$. Hence,

$$\gamma^{j+k,i}_{j,i+l}(v,y)=\sum_{\substack{0\le a\le k\\0\le b\le l}}\left[\big(U(v,y)\big)_{(n-a,l-b+1)}\cdot\sum_{\chi=0}^{k-a+b}\sum_{\mathcal{W}^{\chi}_{(k-a,b)}}\frac{(-1)^{\chi}}{\bar\theta_v(0)^{\chi+1}}\prod_{s=1}^{\chi}\big(U(0)\big)_{n-\alpha_s,\beta_s+1}\right]$$

From Proposition 18.3(1) it follows that for all $0\le\alpha<n$, $1\le\beta\le n$ we have:

$$\big(U(v,z)\big)_{n-\alpha,\beta}=e^t_{n(n-\alpha-1)+\beta}\exp(\nabla_z N)\bar\theta_v(z)\,e_{n(n-1)+1}=\sum_{r=0}^{2n-1}e^t_\beta\,N^{(r)}_\alpha\,e_1\,\frac{\nabla^r_z\big(\bar\theta_v(z)\big)}{r!}.$$

Recall that $\bar\theta_v(z)=\bar\theta(z+v+\frac{1+\tau}{2})$. Note that we have:

$$\nabla^r_z\big(\bar\theta_v(z)\big)=\nabla^r_v\big(\bar\theta_v(z)\big)=\left(\frac{\partial}{\partial v}\right)^r\bar\theta\left(z+\frac{\tau+1}{2}+v\right).$$

Therefore, we can rewrite the expression for $\gamma^{j+k,i}_{j,i+l}(v,y)$ as follows:

$$\gamma^{j+k,i}_{j,i+l}(v,y)=\sum_{\substack{0\le a\le k\\0\le b\le l}}\left(\left(\sum_{r=0}^{2n-1}e^t_{l-b+1}\,N^{(r)}_a\,e_1\,\frac{\nabla^r_v\big(\bar\theta_v(y)\big)}{r!}\right)\cdot\right.\tag{18.11}$$

$$\left.\cdot\sum_{\chi=0}^{k-a+b}\sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}}\frac{(-1)^{\chi}}{\bar\theta_v(0)^{\chi+1}}\prod_{s=1}^{\chi}\left(\sum_{r_s=0}^{2n-1}e^t_{\beta_s+1}\,N^{(r_s)}_{\alpha_s}\,e_1\,\frac{\nabla^{r_s}_v\big(\bar\theta_v(0)\big)}{r_s!}\right)\right).$$

Next, we need the following generalization of the Leibniz formula. The proof, consisting of some lengthy combinatorial argument, is postponed to Subsection 19.2.

**Lemma 18.20.** *Let $f, g$ be any meromorphic functions on $\mathbb{C}$ and $\nabla = -\frac{1}{2\pi i}\frac{d}{dz}$. Then in the notations of Definition 18.4 the following formula is true:*

$$\nabla_{k,l}\Big(\frac{f}{g}\Big) = \sum_{\substack{0\le a\le k\\ 0\le b\le l}} \Big[\Big(\sum_{r=0}^{2n-1} e^t_{l-b+1}\, N_a^{(r)}\, e_1\, \frac{\nabla^r(f)}{r!}\Big)\cdot\Big(\sum_{\chi=0}^{k-a+b} \sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}} \frac{(-1)^\chi}{g^{\chi+1}}\cdot$$
$$\cdot\prod_{s=1}^{\chi}\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s}(g)}{r_s!}\Big)\Big].$$

Applying Lemma 18.20 to equality (18.11), we finally get:

$$\frac{1}{\bar\theta\left(\frac{1+\tau}{2}+y\right)}\gamma^{j+k,i}_{j,i+l}(v,y) = \big(\nabla_{k,l}\big)_v\left(\frac{\bar\theta(y+\frac{\tau+1}{2}+v)}{\bar\theta\left(\frac{1+\tau}{2}+y\right)\cdot\bar\theta(\frac{\tau+1}{2}+v)}\right).$$

Recall that the first and third theta functions $\theta$ and $\bar\theta$ are related by the equality $\bar\theta\left(z+\frac{1+\tau}{2}\right) = i\,q(z)\,\theta(z)$, where $q(z) = \exp(-\pi i(z+\frac{\tau}{4}))$. Thus, up to the constant $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}$, the coefficient of the tensor $e_{i,j+k}\otimes e_{j,i+l}$ in the expansion (18.10) is $(\nabla_{k,l})_v\big(\sigma(v,y)\big)$. This finishes the proof of Theorem 18.5.

*Remark* 18.21. The algorithm from Subsection 14.1 assigning to a matrix $B \in \mathrm{GL}_n(\mathbb{C})$ and a complex torus $E$ a solution of the associative Yang-Baxter can be generalized to the case when $E$ is a singular Weierstrass cubic curve. In this case, one can use a description of semi-stable vector bundles on $E$ following the approach of [6], see also [14]. However, all solutions produced in this way turn out to be degenerations of the constructed elliptic solutions, where we replace the Kronecker function $\sigma(u,x)$ by its trigonometric or rational degenerations $\cot(u)+\cot(x)$ or $\dfrac{1}{u}+\dfrac{1}{x}$.

## 19. Combinatorial proofs

### 19.1. **Proof of Proposition** 18.3.

**1)** We proceed by induction on $r$. For $r=0$ we have $N^r = \mathbb{1}_{n^2\times n^2}$, hence the statement is correct. So assume the statement is correct for $r$. Note that for two $n\times n$ matrices of the form

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & c_1\\ 0 & \cdots & 0 & c_0\end{pmatrix},\; C' = \begin{pmatrix} c'_0 & c'_1 & \cdots & c'_{n-1}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & c'_1\\ 0 & \cdots & 0 & c'_0\end{pmatrix}$$

we have

$$(19.1) \qquad (CC')_{ij} = \sum_{k=1}^{n} (C)_{i,k}(C')_{k,j} = \sum_{k=i}^{j} c_{k-i}\, c'_{j-k} = \sum_{t=0}^{j-i} c_t\, c'_{j-i-t}.$$

Hence $N^{r+1}$ is indeed a $n\times n$ upper triangular block matrix with entries $n\times n$ matrices depending on the difference between column and row index. Next the formula above implies

$$(19.2) \qquad N_i^{(r+1)} = \sum_{t=0}^{i} A_{i-t}N_t^{(r)} = \begin{cases} A_0 N_0^{(r)} & i=0 \\ A_0 N_i^{(r)} + A_i N_0^{(r)} + \sum_{t=1}^{i-1} A_{i-t}N_t^{(r)} & \text{else} \end{cases}.$$

Let us focus on the case $i = 0$. Since $S_0^k = \emptyset$ for all $k \neq 0$, the statement we have to show reads

$$N_0^{r+1} = A_0^{r+1}.$$

Combining (19.2) with the inductive assumption settles the case $i = 0$ immediately.

Hence, for the rest of this proof, assume that $i \neq 0$. Applying the inductive assumption to (19.2) yields

$$N_i^{(r+1)} = \sum_{k=1}^{r} \binom{r}{k} A_0^{r+1-k} \sum_{s\in S_i^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}} + A_i A_0^r + \sum_{k=1}^{r} \binom{r}{k} A_0^{r-k} \sum_{t=1}^{i-1} A_{i-t} \sum_{s\in S_t^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}}.$$

By definition of $S_i^k$, the first summand of this equals

$$\binom{r}{1} A_0^r A_i + \sum_{k=1}^{r-1} \binom{r}{k+1} A_0^{r-k} \sum_{s\in S_i^{k+1}} \prod_{l=0}^{k} A_{s_l - s_{l+1}}$$

while the last equals

$$\sum_{k=1}^{r} \binom{r}{k} A_0^{r-k} \sum_{s\in S_i^{k+1}} \prod_{l=0}^{k} A_{s_l - s_{l+1}}.$$

Thus

$$N_i^{(r+1)} = A_0^r A_i \left(\binom{r}{1} + 1\right) + \sum_{k=1}^{r} \left(\binom{r}{k+1} + \binom{r}{k}\right) A_0^{r-k} \sum_{s\in S_i^{k+1}} \prod_{l=0}^{k} A_{s_l - s_{l+1}}$$

where we use that $\binom{r}{r+1} = 0$. Since $\binom{r}{k} + \binom{r}{k+1} = \binom{r+1}{k+1}$ this equals

$$A_0^r A_i (r+1) + \sum_{k=1}^{r} \binom{r+1}{k+1} A_0^{r-k} \sum_{s\in S_i^{k+1}} \prod_{l=0}^{k} A_{s_l - s_{l+1}}$$

$$= A_0^r A_i(r+1) + \sum_{k=2}^{r+1} \binom{r+1}{k} A_0^{r+1-k} \sum_{s\in S_i^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}}$$

$$= \sum_{k=1}^{r+1} \binom{r+1}{k} A_0^{r+1-k} \sum_{s\in S_i^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}}.$$

**2)** Obviously, nearly the same arguments as in (1) yield that

$$A_0^r = \begin{pmatrix} a_0^{(r)} & 0 & \cdots & 0 \\ a_1^{(r)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{m-1}^{(r)} & \cdots & a_1^{(r)} & a_0^{(r)} \end{pmatrix}$$

with

$$a_j^{(r)} = \sum_{k=0}^{r} \binom{r}{k} a_0^{r-k} \sum_{s\in S_j^k} \prod_{l=0}^{k-1} a_{s_l - s_{l+1}}$$

where $a_0 = 0$. it follows from the definitions that

$$a_j^{(r)} = \sum_{s\in S_j^r} \frac{(-1)^j}{\prod_{l=0}^{r-1}(s_l - s_{l+1})}.$$

By definition we also have

$$\sum_{s\in S_i^k} \prod_{l=0}^{k-1} A_{s_l - s_{l+1}} = \mathbb{1}_{n\times n} \cdot \sum_{s\in S_i^k} \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1}(s_l - s_{l+1})}$$

hence by (1) we get

$$e_j^t\, N_i^{(r)}\, e_1 = \sum_{k=0}^{r} \binom{r}{k} \left(e_j^t\, A_0^{r-k} e_1\right) \sum_{s\in S_i^k} \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1}(s_l - s_{l+1})}$$

$$= \sum_{k=0}^{r} \binom{r}{k} \left( \sum_{t\in S_{j-1}^{r-k}} \frac{(-1)^{j-1}}{\prod_{l=0}^{r-k-1}(t_l - t_{l+1})} \right) \left( \sum_{s\in S_i^k} \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1}(s_l - s_{l+1})} \right).$$

**3)** We use (1) to show that each summand of $N_i^{2n-1}$ equals zero. Obviously, the $k$-th summand in formula (1) is non-zero only if both $A_0^{2n-1-k} \neq 0$ and $S_i^k \neq \emptyset$. Since $A_0^{n-1} = 0$ the former implies $k > n-1$, while the latter implies $k \leq i$. But $i \leq n-1$.

**4)** We show that for each $0 \leq k \leq i+j+r$, the $k$-th summand in formula (2) is zero. Indeed, for fixed $i, j, k$, we show that at least one of $S_{j-1}^{i+j+r-k}$ and $S_i^k$ is empty. Assume $S_i^k \neq \emptyset$. Then $k \leq i$, so $i+j+r-k \geq j+r > j-1$. Hence $S_{j-1}^{i+j+r-k} = \emptyset$ in that case.

**5)** Note that $\exp(N) = J \otimes \tilde{J}$ if and only if $\log(J \otimes \tilde{J}) = N$. In order to apply the power series formula for the logarithm, we set $M = J \otimes \tilde{J} - \mathbb{1}_{n \times n}$, hence

$$\log(J \otimes \tilde{J}) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} M^r. \tag{19.3}$$

This makes sense as we will see immediately that $M$ is nilpotent. Copying the proof of (1) we see that

$$M^r = \begin{pmatrix} M_0^{(r)} & M_1^{(r)} & \cdots & M_{n-1}^{(r)} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_1^{(r)} \\ 0 & \cdots & 0 & M_0^{(r)} \end{pmatrix}$$

with

$$M_i^{(r)} = \sum_{k=0}^{r} \binom{r}{k} B_0^{r-k} \sum_{s \in S_i^k} \prod_{l=0}^{k-1} B_{s_l - s_{l+1}} \tag{19.4}$$

where $B_i = M_0^{(1)}$ i.e.

$$B_i = \begin{cases} \tilde{J} - \mathbb{1}_{n \times n} & i = 0 \\ \tilde{J} & i = 1 \\ 0 & \text{else.} \end{cases}$$

Thus we may omit in (19.4) any $s = (s_l)_{0 \leq l \leq k} \in S_i^k$ such that $s_l - s_{l+1} > 1$ for some $0 \leq l \leq k-1$. Since $s_l - s_{l+1} > 0$ anyway, this implies that only the unique element in $S_i^i$ contributes non trivially, hence

$$M_i^{(r)} = \binom{r}{i} B_0^{r-i} B_1^i. \tag{19.5}$$

Since $B_0^n = 0$ , this implies that $M$ is nilpotent, therefore (19.3) makes sense and is indeed equivalent to

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} M_i^{(r)} = A_i,\ 0 \leq i \leq n-1.$$

Applying (19.3) once more, this means that we have to prove

$$A_i = \begin{cases} \sum_{r=1}^{n-1} \frac{(-1)^{r-1}}{r} B_0^r & i = 0 \\ B_1^i \sum_{r=0}^{n-1} \frac{(-1)^{r+i-1}}{r+i} \binom{r+i}{i} B_0^r & \text{else.} \end{cases} \tag{19.6}$$

In any of these two cases, we have to compare two lower triangular matrices whose entries depend only on the difference between row and column index, cf. (19.1). Analogous to 2), we see that

$$B_0^r = \begin{pmatrix} b_0^{(r)} & 0 & \cdots & 0 \\ b_1^{(r)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{n-1}^{(r)} & \cdots & b_1^{(r)} & b_0^{(r)} \end{pmatrix}$$

with

$$b_j^{(r)} = \sum_{k=0}^{r} \binom{r}{k} b_0^{r-k} \sum_{s\in S_j^k} \prod_{l=0}^{k-1} b_{s_l - s_{l+1}}.$$

But now the situation simplifies considerably in comparison to that of 2): firstly, $b_0 = 0$ hence

$$b_j^{(r)} = \sum_{s\in S_j^r} \prod_{l=0}^{k-1} b_{s_l - s_{l+1}}. \tag{19.7}$$

Secondly, for all $j, k$ and $s \in S_j^k$

$$\prod_{l=0}^{k-1} b_{s_l - s_{l+1}} = \prod_{l=0}^{k-1} (-1)^{s_l - s_{l+1}} = (-1)^j.$$

Thus

$$\sum_{s\in S_j^k} \prod_{l=0}^{k-1} b_{s_l - s_{l+1}} = \begin{cases} 1 & (j,k) = (0,0) \\ \binom{j-1}{k-1} (-1)^j & \text{else.} \end{cases}$$

Combining this with (19.7), we infer that

$$b_j^{(r)} = \begin{cases} 1 & (j,r) = (0,0) \\ \binom{j-1}{r-1} (-1)^j & \text{else.} \end{cases} \tag{19.8}$$

Now we can prove (19.6). Let us start with $i = 0$. We have to show

$$a_j = \sum_{r=1}^{m-1} \frac{(-1)^{r-1}}{r} b_j^{(r)},\ 0 \le j \le n-1.$$

For $j = 0$ the left hand side is zero by definition as is the right hand side by (19.8). For $j \neq 0$, using (19.8) once more, we may rewrite the claim as

$$\frac{(-1)^j}{j} = \sum_{r=1}^{j} \frac{(-1)^{r-1}}{r} \binom{j-1}{r-1} (-1)^j$$

which is true if and only if

$$1 = \sum_{r=1}^{j} (-1)^{r-1} \frac{j}{r} \binom{j-1}{r-1}.$$

Now

$$\sum_{r=1}^{j} (-1)^{r-1} \frac{j}{r} \binom{j-1}{r-1} = \sum_{r=1}^{j} (-1)^{r-1} \binom{j}{r} = (-1)^{j-1} + \sum_{r=1}^{j-1} (-1)^{r-1} \left( \binom{j-1}{r-1} + \binom{j-1}{r} \right) =$$

$$= (-1)^{j-1} + \binom{j-1}{0} + \binom{j-1}{j-1} (-1)^{j-2} = (-1)^{j-1} + 1 + (-1)^{j-2}$$

which proves the claim for $j > 0$.

So assume $i \neq 0$ in (19.6). Then the assertion is that

$$-a_i \cdot \mathbb{1}_{n \times n} = B_1^i \sum_{r=0}^{n-1} \frac{(-1)^{r+i-1}}{r+i} \binom{r+i}{i} B_0^r$$

or, setting $H = J^t$, equivalently

$$-a_i \cdot H^i = \sum_{r=0}^{m-1} \frac{(-1)^{r+i-1}}{r+i} \binom{r+i}{i} B_0^r. \tag{19.9}$$

Note that analogous to the case of $M^r$, the powers of $H$ are of the form

$$H^r = \begin{pmatrix} h_0^{(r)} & 0 & \cdots & 0 \\ h_1^{(r)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1}^{(r)} & \cdots & h_1^{(r)} & h_0^{(r)} \end{pmatrix}$$

with

$$h_i^{(r)} = \binom{r}{i} h_0^{r-i} h_1^i = \binom{r}{i}.$$

Thus (19.9) holds if and only if for all $0 \leq j \leq n-1$ we can show that

$$\frac{(-1)^{i-1}}{i} h_j^{(i)} = \sum_{r=0}^{n-1} \frac{(-1)^{r+i-1}}{r+i} \binom{r+i}{i} b_j^{(r)}$$

i.e.

$$\binom{i}{j} = \sum_{r=0}^{n-1} (-1)^r \binom{r+i-1}{r} b_j^{(r)}.$$

For $j = 0$ both sides evaluate to one by (19.8), hence we assume $j > 0$. Again by (19.8) we may reformulate the above as

$$\binom{i}{j} = (-1)^j \sum_{r=0}^{j} (-1)^r \binom{r+i-1}{r} \binom{j-1}{r-1}. \tag{19.10}$$

Note the following easy combinatorial fact:

$$\binom{i+r-1}{r} = \sum_{l=1}^{r} \binom{i}{l} \binom{r-1}{r-l}.$$

Applying this to the right-hand side of (19.10) yields

$$(-1)^j \sum_{r=1}^{j} (-1)^r \binom{j-1}{r-1} \sum_{l=1}^{r} \binom{i}{l} \binom{r-1}{r-l} = (-1)^j \sum_{l=1}^{j} \binom{i}{l} \sum_{r=l}^{j} (-1)^r \binom{j-1}{r-1} \binom{r-1}{r-l}.$$

We claim that

$$\sum_{r=l}^{j} (-1)^r \binom{j-1}{r-1} \binom{r-1}{r-l} = \begin{cases} (-1)^j & l = j \\ 0 & \text{else} \end{cases}.$$

Note that this would prove (19.10), hence finally (19.9). Hence, let us proceed to verify the above claim:

$$\sum_{r=l}^{j} (-1)^r \binom{j-1}{r-1} \binom{r-1}{r-l} = \frac{(j-1)!}{(l-1)!} \sum_{r=l}^{j} (-1)^r \frac{1}{(j-r)!\,(r-l)!} =$$

$$= \frac{(j-1)!}{(l-1)!} (-1)^l \sum_{t=0}^{j-l} (-1)^t \frac{1}{(j-l-t)!\,t!} =$$

$$= \frac{(j-1)!}{(l-1)!} \frac{(-1)^l}{(j-l)!} \sum_{t=0}^{j-l} (-1)^t \binom{j-l}{t} = \frac{(j-1)!}{(l-1)!} \frac{(-1)^l}{(j-l)!} (1-1)^{j-l}$$

which proves the claim. As mentioned above, this implies (19.9), which finishes the proof of (5).

**6)** Let us denote the right-hand side of the statement by $R$, the left-hand side by $L$. By (2) we have

$$R = \sum_{k=0}^{u} \binom{u}{k} \left( \sum_{t \in S_j^{u-k}} \frac{(-1)^j}{\prod_{l=0}^{u-k-1}(t_l - t_{l+1})} \right) \left( \sum_{s \in S_i^k} \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1}(s_l - s_{l+1})} \right).$$

Since we will compare $R$ to $L$ by examining the single summands on each side, we shall need the notation

$$\rho_{k,t,s} = \underbrace{\left( \frac{(-1)^j}{\prod_{l=0}^{u-k-1}(t_l - t_{l+1})} \right)}_{\rho'_{k,t,s}} \cdot \underbrace{\left( \frac{(-1)^{i+k}}{\prod_{l=0}^{k-1}(s_l - s_{l+1})} \right)}_{\rho''_{k,t,s}}.$$

Fixing $0 \leq k \leq u$, we introduce the following disjoint union:

$$R_k = \sqcup_{(t,s) \in S_j^{u-k} \times S_i^k} \left\{ \rho_{(k,t,s)} \right\}.$$

Then clearly

$$R = \sum_{k=0}^{u} \binom{u}{k} \sum_{\rho \in R_k} \rho. \tag{19.11}$$

On the other hand, applying (2) to $L$, we see that $L$ equals

$$\sum_{(\alpha_p,\beta_p)_{1 \leq p \leq \chi} \in \mathcal{W}_{(i,j)}^{\chi}} \prod_{p=1}^{\chi} \left[ \sum_{k_p=0}^{u_p} \binom{u_p}{k_p} \cdot \right.$$

$$\left. \cdot \left( \sum_{t_p \in S_{\beta_p}^{u_p-k_p}} \frac{(-1)^{\beta_p}}{\prod_{l=0}^{u_p-k_p-1}((t_p)_l - (t_p)_{l+1})} \right) \left( \sum_{s_p \in S_{\alpha_p}^{k_p}} \frac{(-1)^{\alpha_p+k_p}}{\prod_{l=0}^{k_p-1}((s_p)_l - (s_p)_{l+1})} \right) \right].$$

Analogous to above, we denote

$$\lambda_{(\alpha_p,\beta_p,k_p,t_p,s_p)} = \underbrace{\left( \frac{(-1)^{\beta_p}}{\prod_{l=0}^{u_p-k_p-1}((t_p)_l - (t_p)_{l+1})} \right)}_{\lambda'_{(\alpha_p,\beta_p,k_p,t_p,s_p)}} \cdot \underbrace{\left( \frac{(-1)^{\alpha_p+k_p}}{\prod_{l=0}^{k_p-1}((s_p)_l - (s_p)_{l+1})} \right)}_{\lambda''_{(\alpha_p,\beta_p,k_p,t_p,s_p)}}.$$

We also need, for any $0 \leq k \leq u$, the following set

$$U_k = \left\{ (k_p)_{1 \leq p \leq \chi} \in ([0, u_p])_{1 \leq p \leq \chi} \middle| \sum_{p=1}^{\chi} k_p = k \right\}.$$

For fixed $(k_p)_{1\le p\le\chi} \in U_k$, we let $L_{(k_p)_{1\le p\le\chi}}$ denote the disjoint union

$$\sqcup_{(\alpha_p,\beta_p)_{1\le p\le\chi}\in\mathcal{W}^\chi_{(i,j)}} \left\{ \sqcup_{(t_p,s_p)_{1\le p\le\chi}\in\left(S^{u_p-k_p}_{\beta_p}\times S^{k_p}_{\alpha_p}\right)_{1\le p\le\chi}} \left\{ \prod_{p=1}^{\chi} \lambda_{(\alpha_p,\beta_p,k_p,t_p,s_p)} \right\} \right\}.$$

If we keep track of all the notations, it's easy to see that

$$L = \sum_{k=0}^{u} \sum_{(k_p)_{1\le p\le\chi}\in U_k} \left[ \prod_{p=1}^{\chi} \binom{u_p}{k_p} \right] \sum_{\lambda\in L_{(k_p)_{1\le p\le\chi}}} \lambda. \tag{19.12}$$

We shall prove that, fixing $0 \le k \le u$, for any $(k_p)_{1\le p\le\chi} \in U_k$ we actually have $L_{(k_p)_{1\le p\le\chi}} = R_k$. Once this is verified, the equality $L = R$ immediately follows by applying the easy combinatorial fact

$$\sum_{(k_p)_{1\le p\le u}\in U_k} \prod_{p=1}^{\chi} \binom{u_p}{k_p} = \binom{\sum_{p=1}^{\chi} u_p}{\sum_{p=1}^{\chi} k_p} = \binom{u}{k}$$

to (19.12) and comparing this to (19.11).

