Exercises in Geometry II
University of Bonn, Summer Semester 2018
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Sheet 5

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1. First variation of arc length [4 points]

Let $\gamma : [a, b] \to M$ be a unit speed curve in a Riemannian manifold (M, g). Further, let Γ be a proper variation of γ with variation field V, i.e. $\frac{d}{ds}\big|_{s=0} \Gamma_s = V$.
Show that

$$\frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) = -\int_a^b \langle V, D_t\dot{\gamma}\rangle dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i\dot{\gamma}\rangle,$$

where $\Delta_i\dot{\gamma} = \dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-)$ is the "jump" in the tangent vector field $\dot{\gamma}$ at a_i.

2. Unit speed curves [4 points]

Let $\gamma : I \to M$ be a smooth unit speed curve.

a) Show that $D_t\dot{\gamma}(t)$ is orthogonal to $\dot{\gamma}(t)$ for all $t \in I$.

b) Let Γ be a proper variation of γ such that for all s, Γ_s is a reparametrization of γ. Show that the first variation of $L(\Gamma_s)$ vanishes.

3. First variation of arc length for non-proper variations [4 points]

Generalize the first variation formula from Exercise 1 to the case of a variation that is not proper.

4. Distance to a submanifold [4 points]

Let N be a closed, embedded submanifold of a Riemannian manifold (M, g). For any point $p \in M \setminus N$, we define the *distance from p to N* to be

$$d(p, N) := \inf\{d(p, x) : x \in N\}.$$

Now let $q \in N$ be a point such that $d(p, q) = d(p, N)$ and let γ be any minimizing geodesic from p to q. Show that γ intersects N orthogonally.
Hint: Use Exercise 3.

5. Manifolds with constant negative sectional curvature [4 points]

Let M be a Riemannian manifold with constant sectional curvature equal to $-b$, for some $b > 0$. Recall from Exercise Sheet 4, Exercise 4, that M has no conjugate points. Let $\gamma : [0, l] \to M$ be a unit speed geodesic and let $v \in T_{\gamma(l)}M$ such that $\langle v, \dot{\gamma}(l)\rangle = 0$ and $|v| = 1$.

Show that the Jacobi field J along γ with $J(0) = 0$ and $J(l) = v$ is given by

$$J(t) = \frac{\sinh(t\sqrt{b})}{\sinh(l\sqrt{b})} w(t),$$

where $w(t)$ is the parallel transport along γ of the vector

$$w(0) = \frac{u_0}{|u_0|},$$
$$u_0 = (d\exp_{\gamma(0)})^{-1}_{l\dot{\gamma}(0)}(v),$$

and where u_0 is considered as a vector in $T_{\gamma(0)}M$ by the identification $T_{\gamma(0)} \cong T_{l\dot{\gamma}(0)}(T_{\gamma(0)}M)$.
Hint: Use Exercise 1 from Exercise Sheet 4. Further, you can use that any Jacobi field J_1 along γ with $J_1(0) = 0$ and $\dot{J}_1(0) = w(0)$ satisfies

$$J_1(l) = (d\exp_{\gamma(0)})_{l\dot{\gamma}(0)}(lw(0)).$$

Now express J in terms of J_1 using the condition $J(l) = v$, the definition of u_0 and $1 = \|v\|$.

6. Locally but not globally isometric [4 points]

Give an example of two compact Riemannian manifolds without boundary and constant sectional curvature that are locally isometric but not isometric.

Due on Monday, June 4.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`