Arthur's trace formula for GL(2) and GL(3) and non-compactly supported test functions

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I Introduction

The trace formula was initially introduced by Selberg in 1956 to study the spectrum of the Laplace Beltrami operator on the locally symmetric spaces $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ for lattices $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$. Subsequently Arthur developed in a long and involved procedure a generalised version for studying the spectrum of $G(F)\backslash G(\mathbb{A})$ for general reductive groups G over a number field F. Roughly, the trace formula is an identity between sums of distributions

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f) = \sum_{\chi\in\mathfrak{X}} J_{\chi}(f),$$

the so-called geometric and spectral sides. It was often used to establish specific cases of Langlands functoriality like the Jacquet-Langlands correspondence [JaLa70]; see also [Ar97] for more general cases in the context of the theory of endoscopy. For such applications the comparison of two trace formulas for different groups is necessary. However, the consideration of the trace formula for an individual group may as well lead to some interesting results, as, for example, in [LaMu09], where a higher rank version of Weyl's law was shown. We shall concentrate on the trace formula for the group $G = \mathrm{GL}(n)$, or even mostly on $G = \mathrm{GL}(2)$ or $G = \mathrm{GL}(3)$.

In general, Arthur's trace formula allows only smooth, compactly supported test functions, but for some applications non-compactly supported test functions would be desirable. Viewing the trace formula as some kind of non-abelian generalisation of the Poisson summation formula, it is natural to ask for a class of test functions resembling the test functions allowed in the Poisson formula. For a natural class the absolute convergence of the spectral side was shown in [FiLaMu09, FiLaMu11], and the convergence of the geometric side for $G = \mathrm{GL}(2)$ in [FiLa11a]. In general, the convergence of the geometric side is not known, but the convergence of the semisimple part is shown in [FiLa11b]. Therefore our first goal is to establish the absolute convergence of a particular expansion of the geometric side for $G = \mathrm{GL}(3)$ for all such test functions (see Theorem 1). By the results of [FiLa11b] this amounts to the analysis of the non-semisimple part.

For $G = \mathrm{GL}(n)$ this space of test functions contains in particular functions of the form

$$f_s(g) = \int_{\mathbb{A}^\times} |\det(ag)|^{s+\frac{n-1}{2}} \Phi(ag) d^\times a, \qquad g \in \mathrm{GL}_n(\mathbb{A}),$$

for $s \in \mathbb{C}$, $\Re s > \frac{n+1}{2}$, where Φ is a Schwartz-Bruhat function on the space $\mathrm{Mat}_{n\times n}(\mathbb{A})$ of $n \times n$ matrices over the ring of adeles of the base field. Hence the geometric and spectral side can be considered as functions of s. Consequently there appear important arithmetical objects on the spectral side: By the theory of Godement-Jacquet [GoJa72] the discrete spectrum essentially yields a sum of completed L-functions $L^*(\pi, s)$ associated with a particular class of automorphic representations π so that the discrete part has a meromorphic continuation to the whole complex plane as a function of s (see also

Theorem 33). There exist replacements for the space of Schwartz-Bruhat functions on $\mathrm{Mat}_{n\times n}(\mathbb{A})$ for other classical groups having similar features, which are constructed in [GePSRa87] and [BrKa02], leading to new allowed test functions.
Such test functions suggest to study the trace formula from a function theoretic point of view, and we shall do this for the spectral side for $\mathrm{GL}(n)$, as well as the geometric side for GL(2) and GL(3). This approach is also connected to the ideas in [La02]. In general, one expects that the spectral side is easier to handle, but the geometric side might contain the objects one wants to study. Hence one exploits the analytic behaviour of the spectral side. We shall see what this means for the cases GL(2) and GL(3).
More precisely, certain Φ are expected to yield Dirichlet series on the geometric side which contain information on certain objects in its coefficients as, e.g., the number of orders in field extensions of fixed degree. Together with further analysis of the spectral side and application of Tauberian theorems one hopes to obtain asymptotic formulas of these quantities. In the case of $G = \mathrm{GL}(2)$ the main part of the geometric side yields the Shintani zeta function (see Theorem 61) introduced in [Sh75] to study class numbers of binary quadratic forms. Shintani succeeded in proving asymptotic formulas for the mean value of the number of equivalence classes of quadratic forms and regulators of the number fields associated with them. For them first results were already stated by Gauss. Such asymptotics had previously been obtained by Siegel [Si44] by other methods with less precise error terms. This zeta function later was studied in an adelic framework by Yukie [Yu92] and Datskovsky [Da93], and it provides an example of a zeta function associated with a prehomogeneous vector space.

The organisation is as follows:

§II: We are going to show that a particular expansion of the geometric side of the trace formula for GL(3) holds for a large class of test functions, see Theorem 1. This is a generalisation of the results of [FiLa11a] for GL(2), and complements the results of [FiLa11b], where the absolute convergence of the semisimple part was shown for general reductive groups. Hence we are left with distributions associated with non-semisimple data, which leads to subtle convergence issues.

§III: The rest of this thesis is dedicated to the study of various parts of the trace formula as functions of s, when the test functions f_s are inserted. We first prove facts about the analytic behaviour of the geometric part for GL(3) for such test functions. In particular, we show that all distributions except for those associated with the regular elliptic elements have holomorphic continuations at least up to $\Re s > \frac{3}{2}$, see Propositions 29 and 30.

§IV: This chapter deals with the spectral side of the trace formula for $\mathrm{GL}(n)$, $n \geq 2$. It was shown in [MuSp04, FiLaMu11] that a certain expansion holds for a large class of test functions. We are going to plug in the test functions f_s again. This yields a holomorphic function of $s \in \mathbb{C}$ for $\Re s > \frac{n+1}{2}$ by the results of [MuSp04, FiLaMu11]. We show that particular distributions, namely those associated with a Levi subgroup of co-rank 0 or 1 in $\mathrm{GL}(n)$, can be continued meromorphically to all $s \in \mathbb{C}$ (see Theorem 32). In particular, this implies that each spectral term for $\mathrm{GL}(2)$ has a meromorphic continuation to the whole complex plane. This continuation is essentially obtained by deforming contours of integrals. For the remaining distributions we show that they can be holomorphically continued to a larger half plane (Theorem 33), each at least in $\Re s > \frac{n}{2}$. We need to analyse the pole structure of local normalised intertwining operators and their growth on certain subspaces in complexified root spaces for our approach. Moreover, we locate the first poles (the first one already occurs at $s = \frac{n+1}{2}$) and compute the first residue. For $n = 2$ we give a more detailed account of the spectral side connecting it to its well-known form given in [GeJa79].
One could try to use the deformation of contours in general to continue the distribution further. For that one would need to consider integrals in at most $n-1$ variables and one could try to use the method of iterated residues as introduced by Langlands for the analytic continuation of Eisenstein series in [La76]. However, it seems doubtful that all terms possess continuations to all $s \in \mathbb{C}$. As in general the singular hyperplanes are not "admissible" in the sense of [MoWa95], one is led to serious convergence issues, see also the example at the end of IV.iii.iii.

§V: The last chapters purpose is twofold. First, we analyse the geometric parts of the trace formula for $\mathrm{GL}(2)$, and thereby find the Shintani zeta function as a part of it (Theorem 61). We include a quite detailed analysis, since $\mathrm{GL}(2)$ is supposed to serve as a model for more general groups. Motivated by this result, we turn to the analysis of the geometric side for $\mathrm{GL}(3)$ for the test functions f_s, which we are allowed to do by the results of the first chapter. Using the results of the second and third chapter together with some supplementary analysis, we obtain an asymptotic for the sum of certain orbital integrals as a consequence of a Tauberian Theorem (Proposition 68). More precisely, we show that for certain Schwartz-Bruhat functions $\Phi_f \neq 0$ on $\mathrm{Mat}_{3\times 3}(\mathbb{A}_f)$, there exists $\alpha > 0$ such that

$$\sum_{\substack{E/\mathbb{Q} \text{ totally real,} \\ [E:\mathbb{Q}]=3}} \frac{\mathrm{res}_{s=1} \zeta_E(s)}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\substack{\xi \in \mathcal{O}_E \setminus \mathbb{Z} \\ \mathrm{tr}_{E/\mathbb{Q}} \xi^2 \leq X}} \frac{I(\Phi_f, \xi)}{[\mathcal{O}_E : \mathbb{Z}[\xi]]} = \alpha X^{\frac{5}{2}} + o(X^{\frac{5}{2}}) \quad (1)$$

as $X \to \infty$. Here $I(\Phi_f, \xi) \geq 0$ are coefficients associated with Φ_f and ξ by means of orbital integrals. To deduce such an asymptotic, one has to separate the totally real extensions from those having a complex place, which turns out to be conceptionally more difficult than for $\mathrm{GL}(2)$ due to the absence of any prehomogeneous vector space structure. From (1) one readily obtains Corollary 73: There is an $\tilde{\alpha} > 0$ such that

$$\limsup_{X\to\infty} X^{-\frac{5}{2}} \sum_{m_1(E)\leq X} \operatorname*{res}_{s=1} \zeta_E(s) \leq \tilde{\alpha}. \tag{2}$$

Here E ranges over all totally real cubic extensions of $\mathbb{Q}$, whose second successive minimum $m_1(E)$ of the positive definite quadratic form $\xi \mapsto \operatorname{tr}_{E/\mathbb{Q}} \xi^2$ on $\mathcal{O}_E$ is bounded by X. In fact, one expects that the limit of the left hand side in (2) actually exists and equals some suitable $\alpha > 0$. This, at least, is not too far from the truth (see Proposition 74): For any $\varepsilon > 0$, the limit inferior of $X^{-\frac{5}{2}+\varepsilon} \sum_{m_1(E)\leq X} \operatorname{res}_{s=1} \zeta_E(s)$ tends to ∞ as $X \to \infty$. However, we were not yet able to prove the existence and equality of the limit in (2). At least, we construct a sequence of Schwartz-Bruhat functions Φ_f (see Proposition 75), for which the coefficients $I(\Phi_f, \xi)[\mathcal{O}_E : \mathbb{Z}[\xi]]^{-1}$ tend to 1 for all ξ. To prove the equality, it would then suffice to show that this sequence of coefficients converges uniformly in ξ.

There are a lot of questions left unanswered. The probably most obvious one is, whether one can find certain expansions of the geometric side of the trace formula, which converge for our large class of test functions, for more general groups than GL(3), or even at least for GL(n), $n \geq 4$. Although this presumably is true, the present approach of writing down such an expression in an explicit way, is probably not managable in the higher rank cases, as it gets cumbersome even for GL(3).

Restricting our attention to the case GL(3), there are as well a lot of problems left. The first one is to show that the limit of (2) exists and is non-zero. As indicated above, it should therefore suffice to show that the sequence of constructed coefficients converges uniformerly. An alternative approach would be to compute the occuring p-adic orbital integrals for the characteristic function of $\operatorname{Mat}_{3\times3}(\mathbb{Z}_p) \subseteq \operatorname{Mat}_{3\times3}(\mathbb{Q}_p)$. For $n = 2$ this can be done by counting lattices or orders with certain multiplier rings, see, e.g. the example in V.i.iii for GL(2) or also [Fl06, §II.1], but for $n = 3$ this becomes considerably more difficult. If one has overcome such problems, it should as well be possible to obtain asymptotics for cubic fields splitting in a prescribed manner at finitely many places. The example one has in mind is that for quadratic fields and GL(2) for which Datskovsky showed in [Da93] that

$$\lim_{X\to\infty} X^{-1} \sum_{\substack{[E:\mathbb{Q}]=2,\ E\sim x_S \\ m_1(E)\leq X}} \operatorname*{res}_{s=1} \zeta_E(s) = \frac{\zeta(2)}{2} R_{x_S} \prod_{p\notin S} (1 - p^{-2} - p^{-3} + p^{-4}).$$

Here E runs over all quadratic extensions of $\mathbb{Q}$ of splitting signature x_S for a finite set of places S of $\mathbb{Q}$ and R_{x_S} is a suitable constant explicitely given as a product over all places in S. One hopes to show similar results for n-dimensional field extensions in general, i.e. one hopes to obtain asymptotics of the form

$$\sum_{\substack{[E:\mathbb{Q}]=n,\ E\sim_{X_S} \\ m_1(E)\leq X}} \operatorname*{res}_{s=1} \zeta_E(s) \sim \alpha X^{\frac{n(n+1)-2}{4}}$$

for totally real extension for a suitable constant $\alpha > 0$.
For the spectral side the most noticable question is, whether there are indeed distributions, which can not be continued to a meromorphic function on all of $\mathbb{C}$. If the the answer to this question is affirmative, it still might be possible to continue the distributions to a larger half plane than we did, and one could attempt to determine the natural boundary of continuation. However, it seems that at least with our approach the half plane can not be enlarged: To continue the function further along our lines, one is led to the concept of multidimensional residues. But in general the singular hyperplanes of the considered functions are not admissible, which seems to lead to serious difficulties here. Looking more closely at our example in IV.iii.iii, it seems that the terms having no continuation are given as certain distributions supported on singular matrices. Hence one could try to use a trace formula for the whole Lie algebra $\mathfrak{gl}_n(\mathbb{A}) \simeq \mathrm{Mat}_{n\times n}(\mathbb{A})$ as developed in [Ch02], which then also includes distributions associated with singular matrices. This phenomenon appears to be existent already for GL(2) (but does not yield any convergence issues there), see also Remark 56.

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It is always the unreadable that occurs.
(Oscar Wilde, *The Decay of Lying*)

II Absolute convergence of the geometric side of the trace formula for GL(3)

II.i The geometric side of the trace formula for GL(3)

II.i.i Introduction

Let G be a reductive algebraic group over some number field F, and S a finite set of places of F containing all archimedean valuations. The geometric side of the trace formula for G is an absolutely convergent sum over distributions

$$J_{\text{geom}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f)$$

where $f \in C_c^\infty(G(F_S)^1)$ is one of Arthur's test functions, and $\mathcal{O}$ is the set of equivalence classes $\mathfrak{o} \subseteq G(F)$, which are defined via the Jordan decomposition: $\gamma, \gamma' \in G(F)$ are equivalent if and only if their semisimple parts are conjugate in $G(F)$. For $G = \mathrm{GL}(2)$ it was shown in [FiLa11a] that a modification of the fine geometric expansion converges absolutely for a large class of test functions $\mathcal{C}(\mathrm{GL}_2(\mathbb{A})^1, K)$, which now also contains non-compactly supported functions. It was shown in [Ho08] that the coarse geometric expansion converges for rapidly decreasing test functions from the Harish-Chandra space, but the space of test functions considered in [FiLa11a] or [FiLa11b] is more natural in the sense of viewing the trace formula as a non-abelian generalisation of the Poisson summation formula. If $\mathfrak{o}$ consists only of semisimple elements, $J_{\mathfrak{o}}(f)$ for $f \in C_c^\infty(G(F_S)^1)$ can be written as a weighted orbital integral [Ar05, Theorem 11.2]. The sum over all such classes converges absolutely for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$ as shown in [FiLa11b].

For non-semisimple classes the distribution can still be expanded as a sum over distributions resulting from a quite involved limiting process involving weighted orbital integrals, see [Ar05, Theorem 19.2]. For $G = \mathrm{GL}(3)$ and compactly supported test functions all the weights and constants appearing in this expansion have been computed in [Fl82, Lemma 3. and Lemma 4.]. Instead of this expansion one can use Arthur's semisimple descent formula [Ar86, Lemma 6.2] expressing an arbitrary distribution in terms of unipotent distributions to study $J_{\mathfrak{o}}(f)$. If $\mathfrak{o} \in \mathcal{O}$, let σ denote the semisimple part of this class. Let $G_\sigma \subseteq G$ be the centraliser of σ in G, and let $\mathcal{F}^\sigma$ be the set of all parabolic subgroups in G_σ which contain the maximal split torus T of diagonal matrices. If $R \in \mathcal{F}^\sigma$, let M_R be the unique Levi component of R containing T. Then Arthur's formula for the semisimple descent is

$$J_{\mathfrak{o}}(f) = \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} \sum_{R\in\mathcal{F}^\sigma} \frac{|W^{M_R}|}{|W^{G_\sigma}|} J_{\text{unip}}^{M_R}(\Phi_{R,y,T})dy,$$

where $\Phi_{R,y,T}$ is a function on M_R defined in [Ar86, p. 201]. Here $T \in \mathfrak{a}_0$ chosen according to [Ar81, Lemma 1.1].

If not specified otherwise, $G = \mathrm{GL}(3)$ for the rest of this chapter. For this, the equivalence relation above reduces to a simple linear algebra criterion: $\gamma, \gamma' \in G(F)$ are contained in the same equivalence class $\mathfrak{o}$ if and only if their characteristic polynomials are the same (cf. [Ar05, pp. 53-54]). Hence the equivalence classes $\mathfrak{o} \in \mathcal{O}$ are parametrised by monic cubic polynomials $\chi_\mathfrak{o} \neq 0$ with coefficients in F. We may subdivide $\mathcal{O}$ according to the splitting properties of the polynomials $\chi_\mathfrak{o}$, $\mathfrak{o} \in \mathcal{O}$: Write $\mathcal{O} = \coprod_{i=1}^{3} \mathcal{O}^i$ with $\mathcal{O}^i$ the set of all $\mathfrak{o} \in \mathcal{O}$, whose characteristic polynomial factorises in i irreducible polynomials over F. The classes in $\mathcal{O}^1$ and $\mathcal{O}^2$ consist entirely of semisimple elements, and thus the distributions corresponding with such classes were already treated in [FiLa11b] The set $\mathcal{O}^3$ decomposes disjointly into three sets $\mathcal{O}^3_{\text{cent}}$, $\mathcal{O}^3_{\text{quad}}$, $\mathcal{O}^3_{\text{reg}}$, according to whether the roots of $\chi_\mathfrak{o}$ build a set of one, two or three elements. In the last the equivalence classes again are semisimple. For the other two cases each $\mathfrak{o}$ also contains non-semisimple elements, and $\mathfrak{o}$ is no longer an actual conjugacy class, but splits in a disjoint union of finitely many conjugacy classes, which bijectivly correspond to the minimal polynomial of the respective orbit. Thus we will only need to consider those equivalence classes $\mathfrak{o}$ which contain some non-semisimple element, i.e. the classes in $\mathcal{O}^3_{\text{quad}}$ and $\mathcal{O}^3_{\text{cent}}$.

We partly use the results from [Fl82] directly and modify them analogous to [FiLa11a] for the unipotent orbits. For the equivalence classes in $\mathcal{O}^3_{\text{quad}}$ this approach turns out to be a tediuos task. Instead the semisimple descent allows us to separate the terms which need modifications from those, which already converge for our functions.

II.i.ii Notation

We keep the notation introduced so far and additionally use the following. Most of it coincides with the notation Arthur uses in [Ar86, Ar05]. Let $G = \mathrm{GL}(n)$ for some $n \in \mathbb{N}$ and let $T \subseteq \mathrm{GL}(n)$ be the torus of diagonal matrices. $\mathcal{L}$ denotes the set of all Levi subgroups containing T, and for any $M \in \mathcal{L}$, $\mathcal{F}(M) = \mathcal{F}^G(M) = \{P \subseteq G \mid \text{parabolic}, M \subseteq P\}$ and $\mathcal{P}(M) = \mathcal{P}^G(M) = \{P \in \mathcal{F}(M) \mid M \text{ is Levi component of } P\}$. Fix a Borel subgroup $P_0 \in \mathcal{F}(T)$. For $\mathrm{GL}(n)$ we shall always take P_0 to be the group of upper triangular matrices. $P \in \mathcal{F}(T)$ is called a standard parabolic subgroup if $P_0 \subseteq P$. Write $\mathcal{L}(M) = \{L \in \mathcal{L} \mid M \subseteq L\}$ and for any $L \in \mathcal{L}(M)$, $\mathcal{P}^L(M)$ and $\mathcal{F}^L(M)$ are defined as above, but with G replaced by L. If $P \in \mathcal{F}(T)$, denote by $M_P \in \mathcal{L}$ the Levi component and by U_P the unipotent radical of P. If $P \in \mathcal{F}(T)$, let $\overline{P} \in \mathcal{F}(T)$ be the opposite parabolic. For any $M \in \mathcal{L}$, $Z^M \subseteq M$ denotes the center of M, $Z := Z^G$. Let $A_M \subseteq Z^M(F_\infty)$ denote the component of $1 \in Z^M(F_\infty)$ so that in particular, $A_G \simeq \mathbb{R}_{>0}$. Let $\mathfrak{a}_M = \mathrm{Hom}_\mathbb{R}(X(M), \mathbb{R})$

for $X(M)$ the group of characters $Z^M \longrightarrow \mathbb{C}$, defined over F, and $\mathfrak{a}_P = \mathfrak{a}_{M_P}$, $P \in \mathcal{F}(T)$. Let $\mathfrak{a}_M^*$ and $\mathfrak{a}_P^*$ be the respective dual spaces, and denote by $< \cdot, \cdot >$ the pairing $\mathfrak{a}_M \times \mathfrak{a}_M^* \longrightarrow \mathbb{C}$. There is a map $H_M : M(\mathbb{A}) \longrightarrow \mathfrak{a}_M$, defined by $< H_M(m), \lambda >= |\log \lambda(m)| =: \lambda(H_M(m))$. If $L, M \in \mathcal{L}$, $M \subseteq L$, there is a canonical surjection $\mathfrak{a}_M \longrightarrow \mathfrak{a}_L$ whose kernel is denoted by $\mathfrak{a}_M^L$. Similarly, there is a surjection $\mathfrak{a}_M \longrightarrow \mathfrak{a}_L$ for which the kernel is denoted by $(\mathfrak{a}_M^L)^*$. For $\mathrm{GL}(n)$ there are unique standard parabolic subgroups and Levi subgroups in $\mathcal{L}$ associated with any partition $(n_1, \ldots, n_r)$ of n. Let $\beta_1, \ldots, \beta_{n-1} \in \mathfrak{a}_T^*$ be the simple roots of (P_0, A_{P_0}). To shorten notation for the case $\mathrm{GL}(3)$, we shall denote by P_1, M_1, U_1 the standard parabolic subgroup, respectively Levi component, respectively unipotent subgroup associated with β_1, i.e. such that $(\mathfrak{a}_T^M)^* = \mathbb{R}\beta_1$. The subscript 2 indicates the respective groups for β_2, and the subscript 3 for $\beta_1 + \beta_2$.

Fix a finite set S of places of F containing all archimedean places. ζ_F is the Dedekind zeta function associated with F. For some place v of F, $\zeta_{F,v}(s)$ denotes the local, and $\zeta_F^*(s)$ completed Dedekind zeta function associated with F, for which we define the factors at the archimedean places as follows: $\zeta_v(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ if v is real, and $\zeta_v(s) = \pi^{-s+1}\Gamma(s)$ if v is complex. For a finite place v of F, we denote by $\mathcal{O}_v \subseteq F_v$ the ring of v-adic integers with uniformiser ϖ_v and corresponding norm $|\cdot|_v$ such that $|\varpi_v|_v^{-1} =: q_v \in \mathbb{Z}$. For any finite set of places S, let $|a|_S = \prod_{v \in S} |a|_v$, $a \in \mathbb{A}$, in particular $|a|_\infty = \prod_{v|\infty} |a|_\infty$, and let $|a| = \prod_{v \leq \infty} |a|_v$. Let $\mathcal{N}_{F/\mathbb{Q}} : F \longrightarrow \mathbb{Q}$ be the norm map, i.e. $\mathcal{N}_{F/\mathbb{Q}}(a) = |a|_\infty$ for any $a \in F$.

If S' is any set of places, $\zeta_F^{S'}(s) = \prod_{v \notin S'} \zeta_{F,v}(s)$, and $\zeta_{F,S'}(s) = \prod_{v \in S'} \zeta_{F,v}(s)$. $\mathbb{A} = \mathbb{A}_F$ is the ring of adeles of F, and for any finite set S', $\mathbb{A}_{S'} = \prod_{v \in S'} F_v$ and $\mathbb{A}^{S'} = \prod'_{v \notin S'} F_v$ with $\prod'$ denoting the restricted product with respect to $\{\mathcal{O}_v\}_v$ for $\mathcal{O}_v$ the ring of integers of F_v. Let $\mathbb{A}^1 := \{a \in \mathbb{A}^\times \mid |a| = 1\}$. We shall use the standard maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ with

$$\mathbf{K}_v = \begin{cases} G(\mathcal{O}_v) & \text{if } v < \infty, \\ \mathrm{O}(3) & \text{if } v \text{ is real} \\ \mathrm{U}(3) & \text{if } v \text{ is complex.} \end{cases}$$

We choose measures as follows: For non-archimedean v we normalise the additive and multiplicative measures on F_v and $F_v^\times$ such that $\mathcal{O}_v$ and $\mathcal{O}_v^\times$ both have measure 1. On $\mathbb{R}$ and $\mathbb{C}$ we take the usual Lebesgue measure. Then the measures on $\mathbb{A}$ and $\mathbb{A}^\times$ are the product measures. The measures on $\mathbf{K}$ and $\mathbf{K}_v$ for any v are then normalised such that both groups have volume 1. On any unipotent subgroup of G we take the measure induced by the additive measure on $\mathbb{A}$ or F_v, and on $T(\mathbb{A})$ or $T(F_v)$ the multipicative measure of $\mathbb{A}^\times$ or $F_v^\times$. The remaining measures of $G(\mathbb{A}), G(F_v)$, and all parabolic and Levi subgroups are then chosen such that they are compatible with the Iwasawa decomposition. With such measures we have $\mathrm{vol}(F^\times\backslash\mathbb{A}^1) = \mathrm{res}_{s=1}\,\zeta_F(s)|D_F|^{\frac{1}{2}}$ and $\mathrm{vol}(F\backslash\mathbb{A}) = 1$ (see [La86, Chapter XIV], but take account of the different normalisation of $\mathcal{O}_v$ there). We choose measures on $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$ compatible with the pairing above, normalised such that the polytope spanned by the simple roots in $\mathfrak{a}_M^*$ has volume 1.