So let us prove $L_{(k_p)_{1\le p\le\chi}} = R_k$ for fixed $0 \le k \le u$ and $(k_p)_{1\le p\le\chi} \in U_k$. We will construct a bijection

$$\sigma : \sqcup_{(\alpha_p,\beta_p)_{1\le p\le\chi}\in\mathcal{W}^\chi_{(i,j)}} \left\{ (\alpha_p,\beta_p)_{1\le p\le\chi} \times \left( S^{u_p-k_p}_{\beta_p} \times S^{k_p}_{\alpha_p} \right)_{1\le p\le\chi} \right\} \to S^{p-k}_j \times S^k_i$$

such that

$$\left( \prod_{p=1}^{\chi} \lambda'_{(\alpha_p,\beta_p,t_p,s_p)}, \prod_{p=1}^{\chi} \lambda''_{(\alpha_p,\beta_p,k_p,t_p,s_p)} \right) = \tag{19.13}$$

$$= \left( \rho'_{\sigma\left( (\alpha_p,\beta_p)_{1\le p\le\chi},(t_p,s_p)_{1\le p\le\chi} \right)}, \rho''_{\sigma\left( (\alpha_p,\beta_p)_{1\le p\le\chi},(t_p,s_p)_{1\le p\le\chi} \right)} \right).$$

Note that we may assume $k_p \le \alpha_p$ and $u_p - k_p \le \beta_y$ for $1 \le p \le \chi$, else the corresponding set $S^{u_p-k_p}_{\beta_p} \times S^{k_p}_{\alpha_p}$ is empty anyway. Clearly, showing existence and bijectivity of $\sigma$ immediately implies $L_{(k_p)_{1\le p\le\chi}} = R_k$.

As to the definition of $\sigma$, we first need to define an auxiliary map

$$\prod : \left( S^{a_p}_{b_p} \right)_{1\le p\le\chi} \to S^a_b,\ a = \sum_{p=1}^{\chi} a_p \text{ and } b = \sum_{p=1}^{\chi} b_p.$$

To any $(s_p)_{1\leq p\leq\chi} \in \left(S^{a_p}_{b_p}\right)_{1\leq p\leq\chi}$ this map $\prod$ simply assigns the concatenation of $s_\chi$, ..., $s_1$, after removing the entry 0 from all $s_q$ with $2\leq q\leq\chi$ and subsequent addition of the value $\sum_{1\leq p\leq q-1} b_p$ to all entries of $s_q$. In short, for any $1\leq q\leq\chi$ and any $0\leq w\leq a_q$, we let

$$\left(\prod_{p=1}^{\chi} s_p\right)_{a-\sum_{1\leq p\leq q} a_p+w} = (s_q)_w + \sum_{1\leq p\leq q-1} b_p.$$

Note that this is well defined even though there are two different ways of expressing $c = a - \sum_{1\leq p\leq q'} a_p$ for chosen $1\leq q'<\chi$: we could choose $q=q'$ and $w=0$ or choose $q=q'+1$ and $w=a_{q'+1}$.
Now we can define the map $\sigma$ as follows:

$$\sigma\Big((\alpha_p,\beta_p)_{1\leq p\leq\chi},(t_p,s_p)_{1\leq p\leq\chi}\Big) = \left(\prod_{p=1}^{\chi} t_p, \prod_{p=1}^{\chi} s_p\right).$$

It follows immediately from the definitions that we have (19.13) if we restrict to absolute values. Indeed, the absolute value of say $\rho''_{\sigma\left((\alpha_p,\beta_p)_{1\leq p\leq\chi},(t_p,s_p)_{1\leq p\leq\chi}\right)}$ is given by the inverse of the product of all differences between two subsequent entries of $\prod_{p=1}^{\chi} s_p$. But each such difference is just the the difference between two subsequent elements of some $s_p, 1\leq p\leq\chi$. As to signs, observe that for $(k_p)_{1\leq p\leq\chi}\in U_k$ the sign of $\prod_{p=1}^{\chi}\lambda''_{(\alpha_p,\beta_p,k_p,t_p,s_p)}$ equals $(-1)^{\sum_{p=1}^{\chi}(\alpha_p+k_p)} = (-1)^{i+k}$ , which is exactly the sign of $\rho''_{\sigma\left((\alpha_p,\beta_p)_{1\leq p\leq\chi},(t_p,s_p)_{1\leq p\leq\chi}\right)}$ and similarly for the first entry on either side of (19.13).

All that remains is to proof bijectivity of $\sigma$. We do this by constructing an inverse of $\sigma$. First, we need to fix the notations. Given an element $(c,d)\in S^{p-k}_j\times S^k_i$, we determine a unique element $(\alpha_p,\beta_p)_{1\leq p\leq\chi}\in\mathcal{W}^{\chi}_{(i,j)}$ and then a unique element $(t_p,s_p)_{1\leq p\leq\chi}\in\left(S^{u_p-k_p}_{\beta_p}\times S^{k_p}_{\alpha_p}\right)_{1\leq p\leq\chi}$ such that $\sigma\left((t_p,s_p)_{1\leq p\leq\chi}\right)=(c,d)$. Note that since $(k_p)_{1\leq p\leq\chi}$ is fixed, determining the $\alpha_p$'s and $s_p$'s from $d$ is independent from determining the $\beta_p$'s and $t_p$'s from $c$. Since both tasks are obviously alike, we will only show how, starting with $d\in S^k_i$, we determine $(\alpha_p)_{1\leq p\leq\chi}$ and $(s_p)_{1\leq p\leq\chi}\in\left(S^{k_p}_{\alpha_p}\right)_{1\leq p\leq\chi}$ such that $\prod_{1\leq p\leq\chi} s_p = d$. Actually, this is pretty easy if we recall the definition of $\prod$. The details are as follows: we let $\bar{k}_1 = k-k_1$, $\alpha_1 = d_{\bar{k}_1}$ and set $(s_1)_w = d_{\bar{k}_1+w}$ for all $0\leq w\leq k_1$. Obviously, $s_1\in S^{k_1}_{\alpha_1}$. Next, we let $d^{(2)} = (d_w-\alpha_1)_{0\leq w\leq\bar{k}_1}\in S^{\bar{k}_1}_{i-\alpha_1}$. Assuming $\bar{k}_{l-1}$ and $d^{(l)}\in S^{\bar{k}_{l-1}}_{i-\sum_{j=1}^{l-1}\alpha_j}$ have been defined, we set $\bar{k}_l = \bar{k}_{l-1}-k_l$, $\alpha_l = d^{(l)}_{\bar{k}_l}$ and $(s_l)_w = d^{(l)}_{\bar{k}_l+w}$ for all $0\leq w\leq k_l$. We leave it to the reader to check that this construction is inverse to $\prod$.

19.2. **Proof of Lemma 18.20.**

First, let us manipulate the left-hand side (denoted $LHS$), using Proposition 18.3 (1):

$$\begin{aligned} e^t_{n(n-k-1)+l+1}\, \exp(\nabla N) \left(\tfrac{f}{g}\right) e_{n(n-1)+1} &= \textstyle\sum_{a=0}^{2n-1} e^t_{l+1}\, \frac{N_k^{(a)}}{a!}\, e_1\, \nabla^a \left(\tfrac{f}{g}\right) = \\ &= \textstyle\sum_{a=0}^{2n-1} e^t_{l+1}\, \frac{N_k^{(a)}}{a!}\, e_1 \sum_{i=0}^{a} \binom{a}{i}\, \nabla^{(a-i)}\,(f)\, \nabla^i \left(\tfrac{1}{g}\right). \end{aligned} \tag{19.14}$$

Recall the Faá di Bruno formulae, which states that

$$\left(\frac{d}{dz}\right)^n (f\circ g) = \sum_{(k_1,\dots,k_n)\in K_n} \frac{n!}{\prod_{i=1}^n (k_i!)} \left(\left(\frac{d}{dz}\right)^{\sum_{i=1}^n k_i} f\right)(g)\cdot \prod_{i=1}^n \left(\frac{1}{i!}\left(\frac{d}{dz}\right)^i g\right)^{k_i}.$$

where $K_n = \left\{(k_1,\dots,k_n)\in\mathbb{N}^n \,\middle|\, \sum_{j=1}^n k_j\cdot j = n\right\}$. Hence

$$\nabla^n\left(\frac{1}{g}\right) = \sum_{(k_1,\dots,k_n)\in K_n} \frac{n!}{\prod_{i=1}^n (k_i!)}\, \frac{(\sum_{i=1}^n k_i)!\,(-1)^{\sum_{i=1}^n k_i}}{g^{1+\sum_{i=1}^n k_i}} \prod_{i=1}^n \left(\frac{\nabla^i g}{i!}\right)^{k_i}.$$

Combining this with (19.14), $LHS$ equals

$$\sum_{a=0}^{2n-1} e^t_{l+1}\, N_k^{(a)}\, e_1 \sum_{i=0}^{a} \binom{a}{i} \frac{i!}{a!} \nabla^{(a-i)}\,(f)\cdot \sum_{(k_1,\dots,k_i)\in K_i} \frac{(\sum_{j=1}^i k_j)!}{\prod_{j=1}^i (k_j!)}\, \frac{(-1)^{\sum_{j=1}^i k_j}}{(g)^{1+\sum_{j=1}^i k_j}} \prod_{j=1}^i \left(\frac{\nabla^j g}{j!}\right)^{k_j}.$$

Setting $r = i-a$ and introducing $K_i^\chi = \left\{(k_1,\dots,k_i)\in K_i \,\middle|\, \sum_{j=1}^i k_j = \chi\right\}$, we deduce that $LHS$ is equal to

$$\sum_{i=0}^{2n-1}\sum_{r=0}^{2n-1-i} e^t_{l+1}\, N_k^{(r+i)}\, e_1\, \frac{\nabla^r f}{r!} \sum_{\chi=0}^{i} \frac{\chi!\,(-1)^\chi}{(g)^{1+\chi}} \sum_{(k_1,\dots,k_i)\in K_i^\chi} \frac{\prod_{j=1}^i \left(\frac{\nabla^j g}{j!}\right)^{k_j}}{\prod_{j=1}^i (k_j!)}.$$

Finally, some further reordering and application of Proposition 18.3 (3) yields the following form of $LHS$

$$\sum_{r=0}^{2n-1} \frac{\nabla^r f}{r!} \sum_{i=0}^{2n-1} e^t_{l+1}\, N_k^{(r+i)}\, e_1 \sum_{\chi=0}^{i} \frac{\chi!\,(-1)^\chi}{(g)^{1+\chi}} \sum_{(k_1,\dots,k_i)\in K_i^\chi} \frac{\prod_{j=1}^i \left(\frac{\nabla^j g}{j!}\right)^{k_j}}{\prod_{j=1}^i (k_j!)}. \tag{19.15}$$

Next let us work on the right-hand side, which we will denote by $RHS$. First, note that

$$\prod_{s=1}^{\chi}\left(\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s} g}{r_s!}\right) = \sum_{(r_s)_{1\le s\le\chi}\in[0,2n-1)]^{\chi}} \left(\prod_{s=1}^{\chi} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1 \frac{\nabla^{r_s} g}{r_s!}\right).$$

Thus

$$\sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}} \prod_{s=1}^{\chi}\left(\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s} g}{r_s!}\right) =$$

$$= \sum_{(r_s)_{1\le s\le\chi}\in[0,2n-1)]^{\chi}} \left(\prod_{s=1}^{\chi} \frac{\nabla^{r_s} g}{r_s!}\right) \sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}} \prod_{s=1}^{\chi} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1.$$

By Proposition 18.3 (6), we moreover have

$$\sum_{\mathcal{W}^{\chi}_{(k-a,b)}} \prod_{s=1}^{\chi} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1 = e^t_{b+1}\, N^{(\sum_{s=1}^{\chi} r_s)}_{k-a}\, e_1$$

and therefore

$$\sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}} \prod_{s=1}^{\chi}\left(\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s} g}{r_s!}\right) =$$

$$= \sum_{(r_s)_{1\le s\le\chi}\in[0,2n-1)]^{\chi}} \left(\prod_{s=1}^{\chi} \frac{\nabla^{r_s} g}{r_s!}\right) e^t_{b+1}\, N^{(\sum_{s=1}^{\chi} r_s)}_{k-a}\, e_1.$$

Next, let us write $\sum_{s=1}^{\chi} r_s = i$. Then $i \le 2n-1$ by Proposition 18.3 (3). Moreover, if we order the $r_s$ by their multiplicities using $K^{\chi}_i$, we derive that the expressions above equal

$$\sum_{i=0}^{2n-1} e^t_{b+1}\, N^{(i)}_{k-a}\, e_1 \sum_{(k_1,\dots,k_i)\in K^{\chi}_i} \left(\prod_{j=1}^{i}\left(\frac{\nabla^j g}{j!}\right)^{k_j}\right) \frac{\chi!}{\prod_{j=1}^{i}(k_j!)}.$$

Hence, $RHS$ is equal to

$$\sum_{\substack{0\le a\le k\\ 0\le b\le l}} \left[\left(\sum_{r=0}^{2n-1} e^t_{l-b+1}\, N^{(r)}_{a}\, e_1\, \frac{\nabla^r f}{r!}\right)\right.$$

$$\cdot \left( \sum_{i=0}^{2n-1} e_{b+1}^t \, N_{k-a}^{(i)} \, e_1 \sum_{\chi=0}^{i} \frac{\chi! \, (-1)^\chi}{(\ g\ )^{\chi+1}} \sum_{(k_1,\ldots,k_i) \in K_i^\chi} \frac{\left( \prod_{j=1}^{i} \left( \frac{\nabla^j g}{j!} \right)^{k_j} \right)}{\prod_{j=1}^{i} (k_j!)} \right) \Bigg] .$$

Note that we did some reordering and also made use of the fact that $K_i^\chi = \emptyset$ for $\chi > i$. Next, we reorder $RHS$ again and obtain

$$\sum_{r=0}^{2n-1} \frac{\nabla^r f}{r!} \sum_{i=0}^{2n-1} \sum_{\chi=0}^{i} \frac{\chi! \, (-1)^\chi}{(\ g\ )^{\chi+1}} \cdot$$

$$\cdot \sum_{(k_1,\ldots,k_i) \in K_i^\chi} \frac{\left( \prod_{j=1}^{i} \left( \frac{\nabla^j g}{j!} \right)^{k_j} \right)}{\prod_{j=1}^{i} (k_j!)} \left[ \sum_{\substack{0 \le a \le k \\ 0 \le b \le l}} \left( e_{l-b+1}^t \, N_a^{(r)} \, e_1 \right) \left( e_{b+1}^t \, N_{k-a}^{(i)} \, e_1 \right) \right] .$$

By Proposition 18.3 (6), the sum in the square brackets equals $e_{l+1}^t N_k^{(r+i)} e_1$. Comparing the resulting formula for $RHS$ to the form (19.15) of $LHS$ finishes the proof.

## Part 6. Theory of rational solutions

This part contains the main results on rational solutions of this thesis. In a condensed form, these results are presented in [13]. We start by recalling the classification of rational solutions of the classical Yang-Baxter equation by Stolin in Section 20. In Section 21 we will then show how Stolin's results can be used to obtain a concrete algorithm that attaches a rational solution $s_{(n,n-d)}$ of the CYBE to any pair of coprime integers $0 < d < n$. Notably, we will employ the matrix $J = \mathcal{J}_0(n-d, d)$ occurring in the classification of simple vector bundles on the cuspidal cubic curve for the construction of a Frobenius functional for a Stolin triple describing $s_{(n,n-d)}$ (hence $s_{(n,n-d)}$ was denoted $r_{(\mathfrak{g},B_J,n-d)}$ in the introduction). This is done by using the work of Elashvili [21] in order to identify the recursive construction of $J$ (see Subsection 13.2) with the recursive construction of Frobenius functionals for the parabolic subalgebra $\mathfrak{P}_{n-d} \subset \mathfrak{sl}_n(\mathbb{C})$.

In Section 22 we will then be able to prove gauge equivalence between the rational solutions $s_{(n,n-d)}$ and the $c_{(n,d)}$. Moreover, we will apply this gauge equivalence in order to exhibit additional structural results for both $s_{(n,n-d)}$ and $c_{(n,d)}$, see Subsection 22.3.

Finally, in Section 23 we will carry out the algorithm from Section 21 for the tuple $(n, d) = (n, n-1)$ explicitly. Thus we will obtain a concrete formula for $s_{(n,1)}$ for any $n \geq 2$.

## 20. Classification of rational solutions

In this section, we recall the results of Stolin [42] concerning the classification of rational solutions of the CYBE

$$[r^{12}(u_1,u_2), r^{23}(u_2,u_3)] + [r^{12}(u_1,u_2), r^{13}(u_1,u_3)] + \\ + [r^{13}(u_1,u_3), r^{23}(u_2,u_3)] = 0, \tag{20.1}$$

where $r : \mathbb{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ is a meromorphic function. Since we discuss the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ only, we would like to mention that [42] contains Theorems 20.2 and 20.4 for the case where $\mathfrak{g}$ is any simple finite dimensional Lie algebra.

**Definition 20.1.** For the remainder of this part, we fix the following notations:

(1) Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{g}\,[u] = \mathfrak{g} \otimes \mathbb{C}\,[u]$, $\mathfrak{g}\,[[u^{-1}]] = \mathfrak{g} \otimes \mathbb{C}\,[[u^{-1}]]$ and $\mathfrak{g}\,((u^{-1})) = \mathfrak{g} \otimes \mathbb{C}\,((u^{-1}))$.
(2) A solution $r$ of the CYBE (20.1) is called rational if it is non-degenerate, unitary and of the form
$$r(u,v) = \frac{\Omega}{v-u} + r'(u,v)$$
where $r'(u,v) \in \mathfrak{g}[u] \otimes \mathfrak{g}[v]$ is non-constant.

Note that the form $(\cdot,\cdot) : \mathfrak{g}\,((u^{-1})) \times \mathfrak{g}\,((u^{-1})) \to \mathbb{C}$ given by $(f,g) = \mathrm{Res}_u\, \mathrm{tr}\,\big(f(u) \cdot g(u)\big)$ is non-degenerate. Here $\mathrm{Res}_u \sum_{i\in\mathbb{Z}} a_i u^i = a_{-1}$ for any $\sum_{i\in\mathbb{Z}} a_i u^i \in \mathrm{Mat}_{n\times n}(\mathbb{C})\,((u^{-1}))$.

