Sixth exercise sheet "Algebra II" winter term 2024/5.

Problem 1 (3 points). *Let $\hat{R}$ be a DVR with valuation v. Moreover, let $\mathfrak{M}$ be a non-empty set of subrings of $\hat{R}$ such that all $S \in \mathfrak{M}$ are DVR with valuation $v|_S$ and such that for every finite subset $F \subseteq \mathfrak{M}$, $\bigcup_{S\in F} S$ is contained in some element of $\mathfrak{M}$. Show that $R = \bigcup_{S\in\mathfrak{M}} S$ is a DVR with valuation $v|_R$.*

The following two problems can for instance be applied with $F = \mathfrak{k}(X_i \mid i \in \mathbb{N})$, the field of rational functions in countably many variables over a field $\mathfrak{k}$ of characteristic $p > 0$, showing that Proposition 2.5.4 of the lecture may fail without the assumption that B is a finitely generated A-module. Therefore this also gives an example where the integral closure B of a DVR A in a finite purely inseparable field extension L of its field of quotients K fails to be a finitely generated A-module.

Note that for an arbitrary field F of positive characteristic p, $F^p = \{f^p \mid f \in F\}$ is a subfield of F.

Problem 2 (3 points). *Let F be field of characteristic $p > 0$ such that $[F : E]$ is infinite, where $E = F^p$. Let $\hat{R} = F[\![T]\!]$, and let $R \subset \hat{R}$ be the subring containing all $f = \sum_{k=0}^{\infty} f_k T^k$ such that the subfield of F generated by E and $\{f_k \mid k \in \mathbb{N}\}$ is a finite field extension of E. By Problem 1 of sheet 4 we know that $\hat{R}$ is a DVR with residue field $\hat{R}/T\hat{R} \cong F$. Let v be its valuation. Show that R is a DVR with valuation $v|_R$, and calculate the residue field of R.*

Problem 3 (4 points). *In the situation of the previous problem, let K be the field of quotients of R, and let $f = \sum_{k=0}^{\infty} \phi_k^p T^{kp} \in R$ where the subfield of F generated by E and $\{\phi_k \mid k \in \mathbb{N}\}$ has infinite degree over E. Let $L = K(\sqrt[p]{f})$. Identify the integral closure of R in L with a subring $S \subseteq \hat{R}$ which is a DVR with valuation $v|_S$. Moreover, calculate the residue field of S and show that the fundamental equality from Proposition 2.5.4 of the lecture is violated for the extension S/R and the maximal ideal of R!*

Problem 4 (3 points). *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \neq 1$ is a square-free integer, and let p be an odd prime number dividing D. Calculate the prime ideal decomposition of $p\mathcal{O}_K$!*

Problem 5 (3 points). *Use the result of the previous problem to show that for every natural number n there is a number field K such that the dimension of the $\mathbb{F}_2$-vector space $\{x \in \mathrm{Cl}(\mathcal{O}_K) \mid x^2 = 1\}$ exceeds n!*

Problem 6 (8 points)**.** *Let p be a prime number, $\mathfrak{k}$ a field of characteristic $\neq p$, $A = \mathfrak{k}[X]$, K the field of quotients of A and $L = K(\sqrt[p]{X})$.*

- *Describe the condition on $\mathfrak{k}$ under which L/K is a Galois extension.*
- *Calculate the integral closure B of A and describe how the maximal ideals of A decompose into prime ideals in B!*

Remark 1.

- *Your answer to first point must be correct but it is not necessary to prove it as this is essentially a matter of Basic Galois Theory.*
- *When solved correctly this gives an example where different $\mathfrak{q} \in \mathrm{Spec} B$ lying over the same $\mathfrak{p} \in \mathrm{Spec} A$ may have different degrees over $\mathfrak{k}(\mathfrak{p})$ of their residue fields.*

Four of the 24 points available from this exercise sheet are bonus points which are disregarded in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted to the tutor by e-mail before Friday November 22 24:00.