Denote by $C_c^\infty(G(F_S)^1)$ the space of all smooth, compactly supported functions $G(\mathbb{A})^1 \longrightarrow \mathbb{C}$ such that $f = f_S f^S$ with a smooth compactly supported function $f_S : G(F_S)^1 \longrightarrow \mathbb{C}$, and $f^S : G(\mathbb{A}^S) \longrightarrow \mathbb{C}$ the characteristic function of $\mathbf{K}^S \subseteq G(\mathbb{A}^S)$. This is the class of test functions usually used by Arthur.

The space of test functions $\mathcal{C}(G(\mathbb{A})^1, K)$ we are interested in is defined as follows. If G is an arbitrary reductive group over F with maximal compact subgroup $\mathbf{K} \subseteq G(\mathbb{A}_f)$ and maximal torus T, $P \subseteq G$ a parabolic subgroup with Levi subgroup $M \supseteq T$ and unipotent radical U such that $P = MU$, then we denote by $\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}$, and $\mathfrak{k}_\mathbb{C}$ the complexified Lie-algebras of $G(F_\infty), T(F_\infty)$, and $\mathbf{K}_\infty$, respectively. For a Lie algebra $\mathfrak{g}$ let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ with basis $\mathcal{B}_\mathfrak{g}$. For a compact subgroup $K \subseteq G(\mathbb{A}_\mathrm{f})$ with $K_v = \mathbf{K}_v$ for almost all v, and $N \in \mathbb{N} \cup \{\infty\}$, $N > \dim \mathfrak{g}_\mathbb{C}$, let $\mathcal{C}^N(G(\mathbb{A})^1, K)$ be the space of all continuous functions $f : K\backslash G(\mathbb{A})^1/K \longrightarrow \mathbb{C}$, which are differentiable up to degree N and for which $||X * f||_{L^1(K\backslash G(\mathbb{A})^1/K)}$ is finite for all $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C})_{\leq N}$. Let $\mathcal{C}(G(\mathbb{A})^1, K) = \mathcal{C}^\infty(G(\mathbb{A})^1, K)$. Let $\mathcal{C}(G(F_\infty)^1)$ be the space of all smooth functions $h : G(F_\infty)^1 \longrightarrow \mathbb{C}$ such that $||X * h||_{L^1(G(F_\infty)^1)} < \infty$ for all $X \in \mathcal{U}(\mathfrak{g})$. Denote by $\mu(h)$ the semi-norm on $\mathcal{C}^N(G(F_\infty)^1)$ defined by

$$\sum_{X \in \mathcal{B} \subseteq \mathcal{U}(\mathfrak{g}_\mathbb{C})_{\leq \dim \mathfrak{g}_\mathbb{C}}} ||X * h||_{L^1(G(F_\infty)^1)}.$$

In the following we fix an open-compact subgroup $K \subseteq \mathbf{K}_f \subseteq \mathrm{GL}_3(\mathbb{A}_f)$ such that $K_v = \mathbf{K}_v$ for almost all places v. For an arbitrary Levi subgroup M of G denote by W_0^M the Weyl group of M with respect to T.

II.i.iii Results for the geometric side for GL(3)

We want to give a modification of the geometric side of the trace formula for GL(3) such that it still coincides with the geometric side of the trace formula for smooth compactly supported test functions, but also converges absolutely for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$. We start with the coarse geometric expansion and modify the distributions $J_\mathfrak{o}$ individually. As mentioned above, we only need to treat those $\mathfrak{o}$ which contain non-semisimple elements. To state the main result, we need some more definitions. Recall that there is a map $H_P : G(\mathbb{A}) \longrightarrow \mathfrak{a}_P$, which is defined via Iwasawa decomposition by $H_P(muk) = H_{M_P}(m)$, $m \in M_P(\mathbb{A})$, $u \in U_P(\mathbb{A})$, $k \in \mathbf{K}$. If $M \in \mathcal{L}$, there is a function

$v_M : G(\mathbb{A}) \longrightarrow \mathbb{C}$ associating with $x \in G(\mathbb{A})$ the volume of the convex hull in $\mathfrak{a}_M^G$ of the points

$$\{-H_P(x) \mid P \in \mathcal{P}(M)\}.$$

There are other equivalent descriptions of v_M, in particular, as a multidimensional derivative of a certain (G, M)-family $\{v_P \mid P \in \mathcal{P}(M)\}$, which we need later. The volume v_M occurs as a weight function in the invariant orbital integrals belonging to the semisimple terms on the geometric side. For the non-invariant integrals, there are certain modifications necessary, and even further modifications are necessary to obtain convergent expressions for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$.
With the (G, M)-family $\{v_P\}$ there is associated a (G, M) family $\{v'_P\}$ as in [Ar81]. If $\sigma \in G(F)$ is some semisimple element whose centraliser equals a Levi subgroup $M \in \mathcal{L}$, and if $R \in \mathcal{F}^M(T)$, let as in [Ar86, §6] $\nu'_R = \sum_Q v'_Q$ with $Q \in \mathcal{F}(T)$ running over all parabolics such that $M_Q = M_R$ and the centraliser of σ in Q equals R.

We recall the definition of the functions $\lambda_{t,S}$, $t \in T^{\mathrm{GL}(2)}(F)$, and $\omega^{\mathrm{GL}(2),S} : T^{\mathrm{GL}(2)}(F) \times N(\mathbb{A}) \longrightarrow \mathbb{C}$ from [FiLa11a] ($\omega^{\mathrm{GL}(2),S}$ was denoted by ω there). Here $T^{\mathrm{GL}(2)} \subseteq \mathrm{GL}(2)$ is the torus of diagonal matrices, and $N \subseteq \mathrm{GL}(2)$, the unipotent subgroup of upper triangular matrices. Let $\zeta_F^S(1+s) = \lambda_{-1}^S s^{-1} + \lambda_0^S + \lambda_1^S s + \dots$ be the Laurent expansion of the truncated Dedekind zeta function $\zeta_F^S(s)$ around $s = 1$. For $t = \mathrm{diag}(t_1, t_2) \in T^{\mathrm{GL}(2)}(F) \backslash Z^{\mathrm{GL}(2)}(F)$, let

$$\lambda_{t,S} = -\sum_{v \notin S,\ |t_1|_v = |t_2|_v} \frac{1 - |1 - \frac{t_2}{t_1}|_v}{q_v - 1} \log q_v,$$

and for $t = \mathrm{diag}(t, t) \in Z^{\mathrm{GL}(2)}(F)$,

$$\lambda_{t,S} = \lambda_{1,S} = \frac{\lambda_0^S}{\lambda_{-1}^S}.$$

The other weight is defined as a sum of local functions $\omega^{\mathrm{GL}(2),S}(t, u) = \sum_{v \leq \infty} \omega_v^{\mathrm{GL}(2),S}(t, u)$, $t = \mathrm{diag}(t_1, t_2) \in T^{\mathrm{GL}(2)}(F)$, $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N(\mathbb{A})$, each defined by

$$\omega_v^{\mathrm{GL}(2),S}(t, u) = \begin{cases} \log \max\{|1 - \frac{t_2}{t_1}|_v, |x_v|_v\} & \text{if } v < \infty, v \in S, \\ \log \sqrt{|1 - \frac{t_2}{t_1}|_v^2 + x_v^2} & \text{if } v|\infty, v \text{ real}, \\ \log(|1 - \frac{t_2}{t_1}|_v + |x_v|_v) & \text{if } v|\infty, v \text{ complex}, \end{cases}$$

for $v \in S$, and for $v \notin S$, by

$$\omega_v^{\mathrm{GL}(2),S}(t, u) = \begin{cases} \omega_v^{\mathrm{GL}(2),S}(t, u) & \text{if } |t_1|_v \neq |t_2|_v, \\ \omega_v^{\mathrm{GL}(2),S}(t, u) + \frac{1 - |1 - \frac{t_2}{t_1}|_v}{q_v - 1} \log q_v & \text{if } |t_1|_v = |t_2|_v > 1, \\ 0 & \text{if } |t_1|_v = |t_2|_v \leq 1. \end{cases} \tag{3}$$

We shall also write $\omega^{\mathrm{GL}(2),S}(t,u)=\omega^{\mathrm{GL}(2),S}(t,x)$, and similarly for the local functions. There are a few more functions occuring in the statement of the main result. They are all arising as modifications of the weight functions in higher dimensional cases as we will see later.

Put
$$\Lambda_S=\left(\frac{\lambda_0^S}{\lambda_{-1}^S}\right)^2+\frac{\lambda_1^S}{\lambda_{-1}^S}.$$
For each place v define local functions $\Omega_{M_0,v}^S: U_0(F_v)\longrightarrow\mathbb{C}$, $\alpha_v^S: F_v^\times\longrightarrow\mathbb{C}$, by
$$\Omega_{M_0,v}^S(u)=\frac{3}{2}\frac{\lambda_0^S}{\lambda_{-1}^S}(\omega_v^{\mathrm{GL}(2),S}(1,u_1)+\omega_v^{\mathrm{GL}(2),S}(1,u_3))$$
for $u=\left(\begin{smallmatrix}1&u_1&u_2\\&1&u_3\\&&1\end{smallmatrix}\right)\in U_0(F_v)$, and
$$\alpha_v^S(x)=\begin{cases}\log|x|_v & \text{if } v\in S\\ -\frac{\zeta_{F,v}'(2)}{\zeta_{F,v}(2)}+\log|x|_v & \text{if } v\notin S, \text{ and } |x|_v>1\\ 0 & \text{if } v\notin S, \text{ and } |x|_v\le 1\end{cases}$$
for $x\in F_v$. Then we put $\Omega_{M_0}^S(t,u)=\sum_{v\le\infty}\Omega_{M_0,v}^S(t,u_v)$, and $\alpha^S(x)=\sum_{v\le\infty}\alpha_v^S(x)$. The last weight function we need to define is $\omega_{M_0}: U(\mathbb{A})\longrightarrow\mathbb{C}$, which is given by
$$\begin{aligned}\omega_{M_0}^S(u)=\omega^{\mathrm{GL}(2),S}(u_1)\omega^{\mathrm{GL}(2),S}(u_3)+\frac{1}{4}(\omega^{\mathrm{GL}(2),S}(u_1)^2+\omega^{\mathrm{GL}(2),S}(u_3)^2)\\ -\frac{1}{4}\sum_{w\notin S}(\eta_w(u_1)+\eta_w(u_3)),\end{aligned}$$
where
$$\eta_w(x)=\begin{cases}\frac{(\log q_w)^2}{q_w-1}+\left(\frac{\log q_w}{q_w-1}\right)^2 & \text{if } |x|_w>1\\ 0 & \text{if } |x|_w\le 1.\end{cases}$$
Note that $\omega_{M_0}(u)=\sum_{v,w\le\infty}\omega_{M_0,v,w}(u)$ with $\omega_{M_0,v,w}(u)$ only depending on the local components u_v, u_w.

Collecting all the partial results from Propositions 10, 16, 20, and 26 the next sections contain the following theorem.

Theorem 1. *Let $f\in C_c^\infty(\mathrm{GL}_3(\mathbb{A})^1)$ such that f_v is the characteristic function of $\mathbf{K}_v$ for $v\notin S$, and set $f_{\mathbf{K}}(g)=\int_{\mathbf{K}}f(k^{-1}gk)dk$. Then the geometric side of the trace formula $J_{geom}(f)$ equals the sum of the following:*

(i) The semisimple part which is given by
$$\sum_{[\gamma]\subseteq G(F)_{ss}}\int_{A_{M(\gamma)}C(\gamma,F)\backslash G(\mathbb{A})}f(x^{-1}\gamma x)v_{M(\gamma)}(x)dx,$$

here $G(F)_{ss} \subseteq G(F)$ *denotes the set of semisimple elements, and* $[\gamma] \subseteq G(F)_{ss}$ *some conjugacy class, and* $C(\gamma, F)$ *denotes the centraliser of* γ *in* $G(F)$*,*

(ii) the part belonging to the regular unipotent orbit in $\mathrm{GL}(3)$*:*

$$\nu(T) \sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} f_{\mathbf{K}}(tu)\omega^S_{M_0}(u)du + \nu(T) \sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} f_{\mathbf{K}}(tu)\Omega^S_{M_0}(u)du + \nu(T)\Lambda_S \sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} f_{\mathbf{K}}(tu)du,$$

where $\nu(T) = \mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)$*,*

(iii) the part belonging to the minimal unipotent orbit in $\mathrm{GL}(3)$*:*

$$3\,\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^2 \sum_{t\in Z(F)} \int_{\mathbb{A}^\times} f_{\mathbf{K}}(t \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1\end{pmatrix})|x|^2\alpha^S(x)d^\times x - 3\,\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^2 \frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} \sum_{t\in Z(F)} \int_{\mathbb{A}^\times} f_{\mathbf{K}}(t \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1\end{pmatrix})|x|^2 d^\times x,$$

(iv) the part belonging to the regular unipotent orbit in M_1*:*

$$\nu(T) \sum_{t\in Z^{M_1}(F)_{reg}} \int_{N(\mathbb{A})} \int_{U_1(\mathbb{A})} f_{\mathbf{K}}(u^{-1}tnu)\omega^{\mathrm{GL}_2,S}(t_1,n)v_{M_1}(u)dudn + \nu(T)\lambda_{1,S} \sum_{t\in Z^{M_1}(F)_{reg}} \int_{N(\mathbb{A})} \int_{U_1(\mathbb{A})} f_{\mathbf{K}}(u^{-1}tnu)v_{M_1}(u)dudn,$$

where $N = \{\begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1\end{pmatrix}\} \subseteq \mathrm{GL}(3)$*, and* $Z^{M_1}(F)_{reg} = Z^{M_1}(F)_{reg}\backslash Z(F)$*.*

(v) and the remaining part:

$$\nu(T) \sum_{R\in\{B\times\mathrm{GL}(1), \bar{B}\times\mathrm{GL}(1)\}} \sum_{t\in Z^{M_1}(F)_{reg}} \int_{U_1(\mathbb{A})} \int_{N_R(\mathbb{A})} f_{\mathbf{K}}(u^{-1}tnu)\nu'_R(u)dndu$$

with N_R *the unipotent radical of* R *considered as a parabolic subgroup in* M_1*.*

Moreover, each of the following sum-integrals converges for all test functions $f \in \mathcal{C}(\mathrm{GL}_3(\mathbb{A})^1, K)$*:*

(i)

$$\sum_{[\gamma]\subseteq G(F)_{ss}} \int_{A_{M(\gamma)}C(\gamma,F)\backslash G(\mathbb{A})} |f(x^{-1}\gamma x)||v_{M(\gamma)}(x)|dx,$$

(ii)

$$\sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} |f_{\mathbf{K}}(tu)| \sum_{v,w} |\omega^S_{M_0,v,w}(u)|du,$$

$$\sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} |f_{\mathbf{K}}(tu)| \sum_{v} |\Omega^S_{M_0,v}(u)|du,$$

$$\sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} |f_{\mathbf{K}}(tu)|du$$

(iii)

$$\sum_{t\in Z(F)} \int_{\mathbb{A}^\times} |f_{\mathbf{K}}(t \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix})||x|^2 \sum_{v} |\alpha^S_v(x)|d^\times x,$$

$$\sum_{t\in Z(F)} \int_{\mathbb{A}^\times} |f_{\mathbf{K}}(t \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix})||x|^2 d^\times x,$$

(iv)

$$\sum_{t\in Z^{M_1}(F)_{reg}} \int_{N(\mathbb{A})} \int_{U_1(\mathbb{A})} |f_{\mathbf{K}}(u^{-1}tnu)| \sum_{v} |\omega^{\mathrm{GL}_2,S}_v(t_1\mathbf{1}_2,n)||v_{M_1}(u)|dudn,$$

$$\sum_{t\in Z^{M_1}(F)_{reg}} \int_{N(\mathbb{A})} \int_{U_1(\mathbb{A})} |f_{\mathbf{K}}(u^{-1}tnu)||v_{M_1}(u)|dudn,$$

(v) and

$$\sum_{R\in\{B\times\mathrm{GL}(1),\bar{B}\times\mathrm{GL}(1)\}} \sum_{t\in Z^{M_1}(F)_{reg}} \int_{U_1(\mathbb{A})} \int_{N_R(\mathbb{A})} |f_{\mathbf{K}}(u^{-1}tnu)||\nu'_R(u)|dndu.$$

The assertion about (i) is shown in [FiLa11b], and as mentioned before, we only need to consider the non-semisimple parts. To prove the theorem we now proceed as follows: We first show that we only need to consider very special test functions f. Then we continue by considering each type of equivalence class and any conjugacy class therein separately.

Remark 2. *We should note that we shall not make use of the smoothness of the test functions in $\mathcal{C}(G(\mathbb{A})^1, K)$. In fact, the theorem stays true for test functions in $\mathcal{C}^N(G(\mathbb{A})^1, K)$ for N sufficiently large, since the results from [FiLa11b] stay true for $f \in \mathcal{C}^N(G(\mathbb{A})^1, K)$, $N \gg 0$ as well.*

II.ii Preparations for the proof of the absolute convergence

II.ii.i Reduction of the proof to special functions

To show the absolute convergence of the geometric side we do not need to consider a general $f \in \mathcal{C}(G(\mathbb{A})^1, K)$, but rather functions of a quite special form as those considered in [FiLa11a] for GL(2), which allows us to find good bounds for the integrals, see (4).

Denote by $\operatorname{Div}(F)$ the divisor group of $\mathcal{O}_F$, i.e. all formal sums of the form $D = \sum_{v<\infty} D_v \mathfrak{p}_v$ with $D_v \in \mathbb{Z}$ and almost all $D_v = 0$. This group is canonically isomorphic to the class of all fractional ideals in $\mathcal{O}_F$ (cf. [Ne99, Chapter 1, §12]), but here the language of divisors seems to be more suitable. Write 1 for the unit element having $D_v = 0$ at all places. We denote by $\operatorname{Div}^+(F)$ the subgroup of all non-negative divisors, i.e. those divisors with $D_v \geq 0$ for all v. (As we draw our intuition from the case $F = \mathbb{Q}$, it seems most natural to denote the unit in $\operatorname{Div}(F)$ by 1.) If D_1, D_2 are two divisors, we write $D_1 \geq D_2$ if $D_{1,v} \geq D_{2,v}$ for all places v. By [Gr98, Proposition 2.6] there is for any place $v < \infty$ a bijection

$$\mathbb{N}_0^3 \longrightarrow \mathbf{K}_v \backslash G(F_v)/\mathbf{K}_v,\ (\eta_1, \eta_2, \eta_3) \mapsto \mathbf{K}_v \varpi_v^{\eta_1} \operatorname{diag}(\varpi_v^{\eta_2+\eta_3}, \varpi_v^{\eta_2}, 1)\mathbf{K}_v,$$

and hence there is also a bijection

$$\begin{aligned}&\operatorname{Div}(F)^3 \longrightarrow \mathbf{K}_f \backslash G(\mathbb{A}_f)/\mathbf{K}_f,\\ &\quad (D_1, D_2, D_3) \mapsto \mathbf{K}_f (\varpi_v^{D_{1,v}})_v \operatorname{diag}((\varpi_v^{D_{2,v}+D_{3,v}})_v, (\varpi_v^{D_{2,v}})_v, 1)\mathbf{K}_f.\end{aligned}$$

If $r, N_1, N_2 \in \operatorname{Div}(F)$, $N_1 \geq N_2 \geq 0$, $r \geq 0$, we write $T_{r,N_1,N_2} : G(\mathbb{A}_f) \longrightarrow \mathbb{C}$ for the characteristic function of the double coset

$$\mathbf{K}_f (\varpi_v^{r_v})_v \operatorname{diag}((\varpi_v^{N_{1,v}})_v, (\varpi_v^{N_{2,v}})_v, 1)\mathbf{K}_f.$$

If v is a finite place and $\eta, \varepsilon_1, \varepsilon_2 \in \mathbb{Z}$, $\varepsilon_1 \geq \varepsilon_2 \geq 0$, let $f_{\eta,\varepsilon_1,\varepsilon_2} : G(F_v) \longrightarrow \mathbb{C}$ be the characteristic function of the double coset

$$\varpi_v^{\eta} \mathbf{K}_v \operatorname{diag}(\varpi_v^{\varepsilon_1}, \varpi_v^{\varepsilon_2}, 1)\mathbf{K}_v$$

so that $T_{r,N_1,N_2} = \prod_{v:\ r_v+N_{1,v}\neq 0} f_{r_v,N_{1,v},N_{2,v}}$. Let $\mathrm{P}(F) \subseteq \operatorname{Div}(F)$ be the subgroup of principal divisors, i.e. the group of divisor of the form $D(a) := \sum_{v<\infty} v_{\mathfrak{p}_v}(a)\mathfrak{p}_v$ for some $a \in F^\times$, and let $\mathrm{P}^+(F) = \mathrm{P}(F) \cap \operatorname{Div}^+(F)$. This last semigroup is canonically isomorphic to the semigroup of integral principal ideals in $\mathcal{O}_F$. The canonical map

$$F^\times \longrightarrow \mathrm{P}(F),\ a \mapsto D,$$

is surjective and each fibre is isomorphic to $\mathcal{O}_F^\times$. This is clear by using that $\mathrm{P}(F)$ is isomorphic to the group of all fractional principal ideals.

Let $f \in \mathcal{C}(G(\mathbb{A})^1, K)$. As each term on the geometric side is invariant under replacing the test function f by $\int_{\mathbf{K}} f(k^{-1} \cdot k)dk$, we may assume from the outset that f is $\mathbf{K}$-central, i. e. invariant under conjugation with $\mathbf{K}$, and we may also suppose $f(g) \geq 0$ for all $g \in G(\mathbb{A})^1$ by [FiLa11b, Lemma 3.4]. As explained in [FiLa11a, §4], we now reduce to test functions of a very special form. Consider the map

$$\varphi : G(\mathbb{A})^1 \longrightarrow K\backslash G(\mathbb{A}_f)/K, \quad g \mapsto Kg_fK,$$

$g_f \in G(\mathbb{A}_f)$ denotes the finite part of g. As f is K-biinvariant,

$$f = \sum_{x \in K\backslash G(\mathbb{A}_f)/K} f_{|\varphi^{-1}(KxK)},$$

where for any $g \in G(\mathbb{A})^1$, $f_{|\varphi^{-1}(KxK)}(g) = f(g)\chi_{KxK}(g_f)$, and $\chi_{KxK} : G(\mathbb{A}_f) \longrightarrow \mathbb{C}$ denotes the characteristic function of KxK. Hence if we write $g = g_\infty g_f$ with $g_\infty \in G(F_\infty)^1$, and $x = sx^1$ with $x^1 \in G(\mathbb{A})^1$ and $s \in \mathbb{R}_{>0}$, we have $f_{|\varphi^{-1}(KxK)}(g) = f(g_\infty x^1)\chi_{KxK}(g_f)$. Defining $f_{\infty,x} : G(F_\infty)^1 \longrightarrow \mathbb{C}$ by $g_\infty \mapsto f(g_\infty x^1)$, we obtain a function in $\mathcal{C}(G(F_\infty^1))$. By the above isomorphism, there are divisors $r, N_1, N_2 \in \mathrm{Div}(F)$ such that $T_{r,N_1,N_2}(x) \neq 0$. Since $[\mathbf{K}_f : K] < \infty$, we get

$$||f_{|\varphi^{-1}(KxK)}||_{L^1(K\backslash G(\mathbb{A})^1/K)} \geq [\mathbf{K}_f : K]^{-2} \deg T_{r,N_1,N_2} ||f_{\infty,x}||_{L^1(G(F_\infty)^1)}$$

with the degree of the Hecke operator defined by

$$\deg T_{r,N_1,N_2} = \int_{G(\mathbb{A}_f)} T_{r,N_1,N_2}(x)dx,$$

and the inequality

$$||f_{|\varphi^{-1}(KxK)}||_{L^1(K\backslash G(\mathbb{A})^1/K)} \leq \deg T_{r,N_1,N_2} ||f_{\infty,x}||_{L^1(G(F_\infty)^1)}$$

holds trivially. By definition,

$$||f||_{L^1(K\backslash G(\mathbb{A})^1/K)} = \sum_{x \in K\backslash G(\mathbb{A}_f)/K} ||f_{|\varphi^{-1}(KxK)}||_{L^1(K\backslash G(\mathbb{A})^1/K)},$$

and in all the considerations we may replace f with $X * f$ for $X \in \mathcal{U}(\mathfrak{g})$. Hence it suffices to consider functions f of the form

$$f(g) = f_\infty(g_\infty) T_{r,N_1,N_2}(g_f) \tag{4}$$

with $f_\infty \in \mathcal{C}(G(F_\infty)^1)$, and to show that each of the terms in Theorem 1 is bounded by $O(\deg T_{r,N_1,N_2})\mu(f_\infty)$. We may even suppose that $r = 1$ as it only shifts our function.