**Theorem 20.2.** [42, Theorem 1.1] *There exists a bijection between rational solutions of the CYBE* (20.1) *and Lie subalgebras $W \subset \mathfrak{g}\,((u^{-1}))$ satisfying*

(1) *$W \oplus \mathfrak{g}\,[u] = \mathfrak{g}\,((u^{-1}))$ as complex vector spaces.*
(2) *$W \subset u^N \mathfrak{g}\,[[u^{-1}]]$ for some $N \in \mathbb{N}$.*
(3) *$W$ is a Lagrangian subspace with respect to the form $(\cdot,\cdot)$, i.e. $W = W^\perp$.*

**Fact 20.3.** It follows from the proof of Theorem 20.2 that given a Lie subalgebra $W \subset \mathfrak{g}\,((u^{-1}))$ satisfying the conditions of Theorem 20.2, the corresponding non-degenerate rational solution $X(u,v)$ of the CYBE (20.1) is obtained as follows (see also [43, p. 286]). By conditions (1) and (3) we deduce that the form $(\cdot,\cdot)$ is non-degenerate on $\mathfrak{g}\,[u] \times W$. Let $\{I_m\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the trace form and, for any $k \in \mathbb{N}$, let $\left\{\check{I}_{m,k}(u)\right\} \subset W$ be the dual basis elements of $\{I_m u^k\}$ with respect to $(\cdot,\cdot)$. Then

$$X(u,v) = \sum_{k\in\mathbb{N}} u^k \left( \sum_{m=1}^{n^2-1} I_m \otimes \check{I}_{m,k}(v) \right). \tag{20.2}$$

The notion of gauge equivalence for solutions of the CYBE has the following counterpart in the world of Lie subalgebras of $\mathfrak{g}\,((u^{-1}))$ as above:

**Theorem 20.4.** [42, Theorem 1.2] *Let $X(u,v)$ and $\tilde{X}(u,v)$ be rational solutions of* (20.1) *with corresponding Lie subalgebras $W$ respectively $\tilde{W}$ of $\mathfrak{g}\left((u^{-1})\right)$. Then for any polynomial $\sigma(u) : \mathbb{C} \to \mathrm{Aut}\,(\mathfrak{g})$ we have $X = (\sigma(u) \otimes \sigma(v))\left(\tilde{X}\right)$ if and only if $W = \sigma(u)\left(\tilde{W}\right)$.*

**Corollary 20.5.** *In formula* (20.2)*, we may replace the basis $\{I_m\}_{1 \le m \le n-1}$ by any basis $\{b_m\}_{1 \le m \le n-1}$ of $\mathfrak{g}$ if we simultaneously replace $\left\{\check{I}_{m,k}(u)\right\} \subset W$ by the dual basis $\left\{\check{b}_{m,k}\right\} \subset W$ of $\{b_m u^k\}$ with respect to $(a,b) \mapsto \mathrm{Res}_u\,\mathrm{tr}(a \cdot b)$.*

*Proof.* Let $\eta \in \mathrm{Aut}\mathfrak{g}$ be defined by $\eta(I_m) = b_m$ for all $m$. Then $(\eta^{-1} \otimes \eta^{-1})\,(X(u,v))$ is a rational solution $\tilde{X}(u,v) = \sum_{k\in\mathbb{N}} u^k \left(\sum_{m=1}^{n^2-1} I_m \otimes \check{I}_{m,k}(v)\right)$. Here $\{\check{I}_{m,k}(u)\}$ is the dual basis of $\{I_m u^k\}$ with respect to $(a,b) \mapsto \mathrm{Res}_u\,\mathrm{tr}(a \cdot b)$ in the Lie subalgebra $\tilde{W} \subset \mathfrak{g}\left((u^{-1})\right)$ corresponding to $\tilde{X}(u,v)$. From Theorem 20.4 it follows that $\eta\left(\tilde{W}\right) = W$, hence $\eta\left(\check{I}_{m,k}\right) \in W$ for all $m$ and $k$. Also, $(a,b) \mapsto \mathrm{Res}_u\,\mathrm{tr}(a \cdot b)$ is invariant under base change. Thus $X(u,v) = (\eta \otimes \eta)\left(\tilde{X}(u,v)\right) = \sum_{k\in\mathbb{N}} u^k \left(\sum_{m=1}^{n^2-1} b_m \otimes \check{b}_{m,k}(v)\right)$ indeed. □

The following statement is crucial for the classification of rational solutions by Stolin.

**Theorem 20.6.** [42, Section 2] *For any rational solution $X(u,v)$ there exists a gauge equivalent solution $\tilde{X}(u,v)$ such that the corresponding Lie subalgebra $\tilde{W}$ satisfies $\tilde{W} \subset \eta_k(u)^{-1}\mathfrak{g}\left[[u^{-1}]\right]\eta_k(u)$, where $\eta_k(u) = \mathrm{diag}(\underbrace{1,...,1}_{k}, u, ...u) \in \mathrm{GL}_n(\mathbb{C}\left((u^{-1})\right))$.*

**Definition 20.7.** In the notation of Theorem 20.6, $\tilde{W}$ is said to be of class $k$.

**Proposition 20.8.** [42, Corollary 2.3] *For any Lie subalgebra $W \subset \mathfrak{g}\left((u^{-1})\right)$ with corresponding rational solution $X(u,v)$, there exists a gauge equivalent solution $\tilde{X}(u,v)$ with corresponding Lie subalgebra $\tilde{W} \subset \mathfrak{g}\left((u^{-1})\right)$ of class $1 \le k \le n/2$.*

For practical purposes, the above result is much less useful than it might seem, see Remark 20.11. The next step in Stolin's classification is to translate Lie subalgebras $W \subset \mathfrak{g}\left((u^{-1})\right)$ of a given class $k$ to certain triples of Lie algebraic data which we shall call Stolin triples. In order to explain these triples, we need the following notion:

**Definition 20.9.** For any $1 \le k \le n$, let $\mathfrak{P}_k$ denote the parabolic subalgebra of $\mathfrak{g}$ corresponding to the $k$-th simple root, i.e. the Lie subalgebra spanned by all root spaces of those roots that do not contain the negative $k$-th simple root as a summand. That is,

$$\mathfrak{P}_k = \left\{ \begin{pmatrix} A & B \\ 0_{(n-k)\times k} & D \end{pmatrix} \in \mathfrak{g} \right\}.$$

Now we can give the definition of Stolin triples:

**Definition 20.10.** A Stolin triple $(\mathfrak{L}, B, k)$ is given by the following data

- a Lie subalgebra $\mathfrak{L} \subseteq \mathfrak{g}$.
- a skew symmetric bilinear form $B : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C}$ which is a 2-cocycle, i.e.
$$B\big([x,y],z\big) + B\big([y,z],x\big) + B\big([z,x],y\big) = 0.$$
- an integer $1 \le k \le n-1$.

satisfying the following conditions

(1) $\mathfrak{L} + \mathfrak{P}_k = \mathfrak{g}$.
(2) $B$ is non-degenerate on $(\mathfrak{L} \cap \mathfrak{P}_k) \times (\mathfrak{L} \cap \mathfrak{P}_k)$.

*Remark* 20.11. a) Condition (2) in the above definition is equivalent to requiring that $\mathfrak{L} \cap \mathfrak{P}_k$ is a quasi-Frobenius Lie algebra with respect to $B$.

b) Proposition 20.8 is not useful for practical purposes, because replacing a Stolin triple $(\mathfrak{L}, B, k)$ with $1 \le k < n$ by a Stolin triple $(\mathfrak{L}', B', k')$ satisfying $1 \le k' \le n/2$ might yield much more complicated data $\mathfrak{L}'$ and $B'$. See also fact 20.15, where we describe the procedure that is applied to obtain a Lie subalgebra $W \subset \mathfrak{g}((u^{-1}))$ corresponding to the triple $(\mathfrak{L}, B, k)$.

**Theorem 20.12.** [42, Theorem 3.1] *For any* $1 \le k \le n-1$*, let* $\mathcal{W}_k$ *be the set of Lie subalgebras* $W \subset \mathfrak{g}((u^{-1}))$ *of class* $k$ *and* $\mathcal{ST}_k = \{(\mathfrak{L}, B)\,|(\mathfrak{L}, B, k)$ *is a Stolin triple*$\}$. *Then there is a bijection between* $\mathcal{W}_k$ *and* $\mathcal{ST}_k$.

We briefly sketch the proof given in [42, section 7] for the procedure that associates a Stolin triple $(\mathfrak{L}, B, k)$ to a Lie subalgebra $W$ of class $k$. Let $W' = \eta_k(u) W \eta_k(u)^{-1} \subset \mathfrak{g}[[u^{-1}]]$ and note that $W^\perp = W$ by condition (3) of Theorem 20.2. It follows that $u^{-2}\mathfrak{g}[[u^{-1}]] = (\mathfrak{g}[[u^{-1}]])^\perp \subset (W')^\perp = W'$. Hence we may consider the image $X_W$ of $W'$ under the canonical projection $\mathfrak{g}[[u^{-1}]] \to \mathfrak{g}[[u^{-1}]]/u^{-2}\mathfrak{g}[[u^{-1}]] = \mathfrak{g}[\epsilon]$.

**Proposition 20.13.** [42, Proposition 7.2] *The assignment* $W \mapsto X_W \subset \mathfrak{g}[\epsilon]$ *is bijective. Moreover,* $X_W$ *is Lagrangian with respect to the inner product on* $\mathfrak{g}[\epsilon]$ *induced by the inner product* $\operatorname{Res}_u \operatorname{tr}$ *on* $\mathfrak{g}((u^{-1}))$ *and* $X_W \oplus (\mathfrak{P}_k + \epsilon\mathfrak{P}_k) = \mathfrak{g}[\epsilon]$.

Let $\mathcal{L}$ denote the image of $X_{W'}$ under the map $\mathfrak{g}[\epsilon] \to \mathfrak{g}$ given by $a + \epsilon b \mapsto a$. Then $X_W \subset \mathfrak{L} + \epsilon\mathfrak{g}$. Since $X_W$ is Lagrangian, it follows that $X_W \supseteq (\mathfrak{L} + \epsilon\mathfrak{g})^\perp = \mathfrak{L}^\perp$, where $\mathfrak{L}^\perp$ is the dual of $\mathfrak{L}$ with respect to the trace form on $\mathfrak{g}$. One shows that $\mathfrak{L} + \epsilon\mathfrak{g}/\epsilon\mathfrak{L}^\perp \cong \mathfrak{L} + \epsilon\mathfrak{L}^*$, where now $\mathfrak{L}^*$ is the dual space of $\mathfrak{L}$. From this it is possible to deduce that $X_W$ is uniquely determined by a Lagrangian subspace $\tilde{X}_W = \{l + \epsilon f_B(l)\}_{l \in \mathfrak{L}} \subset \mathfrak{L} + \epsilon\mathfrak{L}^*$, where $f_B : \mathfrak{L} \to \mathfrak{L}^*$ is given by $f_B(x)(y) = B(x,y)$ for a certain 2-cocycle $B$ on $\mathfrak{L}$. Thus $X_W = \big\{l + \epsilon f_B(l) + \epsilon\mathfrak{L}^\perp\big\}_{l \in \mathfrak{L}}$. Here we use the epimorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{L}^\perp \cong \mathfrak{L}^*$ given by $x \mapsto \operatorname{tr}(x \cdot -)$ in order to identify $\big\{f_B(l) \cup \mathfrak{L}^\perp\big\}_{l \in \mathfrak{L}}$ with a subset of $\mathfrak{g}$. Finally, use

**Lemma 20.14.** [42, Lemma 7.4] *The following conditions are equivalent:*

- $X_W \cap \left(\mathfrak{P}_k + \epsilon\mathfrak{P}_k^{\perp}\right) = \{0\}$.
- $X_W + \mathfrak{P}_k + \epsilon\mathfrak{P}_k^{\perp} = \mathfrak{g}\,[\epsilon]$.
- $\mathfrak{L} + \mathfrak{P}_k = \mathfrak{g}$ *and* $B$ *is non-degenerate on* $\mathfrak{L} \cap \mathfrak{P}_k$.

This finishes the proof for the assignment $W \mapsto (\mathfrak{L}, B, k)$.

**Fact 20.15.** The proof of Theorem 20.12 implies that for a given Stolin triple $(\mathfrak{L}, B, k)$ the corresponding Lie subalgebra $W \subset \mathfrak{g}\,((u^{-1}))$ of class $k$ is constructed as follows. Let $\chi' : \mathfrak{L} \to \mathfrak{L}^*$ denote the linear map given by $\chi'(l)(-) = B\big(l, -\big)$ and let $\mathfrak{a} \subset u^{-1}\mathfrak{g} \oplus \mathfrak{L}$ denote the following pullback:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & u^{-1}\mathfrak{L}^{\perp} & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{L} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \chi'} & & \\ 0 & \longrightarrow & u^{-1}\mathfrak{L}^{\perp} & \longrightarrow & u^{-1}\mathfrak{g} & \xrightarrow{\ \nu\ } & \mathfrak{L}^* & \longrightarrow & 0 \end{array}$$

Here $\nu(u^{-1}x) = \operatorname{tr}(x \cdot -)$ and $\mathfrak{L}^{\perp}$ denotes the dual of $\mathfrak{L}$ with respect to the trace form on $\mathfrak{g} \times \mathfrak{g}$. Then we first consider the following subspace of $\mathfrak{g}\,((u^{-1}))$:

$$W' = u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \mathfrak{a}.$$

Recalling $\eta_k(u) \in \mathrm{GL}_n(\mathbb{C}\,((u^{-1})))$ from Theorem 20.6, we have $W = \eta_k(u)^{-1}W'\eta_k(u)$.

Note that for $\mathfrak{L} = \mathfrak{g}$, we have $\mathfrak{L}^{\perp} = 0$ and thus we obtain a well defined map $\chi = \rho \circ \chi' : \mathfrak{L} \to \mathfrak{g}$, where $\rho : \mathfrak{g}^* \to \mathfrak{g}$ denotes the the isomorphism induced by the trace map. Hence in that case

$$W' = u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \operatorname{span}_{\mathbb{C}}\left(\left\{l + u^{-1}\chi(l)\right\}_{l \in \mathfrak{g}}\right).$$

Combining Theorems 20.2 and 20.12, we obtain

**Corollary 20.16.** *To any Stolin triple* $(\mathfrak{L}, B, k)$ *we may associate a uniquely determined rational solution of the CYBE* (20.1).

*Remark* 20.17. The above assignment

$$\{\text{Stolin triples}\} \to \{\text{ rational solutions of the CYBE}\}$$

is surjective, but not injective. This due to the fact that a Lie subalgebra $W \subset \mathfrak{g}\,((u^{-1}))$ as in Theorem 20.2 can be gauge equivalent to subalgebras $W_k \neq W_{k'}$ of different classes $k \neq k'$.

## 21. The rational solution $s_{(n,n-d)}$

In this section we discuss an algorithm which attaches to any pair $(n,d)$ of coprime integers with $1 \leq d < n$ a rational solution of the classical Yang-Baxter equation $s_{(n,n-d)}$ (denoted $r_{(\mathfrak{g},B_J,n-d)}$ in the introduction). As we will explain in Subsection 21.3, this procedure derives from the classification of rational solutions of the CYBE by Stolin as presented in Section 20. The actual algorithm is presented in Subsection 21.1, while in Subsection 21.2 we develop a formula for $s_{(n,n-d)}$ for a particular choice of basis of $\mathfrak{g}$.

### 21.1. Algorithm: Construction of $s_{(n,n-d)}$.

**Step 1: construction of the space $W$.**

Let $J = \mathcal{J}_0(n-d,d)$ be the matrix constructed in step 1 of Algorithm 13.2. We define the following $\mathbb{C}$-subspace of $\mathfrak{g}[[u^{-1}]]$:

$$W' = u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \operatorname{span}_{\mathbb{C}}\left(\left\{x + u^{-1}\left[J^t, x\right]\right\}_{x\in\mathfrak{g}}\right). \tag{21.1}$$

Moreover, for the block decomposition of $\operatorname{Mat}_{n\times n}(\mathbb{C})$ induced by $J$, let

$$\eta(u) = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & \mathbb{1}\cdot u \end{array}\right) \in \operatorname{GL}_n\Big(\mathbb{C}\left(\left(u^{-1}\right)\right)\Big).$$

Then we obtain the following subspace of $\mathfrak{g}((u^{-1}))$:

$$W = \eta(u)^{-1}W'\eta(u) \subset \mathfrak{g}\left(\left(u^{-1}\right)\right). \tag{21.2}$$

**Lemma 21.1.** *Let* $\operatorname{Res}_u : \mathbb{C}((u^{-1})) \to \mathbb{C}$ *denote the residue map* $\sum_{i\in\mathbb{Z}} a_i u^i \mapsto a_{-1}$. *The bilinear form* $(\cdot\,,\cdot) : \mathfrak{g}[u] \times W \to \mathbb{C}$ *given by* $(x,y) \mapsto \operatorname{Res}_u \operatorname{tr}(x\cdot y)$ *is non-degenerate.*

Hence, for any basis $\{I_m\}$ of $\mathfrak{g}$, there exists a dual basis $\left\{\check{I}_{m,k}(u)\right\} \subset W$ of $\{I_m u^k\} \subset \mathfrak{g}[u]$ with respect to $(\cdot\,,\cdot)$.

**Step 2: definition of the tensor $s_{(n,n-d)}(u,v)$.**

**Proposition 21.2.** *In the same notations as above, let*

$$s_{(n,n-d)}(u,v) = \sum_{k\in\mathbb{N}} u^k \left(\sum_{m=1}^{n^2-1} I_m \otimes \check{I}_{m,k}(v)\right).$$

*Then $s_{(n,n-d)}(u,v)$ is a rational solution of the classical Yang-Baxter equation.*

*Remark* 21.3. For a different basis $\{I'_m\}$ of $\mathfrak{g}$, the resulting rational solution is the same as the one obtained for $\{I_m\}$, see Corollary 20.5.

Correctness of Lemma 21.1 and Proposition 21.2 follow form the discussion in Subsection 21.3.

*Remark* 21.4. In the introduction, the solution $s_{(n,n-d)}$ was denoted $r_{(\mathfrak{g},B_J,n-d)}$. At this point however, it is not clear at all which Stolin triple corresponds to $s_{(n,n-d)}$.

**21.2. The solution $s_{(n,n-d)}$ for a particular choice of basis.**
Choose $\left\{\{e_{i,j}u^k\}_{1\le i\ne j\le n}\cup\{\check{h}_l u^k\}_{1\le l\le n-1}\right\}_{k\in\mathbb{N}}$ as basis of $\mathfrak{g}[u]$, where $\check{h}_l$ denotes the dual of $h_l$ with respect to the trace form. The dual basis elements in $W$ with respect to the non-degenerate form $(x,y)=\operatorname{Res}_u \operatorname{tr}(x\cdot y)$ on $\mathfrak{g}\,[u]\times W$ will be denoted $w_{(i,j,k)}(u)$ and $w_{(l,k)}(u)$ for $e_{i,j}u^k$ and $\check{h}_l u^k$ respectively. The following fact follows easily from the definitions:

**Fact 21.5.** Consider the canonical projection $\operatorname{pr}^-:\mathfrak{g}\,((u^{-1}))\to u^{-1}\mathfrak{g}\,[[u^{-1}]]$. Then

$$\begin{aligned}\operatorname{pr}^-\left(w_{(i,j,k)}(u)\right)&=e_{j,i}u^{-k-1}\quad 1\le i\ne j\le n\\ \operatorname{pr}^-\left(w_{(l,k)}(u)\right)&=h_l u^{-k-1}\quad\ \ 1\le l\le n-1.\end{aligned}$$

In particular, $\operatorname{pr}^-\left(w_{(i,j,k)}(u)\right)\in W$ implies $w_{(i,j,k)}(u)=e_{j,i}u^{-k-1}$ and $\operatorname{pr}^-\left(w_{(l,k)}(u)\right)\in W$ implies $w_{(l,k)}(u)=h_l u^{-k-1}$.

For the block decomposition of $\operatorname{Mat}_{n\times n}(\mathbb{C})$ induced by $J$, let $W_1$ denote the following subspace

$$W_1=\left\{\left(\begin{array}{c|c}u^{-2}A & u^{-1}B\\ \hline u^{-3}C & u^{-2}D\end{array}\right)\cdot\mathbb{C}[[u^{-1}]]\,\middle|\,\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathfrak{g}\right\}$$

of $\mathfrak{g}\,((u^{-1}))$ and let $W_2=u^2W_1$. Note that by (21.1) we have $u^{-2}\mathfrak{g}\,[[u^{-1}]]\subset W'\subset\mathfrak{g}\,[[u^{-1}]]$. Hence from the definition of $W$, see (21.2), it follows that

$$W_1\subset W\subset W_2. \tag{21.3}$$

This already yields that

$$\begin{aligned}w_{(i,j,k)}(u)&=e_{j,i}u^{-k-1}\quad 1\le i\ne j\le n,\\ w_{(l,k)}(u)&=h_l u^{-k-1}\quad\ \ 1\le l\le n-1\end{aligned}\quad\text{for all } k\ge 2. \tag{21.4}$$

The careful reader will observe that we may extract much more information from (21.3). We will investigate all relevant implications later, but for now, let us content ourselves with (21.4). From Proposition 21.2 we deduce that

$$s_{(n,n-d)}(u,v)=\frac{\Omega}{v}\sum_{k\ge 2}\left(\frac{u}{v}\right)^k+\sum_{0\le k\le 1}u^k\left(\sum_{1\le i\ne j\le n}e_{i,j}\otimes w_{(i,j,k)}(v)+\sum_{1\le l\le n-1}\check{h}_l\otimes w_{(l,k)}(v)\right).$$

Next, note that up to permutation of $u$ and $v$ ($s_{(n,n-d)}$ is unitary) we generically have

$$\Omega\cdot\sum_{k\in\mathbb{N}}\frac{u^k}{v^{k+1}}=\Omega\cdot\frac{1}{v}\left(\frac{1}{1-\frac{u}{v}}\right)=\frac{\Omega}{v-u}.$$

At this point, it is convenient to introduce the following notation:

**Definition 21.6.** For any $1 \leq i \neq j \leq n$ and any $1 \leq l \leq n-1$, we let

$$\tilde{w}_{i,j}(u,v) = (v-u)\left[w_{(i,j,0)}(v) - v^{-1}e_{ji} + u\left(w_{(i,j,1)}(v) - v^{-2}e_{ji}\right)\right],$$
$$\tilde{w}_l(u,v) = (v-u)\left[w_{(l,0)}(v) - v^{-1}h_l + u\left(w_{(l,1)}(v) - v^{-2}h_l\right)\right].$$

**Proposition 21.7.** *In the notations as above*

$$s_{(n,n-d)}(u,v) = \frac{1}{v-u}\left[\Omega + \left(\sum_{1\leq i\neq j\leq n} e_{i,j}\otimes \tilde{w}_{i,j}(u,v)\right) + \left(\sum_{1\leq l\leq n-1} \check{h}_l \otimes \tilde{w}_l(u,v)\right)\right].$$

As already mentioned, we have not yet fully studied all implications of (21.3), which will be our next task. This is a quite technical business and the reader might wish to have only a fleeting glance at the results at this moment. However, these implications turn out to be crucial for the proofs in Section 22. Also, they allow us to develop a more concrete formula for $s_{(n,n-d)}$, see Corollary 21.11, which is well suited for explicit computations as we will see in Section 23. In order to formulate the results properly, we fix the block decomposition of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ induced by $J = \mathcal{J}_0(n-d,d)$, see Algorithm 13.2. This block decomposition induces a partition of the set $\{i,j\}_{1\leq i,j\leq n}$ into four quadrants, which we picture enumerated in the following way:

$$\left(\begin{array}{c|c} 4 & 1 \\ \hline 3 & 2 \end{array}\right).$$

For convenience, we introduce "Kronecker delta functions" for these quadrants:

**Definition 21.8.** For any $1 \leq i,j \leq n$ let

$$\delta_1(i,j) = \begin{cases} 1 & 1\leq i\leq n-d,\ n-d<j\leq n \\ 0 & \text{else}\end{cases}$$

$$\delta_2(i,j) = \begin{cases} 1 & n-d<i\leq n,\ n-d<j\leq n \\ 0 & \text{else}\end{cases}$$

$$\delta_3(i,j) = \begin{cases} 1 & n-d<i\leq n,\ 1\leq j\leq n-d \\ 0 & \text{else}\end{cases}$$

$$\delta_4(i,j) = \begin{cases} 1 & 1\leq i\leq n-d,\ 1\leq j\leq n-d \\ 0 & \text{else}\end{cases}$$

**Lemma 21.9.** *The following hold:*

(1) *For any $1 \leq i \neq j \leq n$ and $k \in \mathbb{N}$, we may write*

$$w_{(i,j,k)}(v) - v^{-k-1}e_{j,i} = \left(\begin{array}{c|c} A_k & B_k + vB'_k \\ \hline 0 & D_k \end{array}\right)$$

*where $A_k, B_k, B'_k, D_k$ are uniquely determined complex matrices depending only on $k$ and $(i,j)$. The analogous statement is true for $w_{(l,k)}(v) - v^{-k-1}h_l$ for any $1 \leq l \leq n-1, k \in \mathbb{N}$.*

(2) *If $(\delta_2 + \delta_4)(i,j) = 1$, then $w_{(i,j,0)}(v) = \eta(v)^{-1}(x + v^{-1}[J^t, x])\eta(v)$ where*

$$x = \left(\begin{array}{c|c} A_0 & B'_0 \\ \hline 0 & D_0 \end{array}\right) \text{ and } \left[J^t, x\right] = \left(\begin{array}{c|c} 0 & B_0 \\ \hline 0 & 0 \end{array}\right) + e_{j,i}.$$

*The analogous statement is true for $w_{(l,0)}(v)$ for any $1 \leq l \leq n-1$.*

(3) *Let $\delta_1(i,j) = 1$ and $k \in \{0,1\}$, then $w_{(i,j,k)}(v) = \eta(v)^{-1}(x + v^{-1}[J^t, x])\eta(v)$ where*

$$x = \left(\begin{array}{c|c} A_0 & B'_0 \\ \hline 0 & D_0 \end{array}\right) + e_{j,i} \text{ and } [J^t, x] = \left(\begin{array}{c|c} 0 & B_0 \\ \hline 0 & 0 \end{array}\right) \quad k = 0$$
$$x = \left(\begin{array}{c|c} A_1 & B'_1 \\ \hline 0 & D_1 \end{array}\right) \text{ and } [J^t, x] = \left(\begin{array}{c|c} 0 & B_1 \\ \hline 0 & 0 \end{array}\right) + e_{j,i} \quad k = 1$$

(4) *For any $1 \leq l \leq n-1$ we have $\tilde{w}_l(u,v) = (v-u)\left(w_{(l,0)}(v) - h_l v^{-1}\right)$. Moreover, for any $1 \leq i \neq j \leq n$ we have*

$$\tilde{w}_{i,j}(u,v) = \begin{cases} (v-u) \cdot \left(w_{(i,j,0)}(v) - e_{j,i}v^{-1}\right) & \delta_1(i,j) \neq 1 \\ 0 & \delta_3(i,j) = 1. \end{cases}$$

*Proof.* By Fact 21.5 we have $w_{(i,j,k)}(u) - u^{-k-1}e_{j,i} \in \mathfrak{g}[u]$. Since $w_{(i,j,k)}(u) \in W$, clearly (21.3) implies (1). But then (2) and (3) easily follow from (21.1) and (21.2) by examining the $u$-grading (replaced by a $v$-grading in the statements) of the expressions involved. As to (4), taking a closer look at (21.3), we deduce that $v^{-2}h_l \in W$ and also $v^{-2}e_{j,i} \in W$ if $\delta_1(i,j) \neq 1$, so $w_{(l,1)}(v) = v^{-2}h_l$ and $w_{(i,j,1)}(v) = v^{-2}e_{j,i}$ for $\delta_1(i,j) \neq 1$ respectively by Fact 21.5. Taking an even closer look at (21.3), we see that actually $v^{-1}e_{j,i} \in W$ if $\delta_3(i,j) = 1$ and thus $w_{(i,j,0)}(v) = v^{-1}e_{j,i}$ by Fact 21.5 in that case as well. □

*Remark* 21.10. Note that the cases listed in statement 4 of Lemma 21.9 are not mutually exclusive.

The following formula, though much too technical for theoretical purposes, is important for explicit calculations.

**Corollary 21.11.** *Let $I = \left\{(i,j) \in \{1,...,n\}^2 \middle| i \neq j\right\}$ and, for $1 \leq k \leq 4$, set $I_k = \{(i,j) \in I \,|\delta_k(i,j) = 1\}$. Then*

$$s_{(n,n-d)}(u,v) = \frac{\Omega}{v-u} + u \sum_{(i,j)\in I_1} e_{i,j} \otimes \left(w_{(i,j,1)}(v) - e_{j,i}v^{-2}\right) +$$
$$+ \sum_{(i,j)\in I_1 \cup I_2 \cup I_4} e_{i,j} \otimes \left(w_{(i,j,0)}(v) - e_{j,i}v^{-1}\right) + \sum_{1\leq l\leq n-1} \check{h}_l \otimes \left(w_{(l,0)}(v) - h_l v^{-1}\right).$$

*Proof.* The formula is an easy consequence of Proposition 21.7 and Lemma 21.9. $\square$

21.3. **Verification of the construction of** $s_{(n,n-d)}$ **.** In this subsection we construct a certain Stolin triple and prove that $s_{(n,n-d)}$ is the rational solution determined by this triple via Algorithm 21.1. In order to do so, we start with some generalities.

**Definition 21.12.** Let $F$ be a complex Lie algebra with Lie bracket $[-,-]$. Then $F$ is a quasi Frobenius Lie algebra if there exists a non-degenerate skew-symmetric bilinear form $B : F \times F \to \mathbb{C}$ which is a 2-cocycle, that is

$$B\big([x,y]\,,z\big) + B\big([y,z]\,,x\big) + B\big([z,x]\,,y\big) = 0.$$

Moreover, $F$ is called a Frobenius Lie algebra if there exists a functional $f : F \to \mathbb{C}$ such that $F$ is quasi-Frobenius with respect to the bilinear form defined by $B(x,y) = f\big([x,y]\big)$. In that case, $f$ is called a Frobenius functional for $F$.

*Remark* 21.13. It is was shown by Elashvili [21, 20] that for $1 \le k \le n-1$, $\mathfrak{P}_k \subset \mathfrak{g}$ is a Frobenius Lie algebra if and only if $\gcd(n,k) = 1$. Other examples of Frobenius Lie algebras include the so called *Seaweed algebras* which were studied by Dergachev and A. Kirillov [18] and also Gerstenhaber and Giaquinto [23]. Obviously, if any 2-cocyle in the Koszul complex of a given quasi-Frobenius Lie algebra $F$ is already a coboundary, i.e. $H^2(F) = 0$, then $F$ is automatically a Frobenius Lie algebra.

The following fact follows easily from the definitions:

**Fact 21.14.** Let $1 \le k \le n-1$ with $\gcd(n,k) = 1$ and let $f$ be any Frobenius functional for $\mathfrak{P}_k$. Extend $f$ to all of $\mathfrak{g}$ via $f(x) = f\left(\mathrm{pr}_{\mathfrak{P}_k}(x)\right)$, where $\mathrm{pr}_{\mathfrak{P}_k} : \mathfrak{g} \to \mathfrak{P}_k$ denotes the canonical projection. Let $B(x,y) = f\big([x,y]\big)$ for all $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}, B, k)$ is a Stolin triple.

Next, recall the matrix $\mathcal{J}_\lambda(n-d, d)$ from Algorithm 13.2.

**Definition 21.15.** For any coprime integers $a, b$, let $J(a,b) = \mathcal{J}_0(a,b)$. Also, we set $J = J(n, n-d)$.

**Proposition 21.16.** *$f : \mathfrak{P}_{n-d} \to \mathbb{C}$ defined by $f(x) = \mathrm{tr}\,(J^t \cdot x)$ is a Frobenius functional. Moreover, the same formula gives an extension of $f$ to all of $\mathfrak{g}$ with the property that $f(x) = f\left(\mathrm{pr}_{\mathfrak{P}_{n-d}}(x)\right)$.*

In order to prove this result, we need an auxiliary result which goes back to Elashvili [21, first Lemma] and which was kindly explained to the author by Alexander Stolin. Recall that for any Lie algebra $\mathfrak{L}$ and its dual vector space $\mathfrak{L}^*$, the coadjoint action $\mathrm{ad}^* : \mathfrak{L} \to \mathrm{End}_{\mathbb{C}}\left(\mathfrak{L}^*\right)$ is defined as

$$\mathrm{ad}^*(l)(m)(l') = m\left([l', l]\right) \quad l, l' \in \mathfrak{L},\, m \in \mathfrak{L}^*.$$

**Lemma 21.17.** *Let $\mathfrak{L}$ be any complex Lie algebra, $L \subset \mathfrak{L}$ a Lie subalgebra and $N \subset \mathfrak{L}$ a commutative ideal such that $\mathfrak{L} = L \oplus N$ as vector spaces. Assume that there exists $n^* \in N^*$ such that $N^* \subseteq \mathrm{ad}^*(L)(n^*)$ and let*

$$S = \{l \in L \,|n^* ([l, -]) = 0\} .$$

*If $S$ is a Frobenius Lie algebra with respect to the Frobenius functional $s^* \in S^*$, then $\mathfrak{L}$ is a Frobenius Lie algebra with respect to the Frobenius functional $n^* + s^*$.*

*Proof.* Assume that there exist $l_1 \in L$ and $n_1 \in N$ such that for all $l_2 \in L$ and all $n_2 \in N$ we have $(n^* + s^*) ([l_1 + n_1, l_2 + n_2]) = 0$. Since $N$ is a commutative ideal, this is equivalent to

$$n^* ([l_1, n_2] + [n_1, l_2]) + s^* ([l_1, l_2]) = 0.$$

Choosing $l_2 = 0$ we obtain $0 = n^* ([l_1, n_2])$ for all $n_2 \in N$, hence $l_1 \in S$ by definition of $S$. But $s^*$ is Frobenius functional for $S$, thus for $l_1 \neq 0$ there exists some $s_1 \in S$ such that $s^* ([l_1, s_1]) \neq 0$. Choosing $l_2 = s_1$ and $n_2 = 0$ it follows that

$$n^* ([n_1, s_1]) + s^* ([l_1, s_1]) = 0.$$

But the first term is zero because $s_1 \in S$ and the second is nonzero if $l_1$ is unequal to zero. Therefore $l_1 = 0$, which means that

$$n^* ([n_1, l_2]) = 0$$

for all $l_2 \in L$. Since $N^* \subseteq \mathrm{ad}^*(L)(n^*)$, there exists some $l_2' \in \mathfrak{L}$ such that $\mathrm{ad}(l_2')(n^*)$ equals $n_1^*$, the dual of $n_1$. But then $n^* ([n_1, l_2']) \neq 0$ if $n_1$ was not already equal to zero, which would give a contradiction. Therefore $n_1 = 0$ as well, which shows that $n^* + s^*$ is indeed a Frobenius functional for $\mathfrak{L}$. □

**Fact 21.18.** The following hold:

(1) The trace form is invariant, i.e. for any matrices $A, B, C \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, we have $\mathrm{tr}(A, [B, C]) = \mathrm{tr}([A, B], C)$.

(2) In the situation of Lemma 21.17, let $\mathcal{L} \subseteq \mathfrak{g}$ and $N^t = \{n^t | \, n \in N\}$. Assume that the map $N^t \to N^*$ given by $n^t \mapsto \mathrm{tr}\,(n^t \cdot -)$ is injective. Then $N^* \subseteq \mathrm{ad}^*(L)(n^*)$ if and only if $N^t \subseteq [n^t, L]$. Moreover, $s \in L$ satisfies $\mathrm{ad}^*(s)(n^*) = 0$ if and only if $\mathrm{tr}\,([n^t, s] \cdot l') = 0$ for all $l' \in \mathfrak{L}$.

*Proof.* (1) is obvious. As to (2), let $\tau : \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism induced by the trace map. We claim that for any $l \in L$ we have $\mathrm{ad}^*(l)(n^*) = -\tau\,([n^t, l])$. Indeed, given $l' \in \mathfrak{L}$, the definition of the coadjoint action and (1) yield $\mathrm{ad}^*(l)(n^*)(l') = n^* ([l', l]) = \mathrm{tr}\,(n^t \cdot [l', l]) = -\mathrm{tr}\,([n^t, l] \cdot l')$. This immediately yields the second statement of (2). Since $n^t \mapsto \mathrm{tr}\,(n^t \cdot -)$ is injetive by assumption, we have $\tau^{-1}(N^*) = N^t$. □

Applying Fact 21.18, we obtain the following version of Lemma 21.17:

**Corollary 21.19.** *Let $\mathfrak{L} \subseteq \mathfrak{g}$ be a complex Lie algebra, $L \subset \mathfrak{L}$ a Lie subalgebra and $N \subset \mathfrak{L}$ a commutative ideal with $N^t \to N^*$ given by $n^t \mapsto tr(n^t \cdot -)$ injective such that $\mathfrak{L} = L \oplus N$ as vector spaces. Assume that there exists $n \in N$ such that $N^t \subseteq [n^t, L]$ and let $S \subseteq L$ be the maximal subalgebra of $L$ such that $\operatorname{tr}([n^t, s] \cdot \mathfrak{L}) = 0$ for all $s \in S$. If $S$ is a Frobenius Lie algebra with respect to the Frobenius functional $x \mapsto \operatorname{tr}(s^t \cdot x)$ for some $s \in S$, then $\mathfrak{L}$ is a Frobenius Lie algebra with respect to the Frobenius functional $x \mapsto \operatorname{tr}((n^t + s^t) \cdot x)$.*

Before we prove Proposition 21.16, we need one additional combinatorial result:

**Lemma 21.20.** *Let $a \in \mathbb{N}$ and consider the Lie algebra*

$$S_a = \left\{ \left( \begin{array}{cc} X & Y \\ 0_{(n-a)\times a} & Z \end{array} \right) \middle| 2\mathrm{tr(X)}+\mathrm{tr(Z)} = 0 \right\} \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$$

*with Lie bracket given by the commutator. Then there exists a Lie algebra isomorphism $\nu$ mapping $\mathfrak{P}_a \subset \mathfrak{sl}_n(\mathbb{C})$ to $S_a$ which satisfies $\nu(J) = J$.*

*Proof.* First note that a generating set of $\mathfrak{P}_a$ is given by $\{h_i, e_{i,i+1}, e_{i+1,i}\}_{1\le i\le n-1} \setminus \{e_{a+1,a}\}$ while a generating set of $S_a$ is given by $\left\{\tilde{h}_i, e_{i,i+1},, e_{i+1,i}\right\}_{1\le i\le n-1} \setminus \{e_{a+1,a}\}$ where

$$\tilde{h}_i = \begin{cases} h_i & i \neq a \\ h_i - e_{a+1,a+1} & i = a. \end{cases}$$

We define the $\mathbb{C}$-linear map $\nu : \mathfrak{P}_a \to S_a$ by giving its images on the fixed generators of $\mathfrak{P}_a$ from above as follows. Let $\nu(e_{i,i+1}) = e_{i,i+1}$, $\nu(e_{i+1,i}) = e_{i+1,i}$ and

$$\nu(h_i) = \begin{cases} \tilde{h}_i & i \neq a \\ \sum_{j=1}^{n-1} t_j \tilde{h}_j & i = a. \end{cases}$$

Here $\{t_j\}_{1\le j\le n-1}$ is a set of complex numbers yet to be defined. Namely, we want to ensure that $\nu$ is an isomorphism of Lie algebra. For this it suffices to choose $\{t_j\}_{1\le j\le n-1}$ in a way such that $\nu(h_a) \neq 0$ and such that the relations

$$\begin{array}{ll} \nu\left([h_k, e_{i,i+1}]\right) = \left[\nu\left(h_k\right), \nu\left(e_{i,i+1}\right)\right] & 1 \le i, k \le n-1 \\ \nu\left([h_k, e_{i+1,i}]\right) = \left[\nu\left(h_k\right), \nu\left(e_{i+1,i}\right)\right] & 1 \le i, k \le n-1,\, i \neq a \end{array} \tag{21.5}$$

are satisfied. Indeed, since $e_{a+1,a}$ is not among the generators of $\mathfrak{P}_a$, the remaining relations that need to be checked are immediate.

Now, for fixed $i$, a relation from the first set of conditions of (21.5) reads

$$\alpha_{i,i+1}(h_k)e_{i,i+1} = \left[\nu\left(h_k\right), e_{i,i+1}\right]$$

where $\alpha_{i,j}$ denotes the root of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the root space spanned by $e_{i,j}$. This condition is obviously satisfied for all $k \neq a$ by definition of $\tilde{h}_k$. However, for $k = a$

we obtain the following condition on $\{t_j\}_{1\leq j\leq n-1}$:

$$\alpha_{i,i+1}(h_a)e_{i,i+1} = \left[\sum_{j=1}^{n-1} t_j h_j, e_{i,i+1}\right] - t_a\left[e_{a+1,a+1}, e_{i,i+1}\right]. \tag{21.6}$$

Since $\left[\sum_{j=1}^{n-1} t_j h_j, e_{i,i+1}\right] = \sum_{j=1}^{n-1} t_j \alpha_{i,i+1}(h_j)$ and

$$[e_{a+1,a+1}, e_{i,i+1}] = \begin{cases} -e_{a,a+1} & i = a \\ e_{a+1,a+2} & i = a+1 \\ 0 & \text{else,} \end{cases} \tag{21.7}$$

we may rewrite (21.6) as

$$\tau_i = \sum_{j=1}^{n} t_j \sigma_{ij}$$

with $\tau_i = \alpha_{i,i+1}(h_a)$ and certain integers $\sigma_{ij}$. Indeed,

$$(\sigma_{ij})_{1\leq i,j\leq n} = \begin{pmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 3 & -1 & & & & \\ & & & & -2 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 \end{pmatrix}$$

where the unique entry 3 stands in the $(a,a)$-th position and correspondingly the unique entry $-2$ in the $(a+1,a)$-th position. It is easy to see that this matrix has full rank, thus a (unique) solution $\{t_j\}_{1\leq j\leq n-1}$ for (21.6) exists. We fix this choice in order to complete the definition of the map $\nu$. Note that since $\{\alpha_{i,i+1}\}_{1\leq i\leq n-1}$ is a basis of $\mathfrak{h}^*$, not all $\tau_i$ can be equal to zero, hence there exists a $t_j$ which is not equal to zero. Since the elements of $\left\{\tilde{h}_j\right\}_{1\leq j\leq n-1}$ are linear independent, this implies $\nu(h_a) \neq 0$.

It remains to check the second set of conditions of (21.5), which we rewrite as

$$\alpha_{i+1,i}(h_k)e_{i+1,i} = \left[\nu(h_k), e_{i+1,i}\right].$$

As above, this condition is immediately satisfied for $k \neq a$ by definition of $\tilde{h}_k$. As for $k = a$, note that

$$\left[\nu(h_a), e_{i+1,i}\right] = \alpha_{i+1,i}\left(\sum_{j=1}^{n-1} t_j h_j\right) e_{i+1,i} - t_a\left[e_{a+1,a+1}, e_{i+1,i}\right].$$

Recall from (21.7) that $[e_{a+1,a+1}, e_{i,i+1}] = c_i e_{i,i+1}$ for certain integers $c_i$. It is easy to see that $[e_{a+1,a+1}, e_{i+1,i}] = -c_i e_{i+1,i}$ in all cases to be considered, i.e. $i \neq a$. Hence

$$[\nu(h_a), e_{i+1,i}] = \left( \alpha_{i+1,i} \left( \sum_{j=1}^{n-1} t_j h_j \right) + c_i t_a \right) e_{i+1,i}.$$

On the other hand, since $\alpha_{i,j} = -\alpha_{j,i}$, we have

$$\alpha_{i+1,i}(h_a) = -\alpha_{i,i+1}(h_a) = -\left( \alpha_{i,i+1} \left( \sum_{j=1}^{n-1} t_j h_j \right) - c_i t_a \right).$$

Thus all conditions of (21.5) are satisfied. Hence with respect to the choice of $\{t_j\}_{1 \leq j \leq n-1}$ we made, $\nu$ is a Lie algebra automorphism. The fact that $\nu(J) = J$ is immediate from the definition of $\nu$ and the fact that all diagonal entries of $J$ are zero. □

Now we can prove Proposition 21.16:

*Proof.* Let $a, b$ be two coprime integers and recall the construction of $J = \mathcal{J}_0(n-d, d)$ from Algorithm 13.2. We will prove that $f(x) = \operatorname{tr}(J(a,b)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_a \subset \mathfrak{sl}_{a+b}(\mathbb{C})$, while considered as a functional on $\mathfrak{sl}_{a+b}(\mathbb{C})$ it satisfies $f(x) = f\left(\operatorname{pr}_{\mathfrak{P}_a}(x)\right)$. The proof runs by induction along the order of construction of $J(a,b) = \mathcal{J}_0(a,b)$.

0) Start with $(a,b) = (1,1)$. Then

$$J(1,1)^t = \left( \begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array} \right) \in \operatorname{Mat}_{2\times 2}(\mathbb{C}).$$

and for arbitrary $x = x_1 h + x_2 e_{1,2}, y = y_1 h + y_2 e_{1,2}$ in $\mathfrak{P}_1$ we derive

$$\operatorname{tr}\left(J(1,1)^t \cdot [x,y]\right) = \operatorname{tr}\left(e_{2,1} \cdot [x_1 h + x_2 e_{1,2}, y_1 h + y_2 e_{1,2}]\right) = 2\left(y_2 x_1 - y_1 x_2\right).$$

Hence $f(x) \mapsto \operatorname{tr}(J(1,1)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_1 \subset \mathfrak{sl}_2(\mathbb{C})$ indeed. Moreover $\operatorname{tr}(J(1,1)^t \cdot e_{2,1}) = 0$ immediately yields $f(x) = f\left(\operatorname{pr}_{\mathfrak{P}_1}(x)\right)$.

