Exercises in Geometry II

University of Bonn, Summer Semester 2018
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Sheet 7

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1. Local diffeomorphism between compact manifolds [4 points]

Show that any local diffeomorphism between compact connected manifolds is a covering map.

2. Local diffeomorphism and completeness [4 points]

Let $f : M_1 \to M_2$ be a local diffeomorphism of a manifold M_1 onto a Riemannian manifold M_2.

a) Introduce a Riemannian metric on M_1 such that f is a local isometry.

b) Show by an example that if M_2 is complete, M_1 does not need to be complete.

3. Riemannian submersion [4 points]

Let $f : (\tilde{M}, \tilde{g}) \to (M, g)$ be a surjective submersion between two Riemannian manifolds. For any $y \in M$ the closed preimage $f^{-1}(y) =: \tilde{M}_y$ is called the *fiber* over y. Each fiber $\tilde{M}_y$ is a closed, embedded submanifold by the implicit function theorem.
At each $x \in \tilde{M}$ the tangent space $T_x\tilde{M}$ decomposes into an orthogonal direct sum

$$T_x\tilde{M} = H_x \oplus V_x,$$

where $V_x := \ker(f_*)$ is the *vertical space* and $H_x := V_x^{\perp}$ is the *horizontal space*.
The submersion $f : (\tilde{M}, \tilde{g}) \to (M, g)$ is a *Riemannian submersion* if $\tilde{g}(X, Y) = g(f_*X, f_*Y)$ whenever X and Y are horizontal.

a) Show that any vector field W on $\tilde{M}$ can be written uniquely as $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and W^H and W^V are smooth.

b) If X is a vector field on M, show that there is a unique smooth vector field $\tilde{X}$ on $\tilde{M}$, called the *horizontal lift of* X, that is f-related to X, i.e. $f_*X_q = X_{f(q)}$ for each $q \in \tilde{M}$.

4. Bonnet's theorem [4 points]

Let M be a complete, connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $\frac{1}{R^2} > 0$.

a) Show that the diameter is less than or equal to πR.

   Hint: Assume the contrary, i.e. assume that there is a minimizing geodesic γ of length bigger than πR and show that there is a proper normal vector field V along γ such that $I(V, V) < 0$ (Exercise 1 from sheet 4 could help).

b) Conclude that M is compact.

c) Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be the Riemannian universal cover of (M, g). Show that this cover is finite, i.e. for any $p \in M$ the preimage $\pi^{-1}(p)$ consists of finitely many points.

Due on Monday, June 25.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`