Hence to show that the terms are bounded as asserted we need to know the degree of the Hecke operator.

Lemma 3. *Let $N_1, N_2 \in \mathrm{P}^+(F)$ and identify them with the respective principle ideals. Choose generator of such ideals, and again denote them by $N_1, N_2 \in \mathcal{O}_F$. Then the degree of the Hecke operator T_{1,N_1,N_2} equals*

$$\deg T_{1,N_1,N_2} = \mathcal{N}_{F/\mathbb{Q}}(N_1)^2 \prod_{v:|N_1|_v<1} (1 + q_v^{-1} + q_v^{-2})\kappa(v)$$

where $\kappa(v) = 1$ if either $\frac{N_1}{N_2}$ or N_2 is a unit in $\mathcal{O}_v$, and $\kappa(v) = 1 + q_v^{-1}$ otherwise.

We shall see in Corollary 6 that only $N_1, N_2 \in \mathrm{P}^+(F)$ are relevant. Hence we restricted our intention to such cases. The choice of $N_1, N_2 \in \mathcal{O}_F$ is unique up to multiplication with units in $\mathcal{O}_F^\times$, which do not change anything as T_{1,N_1,N_2} is invariant under $\mathcal{O}_F^\times$.

Proof. As $\deg T_{1,N_1,N_2} = \int_{G(\mathbb{A}_f)} T_{1,N_1,N_2}(x)dx$, the left hand side is the product over the degrees of the local Hecke operators at all places which divide N_1. To compute this local degree we use [Gr98, Lemma 7.4]. Let v be a place with $|N_1|_v < 1$. Let $\lambda = (\varepsilon_1, \varepsilon_2, 0)$, where ε_i is the valuation of N_i at v. In particular $\varepsilon_1 \geq \varepsilon_2 \geq 0$ so that $< \lambda, 2\rho >= 2\varepsilon_1$ for ρ the half-sum of all positive roots of GL(3), and $< \lambda, \alpha >\geq 0$ for all positive roots α of GL(3). Suppose first that neither $\frac{N_1}{N_2}$ nor N_2 is a unit in $\mathcal{O}_v$. This means that $\varepsilon_1 > \varepsilon_2 > 0$ so that $< \lambda, \alpha >< 0$ for all negative roots α. Thus in the notation of [Gr98], $P_\lambda = P_0$ the standard Borel subgroup, and $\dim(G/P_\lambda) = 3$ by [Gr98, (7.3)]. Since $\#\,\mathrm{GL}_3(\mathbb{F}_{q_v}) = (q_v^3 - 1)(q_v^3 - q_v)(q_v^3 - q_v^2)$ and $\#P_0(\mathbb{F}_{q_v}) = q_v^3(q_v - 1)^3$, we get $\frac{\#\,\mathrm{GL}_3(\mathbb{F}_{q_v})/P_0(\mathbb{F}_{q_v})}{q_v^{\dim(G/P_0)}} q_v^{<\lambda,2\rho>} = q_v^{2\varepsilon_1} \frac{(q_v^2+q_v+1)(q_v+1)}{q_v^3}$ as the degree of the local Hecke operator.
Now suppose that either N_2 or $\frac{N_1}{N_2}$ is a unit in $\mathcal{O}_v$, which means that either $\varepsilon_2 = 0$ or $\varepsilon_1 = \varepsilon_2$. (Both cases can not occur simultaneously, since then $\varepsilon_1 = 0$ so that v was not a divisor of N_1.) Then there are exactly two negative roots α for which $< \lambda, \alpha >< 0$ so that $\dim(G/P_\lambda) = 2$. Moreover, P_λ now is either P_2 or P_1. Therefore $\#P_\lambda(\mathbb{F}_{q_v}) = q_v^3(q_v - 1)^3(q_v + 1)$, and $\#\,\mathrm{GL}_3(\mathbb{F}_{q_v})/P_\lambda(\mathbb{F}_{q_v}) = q_v^2 + q_v + 1$. Thus the remaining assertion follows.
It is clear from the proof that the lemma does not depend on the initial choice of generators for the ideals N_1, N_2. □

II.ii.ii Basic estimates

This section is supposed to provide some lemmas which we shall use repeatedly during the next sections to estimate certain integrals.

If H is a real Lie group, let $\mathfrak{h}$ its Lie algebra. Let $\mathcal{B}_H \subseteq \mathcal{U}(\mathfrak{h})$ be a basis for $\mathcal{U}(\mathfrak{h})_{\leq \dim_\mathbb{R} H}$.

Lemma 4. *Suppose $v|\infty$ and $h \in \mathcal{C}(G(F_v)^1)$. Then* *(i)* *If h is $\mathbf{K}_v$-central, $h * X = X * h$ for all $X \in \mathcal{U}(\mathfrak{k}_{v,\mathbb{C}})$.*

(ii) *Let $H \subseteq G(F_v)^1$ be a (real) Lie subgroup, then we have*

$$\sup_{y\in H} |h(yg)| \leq \sum_{X\in\mathcal{B}_H} \int_H |X * h(yg)| dy$$

and

$$\sup_{y\in H} |h(gy)| \leq \sum_{X\in\mathcal{B}_H} \int_H |h * X(gy)| dy.$$

(iii) *If $A \subseteq H$ is a discrete set*

$$\sum_{a\in H} |h(ag)| \leq \sum_{X\in\mathcal{B}_H} \int_H |X * h(yg)| dy.$$

Proof. (i) This is clear, since $h(xk) = h(kx)$ for all $x \in G(F_v)^1$, $k \in \mathbf{K}_v$.

(ii) and (iii) This is [FiLa11b, §3].

□

In the following lemma we collect some neccesary conditions for our functions $f_{\eta,\varepsilon_1,\varepsilon_2}$ to be non-zero. All conditions follow from the elementary divisor theorem.

Lemma 5. *Let $v < \infty$, $t = \operatorname{diag}(t_1, t_2, t_3) \in T(F)$, and $u \in U_0(F_v)$. Then $f_{\eta,\varepsilon_1,\varepsilon_2}(\varpi_v^\eta tu) = 1$ implies the following conditions on t and u:*

(i) $|t_1t_2t_3|_v = q_v^{-(\varepsilon_1+\varepsilon_2)}$,

(ii) $|t_it_j|_v \leq q_v^{-\varepsilon_2}$ *for all* $i \neq j$,

(iii) $|t_1t_3u_1|_v, |t_1t_2u_3|_v \leq q_v^{-\varepsilon_2}$,

(iv) $|t_1t_2(u_1u_3 - u_2)|_v \leq q_v^{-\varepsilon_2}$,

(v) *all entries of tu are in $\mathcal{O}_v$.*

Corollary 6. *Let $t \in T(F)$ and $u \in U_0(\mathbb{A}_f)$. Then $T_{1,N_1,N_2}(tu) = 1$ implies that $N_1, N_2 \in \mathrm{P}^+(F)$ so that the ideals corresponding to N_i are principal and integral, and thus are each generated by an element in $\mathcal{O}_F$ (again denoted by N_1, N_2). Moreover, the following holds:*

(i) $D(t_1t_2t_3) = N_1N_2$, *and all entries of tu are in $\mathcal{O}_F$,*

(ii) $D(t_it_j) \geq N_2$ *for all* $i \neq j$,

(iii) $D(t_1t_3u_1),\ D(t_1t_2u_3),\ D(t_1t_2(u_1u_3 - u_2)) \geq N_2$.

The following is a consequence of the last two lemmas together with the approximation of certain volumes. It is a higher dimensional analogue of [FiLa11a, Lemma 3.1].

Lemma 7. *Let $v < \infty$ and $t \in T(F)$. Then the integral*

$$\int_{U_0(F_v)} f_{\eta,\varepsilon_1,\varepsilon_2}(\varpi_v^\eta tu)du$$

vanishes unless all conditions on t from Lemma 5 $(i),(ii),(v)$ are satisfied. In any case it is bounded by

$$q_v^{-3\varepsilon_2}|t_1|_v^{-2}|t_2|_v^{-1}\prod_{i=1}^{3}\min\{|t_i|_v^{-1}, q_v^{\varepsilon_2}\}.$$

Similarly,the integral $\int_{U_0(\mathbb{A}_{fin})} T_{1,N_1,N_2}(tu)du$ vanishes unless all conditions on N_1, N_2 from Corollary 6 are satisfied, in which case it is bounded by

$$\mathcal{N}_{F/\mathbb{Q}}(t_1^2t_2N_2^{-3}\prod_{i=1}^{3}\gcd(D(t_i),N_2)).$$

Here we again have chosen a generator $N_2 \in \mathcal{O}_F$ of the principal ideal N_2.

The use of gcd in the last part of the lemma has the obvious meaning: For $D_1, D_2 \in \mathrm{Div}^+(F)$, we set $\gcd(D_1,D_2) = \sum_v \min\{D_{1,v}, D_{2,v}\}\mathfrak{p}_v \in \mathrm{Div}^+(F)$. As for t_i, N_2 as in the lemma, the gcd as well corresponds to a principal ideal, taking the norm makes sense.

Proof. We need to bound the volume of the set C of triples $(u_1,u_2,u_3) \in F_v^3$ such that $f(t\begin{pmatrix}1 & u_1 & u_2\\ & 1 & u_3\\ & & 1\end{pmatrix}) = 1$. For such a triple to lie in C we know from Lemma 5 (iii) and (v) that

$$|u_1|_v \leq \min\{q_v^{-\varepsilon_2}|t_1t_3|_v^{-1}, |t_1|_v^{-1}\} = |t_1|_v^{-1}\min\{1, q_v^{-\varepsilon_2}|t_3|_v^{-1}\}$$

and

$$|u_3|_v \leq |t_2|_v^{-1}\min\{1, q_v^{-\varepsilon_2}|t_1|_v^{-1}\}$$

are necessary conditions. Moreover, (v) of Lemma 5 implies that $|u_2|_v \leq |t_1|_v^{-1}$. Let $x \in F_v$ with $|x|_v = q_v^k$. For an integer m consider the set $\{y \in F_v \mid |x-y|_v \leq q_v^{-m}\}$. This set has volume q_v^{-m} independently of the value of k. Therefore,

$$\mathrm{vol}(\{u_2 \in F_v \mid |u_1u_3 - u_2|_v \leq q_v^{-\varepsilon_2}|t_1t_2|_v^{-1}\}) = q_v^{-\varepsilon_2}|t_1t_2|_v^{-1}.$$

Altogether, we obtain that $\mathrm{vol}(C)$ is bounded from above by

$$|t_1|_v^{-2}|t_2|_v^{-1}\min\{1, q_v^{-\varepsilon_2}|t_3|_v^{-1}\}\min\{1, q_v^{-\varepsilon_2}|t_1|_v^{-1}\}\min\{1, q_v^{-\varepsilon_2}|t_2|_v^{-1}\}$$

which proves the lemma. □

Corollary 8. *Assume that $t \in T(F)$ satisfies Corollary 6 (i) and (ii). Let $f \in \mathcal{C}(G(\mathbb{A})^1)$ be of the form $f = f_\infty T_{1,N_1,N_2}$ with $f_\infty \in \mathcal{C}(G(F_\infty)^1)$, $f_\infty \geq 0$. Then*

$$\begin{aligned}
&\sum_{\varepsilon \in T(\mathcal{O}_F)} \int_{U_0(\mathbb{A})} f(\varepsilon t u) du \\
&\leq \min\{\mathcal{N}_{F/\mathbb{Q}}(\frac{t_3}{t_1}), 1\} \mathcal{N}_{F/\mathbb{Q}}(\frac{t_1^2 t_2}{N_2^3} \prod_{i=1}^{3} \gcd(D(t_i), N_2)) \mu(f_\infty) \\
&\quad = \min\{\mathcal{N}_{F/\mathbb{Q}}(t_1), \mathcal{N}_{F/\mathbb{Q}}(t_3)\} \mathcal{N}_{F/\mathbb{Q}}(\frac{t_1 t_2}{N_2^3} \prod_{i=1}^{3} \gcd(D(t_i), N_2)) \mu(f_\infty).
\end{aligned}$$

Proof. For any $\varepsilon \in T(\mathcal{O}_F)$, $u \in U(\mathbb{A}_f)$, we have $T_{r,N_1,N_2}(\varepsilon t u) = T_{r,N_1,N_2}(tu)$ so that the left hand side is

$$\int_{U_0(\mathbb{A}_f)} T_{r,N_1,N_2}(tu) du \sum_{\varepsilon \in T(\mathcal{O}_F)} \int_{U_0(F_\infty)} f_\infty(\varepsilon t u) du,$$

the finite part of which can be bounded by Lemma 7. For the infinite integral we use that $\mathcal{O}_F^\times$ embeds discretely in $F_\infty^\times$, and that $\int_{U_0(F_\infty)} f_\infty(\varepsilon t u) du = \mathcal{N}_{F/\mathbb{Q}}(\frac{t_3}{t_1}) \int_{U_0(F_\infty)} f_\infty(u \varepsilon t) du$. Hence applying Lemma 4 yields the assertion. □

In particular, a (worse) upper bound for the integral in the corollary is

$$\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2} \prod_{i=1}^{3} \gcd(D(t_i), N_2)) \mu(f_\infty)$$

and this is invariant under permutation of the entries of t.

II.iii Equivalence classes in $\mathcal{O}^3_{\text{cent}}$

The equivalence classes $\mathfrak{o} \in \mathcal{O}^3_{\text{cent}}$ correspond bijectively to $t \in F^\times$, i.e. to the semisimple parts $\sigma = \operatorname{diag}(t,t,t)$, and we write $\mathfrak{o}_t$ for such a class. To determine the expansion of the distributions associated with $\mathfrak{o}_t$, it suffices to consider $t = 1$, since the general case follows from the equality $J_{\mathfrak{o}_t}(f) = J_{\mathfrak{o}_1}(f(t\cdot))$. The class $\mathfrak{o}_t$ decomposes into a disjoint union of unipotent orbits: The conjugacy classes of $t\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1\end{pmatrix}$, $t\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1\end{pmatrix}$, and $t\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1\end{pmatrix}$, denoted by $\mathfrak{n}^t$, $\mathfrak{n}^t_{\text{min}}$, and $\mathfrak{n}^t_{\text{reg}}$, respectively. The distribution $J_{\mathfrak{o}_t}(f)$ can be decomposed accordingly, and if $\mathfrak{n}$ is a unipotent orbit, we write $J_{\mathfrak{n}}(f)$ for the corresponding part of $J_{\mathfrak{o}}(f)$. Hence

$$J_{\mathfrak{o}_t}(f) = J_{\mathfrak{n}^t}(f) + J_{\mathfrak{n}^t_{\text{min}}}(f) + J_{\mathfrak{n}^t_{\text{reg}}}(f).$$

The orbit $\mathfrak{n}^t$ consists of only one element and obviously yields $J_{\mathfrak{n}^t}(f) = \operatorname{vol}(A_G G(F)\backslash G(\mathbb{A}))f(t)$. Thus the central contribution equals

$$\sum_{t\in F^\times} \operatorname{vol}(A_G G(F)\backslash G(\mathbb{A}))f(t),$$

which is absolutely convergent also for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$ by [FiLa11b]. Hence we are left with the unipotent regular contribution $J_{\mathfrak{n}_{\text{reg}}}(f) = \sum_{t\in F^\times} J_{\mathfrak{n}^t_{\text{reg}}}(f)$, and the minimal unipotent contribution $J_{\mathfrak{n}_{\text{min}}}(f) = \sum_{t\in F^\times} J_{\mathfrak{n}^t_{\text{min}}}(f)$.

II.iii.i Example: Modifying the regular unipotent contribution for GL(2)

Before we start finding expansions and modifications for the different unipotent distributions for GL(3), we briefly recall how the regular unipotent distribution for GL(2) was modified, as it gives a good impression on the procedure. Each unipotent class $\mathfrak{o}_t \subseteq \mathrm{GL}_2(F)$, $t \in F^\times$, decomposes into the trivial orbit $\{t\}$, and the regular unipotent orbit $\mathfrak{n}^t_{\text{reg}} = \{t\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right),\ x \in F\}$. For $f \in C_c^\infty(G(F_S)^1)$ the sum $\sum_{t\in F^\times} J^{\mathrm{GL}(2)}_{\mathfrak{n}^t_{\text{reg}}}(f)$ is then given by ([FiLa11a, (10)])

$$\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\frac{d}{ds}\left[(s-1)\int_{\mathbb{A}^\times}\sum_{t\in F^\times} f_{\mathbf{K}}(t\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right))|x|^s d^\times x\right]_{s=1}.$$

Computing the derivative, we obtain

$$\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\operatorname*{res}_{s=1}\zeta_F(s)\int_{\mathbb{A}}\sum_{t\in F^\times} f_{\mathbf{K}}(t\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right))(\frac{\lambda_0^S}{\lambda_{-1}^S} + \log|x|_S)dx.$$

To obtain an expression, which is convergent for $f \in \mathcal{C}(\mathrm{GL}(\mathbb{A})^1, K)$ from this, let v be a finite place of F and $\chi_v : F_v \longrightarrow \mathbb{C}$ the characteristic function of $\mathcal{O}_v$. Then for any $t \in \mathcal{O}_v \backslash \{0\}$

$$\begin{aligned}\int_{F_v} \chi_v(tx) \log|x|_v dx &= \int_{\mathcal{O}_v} \log|x|_v dx + \int_{F_v\backslash\mathcal{O}_v} \chi_v(tx)\log|x|_v dx \\ &= \int_{F_v} \chi_v(tx)(\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\chi_v(x) + \log|x|_v\chi_v(x^{-1}))dx\end{aligned}$$

which is the same as

$$\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\int_{F_v}\chi_v(tx)dx + \int_{F_v}\chi_v(tx)\chi_v(x^{-1})(-\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)} + \log|x|_v)dx,$$

which leads to the definition (3). It is then clear that the definition (3) reduces to the usual unipotent contribution for $f \in C_c^\infty(G(F_S)^1)$.

II.iii.ii The regular unipotent orbit

Fix $t \in Z(F)$ temporarily, and define the function Z by

$$Z(s_1, s_2) = \int_{(\mathbb{A}^\times)^2} f_{\mathbf{K},[U_0,U_0]}(t\begin{pmatrix}1 & a & \\ & 1 & c \\ & & 1\end{pmatrix})|a|^{1+s_1}|c|^{1+s_2}d^\times a d^\times c$$

for $f \in C_c^\infty(G(F_S)^1)$ and $s_1, s_2 \in C$, $\Re s_1, \Re s_2 > 0$. Here

$$f_{\mathbf{K},[U_0,U_0]}(g) = \int_{\mathbf{K}}\int_{[U_0(\mathbb{A}),U_0(\mathbb{A})]} f(k^{-1}guk)dudk$$

and $[U_0(\mathbb{A}), U_0(\mathbb{A})] \simeq \mathbb{A}$ is the first derived subgroup of $U_0(\mathbb{A})$. Note that in general if $f_{\mathbf{K},[U_0,U_0]}$ is a Schwartz-Bruhat function in the variables a and c, Z can be continued to a meromorphic function on $\mathbb{C}^2$, and $s_1s_2Z(s_1,s_2)$ continues to an entire function on $\mathbb{C}^2$.

Lemma 9. *Let $f \in C_c^\infty(G(F_S)^1)$. Then $J_{\mathfrak{n}^t_{reg}}(f)$ is given as the value of*

$$\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\left(\partial_1\partial_2 + \frac{1}{4}(\partial_1^2 + \partial_2^2)\right)[s_1s_2Z(s_1,s_2)] \tag{5}$$

at $(s_1, s_2) = (0,0)$, or more explicitly, it is given by the sum of

$$\begin{aligned}&\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^3\int_{\mathbb{A}^2} f(t\begin{pmatrix}1 & u_1 & \\ & 1 & u_2 \\ & & 1\end{pmatrix}) \\ &\cdot\left(\log|u_1|_S\log|u_3|_S + \frac{1}{4}((\log|u_1|_S)^2 + (\log|u_3|_S)^2)\right)du_1du_3,\end{aligned} \tag{6}$$

$$\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^3\frac{3}{2}\frac{\lambda_0^S}{\lambda_{-1}^S}\int_{\mathbb{A}^2} f(t\begin{pmatrix}1 & u_1 & \\ & 1 & u_3 \\ & & 1\end{pmatrix})\log|u_1u_3|_S du_1du_3, \tag{7}$$

and

$$\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^3\left(\left(\frac{\lambda_0^S}{\lambda_{-1}^S}\right)^2+\frac{\lambda_1^S}{\lambda_{-1}^S}\right)\int_{\mathbb{A}^2} f(t\begin{pmatrix}1&u_1&\\&1&u_3\\&&1\end{pmatrix})du_1du_3. \tag{8}$$

Note that the coefficients and weights were already computed in [Fl82, Lemma 4], and up to a misprint in the statement of the term (8) there, they equal our results. Yet we chose to include a derivation of those terms, as our calculation directly yields a form more suitable for our application.

Proof. Let β_1, β_2 be the reduced positive roots of GL(3) wih respect to T, and let ϖ_1, ϖ_2 be the corresponding weights subject to $<\beta_j,\varpi_i^\vee>=\delta_{ij}$. Let

$$Z(f_{\mathbf{K},[U_0,U_0]},\lambda)=\int_{(\mathbb{A}^\times)^2} f_{\mathbf{K},[U_0,U_0]}(t\begin{pmatrix}1&a&\\&1&c\\&&1\end{pmatrix})|a|^{<\lambda,\varpi_1^\vee>}|c|^{<\lambda,\varpi_2^\vee>}d^\times a d^\times c$$

for $\lambda\in(\mathfrak{a}_0^G)^*$. Each $w\in W^G$ defines an isomorphism $(\mathfrak{a}_0^G)^*\longrightarrow(\mathfrak{a}_0^G)^*$, $\lambda\mapsto w(\lambda)$, and $W^G\ni w\mapsto P_w=w^{-1}P_0w\in\mathcal{P}(T)$ defines a bijection between W^G and $\mathcal{P}(T)$. The regular unipotent contribution is then given by[1]

$$\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\lim_{\lambda\to 0}\sum_{w\in W^G} Z(f_{\mathbf{K},[U_0,U_0]},w(\lambda)+\rho_{\le 2})\hat\theta_{P_w}(\lambda)\theta_{P_w}(\lambda)^{-1} \tag{9}$$

with $\rho_{\le 2}=\beta_1+\beta_2$ (we shall suppress the volume factor in the following). (Recall that we normalised the measures on the root spaces to yield 1 for the root lattice so that $\hat\theta_{P_w}(\lambda)=\frac{1}{3}\prod_{\varpi\in\hat\Delta_0}<w(\lambda),\varpi^\vee>$ and $\theta_{P_w}(\lambda)=\prod_{\beta\in\Delta_0}<w(\lambda),\beta^\vee>$.) Since $f_{\mathbf{K},[U_0,U_0]}$ is the characteristic function of $\mathcal{O}_v^2$ outside of the set S, the zeta function $Z(f_{\mathbf{K},[U_0,U_0]},w(\lambda)+\rho_{\le 2})$ can be written as the product of truncated Dedekind zeta functions and local zeta functions:

$$\left(\prod_{\varpi\in\hat\Delta_0}\zeta_F^S(<w(\lambda),\varpi^\vee>+1)\right)Z_S(f_{\mathbf{K},[U_0,U_0]},w(\lambda)+\rho_{\le 2}),$$

where $Z_S(f_{\mathbf{K},[U_0,U_0]},w(\lambda)+\rho_{\le 2})=\prod_{v\in S}Z_v(f_{\mathbf{K},[U_0,U_0]},w(\lambda)+\rho_{\le 2})$,

$$\begin{aligned}&Z_v(f_{\mathbf{K},[U_0,U_0]},\lambda+\rho_{\le 2})\\&\quad=\int_{F_v^\times\times F_v^\times} f_{\mathbf{K},[U_0,U_0],v}(t\begin{pmatrix}1&a&\\&1&c\\&&1\end{pmatrix})|a|_v^{<\varpi_1^\vee,\lambda>+1}|c|_v^{<\varpi_2^\vee,\lambda>+1}d_v^\times a d_v^\times c,\end{aligned}$$

[1] This is a special case of a more general expression for the distributions associated with unipotent orbits as explained in an unpublished work by Tobias Finis and Erez Lapid.