1) Assume $a < b$. Then

$$J(a,b) = \left( \begin{array}{c|c} 0 & \mathbb{1} \quad 0 \\ \hline 0 & J(a,b-a) \end{array} \right) = \left( \begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a) \end{array} \right) + n$$

and by the induction hypothesis $x \mapsto \operatorname{tr}\ (J(a,b-a)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_a \subset \mathfrak{sl}_b(\mathbb{C})$. In order to prove the statement for the tuple $(a,b)$, we want to apply Corollary 21.19. To that end, let us write $\mathfrak{P}_a = L \oplus N$ with

$$L = \left( \begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array} \right),\ N = \left( \begin{array}{c|c} 0 & N_1 \\ \hline 0 & 0 \end{array} \right).$$

Here, we make use of the block decomposition of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ induced by $J(a,b)$. Clearly $L$ is a Lie subalgebra of $\mathfrak{P}_a$ and $N$ is a commutative ideal with the property that $N^t \to N^*$ given by $n^t \mapsto \mathrm{tr}\,(n^t \cdot -)$ is injective. Now

$$\left[n^t, L\right] = \left[\left(\begin{array}{c|c} 0 & 0 \\ \hline \mathbb{1} & 0 \\ 0 & 0 \end{array}\right), \left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & 0 \\ \hline L_1 - L_2^{(1)} & 0 \\ -L_2^{(3)} & 0 \end{array}\right)$$

where, for appropriate sizes of the matrices $L_2^{(i)}, 1 \leq i \leq 4$, we use the notation

$$L_2 = \begin{pmatrix} L_2^{(1)} & L_2^{(2)} \\ L_2^{(3)} & L_2^{(4)} \end{pmatrix}.$$

Hence $N^t \subset [n^t, L]$ and the stabilizer $S$ is of the form

$$S = \left(\begin{array}{c|cc} L_1 & 0 & 0 \\ \hline 0 & L_1 & L_2^{(2)} \\ 0 & 0 & L_2^{(4)} \end{array}\right) \cong \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & L_1 & L_2^{(2)} \\ 0 & 0 & L_2^{(4)} \end{array}\right).$$

Using the isomorphism from Lemma 21.20, we derive that

$$S \cong \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathfrak{P}_a \end{array}\right) \subset \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathfrak{sl}_b(\mathbb{C}) \end{array}\right).$$

Thus the image of $S$ under the first isomorphism is a Frobenius Lie algebra with Frobenius functional $x \mapsto \mathrm{tr}\,(J(a,b-a)^t \cdot x)$ by the induction hypothesis. Since

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right) + n^t = \left(\begin{array}{c|c} 0 & 0 \\ \hline \mathbb{1} & \\ 0 & J(a,b-a)^t \end{array}\right) = J(a,b)^t,$$

it follows from Corollary 21.19 that $x \mapsto \mathrm{tr}\,(J(a,b)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_a \subset \mathfrak{sl}_{a+b}(\mathbb{C})$. Because $\mathrm{pr}_N\,(J(a,b)^t) = 0$, the statement concerning the extension of $f$ easily follows.

2) Assume $a > b$. Then

$$J(a,b) = \left(\begin{array}{c|c} J(a-b,b) & \begin{array}{c} 0 \\ \mathbb{1} \end{array} \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} J(a-b,b) & 0 \\ \hline 0 & 0 \end{array}\right) + n$$

and by the induction hypothesis $x \mapsto \mathrm{tr}\ (J(a-b,b)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_{a-b} \subset \mathfrak{sl}_a(\mathbb{C})$. We proceed exactly as in 1), writing $\mathfrak{P}_{a-b} = L \oplus N$ with the analogous definitions for $L$ and $N$. This time

$$[n^t, L] = \left[\left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 0 & \mathbb{1} & 0 \end{array}\right), \left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array}\right)\right] = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline L_1^{(3)} & L_1^{(4)} - L_2 & 0 \end{array}\right)$$

where, for appropriate sizes of the matrices $L_1^{(i)}, 1 \le i \le 4$, we use the notation

$$L_1 = \begin{pmatrix} L_1^{(1)} & L_1^{(2)} \\ L_1^{(3)} & L_1^{(4)} \end{pmatrix}.$$

Hence $N^t \subset [n', L]$ and the stabilizer $S$ is of the form

$$S = \left(\begin{array}{cc|c} L_1^{(1)} & L_1^{(2)} & 0 \\ 0 & L_2 & 0 \\ \hline 0 & 0 & L_2 \end{array}\right) \cong \left(\begin{array}{cc|c} L_1^{(1)} & L_1^{(2)} & 0 \\ 0 & L_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{c|c} \mathfrak{P}_{a-b} & 0 \\ \hline 0 & 0 \end{array}\right) \subset \left(\begin{array}{c|c} \mathfrak{sl}_a(\mathbb{C}) & 0 \\ \hline 0 & 0 \end{array}\right).$$

Here we use the appropriate analogue of Lemma 21.20 for the second isomorphism. Thus the image of $S$ under the first isomorphism is a Frobenius algebra with Frobenius functional $x \mapsto \operatorname{tr}(J(a-b,b)^t \cdot x)$ by the induction hypothesis. Since

$$\left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right) + n^t = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{J(a-b,b)^t} & 0 \\ \hline 0 & \mathbb{1} & 0 \end{array}\right) = J(a,b)^t,$$

it follows from Corollary 21.19 that $x \mapsto \operatorname{tr}(J(a,b)^t \cdot x)$ is a Frobenius functional for $\mathfrak{P}_a \subset \mathfrak{sl}_{a+b}(\mathbb{C})$. Because $\operatorname{pr}_N(J(a,b)^t) = 0$, the statement concerning the extension of $f$ easily follows. $\square$

Combining Fact 21.14 and Proposition 21.16 finally yields:

**Corollary 21.21.** Let $B_J(x,y) = \operatorname{tr}(J^t \cdot [x,y])$ for all $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}, B_J, n-d)$ is a Stolin triple.

Before presenting the main result of this subsection, let us state the following fact on the map $\chi : \mathfrak{g} \to \mathfrak{g}$ defined in Fact 20.15.

**Fact 21.22.** For any $x \in \mathfrak{g}$ we have $\chi(x) = [J^t, x]$.

*Proof.* Recall from Fact 20.15 that $\chi(x) = \rho(\operatorname{tr}(J^t \cdot [x, -]))$ where $\rho^{-1} : \mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \operatorname{tr}(x \cdot -)$ is the isomorphism induced by the trace form. Moreover

$$\rho\left(\operatorname{tr}\left(J^t \cdot [x, -]\right)\right) = \rho\left(\operatorname{tr}\left(\left[J^t, x\right] \cdot -\right)\right)$$

by Fact 21.18. But $\rho(\operatorname{tr}(a \cdot -)) = a$ for any $a \in \mathfrak{g}$ by definition. $\square$

**Proposition 21.23.** *Given two coprime integers* $1 \le d < n$, let $B_J(x,y) = \operatorname{tr}(J^t \cdot [x,y])$ for any $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}, B_J, n-d)$ *is a Stolin triple, whose corresponding rational solution of* (20.1) *is computed by performing Algorithm* 21.1.

*Proof.* We have seen in Corollary 21.21 that $(\mathfrak{g}, B_J, n-d)$ is a Stolin triple. As we have seen in Section 20, the corresponding rational solution is constructed in two steps. First, Fact 20.15 gives an algorithm to construct the corresponding Lie subalgebra $W \subset \mathfrak{g}((u^{-1}))$. Due to Fact 21.22, this is exactly the same procedure as described in step one of Algorithm 21.1. Next, Fact 20.3 shows how to construct the actual solution from $W$. By Corollary 20.5, we are free to choose any basis of $\mathfrak{g}$ while doing so. Hence step two of Algorithm 21.1 corresponds exactly to this second step of construction. □

## 22. Connections between the solutions $s_{(n,n-d)}$ and $c_{(n,d)}$

As in the previous section, we let $J = \mathcal{J}_0(n-d, d)$, see Algorithm 13.2. This section is devoted to the main result of this thesis:

**Theorem 22.1.** *The matrix $J$ induces an automorphism of Lie algebras $\varphi_J : \mathfrak{g} \to \mathfrak{g}$ such that*

$$(\varphi_J \otimes \varphi_J)\, c_{(n,d)} = s_{(n,n-d)}.$$

*Remark* 22.2. In the introduction, the solutions $c_{(n,d)}$ and $s_{(n,n-d)}$ were denoted $r_{(E,n,d)}$ (for $E$ the cuspidal cubic curve) and $r_{(\mathfrak{g},B_J,n-d)}$ respectively.

Furthermore, we will study certain structural consequences for both $c_{(n,d)}$ and $s_{(n-d,d)}$ which can be obtained by application of this Theorem, see Subsection 22.3. However, we first need to properly define the map $\varphi_J$ occurring in Theorem 22.1.

### 22.1. The map $\varphi_J$.

We start with the following Lie algebraic statement:

**Lemma 22.3.** *Write $J = \sum_{k=1}^{n-1} e_{i_k,j_k}$ and let $\gamma_k$ denote the root corresponding to the root space $\{e_{i_k,j_k}\}_{\mathbb{C}}$ of the standard Cartan $\mathfrak{h} \subset \mathfrak{g}$. Then $\{\gamma_k\}_{1 \le k \le n-1}$ is a basis of $\mathfrak{h}^*$ with the property that any root $\alpha$ of $\mathfrak{g}$ can be written, in a unique way, in the form $\alpha = \sum_{k=1}^{n-1} a_k \gamma_k$ with $a_i \in \mathbb{Z}$.*

*Proof.* For $1 \le i \ne j \le n$, let $\alpha_{i,j}$ denote the root corresponding to the root space $\operatorname{span}_{\mathbb{C}}(\{e_{i,j}\})$. Since $\Pi = \{\alpha_{i,i+1}\}_{1 \le i \le n-1}$ is a simple system of the root system of $\mathfrak{g}$, it clearly suffices to prove the second part of the statement for $\alpha \in \Pi$. The proof of the whole statement runs via a double induction. The first induction is along the order of construction of $J$. The induction starts with $J(1,1) = e_{1,2} \in \mathfrak{sl}_2(\mathbb{C})$. In that case $\gamma_1 = \alpha_{1,2}$ and there is nothing left to prove. So let $a, b$ be two coprime integers.

1) Assume $a < b$. For any $0 \le t \le a$, let $I_t = \left(\begin{smallmatrix} 0 & 0 \\ 0 & J(a,b-a) \end{smallmatrix}\right) + \sum_{s=1}^{t} e_{s,t+s}$. Since $I_a = J(a,b)$, it clearly suffices to prove the statement for each $0 \le t \le a$. So we do another induction, namely via $t$. However, for $t = 0$ we have $I_0 = J(a, b-a)$. Therefore the induction hypothesis of the first induction implies correctness of the basis of induction by $t$. Thus we may assume that the statement is correct for $t - 1 < a$. The

roots $\{\gamma_k\}_{1\le k\le b+(t-1)-1}$, originally roots of $\mathfrak{sl}_{b+t-1}(\mathbb{C})$, are clearly also roots of $\mathfrak{sl}_{b+t}(\mathbb{C})$. But since they derive from a smaller subspace, they are linearly independent from $\gamma_{b+t-1} = \alpha_{1,t+1}$, where $I_t = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{t-1} \end{smallmatrix}\right) + e_{1,t+1}$. Hence $\{\gamma_k\}_{1\le k\le b+t-1}$ is a basis of $\mathfrak{h}^*$ indeed. As to the remaining assertion, we know that we can write $\gamma_{b+t-1} = \sum_{k=1}^{b+t-1} b_k\alpha_{k,k+1}$ with uniquely determined integers $b_k$ with $|b_k| \le 1$ because $\Pi$ is a simple system of $\mathfrak{g}$. Note that $b_1 \ne 0$ by a similar argument as the one given for the linear independence of $\{\gamma_k\}_{1\le k\le b+t-1}$ above. Hence $\alpha_{1,2} = b_1^{-1}\gamma_{b+t-1} - \sum_{k=2}^{b+t-1} b_1^{-1}b_k\alpha_{k,k+1}$. Now this last sum may be interpreted as a root of $\mathfrak{sl}_{b+t-1}(\mathbb{C})$, thus by the induction hypothesis (of the induction via $t$) we may deduce $-\sum_{k=2}^{b+t-1} b_1^{-1}b_k\alpha_{k,k+1} = \sum_{k=1}^{b+t-2} a_k\gamma_k$ with $a_k \in \mathbb{Z}$ uniquely determined for all $2 \le k \le b+t-1$. Setting $a_{b+t-1} = b_1^{-1}$, this finishes the proof of the induction via $t$.

2) The proof for the case $a > b$ runs exactly parallel to that for $a < b$. □

**Definition 22.4.** In the notation of Lemma 22.3, let $\alpha = \sum_{k=1}^{n-1} a_k\gamma_k$ be any root of $\mathfrak{g}$. Let $\operatorname{ht}(\alpha) = \sum_{k=1}^{n-1} |a_k|$ be the height of $\alpha$. For $1 \le i \ne j \le n$, denote the root corresponding to the root space $\operatorname{span}_{\mathbb{C}}(\{e_{i,j}\})$ by $\alpha_{i,j}$. Let $z_{i,j}$ be the image of $\operatorname{ht}(\alpha_{i,j})+1$ in $\mathbb{F}_2$ and set $z_{i,i} = 1$. We define $Z_J \in \operatorname{Mat}_{n\times n}(\mathbb{F}_2)$ by $(Z_J)_{i,j} = z_{i,j}$.

**Fact 22.5.** For all $1 \le i,j,k \le n$ we have $z_{i,j} + z_{j,k} + 1 = z_{i,k}$ in $\mathbb{F}_2$ and $z_{i,j} = z_{j,i}$.

*Proof.* All computations are in $\mathbb{F}_2$. Since $\alpha_{i,j} = -\alpha_{j,i}$, the second statement follows easily from the definitions. As to the first statement, this is obvious at least for $i = k$ because of the validity of the second statement. So assume $i \ne k$. Then $[e_{i,j}, e_{j,k}] = e_{i,k}$, which implies $\alpha_{i,j} + \alpha_{j,k} = \alpha_{i,k}$. Let $\alpha_{i,j} = \sum a_r\gamma_r$, $\alpha_{j,k} = \sum b_r\gamma_r$ and $\alpha_{i,k} = \sum c_r\gamma_r$ as in Lemma 22.3, then that same Lemma yields that $a_r + b_r = c_r$ by uniqueness of the coefficients $c_r$. Hence $\operatorname{ht}(\alpha_{i,j}) + \operatorname{ht}(\alpha_{j,k}) = \operatorname{ht}(\alpha_{i,k})$. The rest follows from the definitions. □

Combining Definition 22.4 and Fact 22.5 gives a concrete recipe to construct the matrix $Z_J$. This recipe is best explained by the following examples:

**Example 22.6.** By Definition 22.4, $z_{i,j} = 0$ whenever $J_{i,j} = 1$ and $z_{i,i} = 1$ for all $1 \le i,j \le n$. Moreover, $z_{i,j} = z_{j,i}$ for all $1 \le i,j \le n$.

- Since for $(n,d) = (2,1)$ we have $J = \mathcal{J}_0(1,1) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, this implies $Z_J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
- For $(n,d) = (3,2)$, we have

$$J = \mathcal{J}_0(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{and thus } Z_J = \begin{pmatrix} 1 & 0 & z_{1,3} \\ 0 & 1 & 0 \\ z_{3,1} & 0 & 1 \end{pmatrix}.$$

But by Fact 22.5 we have $z_{1,3} = z_{1,2} + z_{2,3} + 1$ as an element of $\mathbb{F}_2$. Hence

$$Z_J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

**Lemma 22.7.** *The linear map* $\varphi_J : \mathfrak{g} \to \mathfrak{g}$ *induced by*

$$\begin{cases} e_{i,j} \mapsto (-1)^{z_{i,j}} e_{j,i} & 1 \le i \neq j \le n \\ h_i \mapsto (-1)^{z_{i,i}} h_i = -h_i & 1 \le i \le n-1 \end{cases}$$

*is a Lie algebra automorphism of* $\mathfrak{g}$ *such that* $\varphi_J(J) = J^t$.

*Proof.* The equality $\varphi_J(J) = J^t$ is a direct consequence of Lemma 22.3 and Definition 22.4.

Hence it suffices to check that $\varphi_J$ is a Lie algebra homomorphism. Since $\varphi_J$ stabilizes $\mathfrak{h}$, this means that it suffices to check

$$\begin{cases} \varphi_J([e_{i,j}, e_{k,l}]) = [\varphi_J(e_{i,j}), \varphi_J(e_{k,l})] & 1 \le i \neq j, k \neq l \le n \\ \varphi_J([h_i, e_{k,l}]) = [\varphi_J(h_i), \varphi_J(e_{k,l})] & 1 \le i \le n-1, 1 \le k \neq l \le n. \end{cases}$$

As to the first claim, we need to consider the following cases

$$[e_{i,j} e_{k,l}] = \begin{cases} 0 & i \neq l, j \neq k \\ e_{i,l} & i \neq l, j = k \\ -e_{k,j} & i = l, j \neq k \\ e_{i,i} - e_{j,j} & i = l, j = k, \end{cases} \text{, so } \varphi_J([e_{i,j}, e_{k,l}]) = \begin{cases} 0 & i \neq l, j \neq k \\ (-1)^{z_{i,l}} e_{l,i} & i \neq l, j = k \\ (-1)^{z_{k,j}+1} e_{j,k} & i = l, j \neq k \\ e_{j,j} - e_{i,i} & i = l, j = k. \end{cases}$$

On the other hand

$$[\varphi_J(e_{i,j}), \varphi_J(e_{k,l})] = (-1)^{z_{i,j}+z_{k,l}} [e_{j,i}, e_{l,k}] = (-1)^{z_{i,j}+z_{k,l}} \cdot \begin{cases} 0 & i \neq l, j \neq k \\ -e_{l,i} & i \neq l, j = k \\ e_{j,k} & i = l, j \neq k \\ e_{j,j} - e_{i,i} & i = l, j = k. \end{cases}$$

Applying Fact 22.5 yields the equality in all four cases.

As to the second claim, $\varphi_J([h_i, e_{k,l}]) = \alpha_{k,l}(h_i)(-1)^{z_{k,l}} e_{l,k}$ while $[\varphi_J(h_i), \varphi_J(e_{k,l})] = \alpha_{l,k}(h_i)(-1)^{z_{l,k}+1} e_{l,k}$. Applying Fact 22.5 once more and recalling $\alpha_{k,l} = -\alpha_{l,k}$ shows that $\varphi_J$ is a Lie algebra automorphism indeed. □

22.2. **Gauge equivalence of** $s_{(n,n-d)}$ **and** $c_{(n,d)}$**.** Now that we have defined $\varphi_J$ properly, we can finally attack the proof of Theorem 22.1. As it turns out, the essential parts of the proof are direct consequences of Lemma 21.9.

**Theorem 22.8.** *The following hold:*

(1) $(\varphi_J \otimes \varphi_J)\left(e_{j,i} \otimes G^u_{i,j}(v)\right) = e_{i,j} \otimes \tilde{w}_{i,j}(u,v)$ *for any* $1 \le i \neq j \le n$.
(2) $(\varphi_J \otimes \varphi_J)\left(\check{h}_l \otimes G^u_l(v)\right) = \check{h}_l \otimes \tilde{w}_l(u,v)$ *for any* $1 \le l \le n-1$.

*In particular* $(\varphi_J \otimes \varphi_J)\, c_{(n,d)} = s_{(n,n-d)}$.

*Proof.* 1) Using the definition of $\varphi_J$, the statement reads

$$\varphi_J\left(G^u_{i,j}(v)\right) = (-1)^{z_{j,i}}\tilde{w}_{i,j}(u,v).$$

In order to check this claim, let us recall the defining properties (16.1) of $G^u_{i,j}(z)$. Since $\tilde{w}_{i,j}(u,u) = 0$ by Definition 21.6, it follows that the claim is correct if and only if $(-1)^{z_{j,i}}\left(e_{j,i} + \tilde{w}_{i,j}(u,z)\right) \in \varphi_J\left(\mathrm{Sol}^u_{n,d}\right)$ i.e.

$$e_{j,i} + \tilde{w}_{i,j}(u,z) \in \varphi_J\left(\mathrm{Sol}^u_{n,d}\right). \tag{22.1}$$

In order to prove this, we need to go into the definition of $\mathrm{Sol}^u_{n,d}$. Recalling the nomenclature introduced in Algorithm 21.1, we may write

$$\varphi_J\left(V_{n,d}\right) = \left\{F_J(z) = \left(\begin{array}{c|c} W & Y \\ \hline X & Z \end{array}\right) + \left(\begin{array}{c|c} W' & Y' \\ \hline 0 & Z \end{array}\right) z + \left(\begin{array}{c|c} 0 & Y'' \\ \hline 0 & 0 \end{array}\right) z^2\right\}.$$

For $F_J(z) \in \varphi_J\left(V_{n,d}\right)$, we set

$$P_0\left(F_J\right) = \left(\begin{array}{c|c} W' & Y'' \\ \hline X & Z' \end{array}\right) \text{ and } P_\epsilon\left(F_J\right) = \left(\begin{array}{c|c} W & Y' \\ \hline 0 & Z \end{array}\right).$$

Lemma 22.7 states that $\varphi_J$ is a morphism of Lie algebras such that $\varphi_J(J) = J^t$. Together this implies

$$\varphi_J\left(\mathrm{Sol}^u_{n,d}\right) = \left\{F_J \in \varphi_J\left(V_{n,d}\right) \;\middle|\; \left[P_0\left(F_J\right), J^t\right] + uP_0\left(F_J\right) + P_\epsilon\left(F_J\right) = 0\right\}.$$

On the other hand, it follows from Definition 21.6 and Lemma 21.9 (1) that, setting $A(u) = A_0 + uA_1$, $B(u) = B_0 + uB_1$, $B'(u) = B'_0 + uB'_1$, $D(u) = D_0 + uD_1$, we have

$$\tilde{w}_{i,j}(u,z) = \left(\begin{array}{c|c} -uA(u) & -uB(u) \\ \hline 0 & -uD(u) \end{array}\right) + \left(\begin{array}{c|c} A(u) & B(u) - uB'(u) \\ \hline 0 & D(u) \end{array}\right) z + \left(\begin{array}{c|c} 0 & B'(u) \\ \hline 0 & 0 \end{array}\right) z^2.$$

Thus $\tilde{w}_{i,j}(u,z) \in \varphi_J\left(V_{n,d}\right)$ and hence also $\tilde{w}_{i,j}(u,z) + e_{j,i} \in \varphi_J\left(V_{n,d}\right)$. We have

$$P_0\left(\tilde{w}_{i,j}(u,z) + e_{j,i}\right) = \left(\begin{array}{c|c} A(u) & B'(u) \\ \hline 0 & D(u) \end{array}\right) + \delta_3(e_{j,i})$$

and

$$P_\epsilon\left(\tilde{w}_{i,j}(u,z) + e_{j,i}\right) = \left(\begin{array}{c|c} -uA(u) & B(u) - uB'(u) \\ \hline 0 & -uD(u) \end{array}\right) + (\delta_2 + \delta_4)(e_{j,i}),$$

where, by abuse of notation, we let $\delta_k(e_{i,j}) = \delta_k(i,j) \cdot e_{i,j}$ for any $1 \leq k \leq 4$ and $1 \leq i,j \leq n$.

