Normally hyperbolic operators on Lorentzian manifolds and their quantization

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Aim: quantize the fields coming from the *wave equation*.

1. Well-known facts

Let (M^n, g) be a timeoriented Lorentzian manifold (*spacetime*), let $E \to M$ be a $\mathbb{R}$-vector bundle.

Definition 1 *A* normally hyperbolic *operator on* E *is a* 2^{nd}*-order differential operator* P *on* E *of the form*

$$P := \nabla^*\nabla + B,$$

where $\left| \begin{array}{l} \nabla \text{ is a connection on } E \\ B \in C^\infty(M, \mathrm{End}(E)). \end{array} \right.$

Ex.: d'Alembert $\Box$, $(\text{Dirac})^2$.

Definition 2 *Let* P *be a normally-hyperbolic operator on* E*. The* wave-equation *associated to* P *is*

$$Pu = f$$

for a given $f \in C^\infty(M, E)$ *(and with conditions on* $\mathrm{supp}(u)$*).*

Definition 3 *A (connected) spacetime* (M, g) *is called* globally hyperbolic *iff it contains a smooth spacelike* Cauchy-hypersurface S
[*every inextendible timelike curve in* M *meets* S *exactly once*].
($\Longleftrightarrow$ $(M, g) \cong (\mathbb{R} \times S, -\beta dt^2 \oplus g_t)$ *with smooth* $\beta : M \to \mathbb{R}_+^*$*, smooth* 1*-parameter family of Riemannian metrics* g_t *on* S*, and each* $\{t\} \times S$ *is a spacelike Cauchy hypersurface in* M*.)*

Ex.: $(M, g) := (I \times S, -dt^2 \oplus f(t)^2 g_0)$ where $f : I \to \mathbb{R}_+^*$ smooth and (S, g_0) complete Riemannian manifold
$\Rightarrow$ Minkowski, Robertson-Walker, deSitter spacetimes are globally hyperbolic.

C.-ex.: compact spacetimes, Anti-deSitter spacetime

$$(M, g) := (\mathbb{R} \times S_+^{n-1}, \frac{1}{x_n^2}(-dt^2 \oplus \mathrm{can}_{S_+^{n-1}})).$$

Theorem 4 *Let M be a globally hyperbolic spacetime and let $S \subset M$ be a smooth spacelike Cauchy hypersurface with future-directed (timelike) unit normal vector field ν.*

i) $\forall\,(f, u_0, u_1) \in \mathcal{D}(M,E) \oplus \mathcal{D}(S,E) \oplus \mathcal{D}(S,E)$, $\exists\,!\, u \in C^\infty(M,E)$ *s.t.*

$$\left| \begin{array}{ll} Pu & = f \\ u_{|_S} & = u_0 \\ \nabla_\nu u & = u_1. \end{array} \right. \qquad (1)$$

Moreover, $\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$ *where* $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

ii)
$$\begin{array}{rcl} \mathcal{D}(M,E) \oplus \mathcal{D}(S,E) \oplus \mathcal{D}(S,E) & \longrightarrow & C^\infty(M,E) \\ (f, u_0, u_1) & \longmapsto & u, \end{array}$$

where $u \in C^\infty(M,E)$ *is the solution of* (1), *is linear continuous.*

Definition 5 *A linear map*

$$G_\pm : \mathcal{D}(M,E) \to C^\infty(M,E)$$

is called advanced *(+) resp.* retarded *(-)* *Green's operator for P iff it satisfies:*

i) $P \circ G_\pm = \mathrm{id}_{\mathcal{D}(M,E)}$.

ii) $G_\pm \circ P_{|_{\mathcal{D}(M,E)}} = \mathrm{id}_{\mathcal{D}(M,E)}$.

iii) $\mathrm{supp}(G_\pm\varphi) \subset J_\pm^M(\mathrm{supp}(\varphi))$ *for all* $\varphi \in \mathcal{D}(M,E)$.

Theorem 6 *For any globally hyperbolic space-time M and any normally-hyperbolic operator P there exist unique advanced and retarded Green's operators G_+ and G_- for P.*
They satisfy:

- *If $P = P^*$ then $(G_\pm)^* = G_\mp$.*
- *The sequence*

$$0 \to \mathcal{D}(M,E) \xrightarrow{P} \mathcal{D}(M,E) \xrightarrow{G} C^\infty_{\mathrm{sc}}(M,E) \xrightarrow{P} C^\infty_{\mathrm{sc}}(M,E)$$

is an exact complex, where $G := G_+ - G_-$.