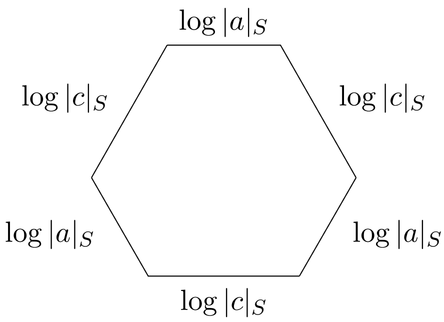
and this local zeta function is absolutely convergent for $<\lambda, \varpi_1^\vee>$, $<\lambda, \varpi_2^\vee>> -1$. Changing the multiplicative to an additive measure, the local function $Z_v(f_{\mathbf{K},[U_0,U_0]}, \lambda + \rho_{\leq 2})$ equals

$$\zeta_{F,v}(1)^2 \int_{F_v \times F_v} f_{\mathbf{K},[U_0,U_0],v}(t \begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1 \end{pmatrix})|a|_v^{<\lambda,\varpi_1^\vee>} |c|_v^{<\lambda,\varpi_2^\vee>} d_v a d_v c.$$

For $P \in \mathcal{P}(T)$ corresponding to $w \in W^G$, let $c_P(\lambda) = Z_S(f_{\mathbf{K},[U_0,U_0]}, w(\lambda) + \rho_{\leq 2})$, and $d_P(\lambda) = \prod_{\varpi \in \hat{\Delta}_0}(\zeta_F^S(< w(\lambda), \varpi^\vee > +1) < w(\lambda), \varpi^\vee >)$. Then $\{c_P(\lambda) \mid P \in \mathcal{P}(T)\}$, and $\{d_P(\lambda) \mid P \in \mathcal{P}(T)\}$ are both (G,T)-families, and the expression (9) equals $(cd)_T(0)$. Since the equation $c_T^L(\lambda) = c_T^Q(\lambda)$ is satisfied for all $Q \in \mathcal{P}(L)$, $L \in \mathcal{L}(T)$, $(cd)_T(0)$ equals by Arthur's splitting formula [Ar81, Corollary 6.5]

$$\sum_{L \in \mathcal{L}(T)} c_T^L(0) d_L(0) = c_T(0) d_G(0) + c_T^T(0) d_T(0) + \sum_{T \subsetneq M \subsetneq G} c_T^M(0) d_M(0),$$

and we have $c_T^T(0) = \zeta_{F,S}(1)^2 \int_{F_S^2} f_{\mathbf{K},[U_0,U_0],v}(t \begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1 \end{pmatrix}) d_S a d_S c$, and $d_G(0) = (\mathrm{res}_{s=1} \zeta_F^S(s))^2$. We first compute $c_T(0)$. For $w \in W^G$ let $X^{P_w}(a,c) = \log|a|_S \varpi_\alpha^\vee + \log|c|_S \varpi_\beta^\vee \in \mathfrak{a}_0^G$ for $\alpha = w(\beta_1)$ and $\beta = w(\beta_2)$, i.e. we have $e^{\lambda(X^{P_w}(a,c))} = |a|_S^{<w(\lambda),\varpi_1^\vee>} |c|_S^{<w(\lambda),\varpi_2^\vee>}$ and $X^{P_w}(a,c) = w(X^{P_0}(a,c))$. The set $\{v_P(\lambda)\}_{P \in \mathcal{P}(T)}$ with $v_P(\lambda) = e^{\lambda(X^P(a,c))}$ is again a (G,T)-family. Since all involved integrals are sufficiently convergent, we may take the limit over λ inside the integral so that $c_T(0) = \int_{(F_v^\times)^2} f_{\mathbf{K},[U_0,U_0],S}(t \begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1 \end{pmatrix}) v_T(0) d_S^\times a d_S^\times c$ so that it remains to compute $v_T(0)$, which is a polynomial in $\log|a|_S$ and $\log|c|_S$. If the points $\{X^P(a,c) \mid P \in \mathcal{P}(T)\}$ constitute a positive (G,T)-orthogonal family (which is the case for $|a|_S, |c|_S > 0$), $v_T(0)$ equals the volume of the convex hull in $\mathfrak{a}_0^G$ spanned by such points. Let $P \in \mathcal{P}(T)$ and let α, β be the reduced positive roots of P. If Q is adjacent to P, $\Delta_Q = \{-\alpha, \alpha+\beta\}$ or $\Delta_Q = \{-\beta, \alpha+\beta\}$ so that $X^P(a,c) - X^Q(a,c) = \log|a|_S \alpha^\vee$ or $X^P(a,c) - X^Q(a,c) = \log|c|_S \beta^\vee$. Hence if $|a|_S, |c|_S > 1$, we have to compute the volume of a polytope, which looks like the following:



and all internal angles are $\frac{2\pi}{3}$. The volume can then be easily computed to be $\log|a|_S \log|c|_S + \frac{1}{4}((\log|a|_S)^2 + (\log|c|_S)^2)$ and this expression remains valid for arbitrary $a, c \neq 0$.

Since $< \alpha, X^P >=< \beta_1, X^{P_0} >= \log|a|_S$ and $< \beta, X^P >=< \beta_2, X^{P_0} >= \log|c|_S$, the volume is the same as $(\partial_1\partial_2 + \frac{1}{4}(\partial_1^2+\partial_2^2))e^{\lambda(X^{P_0}(a,c))}$, where ∂_i denotes the derivative with respect to $< \lambda, \varpi_i^\vee >$.

$$c_T(0) = \zeta_{F,S}(1)^2 \int_{F_S^2} f_{\mathbf{K},[U_0,U_0],S}(t\begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1\end{pmatrix}) \\ \cdot (\log|a|_S \log|c|_S + \frac{1}{4}((\log|a|_S)^2 + (\log|c|_S)^2))d_Sad_Sc.$$

On the other hand, the other extrem case yields

$$d_T(0) = (\partial_1\partial_2 + \frac{1}{4}(\partial_1^2+\partial_2^2))(\zeta_F^S(s_1)\zeta_F^S(s_2)s_1s_2)_{|(s_1,s_2)=(1,1)} \\ = ((\lambda_0^S)^2 + \lambda_1^S\lambda_{-1}^S).$$

Next we compute $c_T^M(0)$ for Levi subgroups M of corank 1. Since again the local integrals are absolutely convergent at 0, we have

$$c_T^M(0) = \zeta_{F,S}(1)^2 \int_{F_S^2} f_{\mathbf{K},[U_0,U_0]}(t\begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1\end{pmatrix})v_T^M(0)d_Sad_Sc.$$

Let $Q \in \mathcal{P}^M(T)$, and $w \in W$ such that $P_w \cap M = Q$. Then $v_Q^M(\lambda_M) = v_{P_w}(\lambda_M) = |a|^{<w(\lambda_M),\varpi_1^\vee>}|c|^{<w(\lambda_M),\varpi_2^\vee>}$ with $\lambda_M \in i(\mathfrak{a}_T^M)^*$ and $\{v_Q^M\}$ is a (T,M)-family. The space $(\mathfrak{a}_T^M)^*$ is spanned by β_1, β_2 or $\beta_1+\beta_2$ depending on whether $M = M_1$, M_2 or M_3. Hence $v_T^{M_1}(0) = \frac{1}{\sqrt{2}}\log|a|_S$, $v_T^{M_2}(0) = \frac{1}{\sqrt{2}}\log|c|_S$, and $v_T^{M_3}(0) = \frac{1}{\sqrt{2}}\log|ac|_S$. On the other hand, the spaces $(\mathfrak{a}_M^G)^*$ are spanned by ϖ_2, ϖ_1, or $\varpi_1 - \varpi_2$ for $M = M_1,\ M_2,\ M_3$ so that $d_{M_1}(0) = \frac{2}{\sqrt{2}}\lambda_0^S\lambda_{-1}^S = d_{M_2}(0)$, and $d_{M_3}(0) = \frac{1}{\sqrt{2}}\lambda_0\lambda_{-1}$. Hence,

$$\sum_{T\subsetneq M\subsetneq G} c_T^M(0)d_M(0) \\ = \frac{3}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)^2\frac{\lambda_0^S}{\lambda_{-1}^S}\int_{F_S^2} f_{\mathbf{K},[U_0,U_0],S}(t\begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1\end{pmatrix})\log|ac|_Sd_Sad_Sc.$$

Since $f_{\mathbf{K},[U_0,U_0],v}$ is the characteristic function of $\mathcal{O}_v^2 \subseteq F_v^2$ for $v \notin S$, and since $\operatorname{vol}(\mathcal{O}_v) = 1$, we may change all integrals into global integrals over $\mathbb{A}^2$ so that the assertion follows. □

For this to yield a convergent expression for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$, one needs to modify the terms by cutting out the too much of the support of f. This is done as follows. We could try to replace $\log|u_i|_S$ by $\omega^{\mathrm{GL}(2),S}(t,u_i)$ (the function from [FiLa11a] for GL(2)), but this gives an expression which is not invariant under enlarging the set S. Nevertheless, this replacement is not to far from the truth: we only need to add an additional term. The polynomial of second order in $\log|u_i|$ will be replaced by a weight function

$\omega^S_{M_0}(t,u) = \omega_{M_0}(t,u)$ (depending on the chosen set S, but not on $t \in Z(F)$), which is the sum $\sum_{v,w} \omega_{M_0,v,w}(t,u)$ of local weight functions $\omega_{M_0,v,w}$. We write $\omega^S(u_i) = \omega^{\mathrm{GL}(2),S}(u_i) = \omega^{\mathrm{GL}(2),S}(t,u_i)$, since $\omega^{\mathrm{GL}(2),S}$ is independent of $t \in Z(F)$, and recall that $\omega^S(x) = \sum_v \omega^S_v(x)$ with

$$\omega^S_v(x) = \begin{cases} \log|x|_v & \text{if } v \in S \\ \log|x|_v + \frac{\log q_v}{q_v - 1} & \text{if } v \notin S \text{ and } |x|_v > 1 \\ 0 & \text{if } v \notin S \text{ and } |x|_v \leq 1. \end{cases}$$

Then define the replacement of the weight function in (6) by

$$\begin{aligned} &\omega^S_{M_0}(u) = \omega^S_{M_0}(t,u) \\ &= \omega^S(u_1)\omega^S(u_3) + \frac{1}{4}(\omega^S(u_1)^2 + \omega^S(u_3)^2) - \frac{1}{4}\sum_{w \notin S}(\eta_w(u_1) + \eta_w(u_3)), \end{aligned}$$

with

$$\eta_w(u_i) = \begin{cases} \frac{(\log q_w)^2}{q_w - 1} + \left(\frac{\log q_w}{q_w - 1}\right)^2 & \text{if } |u_i|_w > 1, \\ 0 & \text{if } |u_i|_w \leq 1. \end{cases}$$

Hence for any v, w we have

$$\begin{aligned} \omega^S_{M_0,v,w}(u) = \omega^S_v(u_1)\omega^S_w(u_3) + \frac{1}{4}(\omega^S_v(u_1)\omega^S_w(u_3) + \omega^S_v(u_3)\omega^S_w(u_3)) \\ - \frac{1}{4}\delta_{v,w}(\eta_w(u_1) + \eta_w(u_3)) \end{aligned}$$

with $\delta_{v,w} = 1$ if $v = w$, and $\delta_{v,w} = 0$ if $v \neq w$.

The weight in (7) will be replaced by $\Omega^S_{M_0}(t,u) = \Omega_{M_0}(t,u) = \sum_v \Omega_{M_0,v}(t,u)$ (again depending on S, but not on $t \in Z(F)$) with

$$\Omega^S_{M_0}(t,u) = \frac{3}{2}\frac{\lambda^S_0}{\lambda^S_{-1}}(\omega^S(u_1) + \omega^S(u_3)),$$

and the prefactor of the integral in (8) stays the same,

$$\Lambda_S = \left(\frac{\lambda^S_0}{\lambda^S_{-1}}\right)^2 + \frac{\lambda^S_1}{\lambda^S_{-1}}.$$

Note that for $T = S \cup \{v\}$ for v some place not contained in S, we have

$$\frac{\lambda_0^{S\cup\{v\}}}{\lambda_{-1}^{S\cup\{v\}}} = \frac{\lambda^S_0}{\lambda^S_{-1}} - \frac{\zeta'_{F,v}(1)}{\zeta_{F,v}(1)},$$

and

$$\frac{\lambda_1^{S\cup\{v\}}}{\lambda_{-1}^{S\cup\{v\}}} = \frac{\lambda_1^S}{\lambda_{-1}^S} - \frac{\lambda_0^S}{\lambda_{-1}^S}\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)} - \frac{1}{2}\frac{\zeta_{F,v}''(1)}{\zeta_{F,v}(1)} + \left(\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\right)^2$$

so that

$$\Lambda_{S\cup\{v\}} - \Lambda_S = -3\frac{\lambda_0^S}{\lambda_{-1}^S}\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)} + 2\left(\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\right)^2 - \frac{1}{2}\frac{\zeta_{F,v}''(1)}{\zeta_{F,v}(1)} \tag{10}$$

If it is clear on which set S the weights depend, we may sometimes drop the index S.

With these weight functions we get:

Proposition 10. *For $f \in C_c^\infty(G(F_S)^1)$ as above the sum $\sum_{t\in F^\times} J_{\mathfrak{n}_{reg}^t}(f)$ is given by the product of $\nu(T)$ with the sum of*

$$\sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} f(k^{-1}tuk)\omega_{M_0}(t,u)dudk,$$

$$\sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} f(k^{-1}tuk)\Omega_{M_0}(t,u),$$

and

$$\Lambda_S \sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} f(k^{-1}tuk)dudk$$

for S large enough. In particular, the expression is invariant under enlarging S. Moreover, each of the sum-integrals

$$\sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} |f(k^{-1}tuk)| \sum_{v,w} |\omega_{M_0,v,w}(t,u)|dudk,$$

$$\sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} |f(k^{-1}tuk)| \sum_{v} |\Omega_{M_0,v}(t,u)|dudk,$$

and

$$\sum_{t\in Z(F)} \int_{\mathbf{K}} \int_{U_0(\mathbb{A})} |f(k^{-1}tuk)|dudk$$

defines a continuous semi-norm on $\mathcal{C}(G(\mathbb{A})^1, K)$.

First we want to show that the expression in the proposition is invariant under enlarging S for f some bi-K-invariant function provided that the sum-integrals do converge.

Lemma 11. *Let $f \in C^\infty(K\backslash G(\mathbb{A})/K)$ be so that the expression*

$$\int_{U_0(\mathbb{A})} f(tu)(\omega_{M_0}^T(t,u) + \Omega_{M_0}^T(t,u) + \Lambda_T)du$$

converges for all finite sets $T \supseteq S$. Then this integral defines a constant map $\{T \mid S \subseteq T,\ T \text{ finite}\} \longrightarrow \mathbb{C}$.

It will certainly suffice to prove the lemma for $T = S \cup \{v\}$ for v some place not contained in S. We may also assume that $t = 1$, and hence we can view f as a function on the affine space $U_0(\mathbb{A})$. It will in particular suffice to prove the lemma for f such that $f(ab_v) = f(a)$ for all $a \in U_0(\mathbb{A})$, $b_v \in U_0(\mathcal{O}_v)$ (i.e. f is a function on the affine space $\mathbb{A}^3$ invariant under adding elements from $\mathcal{O}_v^3$). First we show two integral identities.

Lemma 12. *Suppose S, T are two finite sets of places, $T = S \cup \{v\}$, $v \notin S$, and that $h = \prod_v h_v : \mathbb{A} \longrightarrow \mathbb{C}$ is a sufficiently integrable smooth function with $h(a + b_v) = h(a)$ for all $a \in \mathbb{A},\ b_v \in \mathcal{O}_v$. Then*

$$\int_{\mathbb{A}} h(x)\left((\omega_v^T(x) - \omega_v^S(x))^2 + \eta_v(x)\right)dx = \frac{\zeta_{F,v}''(1)}{\zeta_{F,v}(1)} \int_{\mathbb{A}} h(x)dx,$$

and

$$\int_{\mathbb{A}} h(x)(\omega_v^T(x) - \omega_v^S(x))\omega^S(x)dx = -\frac{\log q_v}{q_v - 1}\int_{\mathbb{A}} h(x)\omega^S(x)dx.$$

(Here "sufficiently integrable" means that all occuring integrals converge absolutely.)

Proof. We start with the first equation for which it suffices to show the assertion for the local integral over F_v. As h_v is invariant under $\mathcal{O}_v$, we can write the left hand side as

$$h_v(0)\int_{\mathcal{O}_v}(\log|x|_v)^2dx + \left(\frac{(\log q_v)^2}{q_v - 1} + 2\left(\frac{\log q_v}{q_v - 1}\right)^2\right)\int_{F_v\backslash\mathcal{O}_v} h_v(x)dx$$
$$= h_v(0)\int_{\mathcal{O}_v}(\log|x|_v)^2dx + \frac{\zeta_{F,v}''(1)}{\zeta_{F,v}(1)}\int_{F_v\backslash\mathcal{O}_v} h_v(x)dx.$$

Since $\int_{\mathcal{O}_v}(\log|x|_v)^2dx = \frac{\zeta_{F,v}''(1)}{\zeta_{F,v}(1)}$ the first assertion follows.

For the second identity write the left hand side as

$$\int_{\mathbb{A}} h(x)(\omega_v^T(x) - \omega_v^S(x))\omega_v^S(x)dx + \int_{\mathbb{A}} h(x)(\omega_v^T(x) - \omega_v^S(x))\sum_{w\neq v}\omega_w^S(x)dx.$$

Since $\omega_v^S(x) = 0$ for $x \in \mathcal{O}_v$, we have

$$\begin{aligned}\int_{F_v} h_v(x)(\omega_v^T(x) - \omega_v^S(x))\omega_v^S(x)dx &= -\frac{\log q_v}{q_v - 1}\int_{F_v \setminus \mathcal{O}_v} h_v(x)\omega_v^S(x)dx \\ &= -\frac{\log q_v}{q_v - 1}\int_{F_v} h_v(x)\omega_v^S(x)\end{aligned}$$

and

$$\begin{aligned}&\int_{F_v} h_v(x)(\omega_v^T(x) - \omega_v^S(x))dx \\ &= h_v(0)\int_{\mathcal{O}_v} \log|x|_v dx - \frac{\log q_v}{q_v - 1}\int_{F_v \setminus \mathcal{O}_v} h_v(x)dx \\ &\qquad = -\frac{\log q_v}{q_v - 1}\int_{F_v} h_v(x)dx\end{aligned}$$

from which the second assertion follows. □

Proof of Lemma 11. Let $T = S \cup \{v\}$ be as before. Then $\omega_{M_0}^T(u) - \omega_{M_0}^S(u)$ is the sum of

$$\begin{aligned}\sum_{(i,j)\in\{(1,3),(3,1)\}} \left(\omega_v^T(u_i) - \omega_v^S(u_i)\right)\omega^S(u_j) + \sum_{i=1,3}\left(\omega_v^T(u_i) - \omega_v^S(u_i)\right)\omega^S(u_i) \\ + \left(\omega_v^T(u_1) - \omega_v^S(u_1)\right)\left(\omega_v^T(u_3) - \omega_v^S(u_3)\right)\end{aligned}$$

and

$$\frac{1}{4}(\left(\omega_v^T(u_1) - \omega_v^S(u_1)\right)^2 + \left(\omega_v^T(u_3) - \omega_v^S(u_3)\right)^2) + \frac{1}{4}(\eta_v(u_1) + \eta_v(u_3)).$$

To compute $\int_{U_0(\mathbb{A})} f(u)(\omega_{M_0}^T(u) - \omega_{M_0}^S(u))du$ note that the integration over u_1 and u_3 is independent of each other so that we may apply the results for GL(2) from [FiLa11a] to the first and third summand. For the second summand we use the first part of Lemma 12, and for the remaining summands the second part of Lemma 12 to conclude that $\int_{U_0(\mathbb{A})} f(u)(\omega_{M_0}^T(u) - \omega_{M_0}^S(u))du$ equals

$$\begin{aligned}&-\frac{3}{2}\frac{\log q_v}{q_v - 1}\int_{U_0(\mathbb{A})} f(u)(\omega^S(u_1) + \omega^S(u_3))du \\ &\qquad + \left(\left(\frac{\log q_v}{q_v - 1}\right)^2 + \frac{1}{2}\frac{(\log q_v)^2}{q_v - 1} + \left(\frac{\log q_v}{q_v - 1}\right)^2\right)\int_{U_0(\mathbb{A})} f(u)du.\end{aligned}$$

Since $\Omega_{M_0}^T(u) - \Omega_{M_0}^S(u)$ equals

$$\begin{aligned}&\frac{3}{2}\frac{\lambda_0^S}{\lambda_{-1}^S}((\omega_v^T(u_1) - \omega_v^S(u_1)) + (\omega_v^T(u_3) - \omega_v^S(u_3))) \\ &+ \frac{3}{2}\frac{\log q_v}{q_v - 1}(\omega^S(u_1) + \omega^S(u_3)) \\ &\qquad + \frac{3}{2}\frac{\log q_v}{q_v - 1}((\omega_v^T(u_1) - \omega_v^S(u_1)) + (\omega_v^T(u_3) - \omega_v^S(u_3))),\end{aligned}$$

the results for GL(2) imply that the integral $\int_{U_0(\mathbb{A})} f(u)(\Omega_{M_0}^T(u) - \Omega_{M_0}^S(u))du$ is

$$\frac{3}{2}\frac{\log q_v}{q_v - 1}\int_{U_0(\mathbb{A})} f(u)(\omega^S(u_1) + \omega^S(u_3))du$$
$$+ \left(-3\frac{\lambda_0^S}{\lambda_{-1}^S}\frac{\log q_v}{q_v - 1} - 3\left(\frac{\log q_v}{q_v - 1}\right)^2\right)\int_{U_0(\mathbb{A})} f(u)du.$$

Adding this to $\int_{U_0(\mathbb{A})} f(u)(\omega_{M_0}^T(u) - \omega_{M_0}^S(u))du$ we obtain

$$\left(-3\frac{\lambda_0^S}{\lambda_{-1}^S}\frac{\log q_v}{q_v - 1} - \left(\frac{\log q_v}{q_v - 1}\right)^2 + \frac{1}{2}\frac{(\log q_v)^2}{q_v - 1}\right)\int_{U_0(\mathbb{A})} f(u)du,$$

which exactly cancels the difference (10) of $(\Lambda_T - \Lambda_S)\int_{U_0(\mathbb{A})} f(u)du$, and thus proves the assertion. □

To prove the assertion about the convergence in Proposition 10 we need some estimates. Recall that it suffices to consider functions f of the form (4), which in particular equal $f_{\eta^v, \varepsilon_1^v, \varepsilon_2^v}$ at the finite places v with $\varepsilon_1^v \geq \varepsilon_2^v \geq 0$, and that we may even assume $\eta^v = 0$. Hence we now establish estimates for the local integrals involving such test functions and the weight factors. As the weight ω_{M_0} is a double sum over all places, we now have to consider integrals over two places at a time.

Lemma 13. *Let v, w be non-archimedean places of F. Let f_v and f_w be the functions $f_{0,\varepsilon_1^v,\varepsilon_2^v}$ and $f_{0,\varepsilon_1^w,\varepsilon_2^w}$, respectively.*

(i) The integrals $\int_{U_0(F_v)} f_v(tu)|\omega_v^S(u_i)|du$, $i = 1, 3$ vanish unless $t \in \mathcal{O}_v$, and unless (i) and (ii) of Lemma 5 are satisfied. There exists a constant $M \in \mathbb{N}$ (independent of v) such that each of the integrals is bounded by

$$M(\log|\det t|_v^{-1} + \delta_v)\int_{U_0(F_v)} f_v(tu)du$$

where δ_v is 0 or 1 depending on whether $v \notin S$ or $v \in S$.