Summarizing, (22.1) and therefore statement (1) is satisfied if and only if

$$\left[\left(\begin{array}{c|c} A(u) & B'(u) \\ \hline 0 & D(u) \end{array}\right) + \delta_3(e_{j,i}), J^t\right] + \left(\begin{array}{c|c} 0 & B(u) \\ \hline 0 & 0 \end{array}\right) + (\delta_2 + \delta_4)(e_{j,i}) + u\delta_3(e_{j,i}) = 0. \tag{22.2}$$

Since $\delta_3(e_{j,i}) = \delta_1(i,j) \cdot e_{j,i}$, Lemma 21.9 (4) implies that the left hand-side of (22.2) equals zero in the case $\delta_3(i,j) = 1$. So (22.2) is immediate for $\delta_3(i,j) = 1$. As to the other cases, further arguments are necessary. From the $u$-grading of (22.2) we deduce that (22.2) holds true already if

$$\left(\begin{array}{c|c} 0 & B_0 \\ \hline 0 & 0 \end{array}\right) + (\delta_2 + \delta_4)(i,j) \cdot e_{j,i} = \left[ J^t, \left(\begin{array}{c|c} A_0 & B_0' \\ \hline 0 & D_0 \end{array}\right) + \delta_1(i,j) \cdot e_{j,i} \right] \tag{22.3}$$

and

$$\left(\begin{array}{c|c} 0 & B_1 \\ \hline 0 & 0 \end{array}\right) + \delta_1(i,j) \cdot e_{j,i} = \left[ J^t, \left(\begin{array}{c|c} A_1 & B_1' \\ \hline 0 & D_1 \end{array}\right) \right]. \tag{22.4}$$

Now for $(\delta_2 + \delta_4)(i,j) = 1$, (22.3) is immediate by Lemma 21.9 (2). Moreover all terms occurring in (22.4) are zero in that case by Lemma 21.9 (4) and Definition 21.6, because they correspond to $w_{(i,j,1)}(v) - e_{j,i}v^{-2} = 0$. Finally, for $\delta_1(i,j) = 1$, both (22.3) and (22.4) are direct consequences of Lemma 21.9 (3) for $k = 0$ and $k = 1$ respectively.

2) This case is treated completely parallel to the case $(\delta_2 + \delta_4)(i,j) = 1$ in 1).

3) As to the remaining statement that $(\varphi_J \otimes \varphi_J)\, c_{(n,d)} = s_{(n,n-d)}$, note that the Casimir element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is invariant under base change. Thus, in view of the formulas given for $c_{(n,d)}$ and $s_{(n,n-d)}$ in Propositions 16.3 and 21.7 respectively, the statement is a direct consequence of (1) and (2). □

22.3. **Structure results.** Let us start with some further notations.

**Definition 22.9.** Let $\psi : \mathfrak{g} \to \mathfrak{g}$ be the Lie algebra automorphism induced by $\psi(e_{i,j}) = (-1)^{z_{i,j}}\, e_{n+1-j,n+1-i}$.

In this subsection, we prove:

**Proposition 22.10.** *The following hold*

1. *We have* $(\psi \otimes \psi)\, s_{(n,n-d)} = s_{(n,d)}$, $(\psi \otimes \psi)\, c_{(n,d)} = c_{(n,n-d)}$.
2. *Let* $\mathrm{pr}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ *denote the projection to the traceless diagonal matrices* $\mathfrak{h} \subset \mathfrak{g}$. *Then*

$$\left(\mathrm{pr}_{\mathfrak{h}} \otimes \mathrm{pr}_{\mathfrak{h}}\right) \left( s_{(n,n-d)}(u,v) - \frac{\Omega}{v-u} \right) = 0 = \left(\mathrm{pr}_{\mathfrak{h}} \otimes \mathrm{pr}_{\mathfrak{h}}\right) \left( c_{(n,d)}(u,v) - \frac{\Omega}{v-u} \right).$$

*Remark* 22.11. In the introduction, the solutions $c_{(n,d)}$ and $s_{(n,n-d)}$ were denoted $r_{(E,n,d)}$ (for $E$ the cuspidal cubic curve) and $r_{(\mathfrak{g},B_J,n-d)}$ respectively.

Let us start with the proof of Proposition 22.10 (1).

**Lemma 22.12.** *We have* $\psi\left(J\left(d,n-d\right)\right) = J\left(n-d,d\right)$.

*Proof.* Let $a, b$ be two coprime integers and for any $k \in \mathbb{N}$ let $\psi_k \in \mathrm{Aut}(\mathfrak{sl}_k(\mathbb{C}))$ be the map induced by $\psi(e_{i,j}) = (-1)^{z_{i,j}} e_{k+1-j,k+1-i}$. We prove that $\psi(J(a,b)) = J(b,a)$. The proof runs by induction along the order of construction of $J(a,b) = \mathcal{J}_0(a,b)$, see Algorithm 13.2.

0) Start with $(a,b) = (1,1)$. Then $\psi_2(J(1,1)) = \psi_2(e_{1,2}) = e_{1,2}$.

1) Assume $a < b$. Then

$$J(a,b) = \left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & \multicolumn{2}{c}{J(a,b-a)} \end{array}\right) \text{ and } J(b,a) = \left(\begin{array}{c||c} J(b-a,a) & \begin{matrix} 0 \\ \mathbb{1} \end{matrix} \\ \hline\hline 0 & 0 \end{array}\right).$$

Note that we are dealing with two different block decompositions of $\mathrm{Mat}_{(a+b)\times(a+b)}(\mathbb{C})$ here. But

$$\psi_{a+b}(J(a,b)) = \left(\begin{array}{c||c} \psi_b(J(a,b-a)) & \begin{matrix} 0 \\ \mathbb{1} \end{matrix} \\ \hline\hline 0 & 0 \end{array}\right) = J(b,a),$$

because the induction hypothesis guarantees $\psi_b(J(a,b-a)) = J(a,b-a)$ and

$$\psi\left(\sum_{i=1}^{a} e_{i,a+i}\right) = \sum_{k=1}^{a} e_{b-a+k,b+k}.$$

Here we use that $z_{i,j} = 0$ whenever $J_{i,j} \neq 0$. This proves the step of the induction for $a < b$.

2) Clearly, the case $a > b$ is treated similarly. $\square$

**Proposition 22.13.** *The following hold:*

(1) $(\psi \otimes \psi)\, s_{(n,n-d)} = s_{(n,d)}$.
(2) $(\psi \otimes \psi)\, c_{(n,d)} = c_{(n,n-d)}$.

*Proof.* 1) Let $W_{(n,n-d)} \subset \mathfrak{g}((u^{-1}))$ be the Lie subalgebra given by (21.2), i.e.

$$W_{(n,n-d)} = \eta_{n-d}^{-1}(u)\left(u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \mathrm{span}_{\mathbb{C}}\left(\left\{l + u^{-1}\chi_{(n-d,d)}(l)\right\}_{l\in\mathfrak{g}}\right)\right)\eta_{n-d}(u).$$

Here $\eta_{n-d}(u) = \mathrm{diag}(1, ..., 1, u, .., u)$ with $n-d$ many 1's and $\chi_{(n-d,d)}(l) = [J(n-d,d)^t, l]$, see Fact 21.22. Analogously, let

$$W_{(n,d)} = \eta_d^{-1}(u)\left(u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \mathrm{span}_{\mathbb{C}}\left(\left\{l + u^{-1}\chi_{(d,n-d)}(l)\right\}_{l\in\mathfrak{g}}\right)\right)\eta_d(u).$$

Then $W_{(n,n-d)}$ corresponds to $s_{(n,n-d)}$ while $W_{(n,d)}$ corresponds to $s_{(n,d)}$, cf. Subsection 20. From Theorem 20.4 we deduce that it suffices to show $\psi\left(W_{(n,n-d)}\right) = W_{(n,d)}$. It is easily verified that $\psi\left(\eta_{n-d}^{-1}(u)\cdot a\cdot \eta_{n-d}(u)\right) = \eta_d^{-1}(u)\cdot\psi(a)\cdot\eta_d(u)$ for any $a\in\mathfrak{g}$ and hence for any $a\in\mathfrak{g}\left((u^{-1})\right)$. Moreover $\psi(\mathfrak{g}) = \mathfrak{g}$ by definition of $\psi$. Thus, it suffices to check $\psi\left(l + u^{-1}\chi_{(n-d,d)}(l)\right) = \psi(l) + u^{-1}\chi_{(d,n-d)}(\psi(l))$ for any $l\in\mathfrak{g}$. But $\psi\circ\chi_{(n-d,d)}(l) = \psi\left([J(n-d,d)^t, l]\right) = \left[\psi\left(J(n-d,d)^t\right), \psi(l)\right] = [J(d,n-d)^t, \psi(l)] = \chi_{(d,n-d)}\left(\psi(l)\right)$. For this last computation we make use of Fact 21.22, the fact that $\psi$ commutes with transposition and Lemma 22.12.

2) Combining (1) and Theorem 22.8 immediately yields the claim. ☐

Let us go on with the following Lemma, which is the crucial step in the proof of Proposition 22.10 (2).

**Lemma 22.14.** *For any $1\le l\le n-1$, there exists a strictly upper triangular element $x_l\in\mathfrak{P}_{n-d}\subset\mathfrak{g}$ such that $\chi(x_l) = h_l + \left(\begin{array}{c|c} 0 & Y_l \\ \hline 0 & 0\end{array}\right)$* for some $Y_l\in\mathrm{Mat}_{(n-d)\times d}(\mathbb{C})$.

*Proof.* Let $a,b$ be two coprime integers. We will prove that for any $1\le l\le n-1$ there exists $x_l\in\mathfrak{sl}_{a+b}(\mathbb{C})$ strictly upper triangular and $X_l\in\mathrm{Mat}_{a\times b}(\mathbb{C})$ such that

$$[x_l, J(a,b)^t] = h_l + \left(\begin{array}{c|c} 0 & Y_l \\ \hline 0 & 0\end{array}\right). \tag{22.5}$$

The statement follows then by Fact 21.22 (after replacing $x_l$ by $-x_l$) and the easy fact that any upper triangular matrix is contained in $\mathfrak{P}_k$ for any $k$. The proof runs by induction along the order of construction of $J(a,b) = \mathcal{J}_0(a,b)$, see Algorithm 13.2.

0) Start with $(a,b) = (1,1)$. Then $[e_{1,2}, J^t] = [e_{1,2}, e_{2,1}] = h_1$. So $x_1 = e_{2,1}$ and $Y_1 = 0$.

1) Assume $a<b$. Then

$$J(a,b) = \left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & \multicolumn{2}{c}{J(a,b-a)}\end{array}\right).$$

By the induction hypothesis there exists, for any $a+1\le l\le a+b-1$, some $x_l'\in\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathfrak{sl}_b(\mathbb{C})\end{array}\right)\subset\mathfrak{sl}_{a+b}(\mathbb{C})$ strictly upper triangular and $Y_l'\in\mathrm{Mat}_{a\times(b-a)}(\mathbb{C})$ such that

$$\left[x_l', \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)\end{array}\right)\right] = h_l + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & Y_l' \\ 0 & 0 & 0\end{array}\right).$$

The proof of the step of the induction goes as follows. First, we will adjust $x'_l$ for $a+1 \le l \le a+b-1$ in an appropriate way to obtain $x_l$ and $B_l$ for the corresponding indices. Then we will construct $x_l$ and $Y_l$ for $1 \le l \le a$.
So let us assume $a+1 \le l \le a+b-1$. Clearly we may write $x'_l = \sum_{i\in I} \alpha_i \, e_{a+p_i,a+q_i}$ for some index set $I$, coefficients $\alpha_i \in \mathbb{C}$ and tuples of integers $(p_i, q_i) \in \{1, ..., b\}^2$. Let $I^a = \{i \in I \,|1 \le p_i, q_i \le a\}$. Then

$$\left[\sum_{i\in I} \alpha_i \, e_{a+p_i,a+q_i}, J(a,b)^t\right] =$$

$$= \left[\sum_{i\in I} \alpha_i \, e_{a+p_i,a+q_i}, \sum_{j=1}^{a} e_{a+j,j}\right] + \left[\sum_{i\in I} \alpha_i \, e_{a+p_i,a+q_i}, \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right)\right] =$$

$$= \left(\sum_{i\in I^a} \alpha_i \, e_{a+p_i,q_i}\right) + \left(h_l + \sum_{r\in R} \beta_r \, e_{a+s_r,2a+t_r}\right),$$

where the sum in the second bracket corresponds to $Y'_l$ by definition. We claim that we may choose $x_l = x'_l + \sum_{i\in I^a} \alpha_i \, e_{p_i,q_i} + \sum_{r\in R} \beta_r \, e_{s_r,2a+t_r}$. Since $x_l$ is strictly upper triangular, we need only check existence of the corresponding $Y_l \in \mathrm{Mat}_{a\times b}(\mathbb{C})$ such that (22.5) is satisfied. This is done by the following two simple computations:

$$\left[\sum_{i\in I^a} \alpha_i \, e_{p_i,q_i}, J(a,b)^t\right] =$$

$$= \left[\sum_{i\in I^a} \alpha_i \, e_{p_i,q_i}, \sum_{j=1}^{a} e_{a+j,j}\right] + \underbrace{\left[\sum_{i\in I^a} \alpha_i \, e_{p_i,q_i}, \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right)\right]}_{=0} =$$

$$= -\sum_{i\in I^a} \alpha_i \, e_{a+p_i,q_i}$$

and secondly

$$\left[\sum_{r\in R} \beta_r \, e_{s_r,2a+t_r}, J(a,b)^t\right] =$$

$$= \left[\sum_{r\in R} \beta_r \, e_{s_r,2a+t_r}, \sum_{j=1}^{a} e_{a+j,j}\right] + \left[\sum_{r\in R} \beta_r \, e_{s_r,a+t_r}, \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right)\right] =$$

$$= -\sum_{r\in R} \beta_r \, e_{a+s_r,2a+t_r} + \sum_{r\in R} \beta_r \, e_{s_r,a+t_r} \cdot \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right).$$

Since $s_r \leq a$, the last product is an element of $\left(\begin{array}{c|c} 0 & \mathrm{Mat}_{a\times b}(\mathbb{C}) \\ \hline 0 & 0 \end{array}\right)$. This yields $Y_l$.
It remains to construct $x_l$ and $Y_l$ for $1 \leq l \leq a$. Start with setting $x_l' = e_{l,a+l}$. Then

$$\left[x_l', J(a,b)^t\right] =$$

$$= \left[e_{l,a+l}, \sum_{j=1}^{a} e_{a+j,j}\right] + \left[e_{l,a+l}, \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & J(a,b-a)^t \end{array}\right)\right] = e_{l,l} - e_{a+l,a+l} + \left(\begin{array}{c|c} 0 & Y_l' \\ \hline 0 & 0 \end{array}\right)$$

for some $Y_l' \in \mathrm{Mat}_{a\times b}(\mathbb{C})$. Note that $e_{l,l} - e_{a+l,a+l} = \sum_{k=0}^{a-1} h_{l+k}$, hence existence of $x_l$ and $Y_l$ follows for $1 \leq l \leq a$ by reverse induction as follows. Setting $x_a = x_a' - \sum_{k=1}^{a-1} x_{a+k}$ we deduce from the computation above that

$$\left[x_a, J(a,b)^t\right] = h_a + \left[\left(\begin{array}{c|c} 0 & Y_l' \\ \hline 0 & 0 \end{array}\right) - \sum_{k=1}^{a-1} \left(\begin{array}{c|c} 0 & Y_{a+k} \\ \hline 0 & 0 \end{array}\right)\right].$$

Hence the square brackets define $Y_a$ in a way such that (22.5) holds. Next, assume that we have already proved existence of $x_i, Y_i$ for $l < i \leq a$ where $1 \leq l$. Then we set $x_l = x_l' - \sum_{k=1}^{a-1} x_{l+k}$ and proceed as for $l = a$ in order to obtain $Y_l$. Thus, having constructed all $x_l, Y_l$ for $1 \leq l \leq a+b-1$ such that (22.5) is satisfied, this finishes the proof in the case $a < b$.

2) Assume $a > b$. Then

$$J(a,b) = \left(\begin{array}{c|c} J(a-b,b) & \begin{matrix} 0 \\ \mathbb{1} \end{matrix} \\ \hline 0 & 0 \end{array}\right).$$

By the induction hypothesis we know that there exists, for any $1 \leq l \leq a-1$, some $x_l' \in \left(\begin{array}{c|c} \mathfrak{sl}_a(\mathbb{C}) & 0 \\ \hline 0 & 0 \end{array}\right) \subset \mathfrak{sl}_{a+b}(\mathbb{C})$ strictly upper triangular and $Y_l' \in \mathrm{Mat}_{(a-b)\times b}(\mathbb{C})$ such that

$$\left[x_l', \left(\begin{array}{c|c} J(a-b,b) & 0 \\ \hline 0 & 0 \end{array}\right)\right] = h_l + \left(\begin{array}{cc|c} 0 & Y_l' & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

The proof of the step of the induction is similar to that in 1). First, we will adjust $x_l'$ for $1 \leq l \leq a-1$ in an appropriate way to obtain $x_l$ and $B_l$ for the corresponding indices. Then we will construct $x_l$ and $Y_l$ for $a \leq l \leq a+b-1$.
So let us assume $1 \leq l \leq a-1$. Clearly we may write $x_l' = \sum_{i\in I} \alpha_i e_{p_i,q_i}$ for some index set $I$, coefficients $\alpha_i \in \mathbb{C}$ and tuples of integers $(p_i, q_i) \in \{1, ..., a\}^2$. Let $I_{a-b} = \{i \in I \,|a-b < p_i, q_i \leq a\}$ and $p_i' = p_i - (a-b)$ as well as $q_i' = q_i - (a-b)$ for all $i \in I_{a-b}$.

Then

$$\left[\sum_{i\in I}\alpha_i\, e_{p_i,q_i},\, J(a,b)^t\right] =$$

$$= \left[\sum_{i\in I}\alpha_i\, e_{p_i,q_i},\, \sum_{j=1}^{b} e_{a+j,a-b+j}\right] + \left[\sum_{i\in I}\alpha_i\, e_{p_i,q_i},\, \left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right)\right] =$$

$$= \left(-\sum_{i\in I_{a-b}}\alpha_i\, e_{a+p'_i,q_i}\right) + \left(h_l + \sum_{r\in R}\beta_r\, e_{s_r,a-b+t_r}\right),$$

where the sum in the second bracket corresponds to $Y_l'$ by definition. We claim that we may choose $x_l = x_l' + \sum_{i\in I_{a-b}}\alpha_i\, e_{a+p'_i a+,q'_i} - \sum_{r\in R}\beta_r\, e_{s_r,a+t_r}$. Since $x_l$ is strictly upper triangular, we need only check existence of the corresponding $Y_l \in \mathrm{Mat}_{a\times b}(\mathbb{C})$ such that (22.5) is satisfied. This is done by the following two simple computations:

$$\left[\sum_{i\in I_{a-b}}\alpha_i\, e_{a+p'_i a+,q'_i},\, J(a,b)^t\right] =$$

$$= \left[\sum_{i\in I_{a-b}}\alpha_i\, e_{a+p'_i,a+q'_i},\, \sum_{j=1}^{b} e_{a+j,a-b+j}\right] + \underbrace{\left[\sum_{i\in I_{a-b}}\alpha_i\, e_{a+p'_i,a+q'_i},\, \left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right)\right]}_{=0} =$$

$$= \sum_{i\in I_{a+b}}\alpha_i\, e_{a+p'_i,q_i}$$

and secondly

$$\left[-\sum_{r\in R}\beta_r\, e_{s_r,a+t_r},\, J(a,b)^t\right] =$$

$$= -\left[\sum_{r\in R}\beta_r\, e_{s_r,a+t_r},\, \sum_{j=1}^{b} e_{a+j,a-b+j}\right] - \left[\sum_{r\in R}\beta_r\, e_{s_r,a+t_r},\, \left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right)\right] =$$

$$= -\sum_{r\in R}\beta_r\, e_{a+s_r,a-b+t_r} + \left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right)\cdot\sum_{r\in R}\beta_r\, e_{s_r,a+t_r}.$$

Since $a+t_r > a$, the last product is an element of $\left(\begin{array}{c|c} 0 & \mathrm{Mat}_{a\times b}(\mathbb{C}) \\ \hline 0 & 0 \end{array}\right)$. This yields $Y_l$.

