Logic and Dynamical Systems

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Introduction

In this talk, I will give an introduction to some recent work in set theory, developed primarily over the last 15 years, and discuss its connections with aspects of dynamical systems and in particular rigidity phenomena in the context of ergodic theory.

- Theory of complexity of classification problems in mathematics.
- "Definable" or Borel cardinality theory of quotient spaces (vs. "classical" or Cantor cardinality theory).

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A classification problem is given by:

- A collection of objects X.
- An equivalence relation E on X.

A complete classification of X up to E consists of:

- A set of invariants I.
- A map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$.

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Example

Classification of finitely generated abelian groups up to isomorphism.
INVARIANTS: finite sequences of integers.

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Classification of Bernoulli automorphisms up to conjugacy (Ornstein).
INVARIANTS: reals

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Classification of torsion-free abelian groups of rank 1 (i.e., subgroups of $(\mathbb{Q}, +)$) up to isomorphism (Baer).
INVARIANTS: subsets of $\mathbb{N}$ modulo finite differences.

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Most often the collection of objects we try to classify can be viewed as forming a “nice” space, namely a standard Borel space, i.e., a Polish (complete separable metric) space with its associated Borel structure and the equivalence relation E turns out to be *Borel* or *analytic* (as a subset of X^2).

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I will concentrate below on Borel equivalence relations.

Borel equivalence relations and reducibility

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Definition

Let $(X, E), (Y, F)$ be Borel equivalence relations. E is (Borel) reducible to F, in symbols

$$E \leq_B F,$$

if there is Borel map $f : X \to Y$ such that

$$x \; E \; y \Leftrightarrow f(x) \; F \; f(y).$$

Intuitive meaning:

- The classification problem represented by E is at most as complicated as that of F.
- F-classes are complete invariants for E.

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E is bi-reducible to F if E is reducible to F and vice versa.

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We also put:

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(Isomorphism of f.g. abelian groups) $\sim_B (=_{\mathbb{N}})$

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(Conjugacy of Bernoulli automorphisms) $\sim_B (=_{\mathbb{R}})$

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(Isomorphism of t.f. abelian groups of rank 1) $\sim_B E_0$,
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$$(x_n)\ E_C\ (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$$

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(Conjugacy of increasing homeomorphisms of $[0,1]$) $\sim_B$
(Isomorphism of countable linear orderings)

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Borel cardinality theory

The preceding concepts can be also interpreted as the basis of a "definable" or Borel cardinality theory for quotient spaces.

- $E \leq_B F$ means that there is a Borel injection of X/E into Y/F, i.e., X/E has Borel cardinality less than or equal to that of Y/F, in symbols

$$|X/E|_B \leq |Y/F|_B$$

- $E \sim_B F$ means that X/E and Y/F have the same Borel cardinality, in symbols

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Structure of $\leq_B$

Below X stands for the equality relation on X, $=_X$.

We clearly have:

$$1 <_B 2 <_B 3 \cdots <_B \mathbb{N} <_B E$$

Theorem (Silver, 1980)

For every Borel E, either $E \leq_B \mathbb{N}$ or $\mathbb{R} \leq_B E$.

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Note that $E \leq_B \mathbb{R}$ means that there is a standard Borel space Y and a Borel map $f : X \to Y$ such that $x \; E \; Y \Leftrightarrow f(x) = f(y)$. Such E are called concretely classifiable or smooth. A canonical example of a non-smooth E is the equivalence relation E_0 of equality of subsets of $\mathbb{N}$ modulo finite. So $\mathbb{R} <_B E_0$.

Theorem (Harrington-K-Louveau, 1990)

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The proofs of these two dichotomies, which are about classical concepts of descriptive set theory, i.e., Borel sets and functions, use methods of so-called effective descriptive set theory, which are based on computability theory, i.e., the theory of algorithms, Turing machines, etc. No "classical" type proofs are known.

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The linearity of $\leq_B$ breaks down after E_0.

Example (K-Louveau)

The following equivalence relations on $\mathbb{R}^{\mathbb{N}}$ are incomparable:

$$(x_n)\ E_1\ (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m)$$

$$(x_n)\ E_2\ (y_n) \Leftrightarrow \lim_{n \to \infty} (x_n - y_n) = 0$$

So the picture is as follows:

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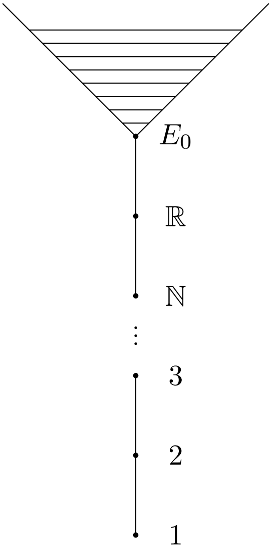

Structure of $\leq_B$

So far all the *known* Borel equivalence relations above E_0 fall into exactly 4 types and it may be that they all do. This is partially supported by a series of results of Hjorth, K, Louveau, ... Below we use the following definitions.