(ii) The integral $\int_{U_0(F_{v,w})} f_{v,w}(tu)|\omega_{M_0,v,w}(t,u)|du$ vanishes unless $t \in \mathcal{O}_v \cap \mathcal{O}_w$, and unless the conditions (i) and (ii) of Lemma 5 are satisfied for v as well as for for w. Then there exists a constant $M' \in \mathbb{N}$ (independent of v and w) such that it is bounded by

$$M'(\log|\det t|_v^{-1} + \delta_v)(\log|\det t|_w^{-1} + \delta_w)\int_{U_0(F_{v,w})} f_{v,w}(tu)du.$$

Note that the first part of the lemma also provides a bound for the integral involving $\Omega_{M_0,v}$, v finite, which is given by

$$\int_{U_0(F_v)} f_v(tu)|\Omega_{M_0,v}^S(u)|du \leq 3M|\frac{\lambda_0^S}{\lambda_{-1}^S}|(\log|\det t|_v^{-1} + \delta_v)\int_{U_0(F_v)} f_v(tu)du.$$

(11)

We still need archimedean estimates, which are given by the following lemma. Define for $w|\infty$, $h\in\mathcal{C}(G(F_w)^1)$, $t\in F^\times$,

$$H(t,h)=\int_{Z(F_w)\backslash G(F_w)}\sum_{X\in\mathcal{B}_{\mathrm{PGL}_3(F_w)}}|X*h(tg)|dg.$$

For $h\in\mathcal{C}(G(F_\infty)^1)$, define $H(t,h)$ analogous.

Lemma 14. *Let w be an archimedean place of F, v an archimedean or non-archimedean place, and assume that f_v has the same form as above if v is non-archimedean. Then there exist constants $M_1,M_2,M_3,M_4>0$ such that the following holds:*

(i) The weighted integrals $\int_{U_0(F_{w,v})}|f_{w,v}(tu)||\omega_{M_0,w,v}(u)|du$ and $\int_{U_0(F_{v,w})}|f_{v,w}(tu)||\omega_{M_0,v,w}(u)|du$ vanish unless all the non-vanishing conditions with respect to v of Lemma 5 are satisfied if v is non-archimedean. In any case they are bounded by

$$M_1q_v^{-3\varepsilon_2^v}|t|_v^{-3}\min\{|t|_v^{-1},q_v^{\varepsilon_2^v}\}^3(-\log|t|_v+\delta_v)H(t,f_w),$$

if v is non-archimedean, and by

$$M_1H(t,f_w)H(t,f_v)$$

if v is archimedean,

(ii)

$$\int_{U_0(F_w)}|f_w(tu)||\omega_{M_0,w,w}(u)|du\le M_2H(t,f_w),$$

(iii)

$$\int_{U_0(F_w)}|f_w(tu)||\Omega_{M_0,w}(u)|du\le M_3H(t,f_w),$$

All inequalities are valid for any $\mathbf{K}_w$-central $f_w\in\mathcal{C}(G(F_w)^1)$ and $\mathbf{K}_v$-central $f_v\in\mathcal{C}(G(F_\infty)^1)$ if $v|\infty$.

(iv) For any $\mathbf{K}_\infty$-central $f_\infty\in\mathcal{C}(G(F_\infty)^1)$, we have

$$\sum_{\varepsilon\in Z(\mathcal{O}_F)}H(\varepsilon t,f_\infty)\le M_4\mu(f_\infty).$$

As the proofs of both lemmas are quite technical, we postpone them till the end of the section, and instead turn our attention to the completion of the proof of Proposition 10 first.

Proof of Proposition 10. The expression given in the proposition is well-defined by Lemma 11. First we show that the expression given in Proposition 10 reduces to the usual contribution from the regular unipotent orbit if we insert $f \in C_c^\infty(G(F_S)^1)$. Then $|t|_v = 1$ for all $v \notin S$ such that $tU_0(\mathbb{A}) \cap \operatorname{supp}(f) \neq \emptyset$, and also $|u_i|_v \leq 1$ whenever $v \notin S$, and there is some t such that $f(tu) \neq 0$. Hence for all t, u such that $tu \in \operatorname{supp}(f)$, we have $\omega^{\mathrm{GL}_2,S}(t, u_i) = \log |u_i|_S$ and $\eta_w(u_i) = 0$ for all $w \notin S$. Hence $\omega_{M_0}^S$ and $\Omega_{M_0}^S$ reduce to the weights in (6) and (7). Since Λ_S is the same scalar as in (8), it follows that the regular unipotent distribution equals the expression given in the proposition.

For the proof of the absolute convergence we may assume that f is of the form (4), i.e. $f = f_\infty T_{1,N_1,N_2}$, and consider each of the three occuring sum-integrals separately. For the term involving Λ_S, we only need to consider $\sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} f_{\mathbf{K}}(tu)du$. The only $t \in Z(F) \simeq F^\times$ for which the integral can be non-zero, have in particular to satisfy $\mathcal{N}_{F/\mathbb{Q}}(t)^3 = \mathcal{N}_{F/\mathbb{Q}}(N_1N_2)$. Thus we may assume that $\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac{1}{3}} \in \mathbb{Z} \subseteq \mathcal{O}_F$, and $t = t_1\varepsilon$ with $t_1 = \mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac{1}{3}}$ and some $\varepsilon \in \mathcal{O}_F^\times$. Using Corollary 8, we obtain

$$\begin{aligned}
&\sum_{t\in Z(F)} \int_{U_0(\mathbb{A})} |f_{\mathbf{K}}(tu)|du \\
&\leq \mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac{1}{3}}, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3] \\
&\qquad\qquad\qquad \cdot \sum_{\varepsilon\in\mathcal{O}_F^\times} \int_{U_0(F_\infty)} |f_\infty(t_1\varepsilon u)|du \\
&\quad \leq \mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac{1}{3}}, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3 \mu(f_\infty).
\end{aligned}$$

Since

$$\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac{1}{3}}, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3 \leq \mathcal{N}_{F/\mathbb{Q}}(N_1)^{\frac{3}{2}},$$

the sum is bounded by $\mathcal{N}_{F/\mathbb{Q}}(N_1)^{\frac{3}{2}}\mu(f_\infty) \leq \deg T_{1,N_1,N_2}\mu(f_\infty)$ by Lemma 3. Consider now the sum-integral involving $\Omega_{M_0}^S = \Omega_{M_0,\infty}^S + \Omega_{M_0,f}^S$. The non-vanishing conditions on t from the last case apply here as well. Hence using the estimate (11) together with Lemmas 7 and 14 (*iv*) again the sum $\sum_{t\in Z(F)} \sum_v \int_{U_0(\mathbb{A})} |f(tu)||\Omega_{M_0,v}^S(tu)|du$ is bounded by

$$\begin{aligned}
&\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(t_1, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3 \sum_{\varepsilon\in\mathcal{O}_F^\times} \int_{U_0(F_\infty)} |f_\infty(\varepsilon t_1 u)||\Omega_{M_0,\infty}^S(u)|du \\
&+ M\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(t_1, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3 (\log \mathcal{N}_{F/\mathbb{Q}}(N_1N_2) + |S|)\mu(f_\infty) \\
&\leq M\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2}) \gcd(t_1, \mathcal{N}_{F/\mathbb{Q}}(N_2))^3 (\log \mathcal{N}_{F/\mathbb{Q}}(N_1N_2) + |S| + M_1)\mu(f_\infty)
\end{aligned}$$

for some absolute constants $M, M_1 > 0$ so that by the same reasoning as before, this is bounded by $O(\deg T_{1,N_1,N_2})\mu(f_\infty)$.
Using Lemmas 5 and 14 similar reasoning applies to $\omega_{M_0}^S = \omega_{M_0,f,f}^S + \omega_{M_0,f,\infty}^S + \omega_{M_0,\infty,f}^S + \omega_{M_0,\infty,\infty}^S$ so that the sum-integral

$$\sum_{t\in Z(F)}\sum_{v,w}\int_{U_0(\mathbb{A})} f(tu)|\omega_{M_0,v,w}^S(u)|du$$

can be bounded by the product of $M\mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2^2})\gcd(t_1,\mathcal{N}_{F/\mathbb{Q}}(N_2))^3$ with

$$((\log\mathcal{N}_{F/\mathbb{Q}}(N_1N_2))^2 + (|S|+M_1)\log\mathcal{N}_{F/\mathbb{Q}}(N_1N_2) + |S|^2 + M_2)\mu(f_\infty)$$

for suitable constants $M, M_1, M_2 > 0$. Hence this is also bounded by $O(\deg T_{1,N_1,N_2})\mu(f_\infty)$. □

Proof of Lemma 13. (i) Assume that $i = 1$ (since conjugation with a certain Weyl group element yields $\omega^S(u_3)$, we only consider $\omega^S(u_1)$). Let $v \notin S$. Then $\omega_v^S(u_1) = 0$ unless $|u_1|_v > 1$, but $|t|_v$ need to be < 1 for such u with $f_v(tu) \neq 0$ to exist. Thus $\int_{U_0(F_v)} f_v(tu)|\omega_v^S(u_1)|du$ is bounded by

$$(\log|t|_v^{-1} + \frac{\log q_v}{q_v - 1})\int_{U_0(F_v)} f_v(tu)du \le a_1\log|\det t|_v^{-1}\int_{U_0(F_v)} f_v(tu)du$$

for some $a_1 \in \mathbb{N}$ independent of f and t, since we assumed $|t|_v < 1$. Now suppose that $v \in S$. Then the integral is bounded by

$$\begin{aligned}(\log|t|_v^{-1})\int_{U_0(F_v)} f_v(tu)du + \int_{U_0(F_v),|u_1|_v\le 1} f_v(tu)|\log|u_1|_v|du\\ \le \log|\det t|^{-1}\int_{U_0(F_v)} f_v(tu)du + \frac{\log q_v}{q_v-1}\int_{U_0(F_v)} f_v(tu)du,\end{aligned}$$

which can be bounded by $(\log|\det t|_v^{-1} + 1)\int_{U_0(F_v)} f_v(tu)du$.

(ii) If $v \neq w$, the assertion is a direct consequence of (i). For $v = w$ similar considerations as before are necessary. The modifications in the case $v = w$ for $\omega_{M_0,v,v}$ do no harm, since the additional terms can be bounded by $-\log|t|_v$ when $|t|_v < 1$. But this is always true when $\omega_{M_0,v,v}$ is supposed to be non-zero.

□

Proof of Lemma 14. For ease of notation we assume for the proof that w is a real place and write $F_w = \mathbb{R}$. The complex case is analogous.

All assertions readily follow from the following statement together with Lemmas 7 and 13: We first show that there exists a constant $M>0$ such that

$$\int_{U_0(\mathbb{R})}|h(tu)|\,|\log|u_1||^{\varepsilon_1}|\log|u_3||^{\varepsilon_3}du\le M\int_{Z(\mathbb{R})\backslash G(\mathbb{R})}\sum_{X\in\mathcal{B}_{\mathrm{PGL}_3(\mathbb{R})}}|X*h(tg)|dg \tag{12}$$

for all $h\in\mathcal{C}(G(\mathbb{R})^1)$ and all $\varepsilon_1,\varepsilon_3\in\{0,1,2\}$. It suffices to consider $\mathbf{K}_\infty$-central h and $t=1$. To show (12) we define the following subsets of $U_0(\mathbb{R})\simeq\mathbb{R}^3$:

$$A_0=\{u\in U_0(\mathbb{R})\mid |u_1|,|u_3|\le e\},$$
$$A_i=\{u\in U_0(\mathbb{R})\mid |u_i|\ge e, |u_k|\le e,\; k\in\{1,3\}\backslash\{i\}\},\; i=1,3,$$

and for $\sigma\in\{(),(13)\}$ a permutation on the two symbols $\{1,3\}$ let

$$B_\sigma=\{u\in U_0(\mathbb{R})\mid |u_{\sigma(1)}|\ge|u_{\sigma(3)}|\ge e\}.$$

This gives a partition of $U_0(\mathbb{R})$ in domains whose intersection form a set of measure 0 (in $\mathbb{R}^3$) so that in particular,

$$\int_{U_0(\mathbb{R})}=\int_{A_0}+\int_{A_1}+\int_{A_3}+\int_{B_{()}}+\int_{B_{(13)}},$$

and it suffices to show (12) for each of these integrals.
Let $\sigma\in\{(),(13)\}$. Then the integral (12) with integration domain restricted to B_σ can be bounded by

$$\int_{B_\sigma}|h(u)|(\log|u_{\sigma(1)}|)^4du\le c_0\int_{B_\sigma}|h(u)||u_{\sigma(1)}|du$$

for some constant $c_0>0$, which is independent of σ and h. Let $\tilde{U}=U_1$ or $\tilde{U}=U_2$ depending on whether $\sigma=()$ or $\sigma=(13)$. Then there is a continuous homomorphism $\Phi:\mathrm{GL}_2(\mathbb{R})\to\mathrm{PGL}_3(\mathbb{R})$ such that for all $u\in U_0(\mathbb{R})$ we have $\Phi(\begin{pmatrix}1&u_{\sigma(1)}\\&1\end{pmatrix})^{-1}u\in\tilde{U}(\mathbb{R})$, and

$$c_0\int_{B_\sigma}|h(u)||u_{\sigma(1)}|du\le 2c_0\int_{N_e(\mathbb{R})}|\tilde{h}(n(x))|xdn \tag{13}$$

for $N_e(R)=\{n(x)=\left(\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right)\mid x\ge e\}\subseteq\mathrm{GL}_2(\mathbb{R})$, and $\tilde{h}(g)=\int_{\tilde{U}(\mathbb{R})}h(\Phi(g)\tilde{u})d\tilde{u}$. As $\tilde{U}(\mathbb{R})$ is centralised by the conjugation with $\Phi(\mathrm{O}(2))$, $\tilde{h}$ is still $\mathrm{O}(2)$-central. Using the KAK decomposition for $\mathrm{PGL}_2(\mathbb{R})$, write for $n(x)\in N_e(\mathbb{R})$ $n(x)=k_1\operatorname{diag}(a,a^{-1})k_2$ with $k_1,k_2\in\mathrm{O}(2)/\{\pm1\}$, $a\in\mathbb{R}_{\ge1}$. Then (13) is bounded by

$$c_1\int_1^\infty\sup_{k_1,k_2\in\mathrm{O}(2)/\{\pm1\}}|\tilde{h}(k_1\operatorname{diag}(a,a^{-1})k_2)|(a^2-a^{-2})d^\times a,$$

where c_1 is suitable constant depending only on the chosen normalisation of measures. Then by [FiLa11b, Lemma 3.2] there is a suitable constant c_2 such that this is bounded by (using that $\tilde{h}$ is O(2)-central)

$$c_2 \int_1^\infty \int_{\mathrm{O}(2)/\{\pm 1\}} \int_{\mathrm{O}(2)/\{\pm 1\}} \sum_{X,Y\in\mathcal{B}_{\mathrm{SO}(2)}}$$
$$|X * Y * \tilde{h}(k_1 \operatorname{diag}(a, a^{-1})k_2)|(a^2 - a^{-2}) dk_1 dk_2 d^\times a$$
$$= c_2 \sum_{X,Y\in\mathcal{B}_{\mathrm{O}(2)}} \int_{\mathrm{PGL}_2(\mathbb{R})} |X * Y * \tilde{h}(g)| dg.$$

Now

$$\int_{\mathrm{PGL}_2(\mathbb{R})} |X * Y * \tilde{h}(g)| dg = \int_{\mathrm{PGL}_2(\mathbb{R})} \int_{\tilde{U}(\mathbb{R})} |\Phi(X) * \Phi(Y) * h(\Phi(g)\tilde{u})| d\tilde{u} dg$$
$$\leq c_3 \int_{\mathrm{PGL}_3(\mathbb{R})} \sum_{Z\in\mathcal{B}_{\mathrm{O}(3)}} |\Phi(X) * \Phi(Y) * h * Z(g)| dg$$

where the last inequality follows from Iwasawa decomposition and [FiLa11b, Lemma 3.2]. Note that $\Phi(X) * \Phi(Y) * h * Z = Z * \Phi(X) * \Phi(Y) * h$ by the O(3)-centrality we imposed on h. Hence (12) follows for $B_{()}, B_{(13)}$.
Now let i, j such that $\{i, j\} = \{1, 3\}$. Then the integral in (12) with integration domain restricted to A_i can be estimated by

$$\int_{A_i} |h(u)||u_i| du + \int_{A_i} |h(u)|| \log |u_j|| du,$$

where for the first integral the same estimates as before apply. For the second one we get an upper bound

$$2 \int_{u_2,u_i\in\mathbb{R}} \sup_{u_j\in[-e,e]} |h(u)| du_2 du_j \int_0^e |\log |x|| dx.$$

Since $\int_0^e |\log |x|| dx$ is a finite constant, and

$$\int_{u_2,u_i\in\mathbb{R}} \sup_{u_j\in[-e,e]} |h(u)| du_2 du_j \leq 2c_4 e \sum_{X\in\mathcal{B}_{U_0(\mathbb{R})}} \int_{U_0(\mathbb{R})} |X * h(u)| du$$
$$\leq 2c_5 e \sum_{X\in\mathcal{B}_{U_0(\mathbb{R})}} \sum_{Y\in\mathcal{B}_{\mathrm{O}(3)}} \int_{U_0(\mathbb{R})} \int_{\mathrm{O}(3)/\{\pm 1\}} |Y * X * h(uk)| dk du,$$

(12) now follows for A_1 and A_3. The integral over A_0 can be bounded by

$$2 \int_{u_2\in\mathbb{R}} \sup_{u_1,u_3\in[-e,e]} |h(u)| du_2 \int_0^e |\log |x|| dx$$
$$\leq 2c_6 \sum_{X\in\mathcal{B}_{U_0(\mathbb{R})}} \sum_{Y\in\mathcal{B}_{\mathrm{O}(3)}} \int_{U_0(\mathbb{R})} \int_{\mathrm{O}(3)/\{\pm 1\}} |Y * X * h(g)| dg \int_0^e |\log |x|| dx,$$

which shows (12) when the integration domain is restricted to A_0.

From (12) and the definition of $\omega_{M_0,v,w}$ and $\Omega_{M_0,w}$, all but the last assertion follow immediately. The last assertion also follows from (12) by additionally using that the embedding $\mathcal{O}_F^\times \hookrightarrow F_\infty^1$ is discrete, applying Lemma 4 (*iii*), and using $Z(F_\infty^1)(Z(F_\infty)\backslash G(F_\infty)) = G(F_\infty)^1$. □

II.iii.iii The minimal unipotent orbit

For the minimal unipotent orbit, we have the following expansion in the case of $f \in C_c^\infty(G(F_S)^1)$.

Lemma 15 ([Fl82], Lemma 4 (2)). *For $f \in C_c^\infty(G(F_S)^1)$ the contribution from the minimal unipotent orbit $J_{\mathfrak{n}_{min}}(f) = \sum_{t\in F^\times} J_{\mathfrak{n}^t_{min}}(f)$ is given by*

$$3\,\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^2 \sum_{t\in F^\times} \int_{\mathbb{A}^\times}\int_{\mathbf{K}} f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)|x|^2\log|x|dkd^\times x,$$

This expression for the minimal unipotent contribution does in general not converge for $f \in \mathcal{C}(G(\mathbb{A})^1, K)$. To obtain a convergent form we modify the log-term again by cutting out some of the support of f: Instead of $\log|x|$ we use a weight function $\alpha(x) = \alpha^S(x) = \sum_w \alpha_w^S(x)$, which is defined by

$$\alpha_w^S(x) = \begin{cases} \log|x|_v & \text{if } v \in S,\\ -\frac{\zeta'_{F,v}(2)}{\zeta_{F,v}(2)} + \log|x|_v & \text{if } v \notin S, \text{ and } |x|_v > 1,\\ 0 & \text{if } v \notin S, \text{ and } |x|_v \le 1,\end{cases}$$

and add

$$-3\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} \sum_{t\in Z(F)} \int_{\mathbf{K}}\int_{\mathbb{A}^\times} f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)|x|^2dkd^\times x.$$

With these weight functions we have

Proposition 16. *For $f \in C_c^\infty(G(F_S)^1)$ the minimial unipotent contribution $J_{\mathfrak{n}_{min}}(f) = \sum_{t\in F^\times} J_{\mathfrak{n}^t_{min}}(f)$ is given by the product of $\mathrm{vol}(F^\times\backslash\mathbb{A}^1)^2$ with*

$$3\sum_{t\in F^\times} \int_{\mathbb{A}^\times}\int_{\mathbf{K}} f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)|x|^2\alpha^S(x)dkd^\times x$$
$$- 3\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} \sum_{t\in F^\times} \int_{\mathbf{K}}\int_{\mathbb{A}^\times} f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)|x|^2dkd^\times x. \quad (14)$$

Moreover, the sum-integrals

$$\sum_{t\in F^\times} \int_{\mathbb{A}^\times}\int_{\mathbf{K}} |f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)||x|^2\sum_v|\alpha_v(x)|dkd^\times x$$

and

$$\sum_{t\in F^\times} \int_{\mathbf{K}}\int_{\mathbb{A}^\times} |f(k^{-1}t\begin{pmatrix}1 & & x\\ & 1 & \\ & & 1\end{pmatrix}k)||x|^2dkd^\times x$$

define continuous semi-norms on $\mathcal{C}(G(\mathbb{A})^1, K)$.

For $f \in C_c^\infty(G(F_S)^1)$ it follows as for the regular unipotent case that this new expansion reduces to the one given in Lemma 15.

First we show that the expression given in the proposition is invariant under enlarging S. For that it will suffice to show the following lemma.

Lemma 17. *Let S be a finite set of places containing all archimedean ones, and let v be a place outside of S. Let $h = \prod_v h_v : \mathbb{A} \to \mathbb{C}$ be sufficiently integrable, and assume further that $h(a + b_v) = h(a)$ for all $a \in \mathbb{A}$, $b_v \in \mathcal{O}_v$. Then*

$$\int_{\mathbb{A}^\times} h(x)|x|^2 \alpha^{S\cup\{v\}}(x) d^\times x - \frac{\zeta_F^{S\cup\{v\}\prime}(2)}{\zeta_F^{S\cup\{v\}}(2)} \int_{\mathbb{A}^\times} h(x)|x|^2 d^\times x$$

equals

$$\int_{\mathbb{A}^\times} h(x)|x|^2 \alpha^S(x) d^\times x - \frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} \int_{\mathbb{A}^\times} h(x)|x|^2 d^\times x.$$

("Sufficiently integrable" again means that all occuring integrals converge absolutely.)

Proof. We have

$$\begin{aligned}\int_{\mathbb{A}^\times} & h(x)|x|^2 \alpha^{S\cup\{v\}}(x) d^\times x \\ &= \int_{\mathbb{A}^\times} h(x)|x|^2 \alpha^S(x) d^\times x + \int_{\mathbb{A}^\times} h(x)|x|^2 (\alpha_v^{S\cup\{v\}}(x) - \alpha_v^S(x)) d^\times x,\end{aligned}$$

and because of the invariance of h,

$$\begin{aligned}\int_{F_v^\times} & h_v(x)|x|_v^2 (\alpha_v^{S\cup\{v\}}(x) - \alpha_v^S(x)) d^\times x \\ &= h(0)\zeta_{F,v}(1) \int_{\mathcal{O}_v} |x|_v \log|x|_v dx + \zeta_{F,v}(1) \frac{\zeta_{F,v}'(2)}{\zeta_{F,v}(2)} \int_{F_v \setminus \mathcal{O}_v} h_v(x)|x|_v dx.\end{aligned}$$

Since

$$\zeta_{F,v}(1) \int_{\mathcal{O}_v} |x|_v \log|x|_v dx = \sum_{k\geq 0} q^{-2k} \log q^{-k} = \zeta_{F,v}'(2),$$

and $\zeta_{F,v}(1) \int_{\mathcal{O}_v} |x|_v dx = \zeta_{F,v}(2)$, this equals

$$\frac{\zeta_{F,v}'(2)}{\zeta_{F,v}(2)} \int_{F_v^\times} h_v(x)|x|_v^2 d^\times x = -\left(\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} - \frac{\zeta_F^{S\cup\{v\}\prime}(2)}{\zeta_F^{S\cup\{v\}\prime}(2)}\right) \int_{F_v^\times} h_v(x)|x|_v d^\times x$$

which proves the assertion. ☐

To prove the second assertion about the absolute convergence for test functions $f \in \mathcal{C}(G(\mathbb{A})^1, K)$ in Proposition 16 we again need estimates for the local integrals. Thus we again assume that $f = f_\infty T_{r,N_1,N_2}$ with $r, N_1, N_2 \in$

$\mathrm{P}^+(F)$, $N_2 \le N_1$, as in (4). We also assume $r = 1$. Fix a place $v < \infty$ such that $f_v = f_{0,\varepsilon_1,\varepsilon_2}$, and write $u(x) = \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix}$. The following estimates are elementary calculations using the conditions under which the function f_v does not vanish.

Lemma 18. *The integral unweighted integral $\int_{F_v^\times} f_{0,\varepsilon_1,\varepsilon_2}(tu(x))|x|_v^2 d^\times x$ as well as the weighted integral $\int_{F_v^\times} f_{0,\varepsilon_1,\varepsilon_2}(tu(x))|x|_v^2|\alpha_v^S(x)|d^\times x$ vanish unless all conditions of Lemma 5 on t are satisfied. In this case the first one is bounded by*

$$\frac{|t|_v^{-2}\min\{q_v^{-2\varepsilon_2}|t|_v^{-2},1\}}{1-q_v^{-2}},$$

and the second one by

$$\frac{|t|_v^{-2}\min\{q_v^{-2\varepsilon_2}|t|_v^{-2},1\}}{1-q_v^{-2}}\log(|t|_v^{-1}) + \frac{q_v^{-2}\log q_v}{1-q_v^{-2}}\left(1+\frac{\delta_v}{1-q_v^{-2}}\right) \tag{15}$$

with $\delta_v = 1$ for $v \in S$ and $= 0$ otherwise.