It remains to construct $x_{a+l}$ and $Y_{a+l}$ for $0 \le l \le b-1$. Start with setting $x'_{a+l} = e_{a+b+l+1,a+l+1}$. Then

$$\left[x'_{a+l},\, J(a,b)^t\right] =$$

$$= \left[e_{a-b+l+1,a+l+1}, \sum_{j=1}^{b} e_{a+j,a-b+j}\right] + \left[e_{a-b+l+1,a+l+1}, \left(\begin{array}{c|c} J(a-b,b)^t & 0 \\ \hline 0 & 0 \end{array}\right)\right]$$

$$= e_{a-b+l+1,a-b+l+1} - e_{a+l+1,a+l+1} + \left(\begin{array}{c|c} 0 & Y'_{a+l} \\ \hline 0 & 0 \end{array}\right)$$

for some $Y'_l \in \mathrm{Mat}_{a\times b}(\mathbb{C})$. Note that $e_{a-b+l+1,a-b+l+1} - e_{a+l+1,a+l+1} = \sum_{k=1}^{b} h_{a-b+l+k}$, hence existence of $x_{a+l}$ and $Y_{a+l}$ follows for $0 \leq l \leq b-1$ by induction as follows. Setting $x_a = x'_a - \sum_{k=1}^{b-1} x_{a-b+k}$ we deduce from the computation above that

$$\left[x_{a+l}, J(a,b)^t\right] = h_a + \left[\left(\begin{array}{c|c} 0 & Y'_l \\ \hline 0 & 0 \end{array}\right) - \sum_{k=1}^{b-1} \left(\begin{array}{c|c} 0 & Y_{a-b+k} \\ \hline 0 & 0 \end{array}\right)\right].$$

Hence the square brackets define $Y_a$ in a way such that (22.5) holds. Next, assume that we have already proved existence of $x_{a+i}, Y_{a+i}$ for $0 \leq i < l$ where $l \leq b-1$. Then we set $x_{a+l} = x'_{a+l} - \sum_{k=1}^{b-1} x_{a-b+l+k}$ and proceed as for $l = 0$ in order to obtain $Y_{a+l}$. Thus, having constructed all $x_l, Y_l$ for $1 \leq l \leq a+b-1$ such that (22.5) is satisfied, this finishes the proof in the case $a > b$. □

**Proposition 22.15.** *Let* $\mathrm{pr}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ *denote the projection to the traceless diagonal matrices* $\mathfrak{h} \subset \mathfrak{g}$*. Then*

$$\left(\mathrm{pr}_{\mathfrak{h}} \otimes \mathrm{pr}_{\mathfrak{h}}\right)\left(s_{(n,n-d)}(u,v) - \frac{\Omega}{v-u}\right) = 0 = \left(\mathrm{pr}_{\mathfrak{h}} \otimes \mathrm{pr}_{\mathfrak{h}}\right)\left(c_{(n,d)}(u,v) - \frac{\Omega}{v-u}\right).$$

*Proof.* Since $\left(\varphi_J \otimes \varphi_J\right)\left(c_{(n,d)}(u,v)\right) = s_{(n,n-d)}(u,v)$ by Theorem 22.8 and $\varphi_J$ stabilizes $\mathfrak{h}$, cf. Lemma 22.7, it clearly suffices to prove that $\left(\mathrm{pr}_{\mathfrak{h}} \otimes \mathrm{pr}_{\mathfrak{h}}\right)\left(s_{(n,n-d)}(u,v) - \frac{\Omega}{v-u}\right) = 0$. By Proposition 21.7, this is equivalent to showing $\mathrm{pr}_{\mathfrak{h}}\left(\tilde{w}_l(u,v)\right) = 0$ for each $1 \leq l \leq n-1$. As stated in Lemma 21.9 (4) respectively (2), $\tilde{w}_l(u,v) = (v-u)\left(w_{(l,0)}(v) - v^{-1}h_l\right)$ and $w_{(l,0)}(v) = \eta^{-1}(v)\left(x + v^{-1}\chi(x)\right)\eta(v)$ where

$$x = \left(\begin{array}{c|c} A_0 & B'_0 \\ \hline 0 & D_0 \end{array}\right) \text{ and } \chi(x) = \left(\begin{array}{c|c} 0 & B_0 \\ \hline 0 & 0 \end{array}\right) + h_l.$$

Now $x$ is uniquely determined by Lemma 21.9 (1), hence $x = x_l, B_0 = Y_l$ by Lemma 22.14. Since $x_l$ is strictly upper triangular, the statement is proved. □

## 23. Explicit computation of $s_{(n,1)}$

In what follows, we are going to work out the essential steps of the algorithm described in Subsection 21.1 for the tuple $(n,d) = (n,n-1)$. Thus, we will make explicit the formula for $s_{(n,1)}$ given in Corollary 21.11.

It is a easy to see that $J = \mathcal{J}_0(1, n-1) = \sum_{k=1}^{n-1} e_{k,k+1}$, see the inductive construction in Algorithm 13.2. Next, we determine

$$W = \eta^{-1}(u)\left(u^{-2}\mathfrak{g}\left[\left[u^{-1}\right]\right] \oplus \operatorname{span}_{\mathbb{C}}\left(\left\{l + u^{-1}\left[J^t, l\right]\right\}_{l\in\mathfrak{g}}\right)\right)\eta(u)$$

where $\eta(u) = \operatorname{diag}(1, u, ..., u) \in \mathrm{GL}_n(\mathbb{C}((u^{-1})))$. The first step is to do so is to determine $[J^t, l]$ on a basis of $\mathfrak{g}$:

**Lemma 23.1.** *For any $1 \leq i \neq j \leq n$ and*

$$\left[J^t, e_{i,j}\right] = \begin{cases} e_{i+1,j} - e_{i,j-1} & i \leq n-1,\, 2 \leq j \\ e_{i+1,j} & i \leq n-1,\, j = 1 \\ -e_{i,,j-1} & i = n,\, 2 \leq j \\ 0 & i = n,\, j = 1. \end{cases}$$

*Moreover, for any $1 \leq l \leq n-1$ we have*

$$\left[J^t, h_l\right] = \begin{cases} 2e_{l+1,l} - (e_{l,l-1} + e_{l+2,l+1}) & 2 \leq l \leq n-2 \\ -e_{l,l-1} + 2e_{l+1,l} & l = n-1 \\ -e_{l+2,l} + 2e_{l+1,l} & l = 1. \end{cases}$$

*Proof.* For $i \leq n-1, j \geq 2$ we have $\left[\sum_{k=1}^{n-1} e_{k+1,k}, e_{i,j}\right] = e_{i+1,i} \cdot e_{i,j} - e_{i,j} \cdot e_{j,j-1}$. The remaining cases for $e_{i,j}$ follow immediately. Let $\alpha_{i,j}$, $1 \leq i \neq j \leq n$, denote the roots corresponding to the roots spaces $\operatorname{span}_{\mathbb{C}}(\{e_{i,j}\})$ of the standard Cartan $\mathfrak{h} \subset \mathfrak{g}$. Note that $\alpha_{i,j} = -\alpha_{j,i}$ and that $\alpha_{k,k+1}(h_l) = 0$ for $k < l-1$ as well as $k > l+1$. Hence for $2 \leq l \leq n-2$ we deduce $\left[\sum_{k=1}^{n-1} e_{k+1,k}, h_l\right] = \sum_{k=1}^{n-1} \alpha_{k,k+1}(h_l)\, e_{k+1,k} = \sum_{k=l-1}^{l+1} \alpha_{k,k+1}(h_l)\, e_{k+1,k}$. Since $\alpha_{l-1,l}(h_l) = \alpha_{l+1,l+2}(h_l) = -1$ and $\alpha_{l,l+1}(h_l) = 2$ the rest is obvious. □

**Corollary 23.2.** *For any $1 \leq i \neq j \leq n$, let $e_{n+1,j} = 0 = e_{i,0}$. Then*

(1) *for $1 \leq i \neq j \leq n$ we have*

$$\eta^{-1}(u)\left(e_{i,j} + u^{-1}\left[J^t, e_{i,j}\right]\right)\eta(u) = \begin{cases} ue_{1,2} - u^{-1}h_1 & (i,j) = (1,2) \\ ue_{1,j} + u^{-1}e_{2,j} - e_{1,j-1} & i = 1,\, 2 \leq j \\ u^{-1}e_{i,1} + u^{-2}e_{i+1,1} & 2 \leq i,\, j = 1 \\ e_{i,2} + u^{-1}e_{i+1,2} - u^{-2}e_{i,1} & 2 \leq i,\, j = 2 \\ e_{i,j} + u^{-1}(e_{i+1,j} - e_{i,j-1}) & 2 \leq i,\, 3 \leq j. \end{cases}$$

(2) *for any $1 \leq l \leq n-1$we have*

$$\eta^{-1}(u)\left(h_l + u^{-1}\left[J^t, h_l\right]\right)\eta(u) = \begin{cases} h_1 + 2u^{-2}e_{2,1} - u^{-1}e_{3,2} & l = 1 \\ h_2 + 2u^{-1}e_{3,2} - u^{-2}e_{2,1} - u^{-1}e_{4,3} & l = 2 \\ h_l + u^{-1}\left(2e_{l+1,l} - e_{l,l-1} - e_{l+2,l+1}\right) & 3 \leq l. \end{cases}$$

According to Corollary 21.11, it remains to determine $w_{(l,0)}(u)$ for all $1 \le l \le n-1$, as well as $w_{(i,j,0)}(u)$ for $\delta_1(i,j) \neq 1$ and $w_{(i,j,1)}(u)$ for $\delta_3(i,j) = 1$. This is done in the next three Lemmata. We shall also need the following fact:

**Fact 23.3.** For all $1 \le l \le n-1$,

$$\check{h}_l = \left(1 - \frac{l}{n}\right) \sum_{j=1}^{n-1} l \cdot h_l - \sum_{j=l+1}^{n-1} (j-l) h_j = \sum_{j=1}^{l} j \frac{n-l}{n} h_j + \sum_{j=l+1}^{n-1} l \frac{n-j}{n} h_j.$$

**Lemma 23.4.** *Let* $1 \le i \neq j \le n$ *such that* $\delta_3(i,j) = 1$. *Then*

$$w_{(i,j,1)}(u) = \begin{cases} u^{-2} e_{2,1} + \check{h}_1 & (i,j) = (1,2) \\ u^{-2} e_{j,1} - \sum_{k=0}^{n-j} e_{j+k,k+2} & i = 1,\ 2 \le j. \end{cases}$$

*Proof.* First, let us prove that both expressions on the right-hand side are elements of $W$. To that end, we claim that

$$u^{-2} e_{2,1} + \check{h}_1 = \frac{n-1}{n} \left(h_1 + 2u^{-2} e_{2,1} - u^{-1} e_{3,2}\right) + \frac{n-2}{n} \cdot \big(h_2 + 2u^{-1} e_{3,2} -$$
$$-u^{-2} e_{2,1} - u^{-1} e_{4,3}\big) + \sum_{l=3}^{n-1} \frac{n-l}{n} \left[h_l + u^{-1} \left(2 e_{l+1,l} - e_{l,l-1} + e_{l+2,l+1}\right)\right].$$

Indeed, for all $l \ge 2$ the coefficient of $e_{l+1,l}$ on the right-hand side equals

$$u^{-1} \left(-\frac{n-l-1}{n} + 2\frac{n-l}{n} - \frac{n-l+1}{n}\right) = 0.$$

Moreover it is easily checked to be equal to zero for $l = 1$ as well. Since $\check{h}_1 = \sum_{l=1}^{n} \frac{n-l}{n} h_l$ by Fact (23.3), this proves our claim. But then, all summands on the right-hand side of this equality are contained in $W$ by Corollary 23.2 2). Thus $u^{-2} e_{2,1} + \check{h}_1 \in W$.
Next, we claim that

$$u^{-2} e_{j,1} - \sum_{k=0}^{n-j} e_{j+k,k+2} = \left(-e_{j,2} - u^{-1} e_{j+1,2} + u^{-2} e_{j,1}\right) -$$
$$-\sum_{k=1}^{n-j} \left(e_{j+k,k+2} + u^{-1}(e_{j+k+1,k+2} - e_{j+k,k+1})\right).$$

Indeed, the coefficient of $e_{j+l,l+1}$ on the right-hand side is $-\left(-u^{-1}\right) - u^{-1} = 0$ for all $l \ge 2$ and also equals zero for $l = 1$. Form the last two cases of Corollary 23.2 1) we deduce that $u^{-2} e_{j,1} - \sum_{k=0}^{n-j} e_{j+k,k+1} \in W$.
It remains to check that the terms given on the right-hand side of the statement are the duals of $e_{i,j} u$ with respect to $(x,y) \mapsto \mathrm{Res}_u \mathrm{tr}(x \cdot y)$ on $\mathfrak{g}[[u]] \times W$. But since the projection of these terms to $u^{-2}\mathfrak{g}[[u^{-1}]]$ is exactly $u^{-2} e_{j,i}$, this is clear. ∎

**Lemma 23.5.** *Let $1 \leq i \neq j \leq n$ such that $\delta_1(i,j) \neq 1$. Then*

$$w_{(i,j,0)}(u) = \begin{cases} u^{-1}e_{j,1} + \sum_{k=1}^{n-j} e_{j+k,k+1} & i = 1,\ 2 \leq j \\ u^{-1}e_{i+1,i} + \check{h}_i & 3 \leq j = i+1 \leq n \\ u^{-1}e_{j,i} - \sum_{k=0}^{n-j} e_{j+k,i+k+1} & 4 \leq i+2 \leq j \\ u^{-1}e_{j,i} + ue_{1,i-j+2} - e_{1,i-j+1} + \sum_{k=1}^{j-2} e_{j-k,i-k+1} & 3 \leq j+1 \leq i. \end{cases}$$

*Proof.* The proof is similar to that of the previous Lemma: first we show that all expressions on the right-hand side are elements of $W$. Only then do we prove that they are the actual duals with respect to $(x,y) \mapsto \mathrm{Res}_u \mathrm{tr}\,(x \cdot y)$.
Let us start with the following claim:

$$e_{j,1}u^{-1} + \sum_{k=1}^{n-j} e_{j+k,k+1} = \left(u^{-1}e_{j,1} + u^{-2}e_{j+1,1}\right) + \left(e_{j+1,2} + u^{-1}e_{j+2,2} - u^{-2}e_{j+1,1}\right) +$$

$$+ \sum_{k=2}^{n-j} \left(e_{j+k,k+1} + u^{-1}\left(e_{j+k+1,k+1} - e_{j+k,k}\right)\right).$$

Indeed, the coefficient at $e_{j+l,l}$ equals $-u^{-1} + u^{-1} = 0$ for all $l \geq 2$ and is obviously zero for $l = 1$ as well. By the last three cases of Corollary 23.2 1), this shows $e_{j,1}u^{-1} + \sum_{k=1}^{n-j} e_{j+k,k+1} \in W$.

The second case is the most tedious. We claim that for $3 \leq i+1 \leq n$:

$$e_{i+1,i}u^{-1} + \check{h}_i = \sum_{j=1}^{i} j\frac{n-i}{n}z_j + \sum_{j=i+1}^{n-1} i\frac{n-j}{n}z_j$$

where $z_1 = h_1 + 2u^{-2}e_{2,1} - u^{-1}e_{3,2}$, $z_2 = h_2 + 2u^{-1}e_{3,2} - u^{-2}e_{2,1} - u^{-1}e_{4,3}$ and $z_j = h_j + u^{-1}\left(2e_{j+1,j} - e_{j,j-1} - e_{j+2,j+1}\right)$ for all $3 \leq j \leq n-1$. Once this equality is established, we know that $e_{i+1,i}u^{-1} + \check{h}_i \in W$ thanks to Corollary 23.2 2). Now, note that by Fact 23.3, the $h_i$ on the right-hand side add up to $\check{h}_i$. Thus, all that remains is to check that the coefficient at $e_{k,k-1}$ is zero if $k \neq i+1$ and $u^{-1}$ else. This is done via a case by case analysis:
Let $i+3 \leq k$. Then the coefficient at $e_{k,k-1}$ is

$$i\frac{n-(k-2)}{n}\left(-u^{-1}\right) + i\frac{n-(k-1)}{n}\left(2u^{-1}\right) + i\frac{n-k}{n}\left(-u^{-1}\right) = 0.$$

while for $i+2 = k$ it is

$$i\frac{n-i}{n}\left(-u^{-1}\right) + i\frac{n-(i+1)}{n}\left(2u^{-1}\right) + i\frac{n-(i+2)}{n}\left(-u^{-1}\right) = 0.$$

For $i+1=k$, the coefficient at $e_{i+1,i}$ is

$$(i-1)\frac{n-i}{n}\left(-u^{-1}\right)+i\frac{n-i}{n}\left(2u^{-1}\right)+i\frac{n-(i+1)}{n}\left(-u^{-1}\right)=u^{-1}.$$

As for the case $k\leq i$, we have to distinguish between the two sub cases $3\leq k$ (which was immediate for $i+1\leq k$) and $k<3$. First, assume that $3\leq k$, then the coefficient at $e_{k,k-1}$ equals

$$(k-2)\frac{n-i}{n}\left(-u^{-1}\right)+(k-1)\frac{n-i}{n}\left(2u^{-1}\right)+k\frac{n-i}{n}\left(-u^{-1}\right)=0$$

The last sub case, $k<3$, simply requires to check the coefficient at $e_{2,1}$, which is zero.

The third case of the statement follows a lot easier from the claim that

$$e_{j,i}u^{-1}-\sum_{k=0}^{n-j}e_{j+k,i+k+1}=-\sum_{k=0}^{n-j}\left(e_{j+k,i+k+1}+u^{-1}\left(e_{j+k+1,i+k+1}-e_{j+k,i+k}\right)\right).$$

To verify this claim, we only need to check that the coefficient at $e_{j+l,i+l}$ equals $-u^{-1}+u^{-1}=0$. Thus $e_{j,i}u^{-1}-\sum_{k=0}^{n-j}e_{j+k,i+k+1}\in W$ by the last case of Corollary 23.2 1).

As for the last case of the statement, we claim that

$$e_{j,i}u^{-1}+ue_{1,i-j+2}-e_{1,i-j+1}+\sum_{k=1}^{j-2}e_{j-k,i-k+1}=$$

$$=\left(ue_{1,i-j+2}+u^{-1}e_{2,i-j+2}-e_{1,i-j+1}\right)+\sum_{k=2}^{j-1}\left(e_{k,i-j+k+1}+u^{-1}\left(e_{k+1,i-j+k+1}-e_{k,i-j+k}\right)\right).$$

To see this, check that the coefficient at $e_{l,i-j+l}$ is $u^{-1}-u^{-1}=0$ for all $2\leq l\leq j-1$, while at $e_{j,i}$ it is just $u^{-1}$. Moreover, note that $\sum_{k=2}^{j-1}e_{k,i-j+k+1}=\sum_{l=1}^{j-2}e_{j-l,i-l}$, which proves the claim, showing $e_{j,i}u^{-1}+ue_{1,i-j+2}-e_{1,i-j+1}+\sum_{k=1}^{j-2}e_{j-k,i-k+1}\in W$ by combining the second and the last case of Corollary 23.2 1).

Finally, we have to check that the terms given on the right-hand side of the statement are the duals of $e_{i,j}$ with respect to $(x,y)\mapsto\mathrm{Res}_u\mathrm{tr}(x\cdot y)$ on $\mathfrak{g}\left[[u]\right]\times W$. But since the projection of these terms to $u^{-1}\mathfrak{g}\left[[u^{-1}]\right]$ is exactly $u^{-1}e_{j,i}$, this is obvious. □

**Lemma 23.6.** *For all* $1\leq l\leq n-1$ *we have*

$$w_{(l,0)}(u)=\begin{cases}-ue_{1,2}+u^{-1}h_1 & l=0\\ -e_{l,l+1}+u^{-1}h_l & 2\leq l.\end{cases}$$

*Proof.* Note that both terms on the right-hand side is are $W$ by the first and last case of Corollary 23.2 1). It remains to check that the terms given on the right-hand side of the statement are the duals of $\check{h}_l u$ with respect to $(x,y)\mapsto \mathrm{Res}_u \mathrm{tr}(x\cdot y)$ on $\mathfrak{g}\,[[u]]\times W$. But since the projection of these terms to $u^{-1}\mathfrak{g}\,[[u^{-1}]]$ is exactly $u^{-1}h_l$, this is clear. □

**Proposition 23.7.** *We have*

$$s_{(n,1)}(u,v)=\frac{\Omega}{v-u}+$$
$$+u\left[e_{12}\otimes\check{h}_1-\sum_{j=3}^{n}e_{1,j}\otimes\left(\sum_{k=1}^{n-j+1}e_{j+k-1,k+1}\right)\right]-$$
$$-v\left[\check{h}_1\otimes e_{12}-\sum_{j=3}^{n}\left(\sum_{k=1}^{n-j+1}e_{j+k-1,k+1}\right)\otimes e_{1,j}\right]+$$
$$+\sum_{j=2}^{n-1}\sum_{k=1}^{n-j}(e_{1,j}\otimes e_{j+k,k+1}-e_{j+k,k+1}\otimes e_{1,j})+\sum_{i=2}^{n-1}\left(e_{i,i+1}\otimes\check{h}_i-\check{h}_i\otimes e_{i,i+1}\right)+$$
$$+\sum_{i=2}^{n-2}\sum_{k=2}^{n-i}\sum_{l=1}^{n-i-k+1}\left(e_{i+k+l-1,l+i}\otimes e_{i,i+k}-e_{i,i+k}\otimes e_{i+k+l-1,l+i}\right).$$

*Proof.* Let us insert the results obtained in Lemmata 23.4, 23.5 and 23.6 into the formula for $s_{(n,n-1)}(u,v)$ given in Corollary 21.11. Firstly, we deduce from Lemma 23.4 that

$$u\sum_{(i,j)\in I_1}e_{i,j}\otimes\left(w_{(i,j,1)}(v)-v^{-2}e_{j,i}\right)=$$
$$=u\left(e_{1,2}\otimes\check{h}_1-\sum_{j=3}^{n}e_{1,j}\otimes\left(\sum_{l=1}^{n-j-1}e_{j+l-1,l+1}\right)\right).$$

Also, Lemma 23.6 yields

$$\sum_{l=1}^{n-1}\check{h}_l\otimes\left(w_{(l,0)}(v)-v^{-1}h_l\right)=-v\check{h}_1\otimes e_{12}-\sum_{l=2}^{n-1}\check{h}_l\otimes e_{l,l+1}.$$

Next, let

$$J=\left\{(i,j)\in\{1,...,n\}^2\,\middle|\,3\le j+1\le i\right\},$$
$$L=\left\{(i,j)\in\{1,...,n\}^2\,\middle|\,4\le i+2\le j\right\}.$$

Then we derive from Lemma 23.5 that

$$\sum_{(i,j)\in I_1\cup I_2\cup I_4}^{n}e_{i,j}\otimes\left(w_{(i,j,0)}(v)-v^{-1}e_{j,i}\right)=$$

$$= \sum_{j=2}^{n} e_{1,j} \otimes \left( \sum_{k=1}^{n-j} e_{j+k,k+1} \right) + \sum_{i=2}^{n-1} e_{i,i+1} \otimes \check{h}_i - \sum_{(i,j)\in L} e_{i,j} \otimes \left( \sum_{k=0}^{n-j} e_{j+k,i+k+1} \right) +$$
$$+ \sum_{(i,j)\in J} e_{i,j} \otimes \left( v e_{1,i-j+2} - e_{1,i-j+1} + \sum_{k=1}^{j-2} e_{j-k,i-k+1} \right).$$

Clearly, the sums over $L$ and $J$ require some shift of indices in order to fit into the formula of the statement. This is done in several steps:

It follows from the definition of $J$ that if we set $l = i - j + 2$ for $(i,j) \in J$, then $3 \le l \le n$. Hence, if we let $k = j - 1$, then

$$v \sum_{(i,j)\in J} e_{i,j} \otimes e_{1,i-j+2} = v \sum_{l=3}^{n} \left( \sum_{k=1}^{n-l+1} e_{l+k-1,k+1} \right) \otimes e_{1,l},$$

where we use that $k \le n - l + 1$ if and only if $i \le n$.