Definition

For a Polish group G, Polish space X, and a continuous or Borel action of G on X, we denote by E_G^X the induced (orbit) equivalence relation.

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S_∞ is the infinite symmetric group.

Definition

Γ denotes an arbitrary countable (discrete) group.

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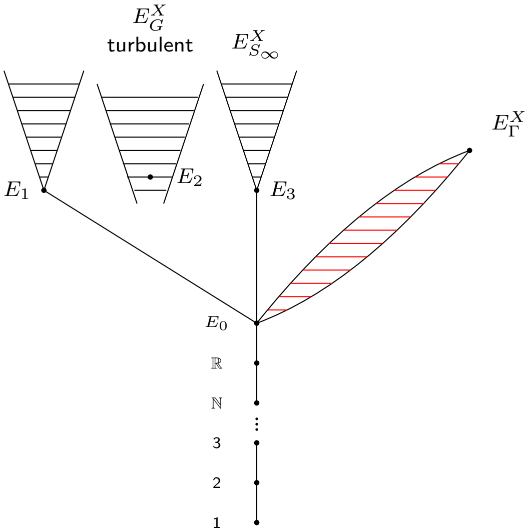

Countable Borel equivalence relations

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E is countable if every E-class is countable.

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Any equivalence relation, E_Γ^X, induced by a Borel action of a countable group Γ on X

We actually have:

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Up to bireducibility they also include:

Example (K)

E_G^X for G second countable locally compact group (e.g., Lie group)

Example (Hjorth-K)

Isomorphism of countable structures that are of "finite type", e.g., finitely generated groups, locally finite trees, finite rank torsion-free abelian groups, finite transcendence degree fields, etc.

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Hyperfinite

We will now consider the structure of $\leq_B$ on the countable Borel equivalence relations. Unless otherwise stated, the results below are due to: Dougherty-Jackson-K (1994) and Jackson-K-Louveau (2002).

The simplest countable equivalence relations are the smooth ones, which have a trivial structure. The next more complicated ones are the so-called hyperfinite ones.

Definition

E is hyperfinite if $E = \bigcup_n E_n$, with E_n increasing and finite (i.e., having equivalence classes that are finite).

Theorem (Slaman-Steel, Weiss)

E is hyperfinite iff it is of the form $E_{\mathbb{Z}}^X$.

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Which groups always give hyperfinite equivalence relations?

Necessary condition: They have to be amenable.

Problem (Weiss, 1984)

If Γ is amenable, is E_Γ^X hyperfinite?

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If Γ is finitely generated of polynomial growth, then E_Γ^X is hyperfinite.

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The hyperfinite equivalence relations have been classified both under bireducibilty and isomorphism.

Theorem

i) Up to Borel bireducibility, there is only one non-smooth, hyperfinite equivalence relation, namely E_0.
ii) Up to Borel isomorphism, there are exactly countably many non-smooth, aperiodic, hyperfinite equivalence relations, namely

$$E_t, E_0, 2E_0, 3E_0, \dots, nE_0, \aleph_0 E_0, E_s.$$

Here E_t is the tail equivalence relation on $2^{\mathbb{N}}$ and E_s is the aperiodic part of the shift equivalence relation on $2^{\mathbb{Z}}$.

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The hyperfinite equivalence relations are the simplest non-trivial countable equivalence relations. At the other end there are the most complex ones, the so-called universal ones.

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There is a universal countable Borel equivalence relation, E_∞. It satisfies $E \leq_B E_\infty$, for all countable E.

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Since the early 1990's a small finite number of intermediate equivalence relations were known and they were in a linear order under $\leq_B$. This lead to the following basic problems:

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Are there infinitely many?

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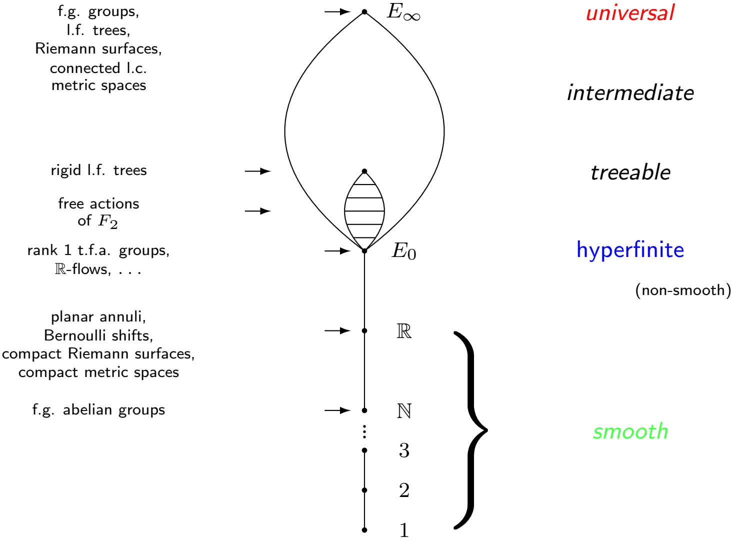
Picture of $\leq_B$ on countable equivalence relations



Set theoretic rigidity

The proof of the preceding theorem of Adams-K used Zimmer's cocycle superrigidity theory for ergodic actions of linear algebraic groups and their lattices.