Note that

$$\sum_{v<\infty}\frac{q_v^{-2}\log q_v}{1-q_v^{-2}}\left(1+\frac{\delta_v}{1-q_v^{-2}}\right) \le 3\left|\frac{\zeta_F'(2)}{\zeta_F(2)}\right|,$$

and

$$\begin{aligned}\frac{|t|_v^{-2}\min\{q_v^{-2\varepsilon_2}|t|_v^{-2},1\}}{1-q_v^{-2}} &\ge \min\{q_v^{-2\varepsilon_2}|t|_v^{-2},1\}\\ &\ge \mathcal{N}_{F/\mathbb{Q}}(N_2)^{-2}\gcd(\mathcal{N}_{F/\mathbb{Q}}(t),\mathcal{N}_{F/\mathbb{Q}}(N_2))^2\end{aligned}$$

for any v. In particular, the sum over all $v < \infty$ of (15) divided by $\frac{|t|_v^{-2}\min\{q_v^{-2\varepsilon_2}|t|_v^{-2},1\}}{1-q_v^{-2}}$ is bounded by

$$\log\mathcal{N}_{F/\mathbb{Q}}(t) + 3\left|\frac{\zeta_F'(2)}{\zeta_F(2)}\right|\frac{\mathcal{N}_{F/\mathbb{Q}}(N_2)^2}{\gcd(\mathcal{N}_{F/\mathbb{Q}}(t),\mathcal{N}_{F/\mathbb{Q}}(N_2))^2}. \tag{16}$$

Again we also need archimedean estimates. Recall that we defined $H(t,h) = \int_{Z(F_v)\backslash G(F_v)}\sum_{X\in\mathcal{B}_{\mathrm{PGL}_3(F_v)}}|X * h_v(g)|dg$ for $v|\infty$, $h \in \mathcal{C}(G(F_v)^1)$ and $t \in F^\times$.

Lemma 19. *Let v be an archimedean place. Then there exist constants $M_1, M_2, M_3 > 0$ such that for all $h \in \mathcal{C}(G(F_\infty)^1)$ being $\mathbf{K}_\infty$-central*

(i)

$$\int_{F_v^\times}|h_v(u(x))||x|_v^2 d^\times x \le M_1 H(t,f_v),$$

(ii)

$$\int_{F_v^\times}|h_v(u(x))||x|_v^2|\log|x|_v|d^\times x \le M_2 H(t,f_v),$$

(iii) and

$$\sum_{\varepsilon\in\mathcal{O}_F^\times}\int_{Z(F_\infty)\backslash G(F_\infty)}\sum_{X\in\mathcal{B}_{\mathrm{PGL}_3(F_\infty)}}|X*h(\varepsilon g)|dg\le M_3\mu(h).$$

We postpone the proof of this lemma until after the proof of Proposition 16.

Proof of Proposition 16. By Lemma 17 (14) is well-defined, i.e. invariant under enlarging S, for all functions f, which are bi-K-invariant and for which the sum-integrals converge. From the definition of α it is clear that for such f with compact support the sum (14) reduces to the distribution associated with the minimal unipotent orbit provided S is chosen sufficiently large.

For the convergence assume that $f=f_\infty T_{1,N_1,N_2}$ as always. As explained in the proof of Proposition 10 we may restrict our attention to the case that N_1N_2 has a cubic root in F, and to those $t\in F^\times$ in (14), which are of the form $t=t_1\varepsilon$ with $t_1=\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)^{\frac13}$ and $\varepsilon\in\mathcal{O}_F^\times$. By Lemmas 18 and 19 we therefore have

$$\begin{aligned}&\sum_{t\in F^\times}\int_{\mathbb{A}^\times}|f_{\mathbf{K}}(tu(x))|d^\times x\\ &\le\int_{\mathbb{A}_f^\times}T_{1,N_1,N_2}(t_1u(x))d^\times x\sum_{\varepsilon\in\mathcal{O}_F^\times}\int_{F_\infty^\times}|f_\infty(t_1\varepsilon u(x))|d^\times x\\ &\qquad\le M\zeta_F(2)\mathcal{N}_{F/\mathbb{Q}}(\frac{t_1}{N_2})^2\gcd(t_1,\mathcal{N}_{F/\mathbb{Q}}(N_2))^2\mu(f_\infty)\end{aligned}$$

for some absolute constant $M>0$. Using some crude estimate for the gcd, we can bound this by a constant multiple of

$$\begin{aligned}&\zeta(2)\mathcal{N}_{F/\mathbb{Q}}(N_1^{\frac23}N_2^{-\frac23}(N_1N_2)^{\frac23})\mu(f_\infty)\\ &\qquad\le\zeta(2)\mathcal{N}_{F/\mathbb{Q}}(N_1)^{\frac43}\mu(f_\infty)=O(\deg T_{1,N_1,N_2})\mu(f_\infty).\end{aligned}$$

Using (16) we similarly obtain for the weighted sum-integral in (14)

$$\begin{aligned}&M'\left(\log t_1+3|\frac{\zeta_F'(2)}{\zeta_F(2)}|\frac{\mathcal{N}_{F/\mathbb{Q}}(N_2)^2}{\gcd(\mathcal{N}_{F/\mathbb{Q}}(t),\mathcal{N}_{F/\mathbb{Q}}(N_2))^2}\right)\\ &\cdot\zeta_F(2)\mathcal{N}_{F/\mathbb{Q}}(\frac{t_1}{N_2})^2\gcd(t_1,\mathcal{N}_{F/\mathbb{Q}}(N_2))^2\mu(f_\infty)\\ &\qquad+M''\zeta_F(2)\mathcal{N}_{F/\mathbb{Q}}(\frac{t_1}{N_2})^2\gcd(t_1,\mathcal{N}_{F/\mathbb{Q}}(N_2))^2\mu(f_\infty)\end{aligned}$$

for some absolute constants $M',M''>0$. This is bounded by a constant multiple of

$$(\log\mathcal{N}_\infty+3|\frac{\zeta_F'(2)}{\zeta_F(2)}|)\mathcal{N}_{F/\mathbb{Q}}(N_1)^{\frac43}\mu(f_\infty)=O(\deg T_{1,N_1,N_2})\mu(f_\infty).$$

□

Proof of Lemma 19. For (i) and (ii) we again assume $F_v = \mathbb{R}$ and write $|\cdot|_v = |\cdot|$.

(i) Changing the multiplicative measure on $\mathbb{R}^\times$ to an additive one on $\mathbb{R}$, we can consider the integral $\int_{\mathbb{R}} |h(u(x))||x|dx$ instead. We have

$$\int_0^\infty |h(\begin{pmatrix} 1 & 0 & x \\ & 1 & 0 \\ & & 1 \end{pmatrix})|xdx \leq \int_0^\infty \int_{\mathrm{O}(2)} \sum_{X \in \mathcal{B}_{\mathrm{O}(3)}} |X * h(\begin{pmatrix} k & k\binom{x}{0} \\ & 1 \end{pmatrix})|xdkdx$$

Changing variables, we get the upper bound

$$\int_0^\infty \int_0^\infty \int_{\mathrm{O}(3)} \sum_{X \in \mathcal{B}_{\mathrm{O}(3)}} |X * h(\begin{pmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{pmatrix} k)|dkdydx$$

which in turn can be bounded by $\int_{U_0(\mathbb{R})} \int_{\mathrm{O}(3)/\{\pm 1\}} \sum_{X \in \mathcal{B}_{\mathrm{O}(3)}, Y \in \mathcal{B}_{U_0(\mathbb{R})}} |Y * X * h(uk)|dkdu$.

(ii) We first show that $\int_{x \in \mathbb{R}, |x| \geq 3} |h(u(x))|x^2 dx$ can be bounded as asserted. For $k_1, k_2 \in \mathrm{O}(3)$, and $a, b \in \mathbb{R}^\times$ let

$$F_{k_1,k_2}(a,b) = h(k_1 \operatorname{diag}(a, a^{-1}b, b)k_2)(a^2 - a^{-2})(b^2 - b^{-2})(ab - a^{-1}b^{-1})$$

so that by the KAK decomposition for $\mathrm{PGL}(3)$,

$$\int_{\mathrm{O}(3)/\{\pm 1\}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_1^\infty \int_1^\infty F_{k_1,k_2}(a,b) d^\times a d^\times b dk_1 dk_2 = c_3 \int_{Z(\mathbb{R}) \backslash G(\mathbb{R})} f(g)dg$$

for some suitable constant c_3 depending only on the chosen normalisation of the measures. By [FiLa11b, Lemma 3.3] there exists a constant $c > 0$ independent of h such that for all $a \in \mathbb{R}^\times$

$$\begin{aligned} |F_{k_1,k_2}(a,a)| &\leq c \int_{\frac{1}{2}}^{\frac{3}{2}} |F_{k_1,k_2}(a,ta)| + |\partial_2 F_{k_1,k_2}(a,ta)|dt \\ &\leq \frac{3c}{2} \int_{\frac{1}{2}}^{\frac{3}{2}} |F_{k_1,k_2}(a,ta)| + |\partial_2 F_{k_1,k_2}(a,ta)|d^\times t. \end{aligned} \tag{17}$$

Here $\partial_2 F_{k_1,k_2}(a,b)$ denotes the partial derivative of F with respective to the variable b, which equals

$$\begin{aligned} &\partial_2 F_{k_1,k_2}(a,b) = \\ &(X_b * h)(k_1 \operatorname{diag}(a, a^{-1}b, b^{-1})k_2)(a^2 - a^{-2})(b^2 - b^{-2})(ab - a^{-1}b^{-1}) \\ &\quad + 2\frac{b^2 + b^{-2}}{b(b^2 - b^{-2})} F_{k_1,k_2}(a,b) + \frac{ab + a^{-1}b^{-1}}{b(ab - a^{-1}b^{-1})} F_{k_1,k_2}(a,b), \end{aligned}$$

where for $A_2 = \{\mathrm{diag}(a, a^{-1}b, b^{-1}) \mid a, b \in \mathbb{R}^\times\}$, $X_b \in \mathcal{L}ie(A_2)$ denotes one of the basis elements $\mathrm{diag}(0, 1, -1)$ or $\mathrm{diag}(1, -1, 0)$. It follows from the identity $h(k_1 g k_2) = h(g k_2 k_1)$ (as we supposed h to be O(3)-central) that the application of X_b yields the derivative with respect to b. Note that for $b > 2$, $|\frac{b^2+b^{-2}}{b(b^2-b^{-2})}| \leq 1$ and $|\frac{ab+a^{-1}b^{-1}}{b(ab-a^{-1}b^{-1})}| \leq 1$.
Suppose that $x \geq 3$. Again as above the KAK decomposition of x is $k_1 \,\mathrm{diag}(a, 1, a^{-1}) k_2$ for suitable $k_1, k_2 \in \Phi(\mathrm{O}(2)) \subseteq \mathrm{O}(3)$ and $a = a(x) \geq 1$ with $3 + x^2 = a^2 + a^{-2} + 1$ so that $a > 2$. For $x \to \infty$, $a(x)$ is asymptotic to x so that $a'(x) = \frac{xa(x)}{a^2-a^{-2}}$ is asymptotic to 1. Thus there is some $c_0 > 0$ such that $|a'(x)| \leq c_0$ for all $x \geq 1$. Therefore

$$\int_3^\infty |h(u(x))| x^2 dx \\ \leq \int_3^\infty |h(k_1 \,\mathrm{diag}(a, 1, a^{-1}) k_2)| a'(x)^2 \frac{(a^2 - a^{-2})^2}{a^2} dx$$

which is bounded by

$$c_1 c_0 \sum_{X,Y \in \mathcal{B}_{\mathrm{O}(3)}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_2^\infty \\ |X * h * Y(k \,\mathrm{diag}(a, 1, a^{-1}) k')| \frac{(a^2 - a^{-2})^2}{a} d^\times a dk dk',$$

where c_0 is an absolute constant. Since $a > 2$ and $a^2 - a^{-2} > 1$, this is bounded by

$$\leq c_1 c_0 \sum_{X,Y \in \mathcal{B}_{\mathrm{O}(3)}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_2^\infty |XFY_{k,k'}(a, a)| d^\times a dk dk'$$

with XFY the function on $\mathbb{R}^\times \times \mathbb{R}^\times$ associated with $X * h * Y$ as above. Using (17) this can be bounded by

$$c_1 c_0 c \frac{3}{2} \sum_{X,Y \in \mathcal{B}_{\mathrm{O}(3)}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_2^\infty \int_{\frac{1}{2}}^{\frac{3}{2}} |XFY_{k,k'}(a, ta)| \\ + |\partial_2 XFY_{k,k'}(a, ta)| d^\times t d^\times a dk dk'$$

$$\leq 9 c_1 c_0 c \sum_{X,Y \in \mathcal{B}_{\mathrm{O}(3)}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_{\mathrm{O}(3)/\{\pm 1\}} \int_1^\infty \int_1^\infty |XFY_{k,k'}(a, b)| \\ + |X_b XFY(a, b)| d^\times a d^\times b dk dk',$$

where $X_b XFY$ is the function associated with $X_b * X * h * Y$. Hence using the O(3)-centrality of h, this last expression equals

$$9c_1c_0cc_3 \sum_{X,Y\in\mathcal{B}_{\mathrm{O}(3)}} \int_{Z(\mathbb{R})\backslash G(\mathbb{R})} |Y * X_b * X * h(g)|dg \\ + 9c_1c_0cc_3 \sum_{X,Y\in\mathcal{B}_{\mathrm{O}(3)}} \int_{Z(\mathbb{R})\backslash G(\mathbb{R})} |Y * X_b * X * h(g)|dg.$$

Since $|\log|x|| \leq |x|$ for $|x| > e$, this gives the desired bound for the integral over $|x| \in [3, \infty)$. For the remaining integral note that

$$\int_{-3}^{3} |h(u(x))||x||\log|x||dx \leq 2 \sup_{x\in[-3,3]} |h(u(x))| \int_0^3 |x||\log|x||dx$$

and the last integral is a finite constant. Applying Lemma 4 to bound $\sup_{x\in[-3,3]} |h(u(x))|$ gives the missing estimate.

(iii) This is the same as Lemma 14 (*iv*).

□

II.iv Equivalence classes in $\mathcal{O}^3_{\text{quad}}$

The classes $\mathfrak{o} \in \mathcal{O}^3_{\text{quad}}$ are in bijective correspondence to tuples $(t_1, t_2) \in F^\times \times F^\times$, $t_1 \neq t_2$, or rather semisimple $\sigma \in Z^{M_1}(F)_{\text{reg}} = Z^{M_1}(F)\backslash Z(F)$. Fix such a σ and let $\mathfrak{o}$ be the corresponding class. Then for $f \in C_c^\infty(G(F_S)^1)$ we have by [Ar86, Lemma 6.2]

$$J_{\mathfrak{o}}(f) = \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} \sum_{R \in \mathcal{F}^\sigma} \frac{|W_0^{M_R}|}{|W_0^{G_\sigma}|} J^{M_R}_{\text{unip}}(\Phi^\sigma_{R,y}) dy, \tag{18}$$

where $\mathcal{F}^\sigma$ denotes the set of all parabolic subgroups in G_σ containing T, and the function $\Phi^\sigma_{R,y} : M_R(\mathbb{A}) \longrightarrow \mathbb{C}$ is defined by

$$\Phi^\sigma_{R,y}(m) = \delta_R(m)^{\frac{1}{2}} \int_{\mathbf{K}_\sigma} \int_{N_R(\mathbb{A})} f(y^{-1}k^{-1}\sigma mnky)\nu'_R(ky, T_0) dn dk$$

with $T_0 \in \mathfrak{a}_0$ given by [Ar81, Lemma 1.1] and δ_R the modulus function of the action of M_R on R. We may assume that $T = 0$ by [Ar81, Lemma 1.1]. The function $\nu'_R(ky, T)$ is defined in [Ar86, §6] as follows: For a parabolic subgroup $P \in \mathcal{F}(T)$ put $v_P(\lambda, x, T) = e^{\lambda(-H_P(x)+T)}$ and $v_M(\lambda, x, T) = \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x, T)\theta_P(\lambda)^{-1}$ as usual. Then $v_M(x, T) = \lim_{\lambda \to 0} v_M(\lambda, x, T)$, and $v'_P(x, T)$ is associated with $v_P(\lambda, x, T)$ as explained in [Ar81, (6.3)]. Then

$$\nu'_R(ky, T) = \sum_{Q \in \mathcal{F}^0_R(T)} v'_Q(ky, T)$$

with $\mathcal{F}^0_R(T) \subseteq \mathcal{F}(T)$ the subset of all parabolics Q such that $Q_\sigma = R$ and $\mathfrak{a}_Q = \mathfrak{a}_R$. The condition that $\mathfrak{a}_R = \mathfrak{a}_Q$ is equivalent to demanding that the Levi components of both groups are equal. Alternatively, $v'_Q(x, T)$ can be defined by

$$v'_Q(x, T) = \int_{\mathfrak{a}^G_Q} \Gamma^G_Q(X, -H_Q(x) + T) dX,$$

where the function $\Gamma^G_Q(X, H)$ equals [Ar81, (2.1)]

$$\sum_{R \in \mathcal{F}(T), Q \subseteq R} (-1)^{\dim \mathfrak{a}^G_R} \tau^R_Q(X)\hat{\tau}_R(X - H)$$

for $\tau^R_Q : \mathfrak{a}_T \longrightarrow \mathbb{C}$ the characteristic function of the set $\{X \in \mathfrak{a}_T \mid \alpha(X) > 0 \ \forall \alpha \in \Delta^R_Q\}$, and $\hat{\tau}_R : \mathfrak{a}_0 \longrightarrow \mathbb{C}$ the characteristic function of $\{X \in \mathfrak{a}_T \mid \varpi(X) > 0, \ \forall \varpi \in \hat{\Delta}_R\}$. Here Δ^R_Q is the set of simple roots of $(Q \cap M_R, A_Q) \subseteq M_R$, and $\hat{\Delta}_R$ the set of simple weights of (R, A_R).

For our σ, $G_\sigma = M_1$ so that $\mathcal{F}^\sigma = \{M_1, B \times \mathrm{GL}(1), \bar{B} \times \mathrm{GL}(1)\}$ for $B \subseteq \mathrm{GL}(2)$ the standard Borel subgroup of upper triangular matrices. It will suffice to consider each of the terms

$$\int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} J^{M_R}_{\text{unip}}(\Phi^\sigma_{R,y}) dy \tag{19}$$

individually for $R \in \mathcal{F}^\sigma$ instead of (18).

II.iv.i The case $R = B \times \mathrm{GL}(1)$

The Levi component $M_{B\times\mathrm{GL}(1)} = M_{\bar{B}\times\mathrm{GL}(1)}$ is the torus $T \subseteq \mathrm{GL}(3)$ so that the trivial conjugacy class $\{\mathbf{1}_3\}$ is the unique unipotent orbit. Hence

$$J_{\mathrm{unip}}^{M_{B\times\mathrm{GL}(1)}}(\Phi^{\sigma}_{B\times\mathrm{GL}(1),y}) = \Phi^{\sigma}_{B\times\mathrm{GL}(1),y}(1)$$

and similarly $J_{\mathrm{unip}}^{M_{\bar{B}\times\mathrm{GL}(1)}}(\Phi^{\sigma}_{\bar{B}\times\mathrm{GL}(1),y}) = \Phi^{\sigma}_{\bar{B}\times\mathrm{GL}(1),y}(1)$. For this distribution we have the following.

Proposition 20. *For $R \in \{B \times \mathrm{GL}(1), \bar{B} \times \mathrm{GL}(1)\}$ the sum*

$$\sum_{\mathfrak{o}\in\mathcal{O}^3_{quad}} \frac{1}{2}\int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} J^T_{unip}(\Phi^{\sigma}_{R,y})dy$$

converges absolutely for all $f \in \mathcal{C}(G(\mathbb{A})^1, K)$.

As R and $\bar{R}$ are conjugate, we only need to consider $J_{\mathrm{unip}}^{M_{B\times\mathrm{GL}(1)}}(\Phi^{\sigma}_{B\times\mathrm{GL}(1),y})$.

To prove this, we use the following simple upper bound for the weight $\nu'_{B\times\mathrm{GL}(1)}$.

Lemma 21. *For all $u \in U_1(\mathbb{A})$ we have*

$$|\nu'_{B\times\mathrm{GL}_1}(u, T_0)| \leq (\log||(1, u_2)||)^2 + (\log||(1, u_3)||)^2.$$

Here $||(x, y)|| = \sum_{v\leq\infty} ||(x, y)||_v$ with

$$||(x,y)||_v = \begin{cases} \max\{|x|_v, |y|_v\} & \text{if } v < \infty \\ \sqrt{|x|_v^2 + |y|_v^2} & \text{if } v|\infty \text{ real} \\ |x|_v + |y|_v & \text{if } v|\infty \text{ complex.} \end{cases}$$

We postpone the proof of this lemma until the end of this section. It will therefore suffice to estimate

$$\int_{U_1(\mathbb{A})}\int_{N_{B\times\mathrm{GL}_1}(\mathbb{A})} f_{\mathbf{K}}(\begin{pmatrix} 1 & 0 & u_2 \\ & 1 & u_3 \\ & & 1\end{pmatrix}^{-1} \sigma \begin{pmatrix} 1 & n & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & u_2 \\ & 1 & u_3 \\ & & 1\end{pmatrix})(\log||(1, u_i)||)^2 dn du$$

for $i = 2, 3$, where we already changed y to kyk^{-1} (which yields only the factor 1), and used $G_\sigma(\mathbb{A})\backslash G(\mathbb{A}) \simeq U_1(\mathbb{A})(\mathbf{K}_\sigma\backslash\mathbf{K})$. Taking into account that $t_1 \neq t_2$ for $\sigma = \mathrm{diag}(t_1, t_1, t_2)$ so that $|\frac{t_1-t_2}{t_1}| = 1$, another change of variables gives

$$\int_{U_0(\mathbb{A})} f_{\mathbf{K}}(\sigma\begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & u_3 \\ & & 1\end{pmatrix})p_i(\sigma, u)^2 du$$

with

$$p_2(\sigma, u) = \log||((1 - \frac{t_2}{t_1})^2, (1 - \frac{t_2}{t_1})u_2 - u_1u_3)||$$

and

$$p_3(\sigma, u) = \log ||(1 - \frac{t_2}{t_1}, u_3)||.$$

If v is a place of F, let $p_{2,v}(\sigma, u) = \log ||(1 - \frac{t_2}{t_1})^2, (1 - \frac{t_2}{t_1})u_2 - u_1 u_3)||_v$, $p_{3,v}(\sigma, u) = \log ||(1 - \frac{t_2}{t_1}, u_3)||_v$.

To show the convergence we will need the following estimates.

Lemma 22. *Let v be a non-archimedean place of F and let f_v be the function $f_{0,\varepsilon_1^v,\varepsilon_2^v}$.*

(i) The integrals $\int_{U_0(F_v)} f_v(tu)|p_{2,v}(t,u)|du$, and $\int_{U_0(F_v)} f_v(tu)|p_{3,v}(t,u)|du$ vanish unless $t_1, t_2 \in \mathcal{O}_v$, and unless (i) and (ii) of Lemma 5 are satisfied. There exists a constant $M \in \mathbb{N}$ (independent of v) such that each of the integrals is bounded by

$$M \left(\log |\det t|_v^{-1} + \log |t_1 - t_2|_v^{-1}\right) \int_{U_0(F_v)} f_v(tu)du.$$

(ii) The integral $\int_{U_1(F_v)} f_v(tu)p_{i,v}(t,u)^2 du$ vanishes unless $t_1, t_2 \in \mathcal{O}_v$, and unless the conditions (i) and (ii) of Lemma 5 are satisfied for v. Then there exists a constant $M' \in \mathbb{N}$ (independent of v) such that it is bounded by

$$M' \left(\log |\det t|_v^{-1} + \log |t_1 - t_2|_v^{-1}\right)^2 \int_{U_0(F_v)} f_v(tu)du.$$

We also need archimedean estimates.

Lemma 23. *For $t_1, t_2 \in F^\times$, $h \in \mathcal{C}(G(F_\infty)^1)$ let the function $H(t_1, t_2\sqrt{|\frac{t_1}{t_2}|})$ be defined by*

$$\int_{Z^{M_2}(F_\infty)\backslash M_2(F_\infty)} \int_{U_2(F_\infty)} \int_{\mathbf{K}_\infty/Z(F_\infty)\cap\mathbf{K}_\infty} \sum_{X \in \mathcal{B}_{M_2(F_\infty)^1}, Y \in \mathcal{B}_{\mathbf{K}_\infty}} |Y * X * h(\operatorname{diag}(t_1, t_2\sqrt{|\frac{t_1}{t_2}|}, t_2\sqrt{|\frac{t_1}{t_2}|})muk)| dk du dm,$$

and let $H_v(t_1, t_2\sqrt{|\frac{t_1}{t_2}|})$ the respective local versions, $v|\infty$.