Next, for $(i,j) \in J$, let $l = i - j + 1$. Then $2 \le l \le n - 1$. Thus, for $k = j - 1$,

$$- \sum_{(i,j)\in J} e_{i,j} \otimes e_{1,i-j+1} = - \sum_{l=2}^{n-1} \left( \sum_{k=1}^{n-l} e_{l+k,k+1} \right) \otimes e_{1,j},$$

where we use that $k \le n - l$ if and only if $i \le n$.

As to the last sum term of the sum over $J$, let $a = j - k$ and $b = i - j + 1$. Then $2 \le a \le n - 2$ due to the bounds of $j$ and $k$, and $2 \le b \le n - a$ where we use that $b \le n - a$ if and only if $i \le n$. Hence

$$\sum_{(i,j)\in J} e_{i,j} \otimes \sum_{k=1}^{j-2} e_{j-k,i-k+1} = \sum_{a=2}^{n-2} \left( \sum_{b=2}^{n-a} \left( \sum_{k=1}^{n-a-b+1} e_{a+b+k-1,k+a} \right) \otimes e_{a,a+b} \right),$$

where we use that $k \le n - a - b + 1$ if and only if $i \le n$.

Finally, for $(i,j) \in L$, let $a = j - i$. Then $2 \le a \le n - i$. Thus, for $b = k + 1$, we deduce that

$$- \sum_{(i,j)\in L} e_{i,j} \otimes \left( \sum_{k=0}^{n-j} e_{j+k,i+k+1} \right) = - \sum_{i=2}^{n-2} \left( \sum_{a=2}^{n-i} e_{i,i+a} \otimes \left( \sum_{b=1}^{n-i-a+1} e_{i+a+b-1,b+i} \right) \right),$$

where we use that $b \le n - i - a + 1$ if and only if $k \le n - j$. □

## Part 7. PC Implementations

In this part we present the source code of the implementations in mathematica for the algorithms producing $r_{(n,d)}(v;y_1,y_2)$ and $c_{(n,d)}(y_1,y_2)$ respectively $s_{(n,n-d)}(u,v)$.

23.1. **The program for $r_{(n,d)}$ and $c_{(n,d)}$.**
The implementation of the algorithm for the construction of $r_{(n,d)}(v; y_1, y_2)$ is straightforward. The input parameters $n$ and $d$ are defined in the first two lines, which are $n = 2$ and $d = 1$ in the form of the source code printed below. Assuming that $r_{(n,d)}(v; y_1, y_2)$ has a Laurent expansion of the form (3.5), the program also computes the corresponding solution $c_{(n,d)}(y_1, y_2)$ of the CYBE. Moreover, it constructs and applies the gauge equivalence $\varphi_J \otimes \varphi_J$ to $c_{(n,d)}(y_1, y_2)$, yielding $s_{(n,n-d)}(y_1, y_2)$, see Theorem 22.1.

```
(*------------------------------------------------------------------*)
  (*-----------------------------------------------------------------*)
  n=2;
  d =1;

  Comm[x_,y_]:=x.y-y.x;

  (*-------------------- construct J(n-d,d) ---------------------*)

  J[a_,b_]:=(

  If [a==b, Return[{{0,1},{0,0}}]];
  If[a>b,(
  A= J[a-b,b];
  U= Table[0, {i,a+b}, {j,a+b}];
  Do[U[[i,j]]= A[[i,j]], {i,a}, {j,a}];
  Do[U[[a-b+k,a+k]]=1, {k,b}];
  Return[U];
  )];

  If[a< b,(
  A= J[a,b-a];
  U= Table[0, {i,a+b}, {j,a+b}];
  Do[U[[i+a,j+a]]= A[[i,j]], {i,b}, {j,b}];
  Do[U[[k,a+k]]=1, {k,a}];
  Return[U];
  )];);

  J = J[n-d,d];

  (*------------------ construct Sol^{v,y1}_{n,d} -----------------*)
  F[z0_,z1_]:= Table[ If[(i <= n-d && j <= n-d) || (i > n-d && j>n-d),
```

```
f[i,j]*z0 + f'[i,j]*z1 , If[i>n-d && j <= n -d ,
f[i,j]*z0^2+f'[i,j]*z0*z1+f''[i,j]*z1^2,f[i,j]]],{i,n},{j,n}];

F0 = F[eps,1] /. eps -> 0;
Feps = Coefficient [F[eps,1], eps];

R = Comm[F0,J]+ Feps+(y1-v)* F0;

(* solve the equations imposed by $Sol_{n,d}^{v,y_1}$ on entries
of a general element of $W_{n,d}$ *)
redlist= Complement[Flatten[ Union[ Feps,
Take[Transpose[Take[F0, n-d]], -d]]],{0}];
V =Transpose[Solve[R==0, redlist]] // FullSimplify;
Vlist =Union[Flatten[V]];

(* impose relations of $Sol_{n,d}^{v,y_1}$ on a general element of $W_{n,d}$ and
evaluate at $z_0 = 1$ *)
G[z_]=F[1,z] //. Vlist // FullSimplify;

(* compute $res_{y_1}^{-1}$ *)
bluelist=Complement[Flatten[Union[F[1,0], F[0,1],
Coefficient[F[z0,z1], z0*z1]]], Union[redlist, {0}]];

T = Solve[G[y1]== Table[ a[i,j], {i,n}, {j,n}], bluelist ];
Transpose[T] // MatrixForm // FullSimplify;

(* compute $\mathrm{ev}_{y_2} \circ \mathrm{res}_{y_1}^{-1}$ *)
PreAMatrix = (G[y2]/(y2-y1) //. Vlist
//. Flatten[T]) // FullSimplify;

(* compute $lim_{x \to 0} \left( ev_{y_2} \circ res_{y_1}^{-1} \right)$ *)
PreR2Matrix=(((( PreAMatrix - Table[ a[i,j] ,{i,n}, {j,n}]/(y2-y1)
-Sum[a[i,i],{i,n}]*IdentityMatrix[n]/(n*v))//FullSimplify)
//. v->0 )) // FullSimplify;

(* ---------------- processing the output ---------------------*)
Do[ e[i,j]=Normal[SparseArray[{i,j}->1,{n,n}]], {i,n}, {j,n}];
AMatrix=Sum[Outer[Times, e[i,j], Coefficient[PreAMatrix,
a[j,i]]],{i,n},{j,n}] // FullSimplify;
```

```
Print["The AMatrix $r_(n,d)(v;y_1,y_2)$"];
AMatrix // MatrixForm

(*construct $c_{(n,d)} = \Omega/(y_2 - y_1) +$ R2Matrix as a tensor*)
pr[X_]:= X-IdentityMatrix[n]*(Tr[X]/n);

R2Matrix = Sum[Outer[Times, e[i,j], pr[Coefficient[PreR2Matrix,
a[j,i]]]],{i,n},{j,n}] // FullSimplify;

(* note that the above algorithm treats a[i,i] as h[i],
which is not correct; hence we have to apply some corrections*)
CorrectionOfDiagonalCoeff={};
Do[
CorrectionOfDiagonalCoeff=Union[CorrectionOfDiagonalCoeff,
{a[i,i]->a[i,i]-1/n*Sum[a[j,j],{j,n}]}],{i,n}];

R2Matrix=R2Matrix //. CorrectionOfDiagonalCoeff// FullSimplify;

Print["R2Matrix in basis $\left\{\{e_{i,j}\}_{1\leq i,j\leq n}\right\}^{\otimes 2}$ of Mat$_{n\times n}(\mathbb{C})\otimes$Mat$_{n\times n}(\mathbb{C})$"];
R2Matrix// MatrixForm

(* --------------- compute $\varphi_J \otimes \varphi_J(c_{n,d})$-------------------*)

(* first, construct $Z_J$ *)
Z=Table[zfake,{i,n},{j,n}];
Z=Z-(J+Transpose[J])*zfake+IdentityMatrix[n]*(1-zfake);
While[Coefficient[Z,zfake]!= IdentityMatrix[n]*0,
Do[ If[Z[[a,c]]== zfake && (Z[[a,b]]/. {zfake->-1})!=-1 &&
(Z[[b,c]]/. {zfake->-1})!=-1,
Z[[a,c]]=Mod[Z[[a,b]]+Z[[b,c]]+1,2];
],{a,n},{b,n},{c,n}];];

(*next, construct $\varphi_J$*)
ApplyGaugeToentry[X_,Z_]:=( GaugeReturn2 =
Table[ X[[i,j]]*(-1)^Z[[i,j]],{i,n},{j,n}];
Return[Transpose[GaugeReturn2]]);

ApplyGauge[X_,Z_]:=( Return[
Sum[Outer[Times,ApplyGaugeToentry[e[i,j],Z],
```

```
ApplyGaugeToentry[X[[i,j]],Z]],{i,n},{j,n}]]);

(* apply φ_J ⊗ φ_J*)
GaugedR2Matrix =ApplyGauge[R2Matrix,Z];
Print["φ_J ⊗ φ_J(R2Matrix) in basis {{e_{i,j}}_{1≤i,j≤n}}^{⊗2}
of Mat_{n×n}(ℂ) ⊗ Mat_{n×n}(ℂ)"];
GaugedR2Matrix// MatrixForm

(*-------------------------------------------------------*)
(*-------------------------------------------------------*)
```

**Example 23.8.** The output of the mathematica program above will consist, along with some explanatory text, of the following three matrices in the case $(n,d) = (2,1)$:

$$\begin{pmatrix} \begin{pmatrix} \frac{1}{2v} + \frac{1}{-\mathrm{y1}+\mathrm{y2}} & 0 \\ \frac{v+\mathrm{y2}}{2} & \frac{1}{2v} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \frac{1}{-\mathrm{y1}+\mathrm{y2}} & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{v-\mathrm{y1}}{2} & \frac{1}{-\mathrm{y1}+\mathrm{y2}} \\ -\frac{1}{2}v(v-\mathrm{y1})(v+\mathrm{y2}) & \frac{1}{2}(-v+\mathrm{y1}) \end{pmatrix} & \begin{pmatrix} \frac{1}{2v} & 0 \\ \frac{1}{2}(-v-\mathrm{y2}) & \frac{1}{2v} + \frac{1}{-\mathrm{y1}+\mathrm{y2}} \end{pmatrix} \end{pmatrix} \tag{23.1}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{\mathrm{y2}}{2} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\frac{\mathrm{y1}}{2} & 0 \\ 0 & \frac{\mathrm{y1}}{2} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -\frac{\mathrm{y2}}{2} & 0 \end{pmatrix} \end{pmatrix} \tag{23.2}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & -\frac{\mathrm{y2}}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{y1}}{2} & 0 \\ 0 & -\frac{\mathrm{y1}}{2} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{\mathrm{y2}}{2} \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{23.3}$$

These matrices correspond to $r_{(2,1)}(v; y_1, y_2)$, $c_{(2,1)}(y_1, y_2)$ and $(\varphi_J \otimes \varphi_J)\, c_{(2,1)}(y_1, y_2)$ respectively. To interpret them correctly, let us agree on the following conventions. First, we choose $\{e_{i,j}\}_{1\le i,j\le n}$ as a basis of $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Hence any tensor $A \otimes B \in \mathrm{Mat}_{n\times n}(\mathbb{C})^{\otimes 2}$ may be identified with the matrix $(a_{i,j} \cdot B)_{1\le i,j\le n}$, where $A = (a_{i,j})_{1\le i,j\le n}$. This explains the presentation of $r_{(2,1)}(v; y_1, y_2)$ in (23.1). Next, we make use of the fact that $c_{(n,d)}$ is of the form $c_{(n,d)}(y_1, y_2) = \frac{\Omega}{v-u} + c'_{(n,d)}(y_1, y_2)$ (in the source code, $c'_{(n,d)}(y_1, y_2)$ is called "R2Matrix"). Note that the canonical embedding $\mathfrak{sl}_n(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ allows to identify any tensor $A \otimes B \in \mathfrak{sl}_n(\mathbb{C})^{\otimes 2}$ with the matrix $(a_{i,j} \cdot B)_{1\le i,j\le n}$ where again

$A = (a_{i,j})_{1 \leq i,j \leq n}$. Via this identification, (23.2) corresponds exactly to $c'_{(2,1)}(y_1, y_2)$. Analogously, (23.3) corresponds to $(\varphi_J \otimes \varphi_J)\, c'_{(2,1)}(y_1, y_2)$.

23.2. **The program for $s_{(n,n-d)}$.**
The mathematica implementation of the algorithm for $s_{(n,n-d)}(u, v)$ is based on the formula given in Proposition 21.7. Thus, only $w_{(i,j,k)}(u)$ and $w_{(l,k)}(u)$ for $k \leq 1$ need to be computed for all $1 \leq i, j \leq n$ respectively $1 \leq l \leq n-1$ (in the source code, these elements are denoted by $w[i, k]$, where $1 \leq i \leq n^2 - 1$ and $k \in \{0, 1\}$). This implies that much of the information contained in $W$ as defined in (21.2) is superflous. Indeed, $V = \{\, w(u) \in W | -2 \leq \deg_u (w(u)) \leq 1\}$ already contains all $w[i, k]$ for $1 \leq i \leq n-1$ and $k \in \{0, 1\}$, see Lemma 21.9. The rest of the implementaion is straightforward. Note however that the integer $n - d$ is replaced by the symbol $p$ in the sourcecode.

```
(*-----------------------------------------------------------*)
  (*-------------------------------------------------------*)
  n=2;
  p=1; (*p <-> n-d*)

  Do[ e[i,j]=Normal[SparseArray[{i,j}->1,{n,n}]], {i,n}, {j,n}];
  Do[h[i] = e[i,i]-e[i+1,i+1], {i,n-1}];

  Comm[x_,y_]:=x.y-y.x;

  (*------------------ construct J(p,n-p) -----------------*)

  J[a_,b_]:=(

  If [a==b, Return[{{0,1},{0,0}}]];
  If[a>b,(
  A= J[a-b,b];
  U= Table[0, {i,a+b}, {j,a+b}];
  Do[U[[i,j]]= A[[i,j]], {i,a}, {j,a}];
  Do[U[[a-b+k,a+k]]=1, {k,b}];
  Return[U];
  )];

  If[a< b,(
  A= J[a,b-a];
  U= Table[0, {i,a+b}, {j,a+b}];
  Do[U[[i+a,j+a]]= A[[i,j]], {i,b}, {j,b}];
```

```
Do[U[[k,a+k]]=1, {k,a}];
Return[U];
)];);

(* frob <-> J(p,n-p)^t *)
frob =Transpose[J[p,n-p]];

(*---- construct V = part of W containng all g(u)
with -2 ≤ deg(g(u)) ≤ 1 ------*)

(* create list of standard basis of 𝔤 *)
Basislist= Array[Dummy,n^2];
Do[ If[i!=j,Basislist[[(i-1)*n+j]]= e[i,j],
If[i<n,Basislist[[(i-1)*n+j]]=h[i],
Basislist[[(i-1)*n+j]]=0]],{i,n}, {j,n}];

(*construct conjugation by η(u) *)
eta=Table[If[i!= j,0,If[i>k, 1/u, 1]] , {i,n}, {j,n} ];
Conjug[x_]=eta.x.Inverse[eta];

(*construct list containing basis of η^-1(u)·(l+1/u·[J^t,l])·η(u) *)
Do[z'[i]=Basislist[[i]]+1/u *Comm[frob,Basislist[[i]] ], {i,n^2-1}];
Do[z[i]=Conjug[z'[i]], {i,n^2-1}];

(* construct list containing basis of η^-1(u)·(u^-2 𝔤)·η(u) *)
Do[z[i+n^2-1]=Conjug[Basislist[[i]]]*1/u^2, {i,n^2-1}]

(*----- for 0 ≤ k ≤ 1 and 1 ≤ i ≤ n^2 - 1, determine w[i,k]-------*)

(* first, create a matrix that contains all values Res_u tr(x,y)
where x runs over basis of V and y over basis of 𝔤+u𝔤 *)
T' = Table[If[i<n^2, Coefficient[z[j].Basislist[[i]]*u,1/u],
Coefficient[z[j].Basislist[[i-(n^2-1)]],1/u]],
{i,2*(n^2-1)}, {j,2*(n^2-1)}];
T = Table[Tr[T'[[i,j]]], {i,2*(n^2-1)}, {j,2*(n^2-1)}];
l= LinearSolve[T];

(*the dual basis of the standard basis of 𝔤 wrt to the trace form*)
Do[f[i,j]=e[j,i], {i,n},{j,n}]
Do[dualcoeff[i]= 1-i/n;
```

```
g[i]= Sum[j*dualcoeff[i]*h[j], {j,n-1}]
-Sum[(j-i)*h[j],{j,i+1,n-1}],{i,n-1}];
DualBasislist= Array[Dummy,n^2];
Do[ If[i!=j,DualBasislist[[(i-1)*n+j]]= f[i,j],
If[i<n,DualBasislist[[(i-1)*n+j]]=
g[i], DualBasislist[[(i-1)*n+j]]=0]],{i,n}, {j,n}];

(*now, determine the w[i,1]; if Basislist[[i]] $\cdot u^{-k}$ is in W
for all $k \geq 1$, the program does only know that Basislist[[i]] $\cdot u^{-1}$ is
in V, so the corresponding row of T is zero; hence in this case
we have to tell the programm that
$w[i,1] =$ Basislist[[i]] $\cdot u^{-2}$ explicitely*)
Do[
If[Sum[Abs[T[[i,j]]],{j,2*(n^2-1)}]== 0,
w[i,1]=DualBasislist[[i]]*1/v^2,
{sol = l[UnitVector[2*(n^2-1),i]];w[i,1]=
Sum[z[j]*sol[[j]] /.  u->v, {j,2*(n^2-1)}]}], {i,n^2-1}]

(* the w[i,0], corresponding to $w_{(i,j,0)}(v)$, are computed formally*)
Do[{sol = l[UnitVector[2*(n^2-1),i]];
w[i-(n^2-1),0]=Sum[z[j]*sol[[j]]
/.  u->v, {j, 2*(n^2-1)}];}, {i,n^2 , 2*(n^2-1)}];

(*---- $s_{(n,k)}(u,v) = \frac{\Omega}{v-u} +$ R2Matrix as a tensor ----------*)

(* determine all y[i] <-> $(v-u) \cdot (\tilde{w}_{i,j}(u,v))$ and $(v-u) \cdot (\tilde{w}_l(u,v)$ *)
Do[y[i]= u*(w[i,1]- DualBasislist[[i]]*1/v^2)
+w[i,0]- DualBasislist[[i]]*1/v; ,{i,n^2-1}];
y[n^2]=0*IdentityMatrix[n];
R2Matrix=Sum[Outer[Times,e[i,j], y[(i-1)*n+j] ], {i,n},{j,n}];

(*--------------- output -------------------*)
Print["R2Matrix in basis $\left\{\{e_{i,j}\}_{1\leq i,j\leq n}\right\}^{\otimes 2}$ of $Mat_{n\times n}(\mathbb{C}) \otimes Mat_{n\times n}(\mathbb{C})$"];
Table[If[i!=j, R2Matrix[[i,j]], If[1<i,-R2Matrix[[i-1,i-1]],
IdentityMatrix[n]*0 ]+R2Matrix[[i,i]]],{i,n},{j,n}] // MatrixForm

(*-------------------------------------------------------*)
(*-------------------------------------------------------*)
```

**Example 23.9.** The output of the mathematica program above will consist, along with some explanatory text, of the following matrix in the case $(n, d) = (2, 1)$:

$$\begin{pmatrix} \begin{pmatrix} 0 & -\frac{v}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{u}{2} & 0 \\ 0 & -\frac{u}{2} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{v}{2} \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{23.4}$$

In order to understand how this matrix represents $s_{(2,1)}(u, v)$, we again have to agree on some conventions. Similar to the treatment of $c_{(n,d)}$ above, we use the fact that $s_{(n,n-d)}$ is of the form $s_{(n,n-d)}(u, v) = \frac{\Omega}{v-u} + s'_{(n,n-d)}(u, v)$. Also, we choose $\{e_{i,j}\}_{1\leq i,j\leq n}$ as a basis of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ again and use the canonical embedding $\mathfrak{sl}_n(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$. This allows to identify any tensor $A \otimes B \in \mathfrak{sl}_n(\mathbb{C})^{\otimes 2}$ with the matrix $(a_{i,j} \cdot B)_{1\leq i,j\leq n}$ where $A = (a_{i,j})_{1\leq i,j\leq n}$. Via this identification, $s'_{(2,1)}(u, v)$ corresponds exactly to (23.4). Note that $s'_{(n,n-d)}(u, v)$ is denoted "R2Matrix" in the source code. Also, since the program works with the standard basis $\left\{\{e_{i,j}\}_{1\leq i\neq j\leq n} \cup \{h_l\}_{1\leq l\leq n-1}\right\}$ of $\mathfrak{sl}_n(\mathbb{C})$, some adjustment is needed at the end in order to convert R2Matrix to the basis $\left\{\{e_{i,j}\}_{1\leq i,j\leq n}\right\}^{\otimes 2}$. This explains the last few lines of the source code.

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