2. Quantization functors

2.1 Categories

Category	Objects	Morphisms
GlobHyp	(M, E, P) where - $E \to M$ (real) v.b. with *indef.* $\langle \cdot\,, \cdot \rangle$ - P norm. hyp. op. and *form. s.-a.*	(f, F) with a) $\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$ b) $\begin{array}{ccc} \mathcal{D}(M_1, E_1) & \xrightarrow{\text{ext}} & \mathcal{D}(M_2, E_2) \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ \mathcal{D}(M_1, E_1) & \xrightarrow{\text{ext}} & \mathcal{D}(M_2, E_2) \end{array}$
LorFund	$(M, E, P, G_\pm)$ with $(G_\pm)^* = G_\mp$	- M_1 glob. hyp. $\Rightarrow$ as above - M_1 *not* glob. hyp. $\Rightarrow$ $\emptyset$ or $\{(\mathrm{id}_{M_1}, \mathrm{id}_{E_1})\}$
SymplVec	(V, ω)	symplectomorphisms
C^*-*Alg*	$(A, \Vert \cdot \Vert, *)$ with 1	C^*-alg.-morphisms inj., preserving 1

2.2 Functors

- Functor SOLVE : $\mathit{GlobHyp} \longrightarrow \mathit{LorFund}$:

$$\left|\begin{array}{ll} \mathsf{SOLVE}(M,E,P) & := (M,E,P,G_\pm) \\[2ex] \mathsf{SOLVE}(f,F) & := (f,F) \end{array}\right.$$

- Functor SYMPL : $\mathit{LorFund} \longrightarrow \mathit{SymplVec}$:

$$\left|\begin{array}{ll} \mathsf{SYMPL}(M,E,P,G_\pm) & := (\mathcal{D}(M,E)/\ker(G), \int_M \langle G\,\cdot,\cdot\rangle dv_g) \\[2ex] \mathsf{SYMPL}(f,F) & := \overline{\mathrm{ext}} \end{array}\right.$$

where $\overline{\mathrm{ext}} : \mathcal{D}(M_1,E_1)/\ker(G_1) \to \mathcal{D}(M_2,E_2)/\ker(G_2)$.

- Functor CCR : $\mathit{SymplVec} \longrightarrow \mathit{C^*\text{-}Alg}$:

$$\left|\begin{array}{ll} \mathsf{CCR}(V,\omega) & := \text{CCR repr. of } (V,\omega) \\ \mathsf{CCR}(\mathcal{S}) & := \widetilde{\mathcal{S}} \end{array}\right.$$

where

$$\begin{array}{ccc} \mathsf{CCR}(V_1,\omega_1) & \xrightarrow{\widetilde{\mathcal{S}}} & \mathsf{CCR}(V_2,\omega_2) \\ {\scriptstyle W_1}\uparrow & & \uparrow{\scriptstyle W_2} \\ V_1 & \xrightarrow{\quad\mathcal{S}\quad} & V_2 \end{array}$$

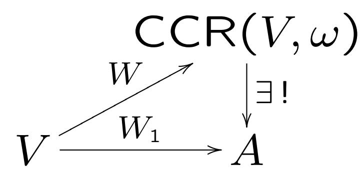
* Here $\mathrm{CCR}(V,\omega) := C^*(\{W(\varphi),\ \varphi \in V\})$, where the map $W : V \longrightarrow \mathcal{L}(L^2(V,\mathbb{C}))$ is defined by

$$(W(\varphi)F)(\psi) := e^{i\frac{\omega(\varphi,\psi)}{2}} F(\varphi+\psi)$$

for all $F \in L^2(V,\mathbb{C})$ and $\varphi, \psi \in V$.

* W is a *Weyl system* for (V,ω), i.e., $W(0) = 1$, $W(-\varphi) = W(\varphi)^*$ and $W(\varphi+\psi) = e^{i\frac{\omega(\varphi,\psi)}{2}} W(\varphi)\cdot W(\psi)$.

* W is the "smallest" Weyl system:



Theorem 7 *Those functors are well-defined.*