(i) Let $v|\infty$. There is a constant $M > 0$ such that for all $h \in \mathcal{C}(G(F_v)^1)$, $t \in Z^{M_1}(F)_{reg} \cap \mathrm{Mat}_{3\times 3}(\mathcal{O}_F)$, $t = \operatorname{diag}(t_1, t_1, t_2)$,

$$\int_{U_0(F_v)} |h(tu)||p_{2,v}(t,u)| du$$

$$\leq M\frac{(|\log|t_1-t_2|_v|+\log|\det t|_v+\log(2+|\frac{t_1}{t_2}|_v))}{\max\{1,|\frac{t_2}{t_1}|_v\}}H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|})$$

and

$$\int_{U_0(F_v)}|h(tu)||p_{3,v}(t,u)|du\leq M\frac{\log(2+|\frac{t_2}{t_1}|_v)}{\max\{1,|\frac{t_2}{t_1}|_v\}}H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|}).$$

Replacing $(|\log|t_1-t_2|_v|+\log|\det t|_v+\log(2+|\frac{t_1}{t_2}|_v))$ *with its square, and* $\log(2+|\frac{t_2}{t_1}|_v)$ *by* $(1+\log(2+|\frac{t_2}{t_1}|_v))^2$ *such estimates stay true for* $p_{i,v}^2$ *in place of* $p_{i,v}$*.*

(ii) *There exists* $M_2>0$ *such that for all* $h\in\mathcal{C}(G(F_\infty)^1)$

$$\sum_\varepsilon|\log|\varepsilon_1t_1-\varepsilon_2t_2|_v|H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|})\leq M_2(\log|t_1t_2|_v+1)\mu(h)$$

and

$$\sum_\varepsilon H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|})\leq M_2\mu(h)$$

where the sum runs over all $\varepsilon\in Z^{M_1}(\mathcal{O}_F)$ *such that* $\varepsilon_1t_1\neq\varepsilon_2t_2$*.*

We again postpone the proof of both lemmas to the end of the section and first complete the proof of Proposition 20. Let $\mathcal{D}(N_1,N_2)$ be the set of all pairs of integers $(t_1,t_2)\in\mathbb{N}^2$, satisfying similar conditions as in Corollary 6: $t_1^2t_2=\mathcal{N}_{F/\mathbb{Q}}(N_1N_2)$, and $\mathcal{N}_{F/\mathbb{Q}}(N_2)$ divides t_1^2 and t_1t_2. For that we have the following:

Lemma 24. *Suppose that* $N_1,N_2\in\mathbb{N}$ *with* $N_2|N_1$*. For any* $\varepsilon>0$ *with* $\varepsilon<\frac{1}{16}$ *we have*

$$\frac{N_1^{1+\varepsilon}}{N_2^2}\sum_{(t_1,t_2)\in\mathcal{D}(N_1,N_2)}\gcd(t_1,N_2)^2\gcd(t_2,N_2)=O(\deg T_{1,N_1,N_2})\qquad(20)$$

with implied constant independent of N_1,N_2*.*

With this lemma we can finish the proof of the proposition.

Proof of Proposition 20. We need to show that

$$\sum_{t\in Z^{M_1}(F)_{\mathrm{reg}}}\int_{U_0(\mathbb{A})}|f_{\mathbf{K}}(tu)|p_i(t,u)^2du$$

can be bounded by $O(\deg T_{1,N_1,N_2})\mu(f_\infty)$ for any $f=f_\infty T_{1,N_1,N_2}$. Write $p_i(t,u)^2=p_{i,f}(t,u)^2+2p_{i,f}p_{i,\infty}+p_{i,\infty}(t,u)^2$. Let $\overline{\mathcal{D}}(N_1,N_2)\subseteq Z^{M_1}(F)_{\mathrm{reg}}$ be the set of all $t\in Z^{M_1}(F)_{\mathrm{reg}}$ satisfying the conditions (i) and (ii) of

Corollary 6. Then the canonical map $\overline{\mathcal{D}}(N_1, N_2) \longrightarrow \mathcal{D}(N_1, N_2)$ mapping t to $\operatorname{diag}(\mathcal{N}_{F/\mathbb{Q}}(t_1), \mathcal{N}_{F/\mathbb{Q}}(t_1), \mathcal{N}_{F/\mathbb{Q}}(t_2))$ has fibres $Z^{M_1}(\mathcal{O}_F) \simeq (\mathcal{O}_F^\times)^2$, if the image is not central in $\mathrm{GL}_3(F)$. (In this case, $(1, \frac{t_1}{t_2})$ is not contained in the fibre.) We can restrict the summation to the set $\overline{\mathcal{D}}(N_1, N_2)$ since all other summands vanish by Corollary 6.
By Lemma 22

$$\sum_{t\in Z^{M_1}(F)} \int_{U_0(\mathbb{A})} |f_{\mathbf{K}}(tu)| p_{i,f}(t,u)^2 du$$
$$\leq MM' \sum_{t\in Z^{M_1}(F)} (\log|\det t|_\infty + |\log|t_1 - t_2|_\infty|)^2 \int_{U_0(\mathbb{A})} f(tu)du.$$

If $t \in Z^{M_1}(F)_{\mathrm{reg}}$ with $|t_1|_\infty \neq |t_2|_\infty$ and $t_1, t_2 \in \mathcal{O}_F$, then $|\log|t_1 - t_2|_\infty| \leq \log|\det t|_\infty$. If $|t_1|_\infty = |t_2|_\infty$ with $t_1, t_2 \in \mathcal{O}_F$, there is $\alpha \in \mathcal{O}_F^\times$ with $t_1 = t_2\alpha$, and thus $\log|t_1 - t_2|_\infty = \log|t_1|_\infty + \log|1-\alpha|_\infty$. Since there is some constant $A > 0$ only depending on F such that $|1-\alpha|_\infty \geq A$ for all $\alpha \in \mathcal{O}_F^\times$, we get in any case $|\log|t_1 - t_2|_\infty| \leq \log|\det t|_\infty + |\log A|$. By Lemma 8 we therefore get for the sum-integral above as an upper bound the product of

$$MM'(\log \mathcal{N}_{F/\mathbb{Q}}(N_1N_2) + |\log A|)^2$$

with

$$\sum_{\tau\in\mathcal{D}(N_1,N_2)} \mathcal{N}_{F/\mathbb{Q}}(\frac{N_1}{N_2}) \gcd(\tau_1, \mathcal{N}_{F/\mathbb{Q}}(N_2))^2 \gcd(\tau_2, \mathcal{N}_{F/\mathbb{Q}}(N_2))\mu(f_\infty)$$

and this can be bounded as asserted by Lemma 24.
Using Lemmas 22, 23 and 8 we can find similar bounds for the sum-integrals involving $p_{i,f}p_{i,\infty}$ and $p_{i,\infty}^2$. Application of Lemma 24 again yields the assertion. □

Proof of Lemma 24. By definition of $\mathcal{D}(N_1, N_2)$, $N_2|t_1^2 = \frac{N_1N_2}{t_2}$ and $N_2|t_1t_2 = \frac{N_1N_2}{t_1}$ so that t_1 and t_2 are divisors of N_1. The product over the gcd's is less than or equal to $N_2^2 \gcd(t_1, N_2) \leq N_2^2(N_1N_2)^{\frac{1}{3}}$. Hence the partial sum on the left hand side subject to the condition $t_1 \leq t_2$ is bounded by

$$N_1^{1+\varepsilon}(N_1N_2)^{\frac{1}{3}} \sum_{t_1:\ N_2|t_1^2|N_1N_2,\ t_1\leq(N_1N_2)^{\frac{1}{3}}} \sum_{t_2|\gcd(t_1^2,N_1)} 1$$
$$\leq N_1^{1+\varepsilon}(N_1N_2)^{\frac{1}{3}} \sum_{a:\ N_2|a|N_1N_2,\ a\leq(N_1N_2)^{\frac{2}{3}}} \sigma_0(\gcd(a, N_1))$$

with σ_0 the divisor function $\sigma(n) = \sum_{d|n} 1$. Since $\sigma_0(n) = O(n^\eta)$ for all $\eta > 0$, there is a constant $C_1 > 0$, which is independent of N_1 and N_2 such that

$\sigma_0(\gcd(a,N_1)) \le C_1 \gcd(a,N_1)^{\frac{1}{16}}$ for all $a \in \mathbb{N}$. Hence this partial sum is bounded by

$$N_1^{1+\varepsilon}(N_1N_2)^{\frac{1}{3}}C_1 \sum_{b\colon\, b|N_1,\ b\le N_1^{\frac{2}{3}}N_2^{-\frac{1}{3}}} N_2^{\frac{1}{16}} \gcd(b,\frac{N_1}{N_2})^{\frac{1}{16}}$$
$$\le N_1^{\frac{5}{3}+\varepsilon}N_2^{\frac{1}{16}}C_1 \sum_{b\colon\, b|N_1,\ b\le N_1^{\frac{2}{3}}N_2^{-\frac{1}{3}}} b^{\frac{1}{16}}.$$

Using Lemma 25 this can be estimated by

$$N_1^{\frac{5}{3}+\varepsilon}N_2^{\frac{1}{16}}C_1N_1^{\frac{1}{24}}N_2^{-\frac{1}{48}}\prod_{p|N_1}(1-p^{-\frac{1}{16}})^{-1} = N_1^{\frac{7}{4}+\varepsilon}C_1\prod_{p|N_1}(1-p^{-\frac{1}{16}})^{-1}. \quad (21)$$

Consider now the partial sum, in which t_1, t_2 are subject to the condition $t_1 > t_2$. This means that $t_1t_2 \le (N_1N_2)^{\frac{2}{3}}$ so that if we set $a = t_1t_2$ this partial sum is bounded by

$$N_1^{1+\varepsilon}(N_1N_2)^{\frac{1}{3}} \sum_{a\colon\, N_2|a|N_1N_2,\ a\le(N_1N_2)^{\frac{2}{3}}} \sum_{t_2|\gcd(a,N_1)} 1.$$

Thus we can proceed as before and eventually arrive at (21) again.
Note that there is some prime $p_0 \in \mathbb{N}$ such that for all $p \ge p_0$ we have $(1-p^{-\frac{1}{16}})^{-1} \le p^{\frac{1}{16}}$ (and in particular, p_0 is indepent of N_1). Hence we can bound the product $\prod\limits_{p|N_1}(1-p^{-\frac{1}{16}})^{-1}$ by $cN_1^{\frac{1}{16}}$ with $c = \prod_{p\le p_0}(1-p^{-\frac{1}{16}})^{-1}$. Hence the whole left hand side can be bounded by $2cC_1N_1^{\frac{7}{4}+\frac{1}{16}+\varepsilon} = 2cC_1N_1^{\frac{29}{16}+\varepsilon}$. Using Lemma 3 and taking into account that $\varepsilon < \frac{1}{16}$, we see that (20) is bounded by

$$2^3 \cdot 3^2 cC_1 \deg T_{1,N_1,N_2}$$

as asserted. □

Lemma 25. *Let $q \in \mathbb{Q}_{>0}, N \in \mathbb{N}$, and $1 \le a \le N$. Then*

$$\sum_{d|N,\ d\le a} d^q \le a^q \prod_{p|N}(1-p^{-q})^{-1}.$$

Proof. The statement is clear if a is a divisor of N. For the other cases just choose a divisor b of N, which is maximal with respect to the property that $b \le a$. □

Proof of Lemma 22. (i) We first consider $p_{3,v}$. By Lemma 5 we may assume $|u_3| \le |t_1|^{-1}$, and $t_1, t_2 \in \mathcal{O}_v$ so that if $|1-\frac{t_2}{t_1}|_v \le 1$, we have

$$0 \ge \log|1-\frac{t_2}{t_1}|_v \ge \log|t_1-t_2|_v,$$

and if $|1-\frac{t_2}{t_1}|_v > 1$,

$$0 < \log|1-\frac{t_2}{t_1}|_v \leq \log|t_1|_v^{-1}.$$

In any case we get $|\log|1-\frac{t_2}{t_1}|_v| \leq \log|t_1|_v^{-1} + \log|t_2-t_1|_v^{-1}$. We can split the integral in two parts, one belonging to the condition $|u_3|_v > |1-\frac{t_2}{t_1}|_v$, and the other one to $|u_3|_v \leq |1-\frac{t_2}{t_1}|_v$. By the preceeding consideration we may bound the second integral by the product of $\log|t_1|_v^{-1} + \log|t_2-t_1|_v^{-1}$ with $\int_{U_0(F_v)} f_v(tu)du$. For the first integral note that $|1-\frac{t_2}{t_1}|_v < |u_3|_v \leq \min\{|t_1|_v^{-1}, q_v^{-\varepsilon_2}|t_1|_v^{-2}\}$ for u_3 such that $f_v(tu)$ is possibly non-zero. Thus for such u_3 we have

$$\begin{aligned}|\log|u_3|_v| &\leq |\log|1-\frac{t_2}{t_1}|_v| + \log|t_1|_v^{-1} + \log q_v^{-\varepsilon_2}|t_1|_v^{-2} \\ &\leq \log|t_1|^{-1} + \log|t_2-t_1|_v^{-1} + \log|t_1|_v^{-1} + \log|t_1|_v^{-2} \\ &\leq 2\log|\det t|_v^{-1} + \log|t_2-t_1|_v^{-1}.\end{aligned}$$

Hence the asserted bound follows for $p_{3,v}$.

Now consider $p_{2,v}$. The term $\log|(1-\frac{t_2}{t_1})^2|_v$ is again bounded by $4\log|\det t|_v^{-1} + 2\log|t_1-t_2|_v^{-1}$.

We now split the integral in one part belonging to $|(1-\frac{t_2}{t_1})u_2-u_1u_3|_v \leq |(1-\frac{t_2}{t_1})^2|_v$, and the other belonging to $|(1-\frac{t_2}{t_1})u_2-u_1u_3|_v > |(1-\frac{t_2}{t_1})^2|_v$. The first integral is then bounded by the product of $4\log|\det t|_v^{-1} + 2\log|t_1-t_2|_v^{-1}$ with $\int_{U_0(F_v)} f_v(tu)du$, whereas for the second integral we have

$$|(1-\frac{t_2}{t_1})^2|_v < |(1-\frac{t_2}{t_1})u_2 - u_1u_3|_v \leq |u_2-u_1u_3|_v + |\frac{t_2}{t_1}u_2|_v.$$

By Lemma 5 if u is such that $f_v(tu)$ is possibly non-zero, the term $|u_2-u_1u_3|_v$ is bounded from above by $q_v^{-\varepsilon_2}|t_1|_v^{-2}$, and $|u_2|_v \leq |t_1|_v^{-1}$. Thus, since $\varepsilon_2 \geq 0$,

$$|(1-\frac{t_2}{t_1})u_2-u_1u_3|_v \leq |t_1|_v^{-2} + |t_1|_v^{-1}|\frac{t_2}{t_1}|_v \leq |t_1|_v^{-2} + |t_1^2t_2|_v^{-1} \leq 2|\det t|_v^{-1}$$

so that for those u which we are interested in

$$|\log|(1-\frac{t_2}{t_1})u_2 - u_1u_3|_v| \leq 2\log|\det t|_v^{-1} + \log|t_1-t_2|_v^{-1},$$

which shows the asserted bound also for $p_{2,v}$.

(ii) This is a direct consequence of the proof for (i).

□

Proof of Lemma 23. (i) For ease of notation we again assume $F_v = \mathbb{R}$, and write $|\cdot| = |\cdot|_v$, and $h = h_v$. We first consider p_3. Write $t = \operatorname{diag}(t_1, t_2\sqrt{|\frac{t_1}{t_2}|}\tau)$ with $\tau = \begin{pmatrix} \sqrt{|\frac{t_1}{t_2}|} & \\ & \sqrt{\frac{t_2}{t_1}} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^1$. Since $p_{3,v}(t,u) = \omega_v^{\mathrm{GL}_2}(\tau, u_3)$, we can bound the weight functions as in [FiLa11a, Lemma 3.6]: if $|u_3| \geq |\frac{t_2}{t_1}|$, then $|p_{3,\infty}(t,u)| \leq c|u_3|^{\frac{1}{2}} \min\{1, |\frac{t_1}{t_2}|\} \log(2 + |\frac{t_2}{t_1}|)$, where we inserted $|u_3|^{\frac{1}{2}}$ instead of $|u_3|$, for which the estimate still holds as can be seen from the proof of the lemma. (In fact, the lemma remains valid if the exponent 1 is replaced by any exponent $r > 0$. We need the smaller exponent to bound $p_{3,v}^2$ linearly in $|u_3|$.) Here c is an absolute constant, which is independent of t and u. Using that the map $h \mapsto \tilde{h}(m) = \int_{U_2(\mathbb{R})} f(mu)du \in \mathcal{C}(Z^{M_2}(\mathbb{R})\backslash M_2(\mathbb{R}))$ is continuous, we can follow along the lines of [FiLa11a, Lemma 3.7] to see that the integral involving $p_{3,v}(t,u)$ is bounded by a constant multiple of the product of $\min\{1, |\frac{t_1}{t_2}|\} \log(2 + |\frac{t_2}{t_1}|)$ with

$$\int_{Z^{M_2}(\mathbb{R})\backslash M_2(\mathbb{R})} \int_{U_2(\mathbb{R})} \int_{\mathrm{O}(3)/\{\pm 1\}} \sum_{X,Y} |Y * X * h(\operatorname{diag}(t_1, t_2\sqrt{|\frac{t_1}{t_2}|}, t_2\sqrt{|\frac{t_1}{t_2}|})muk)| dk\,du\,dm = H(t_1, t_2\sqrt{|\frac{t_1}{t_2}|}),$$

where $X \in \mathcal{B}_{M_2(\mathbb{R})^1}$ and $Y \in \mathcal{B}_{\mathrm{O}(3)}$, and a similar conclusion is true for the integral involving $p_{3,v}^2$.

For $p_{2,v}$ note that $\log|(1 - \frac{t_2}{t_1})^2| = 2\log|t_1 - t_2| + 2\log|t_1|^{-1}$ is a lower bound for all $u \in U_0(\mathbb{R})$. This has absolute value bounded by $2\log|t_1 - t_2| + \log|\det t| \geq 0$ if $t_1, t_2 \in \mathbb{Z}\backslash\{0\}$. Hence we can bound

$$\int_{u \in U_0(\mathbb{R}), |(1-\frac{t_2}{t_1})u_2 - u_1u_3| \leq \max\{1, |\frac{t_2}{t_1}|\}} |h(tu)||p_{2,v}(t,u)|^2 du$$

$$\leq (2\log|t_1 - t_2| + \log|\det t| + \log|t_1| + \log|t_2|)^2 \int_{U_0(\mathbb{R})} |h(tu)| du$$
$$\leq 4\frac{(\log|t_1 - t_2| + \log|\det t|)^2}{\max\{1, |\frac{t_2}{t_1}|\}} H(t_1, t_2\sqrt{|\frac{t_1}{t_2}|}).$$

Now suppose that u is such that $|(1 - \frac{t_2}{t_1})u_2 - u_1u_3| \geq \max\{1, |\frac{t_2}{t_1}|\}$. If $|\frac{t_2}{t_1}| \geq \frac{1}{2}$, we have

$$0 \leq p_{2,\infty}(t,u) \leq |\log|1 - \frac{t_2}{t_1}||^2 + \frac{1}{2}\log|(1 - \frac{t_2}{t_1})^2 + (u_2 - \frac{u_1u_3}{1 - \frac{t_2}{t_1}})^2|.$$

The integral over $|h(tu)|(\log|1-\frac{t_2}{t_1}|)^2$ can be bounded by

$$\frac{(\log|t_1-t_2|+\log|\det t|)^2}{\max\{1,|\frac{t_2}{t_1}|\}}H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|})$$

again. For the second summand note that by the modified version of [FiLa11a, Lemma 3.7] there is an absolute constant $C_1>0$ such that

$$\begin{aligned}\frac{1}{2}\log|(1-\frac{t_2}{t_1})^4+(u_2-\frac{u_1u_3}{1-\frac{t_2}{t_1}})^2| &\le C_1|u_2-\frac{u_1u_3}{1-\frac{t_2}{t_1}}|^{\frac{1}{4}}\frac{\log(2+|\frac{t_1}{t_2}|)}{\max\{1,|\frac{t_2}{t_1}|\}}\\ &\le \frac{3}{2}C_1\frac{\log(2+|\frac{t_1}{t_2}|)}{\max\{1,|\frac{t_2}{t_1}|\}}\max\{1,|u_i|^{\frac{1}{2}}\}.\end{aligned}$$

If $|\frac{t_2}{t_1}|<\frac{1}{2}$, then the proof of [FiLa11a, Lemma 3.7] shows that there is a constant $C_2>0$ independent of t and u such that

$$\begin{aligned}|p_{2,\infty}(t,u)| &\le C_2\min\{1,|\frac{t_1}{t_2}|\}|(1-\frac{t_2}{t_1})u_2-u_1u_3|^{\frac{1}{4}}\\ &\le 2C_2\min\{1,|\frac{t_1}{t_2}|\}\max\{1,|u_i|^{\frac{1}{2}}\}.\end{aligned}$$

What remains to show is that the integral

$$\int_{u\in U_0(\mathbb{R}),|(1-\frac{t_2}{t_1})u_2-u_1u_3|\ge\max\{1,|\frac{t_2}{t_1}|\}}|h(tu)|\max\{1,|u_i|^{\frac{1}{2}}\}du$$

can be bounded by some constant multiple of $H(t_1,t_2\sqrt{|\frac{t_1}{t_2}|})$. But this follows as in the proof of the estimate (12). The assertion for $p_{2,v}$ follows the same way.

(ii) The second assertion follows as in the proof of Lemma 14. For the first assertion note that for any $t_1,t_2\in\mathcal{O}_F$, $\varepsilon_1,\varepsilon_2\in\mathcal{O}_F^\times$, $|\varepsilon_1t_1-\varepsilon_2t_2|\ge||t_1|-|t_2||\ge1$ if $|t_1|\ne|t_2|$, and that in any case $|\varepsilon_1t_1-\varepsilon_2t_2|\le|t_1|+|t_2|\le|t_1t_2|$ is an upper bound. If $|t_1|=|t_2|$, then for all $\varepsilon_1,\varepsilon_2$ with $\varepsilon_1t_1\ne\varepsilon_2t_2$, $|\varepsilon_1t_1-\varepsilon_2t_2|\ge|t_1|\min_{\varepsilon\in\mathcal{O}_F^\times,\varepsilon\ne1}|1-\varepsilon|\ge A|t_1|$ for $A>0$ some constant, which only depends on F. Hence $|\log|\varepsilon_1t_1-\varepsilon_2t_2||\le\tilde{A}+\log|t_1t_2|$ for some $\tilde{A}>0$ only depending on F. Then we can argue as in the proof of Lemma 14 to conclude the assertion.

□

II.iv.ii Proof of Lemma 21

Let $u=\begin{pmatrix}1&0&x\\&1&y\\&&1\end{pmatrix}\in U_1(\mathbb{A})$. We first compute the points $H_P(u)$ for $P\in\mathcal{P}(T)$, which we need to study the functions $\Gamma_P^G(X,T-H_P(u))$, $X\in\mathfrak{a}_P^G$ leading to a proof of Lemma 21.

Recall that $\beta_1 = e_1 - e_2$, $\beta_2 = e_2 - e_3$ are the simple roots of the standard minimal parabolic $P_0 \subseteq \mathrm{GL}(3)$ of upper triangular matrices, and ϖ_1, ϖ_2 are the corresponding weights. The parabolics in $\mathcal{P}(T)$ are enumerated according to the following table.