The key point is that there is a phenomenon of set theoretic rigidity analogous to the measure theoretic rigidity phenomena discovered by Zimmer.

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- **(Measure theoretic rigidity)** Under certain circumstances, when a countable group acts preserving a probability measure, the equivalence relation associated with the action together with the measure "encode" or "remember" a lot about the group (and the action).
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- **(Set theoretic rigidity)** Such information is simply encoded in the Borel cardinality of the (quotient) orbit space.

Set theoretic rigidity

Some set theoretic rigidity results.

Theorem (Adams-K)

$$|\mathbb{T}^m/GL_m(\mathbb{Z})|_B = |\mathbb{T}^n/GL_n(\mathbb{Z})|_B \Leftrightarrow m = n$$

Below $\Gamma_p = SO_7(\mathbb{Z}[1/p])$, p prime. Also E_p is the free part of the shift equivalence relation on 2^{Γ_p}.

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Below let $\cong_n$ be isomorphism of torsion-free abelian groups of rank n, i.e., subgroups of $(\mathbb{Q}^n, +)$. This can be seen to be (up to $\sim_B$) a countable Borel equivalence relation.

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Recently Hjorth-K, 2005 developed a set theoretic rigidity theory for product groups that has several applications in the study of countable Borel equivalence relations – but also in ergodic theory. They also use ergodic theoretic methods, like cocycle reduction techniques, actions on boundaries, etc. (Also, independently, Monod-Shalom and Popa have proved important rigidity results for product groups in the context of ergodic theory – it is yet unclear what is the relationship between these theories.)

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Below, for any group Γ, we let E_Γ be the free part of the shift equivalence relation on 2^Γ.

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$$E_{(\mathbb{Z}_p \star \mathbb{Z}_p)\times\mathbb{Z}} \leq_B E_{(\mathbb{Z}_q \star \mathbb{Z}_q)\times\mathbb{Z}} \Leftrightarrow p = q$$

Note however that:

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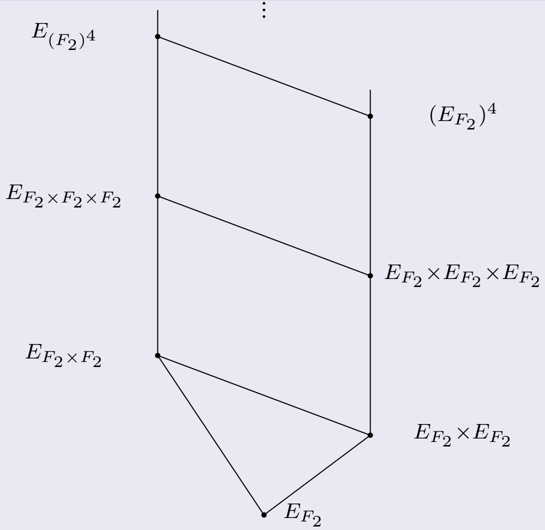
The next result concerns the distinction between the equivalence relation $E_{F_2^n}$ induced by the shift action of the product of n copies of F_2 (shift of the product) and the product equivalence relation of n copies of the shift action of F_2, i.e., $(E_{F_2})^n$ (product of the shift). It can be best summarized in a picture.

Set theoretic rigidity

Theorem



An ergodic theory rigidity result

Theorem

Suppose H_0, H_1 are non-amenable, torsion-free, hyperbolic groups and Δ_0, Δ_1 are infinite amenable groups. Let each $H_i \times \Delta_i$ act freely on X_i with invariant, probability measure, so that the action is ergodic on Δ_i, $i = 1, 2$. If the action of $H_0 \times \Delta_0$ is (stably) orbit equivalent to the action of $H_1 \times \Delta_1$, then $H_0 \cong H_1$.

... and a final comment

The theory of countable Borel equivalence relations points to an interesting phenomenon. Although one is dealing here with very simple set theoretic notions (countable Borel equivalence relations and Borel reducibility) most basic questions about them (like existence of intermediate or incomparable ones) have been answered by using rather sophisticated ergodic theory methods, and this certainly represents an interesting application of ergodic theory to set theory. At this time no other methods to study these problems are known.