Name of parabolic P	Δ_P	$\hat{\Delta}_P$
Q_1	$\{\beta_1, \beta_2\}$	$\{\varpi_1, \varpi_2\}$
Q_2	$\{-\beta_1, \beta_1+\beta_2\}$	$\{\varpi_2 - \varpi_1, \varpi_2\}$
Q_3	$\{-\beta_2, \beta_1+\beta_2\}$	$\{\varpi_1 - \varpi_2, \varpi_1\}$
Q_4	$\{-\beta_1, -\beta_2\}$	$\{-\varpi_1, -\varpi_2\}$
Q_5	$\{\beta_1, -\beta_1-\beta_2\}$	$\{\varpi_1 - \varpi_2, -\varpi_2\}$
Q_6	$\{\beta_2, -\beta_1-\beta_2\}$	$\{\varpi_2 - \varpi_1, -\varpi_1\}$

In this notation we have

$$Q_1|^{\beta_2}Q_3|^{\beta_1+\beta_2}Q_5|^{\beta_1}Q_4|^{-\beta_2}Q_6|^{-\beta_1-\beta_2}Q_2|^{-\beta_1}Q_1,$$

and

$$\mathcal{F}^0_{B\times \mathrm{GL}_1} = \{Q_1, Q_3, Q_5\}, \qquad \mathcal{F}^0_{\bar{B}\times \mathrm{GL}_1} = \{Q_2, Q_4, Q_6\}.$$

Writing $\xi_i = H_{Q_i}(u)$ we get

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6
0	0	$a_y\beta_2$	$a_x\beta_1 + b\beta_2$	$b(\beta_1+\beta_2) - a_y\beta_1$	$a_x(\beta_1+\beta_2)$

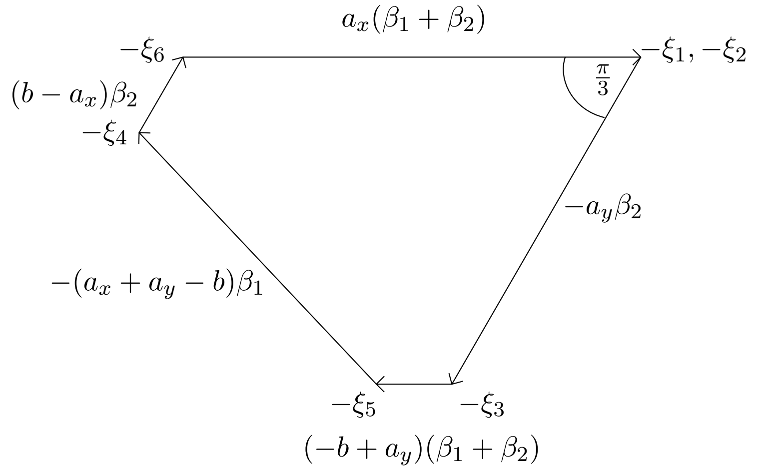
with $a_x = \log||(1,x)||$, $a_y = \log||(1,y)||$, $b = \log||(1,x,y)||$.

We now turn to the proof of Lemma 21. We have

$$\nu'_{B\times \mathrm{GL}_1}(u,T) = v'_{Q_1}(u,T) + v'_{Q_3}(u,T) + v'_{Q_5}(u,T),$$
$$\nu'_{\bar{B}\times \mathrm{GL}_1}(u,T) = v'_{Q_2}(u,T) + v'_{Q_4}(u,T) + v'_{Q_6}(u,T)$$

and $v'_{Q_i}(u,T) = \int_{\mathfrak{a}^G_{Q_i}} \Gamma^G_{Q_i}(X, T-\xi_i)dX$. In particular, $\nu'_{B\times \mathrm{GL}_1}$ depends only on the points ξ_1, ξ_3, ξ_5, and $\nu'_{\bar{B}\times \mathrm{GL}_1}$ only on the points ξ_2, ξ_4, ξ_6. From the form of the ξ_i it is clear that $\nu'_{\bar{B}\times \mathrm{GL}_1}$ is formally obtained from $\nu'_{B\times \mathrm{GL}_1}$ by replacing a_y with a_x.

The sum $\nu'_{B\times \mathrm{GL}_1}(u,T) + \nu'_{\bar{B}\times \mathrm{GL}_1}(u,T) = v_T(u,T)$, is the volume of the polytope in $i(\mathfrak{a}^G_T)^*$ with the five vertices $\{T - \xi_i\}$ (because of the special form of u two of the normally six vertices of the polytope degenerate into one). The order in which the vertices are traversed is given by the order in which the parabolics are adjacent. Thus the pentagon in question looks like the following (or rather its translate by $-T$):



Now the volume of this polytope can be computed to be

$$\frac{1}{4}\left(a_xa_y-\frac{1}{2}(a_x+a_y-b)^2\right)=\frac{1}{4}(a_xb+a_yb-\frac{1}{2}(a_x^2+a_y^2+b^2)),$$

if we normalise the volume of the coroot lattice to be 1. Since $\nu'_{B\times \mathrm{GL}_1}$ depends only on b and a_y, $\nu'_{\bar{B}\times \mathrm{GL}_1}$ only on b and a_x, and one emerges from the other by interchanging x and y, we must have

$$\nu'_{B\times \mathrm{GL}_1}(u,T)=\frac{1}{4}(a_yb-\frac{1}{2}(a_y^2+\frac{1}{2}b^2)),$$

and

$$\nu'_{\bar{B}\times \mathrm{GL}_1}(u,T)=\frac{1}{4}(a_xb-\frac{1}{2}(a_x^2+\frac{1}{2}b^2)).$$

This proves Lemma 21 as $0\le b\le a_x+a_y$.

II.iv.iii The case $R=M_1$

There is a canonical morphism from the variety of unipotent elements in $\mathrm{GL}_2(F)$ to the variety of unipotent elements in $M_1(F)$ preserving conjugacy classes. Hence there are two unipotent orbits in M_1: The trivial orbit $\{\mathbf{1}_3\}$, and the conjugacy class belonging to $\begin{pmatrix}1&1&\\&1&\\&&1\end{pmatrix}$. We denote the orbits by $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively. In particular the unipotent distribution $J^{M_1}_{\mathrm{unip}}$ is the sum of two distributions $J^{M_1}_{\mathfrak{b}_1}$ and $J^{M_1}_{\mathfrak{b}_2}$. Note that there exists a subset $\Sigma_1\subseteq G(F)_{\mathrm{ss}}$ which is closed under conjugation such that

$$\sum_{\sigma\in Z^{M_1}(F)_{\mathrm{reg}}}\int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})}J^{M_1}_{\mathfrak{b}_1}(\Phi^\sigma_{M_1,y})dy$$
$$=\nu(M_1)\sum_{[\gamma]\subseteq\Sigma_1}\int_{U_1(\mathbb{A})}f(u^{-1}\gamma u)v_{M_1}(u)du$$

with $\nu(M_1) = \mathrm{vol}(M_1(F)\backslash M_1(\mathbb{A})^1)$. This is, because by definition,

$$\Phi^\sigma_{M_1,y}(1) = \int_{\mathbf{K}\cap M_1(\mathbb{A})} f(y^{-1}k^{-1}\sigma ky)\nu'_{M_1}(ky,T_0)dk$$
$$= f(y^{-1}\sigma y)\int_{\mathbf{K}\cap M_1(\mathbb{A})} \nu'_{M_1}(ky,T_0)dk.$$

and since $\mathcal{F}^0_{M_1}(T) = \{P_1, \overline{P}_1\}$, $\nu'_{M_1}(ky,T_0) = v'_{P_1}(ky,T_0) + v'_{\overline{P}_1}(ky,T_0) = v_{M_1}(ky)$ by [Ar81, Corollary 6.4]. Moreover, $v_{M_1}(\cdot)$ is bi-invariant under M_1 so that

$$\Phi^\sigma_{M_1,y}(1) = \mathrm{vol}(\mathbf{K}\cap M_1(\mathbb{A}))v_{M_1}(y)f(y^{-1}\sigma y).$$

As $J^{M_1}_{\mathfrak{b}_1}(\Phi^\sigma_{M_1,y})$ equals $\mathrm{vol}(M_1(F)\backslash M_1(\mathbb{A})^1)\Phi^\sigma_{M_1,y}(1)$ and $G_\sigma(\mathbb{A})\backslash G(\mathbb{A}) \simeq U_1(\mathbb{A})(\mathbf{K}\cap M_1(\mathbb{A}))\backslash\mathbf{K}$, the asserted equality above follows.

Hence we only have to consider $\mathfrak{b}_2$, as it contains non-semisimple elements. We apply the regularisation used in [FiLa11a] to obtain a convergent expression for $f\in\mathcal{C}(G(\mathbb{A})^1,K)$.

Proposition 26. *For $f\in C_c^\infty(G(F_S)^1)$ the sum-integral*

$$\sum_{\sigma\in Z^{M_1}(F)_{reg}} \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} J^{M_1}_{\mathfrak{b}_2}(\Phi^\sigma_{M_1,y})dy$$

equals

$$\nu(T)\sum_{\sigma\in Z^{M_1}(F)_{reg}} \int_{\mathbf{K}}\int_{N(\mathbb{A})}\int_{U_1(\mathbb{A})} f(k^{-1}u^{-1}\sigma nuk)\omega^{\mathrm{GL}(2),S}(n)v_{M_1}(u)dudndk$$

$$\nu(T)\lambda_{1,S}\sum_{\sigma\in Z^{M_1}(F)_{reg}} \int_{\mathbf{K}}\int_{N(\mathbb{A})}\int_{U_1(\mathbb{A})} f(k^{-1}u^{-1}\sigma nuk)v_{M_1}(u)dudk$$

with $\nu(T) = \mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)$. Here $\omega^{\mathrm{GL}(2),S}$ denotes the modified weight function for $\mathrm{GL}(2)$ from [FiLa11a], and $\lambda_{1,S}$ is the coefficient also defined in [FiLa11a]. Moreover, N is the unipotent radical of the standard minimal parabolic subgroup in M_1, and if $n = \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1\end{pmatrix}\in N(\mathbb{A})$, we also write $n = \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}$. Then both sum-integrals converge absolutely for $f\in\mathcal{C}(G(\mathbb{A})^1,K)$.

Proof. Let $f\in C_c^\infty(G(F_S)^1)$. Then by the results for $\mathrm{GL}(2)$ from [FiLa11a], the distribution $J^{M_1}_{\mathfrak{b}_2}(\Phi^\sigma_{M_1,y})$ equals

$$\lambda_{1,S}\int_{N(\mathbb{A})}\Phi^\sigma_{M_1,y}(n)dn + \int_{N(\mathbb{A})}\Phi^\sigma_{M_1,y}(n)\omega^{\mathrm{GL}(2),S}(n)dn$$

for S sufficiently large. Inserting the definitions of $\Phi^\sigma_{M_1,y}$ and using the calculations from above, this equals the sum of

$$\lambda_{1,S}\int_{M_1(\mathbb{A})\backslash G(\mathbb{A})}\int_{N(\mathbb{A})}\int_{\mathbf{K}\cap M_1(\mathbb{A})} f(y^{-1}k^{-1}\sigma nky)v_{M_1}(y)dy$$

and

$$\int_{M_1(\mathbb{A})\backslash G(\mathbb{A})}\int_{N(\mathbb{A})}\int_{\mathbf{K}\cap M_1(\mathbb{A})} f(y^{-1}k^{-1}\sigma nky)\omega^{\mathrm{GL}(2),S}(n)v_{M_1}(y)dy.$$

Since $\mathbf{K}\cap M_1(\mathbb{A})$ centralises $U_1(\mathbb{A})$ and v_{M_1} is invariant under $M_1(\mathbb{A})$, using Iwasawa decomposition and changing variables, yields the asserted form given in the proposition. Hence it remains to show the absolute convergence for $f\in\mathcal{C}(G(\mathbb{A})^1,K)$. For that note that for $u\in U_1(\mathbb{A})$, there is $k\in M_1(\mathbb{A})\cap\mathbf{K}$ such that $ku=\left(\begin{smallmatrix}1 & 0 & \|(u_2,u_3)\|\\ & 1 & 0\\ & & 1\end{smallmatrix}\right)$. Hence

$$H_{\bar{P}_1}(u)=(\log\|(1,u_2,u_3)\|,\log\|(1,u_2,u_3)\|,-2\log\|(1,u_2,u_3)\|),$$

and therefore $v_{M_1}(u)$ is a constant multiple of $\log\|(1,u_2,u_3)\|$, which can be bounded by $(\log\|(1,u_2)\|)^2+(\log\|(1,u_3)\|)^2$. For this we already found upper bounds for the local integrals in the last section. The second sum-integral can thus be treated exactly the same way as was the sum in Proposition 20. Moreover, there are bounds for $\omega^{\mathrm{GL}_2,S}$ available in [FiLa11a]. For the first sum-integral, a combination of the estimates for the weights yields upper bounds for the local integrals as usual (for the archimedean places using a partition of the integration domain similar to the one used for (12)). Hence an application of Lemma 24 yields the absolute convergence also for this term. □

III Special test functions

In the following sections we shall be concerned with a class of special test functions. Suppose that $G = \mathrm{GL}(n)$ for some $n \geq 1$, and let $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}))$ be the space of Schwartz-Bruhat functions $\Phi : \mathrm{Mat}_{n\times n}(\mathbb{A}) \longrightarrow \mathbb{C}$. $K \subseteq \mathbf{K}_f$ still is a fixed compact-open subgroup of finite index. We denote by $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K) \subseteq \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}))$ the subspace of all functions being biinvariant under K, $\Phi(k_1 g k_2) = \Phi(g)$ for all $k_1, k_2 \in K$, $g \in \mathrm{Mat}_{n\times n}(\mathbb{A})$. Let $s \in \mathbb{C}$, $\Re s > \frac{n+1}{2}$, and consider

$$f_s : \mathrm{GL}_n(\mathbb{A}) \longrightarrow \mathbb{C}, \quad f_s(g) = \int_{\mathbb{A}^\times} |\det(ag)|^{s+\frac{n-1}{2}} \Phi(ag) d^\times a.$$

This is well-defined and absolutely convergent for $\Re s > \frac{1}{n} - \frac{n-1}{2}$ by Tate's theory of zeta functions. Since for $\Phi \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ we have $X * \Phi * Y \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ for all $X, Y \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$, we get

Lemma 27. *For $\Phi \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$, we have for all $s \in \mathbb{C}$ with $\Re s > \frac{n+1}{2}$ that $f_s \in \mathcal{C}(Z(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A}), K) = \mathcal{C}(\mathrm{PGL}_n(\mathbb{A}), K)$.*

Proof. This follows from

$$\int_{\mathbb{A}^\times \backslash \mathrm{GL}_n(\mathbb{A})} |f_s(g)| dg \leq \int_{\mathrm{GL}_n(\mathbb{A})} |\det g|^{\Re s + \frac{n-1}{2}} |\Phi(g)| dg$$

and [GoJa72, Lemma 12.5]. □

Hence such functions may be used as test functions in the trace formula for GL(3), as the transition from $\mathrm{GL}(3)^1$ to PGL(3) only affects the "counting" of our conjugacy classes, but not the convergence. Thus we have to replace the sum over all conjugacy classes $[\gamma] \subseteq \mathrm{GL}_3(F)_{\mathrm{ell,reg}}$ by one over $[\gamma] \subseteq \mathrm{GL}_3(F)_{\mathrm{ell,reg}}/Z(F) = \mathrm{PGL}_3(F)_{\mathrm{ell,reg}}$

Remark 28. *Suppose $M \subseteq \mathrm{GL}(n)$ is a Levi subgroup, $M \simeq \mathrm{GL}(n_1) \times \ldots \mathrm{GL}(n_r)$, and $P \in \mathcal{P}(M)$ with $P = MU$. Let $\Phi : \mathrm{Mat}_{n_1\times n_1}(\mathbb{A}) \times \ldots \times \mathrm{Mat}_{n_r\times n_r}(\mathbb{A}) \longrightarrow \mathbb{C}$ be a Schwartz-Bruhat function, which is bi-invariant under $M(\mathbb{A}) \cap K$. If we then define for $m \in M(\mathbb{A})^1$*

$$f_s^M(m) = \int_{\mathbb{A}^\times} |\lambda|^{ns + \sum_{i=1}^r \frac{n_i(n_i-1)}{2}} \Phi(\lambda m) d^\times \lambda,$$

then $f_s^M \in \mathcal{C}(M(\mathbb{A})^1, K \cap M(\mathbb{A}))$ for all s with $\Re s > \max_{i=1,\ldots,r} \frac{n_i+1}{2}$. This is because of the following: Suppose for notational purposes that we have $\Phi(m_1, \ldots, m_r) = \Phi_1(m_1) \cdot \ldots \cdot \Phi_r(m_r) \geq 0$ and $s \in \mathbb{R}$, $s > \frac{n+1}{2}$. Then

$$\int_{Z^M(\mathbb{A})\backslash M(\mathbb{A})} \int_{\mathbb{A}^\times} |\lambda|^{ns + \sum_{i=1}^r \frac{n_i(n_i-1)}{2}} |\Phi(\lambda m)| d^\times \lambda dm$$

$$= \int_{\mathbb{A}^\times} \prod_{i=1,\ldots,r} \int_{Z(\mathbb{A})\backslash \mathrm{GL}_{n_i}(\mathbb{A})} |\det \lambda m_i|^{s+\frac{n_i-1}{2}} |\Phi_i(\lambda m_i)| dm_i$$

and if $C \subseteq \mathbb{A}^\times$ is a compact neighbourhood of 1*, we have by [FiLa11b, Lemma 3.3] for any* i*,*

$$\int_{Z(\mathbb{A})\backslash \mathrm{GL}_{n_i}(\mathbb{A})} |\det \lambda m_i|^{s+\frac{n_i-1}{2}} |\Phi_i(\lambda m_i)| dm_i$$
$$\leq c \sum_X \int_{Z(\mathbb{A})\backslash \mathrm{GL}_{n_i}(\mathbb{A})} \int_C |\det \lambda m_i|^{s+\frac{n_i-1}{2}} |X * \Phi_i(\lambda\alpha m_i)| d^\times \alpha dm_i$$

for some suitable constant $c > 0$ depending only on C, and X running over a suitable set of derivatives. But this is then bounded by

$$\tilde{c} \sum_X \int_{Z(\mathbb{A})\backslash \mathrm{GL}_{n_i}(\mathbb{A})} \int_C |\det \lambda\alpha m_i|^{s+\frac{n_i-1}{2}} |X * \Phi_i(\lambda\alpha m_i)| d^\times \alpha dm_i$$
$$\leq \tilde{c} \sum_X \int_{Z(\mathbb{A})\backslash \mathrm{GL}_{n_i}(\mathbb{A})} \int_{\mathbb{A}^\times} |\det \alpha m_i|^{s+\frac{n_i-1}{2}} |X * \Phi_i(\alpha m_i)| d^\times \alpha dm_i$$
$$= \tilde{c} \sum_X \int_{\mathrm{GL}_{n_i}(\mathbb{A})} |\det g_i|^{s+\frac{n_i-1}{2}} |X * \Phi_i(g_i)| dg_i$$

where $\tilde{c} > 0$ is again a constant depending only on C. But this last integral converges for $\Re s > \frac{n_i+1}{2}$.

III.i Special test functions for $G = \mathrm{GL}(3)$

For $\Re s > \frac{3+1}{2} = 2$, $f_s \in \mathcal{C}(G(\mathbb{A})^1, K)$, and inserting f_s into the trace formula we can view the different parts of the formula as functions of s so that we can analyse the analytic behaviour with respect to s. Note that each of the contributions from Theorem 1 defines a holomorphic function for $\Re s > 2$. What we want to show now is that we can continue each part holomorphically up to $\Re s > 2 - \varepsilon$ for some $\varepsilon > 0$ except for the regular elliptic contribution. The regular elliptic contribution amounts for the first pole at $s = 2$, which follows from the results for the spectral side.
We shall assume that $\Phi((a_{ij})_{i,j}) = \prod_{i,j} \Phi_{ij}(a_{ij})$ with $\Phi_{ij} \in \mathcal{S}(\mathbb{A})$, and $\Phi_{ij} \geq 0$, $i, j = 1, 2, 3$, which is enough for our purposes by [Yu93, Lemma (1.2.5)]. Additionally, Φ will be assumed to be $\mathbf{K}$-central, which is no restriction as each of the contributions to the geometric side is invariant under replacing Φ by $\int_{\mathbf{K}} \Phi(k^{-1} \cdot k) dk$.

Define functions $\mathcal{E}(s), \mathcal{S}(s)$ $s \in \mathbb{C}$, $\Re s > 2$, by

$$\mathcal{E}(s) = \sum_{[\gamma] \subseteq G(F)_{\mathrm{ell,reg}}/Z(F)} \nu(G_\gamma) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f_s(x^{-1}\gamma x) dx,$$

and

$$\mathcal{S}(s) = \sum_{[\sigma] \subseteq G(F)_{\mathrm{ss}}/Z(F)} \nu(G_\sigma) \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} f_s(x^{-1}\sigma x) v_{M(\sigma)}(x) dx$$

with $\nu(G_\gamma) = \mathrm{vol}(G_\gamma(F)Z^{G_\gamma}(\mathbb{A})\backslash G_\gamma(\mathbb{A}))$. By [FiLa11b] these functions are well-defined and holomorphic for $\Re s > 2$.

Proposition 29. *The function $\mathcal{S}(s) - \mathcal{E}(s)$ has an analytic continuation in the right half plane $\Re s > \frac{3}{2}$, and is holomorphic there.*

It will be clear from the proof that the sum-integral defining $\mathcal{S}(s) - \mathcal{E}(s)$ not only continues to that larger half plane, but is given in this half plane by the same-sum integral as before, which is still absolutely convergent there.

Proof. For $\Re s > 2$, we have

$$\mathcal{S}(s) - \mathcal{E}(s) = \sum_{[\sigma]\subseteq G(F)_{\mathrm{ss}}\backslash G(F)_{\mathrm{ell,reg}}} \nu(G_\sigma) \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} f_s(x^{-1}\sigma x) v_{M(\sigma)}(x)dx$$

and this equals the sum of

$$\nu(G) \int_{\mathbb{A}^\times} |a|^{3s+3} \Phi(a) d^\times a,$$

$$\sum_{\substack{[\sigma]\subseteq G(F)_{\mathrm{ss}}/F^\times \\ M(\sigma) \text{ conjugate to } M_1}} \nu(M_1) \int_{M_{1,\sigma}(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{A}^\times} |a|^{3s+3} \Phi(ax^{-1}\sigma x) v_{M_1}(x) d^\times a dx,$$

and

$$\sum_{\substack{[\sigma]\subseteq G(F)_{\mathrm{ss}}/F^\times \\ M(\sigma) \text{ conjugate to } T}} \nu(T) \int_{T_\sigma(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{A}^\times} |a|^{3s+3} \Phi(ax^{-1}\sigma x) v_T(x) d^\times a dx$$

where $\nu(M)$ denotes the volume of $M(F)Z^M(\mathbb{A})\backslash M(\mathbb{A})$. Here $M(\sigma)$ denotes the smallest Levi subgroup containing the centraliser of the center of the centraliser of σ in G. The first integral is absolutely convergent for $\Re(3s + 3) > 1$, hence holomorphic there, and in fact has a meromorphic continuation to the whole complex plane.

For $c > 0$ let $\mathcal{S}_c = \{pk \mid p \in P_0(\mathbb{A}), k \in \mathbf{K},\ |\alpha(p)| > c\ \forall \alpha \in \Delta_0\}$. As in [FiLa11b, §6] the second summand can be estimated by the sum over standard parabolics P with $P = MU$ such that M is conjugate to M_1 of

$$\int_{A_M P_0^M(F)\backslash \mathcal{S}_c^M} \sum_{\sigma\in M(F)_{\mathrm{well}}/F^\times} \int_{\mathbb{A}^\times} |a|^{3\Re s+1} \Phi_P(am^{-1}\sigma m) d^\times a dm$$

for c sufficiently small and $M(F)_{\mathrm{well}}$ is the set of all $\gamma \in M(F)$ whose centraliser is not contained in any proper parabolic subgroup of M. Here

$$\Phi_P(m) = \Delta(m)^{-1} \int_{U(\mathbb{A})} \Phi(mu)du$$

and $\Delta(m)$ is the determinant of the linear map $U(\mathbb{A}) \simeq \mathbb{A}^2 \longrightarrow U(\mathbb{A})$, $u \mapsto mu$. Then Φ_P is a Schwartz-Bruhat function on $\mathcal{L}ie(M(\mathbb{A})) \simeq \mathrm{Mat}_{2\times 2}(\mathbb{A}) \oplus \mathbb{A}$, which is invariant under $M(\mathbb{A}) \cap K$. This last sum-integral converges absolutely as long as the function

$$h_s(m) = \int_{\mathbb{A}^\times} |a|^{3\Re s+1} \tilde{\Phi}_{P_1}(am) d^\times a$$

defines an element in $\mathcal{C}(M(\mathbb{A})^1, K \cap M(\mathbb{A}_f))$. By Remark 28, this is the case for $\Re s > \frac{3}{2}$.

The last integral can similarly be bounded by

$$\int_{A_T P_0^T(F)\backslash \mathcal{S}_c^T} \sum_{\sigma \in T(F)/F^\times} \int_{\mathbb{A}^\times} |a|^{3\Re s} \Phi_{P_0}(at^{-1}\sigma t) d^\times a dt$$

with

$$\Phi_{P_0}(t) = \Delta(t)^{-1} \int_{U_0} \Phi(tu) du$$

where now Δ is the discriminant of the linear map $U_0(\mathbb{A}) \simeq \mathbb{A}^3 \longrightarrow U(\mathbb{A})$, $u \mapsto tu$, and Φ_{P_0} is a Schwartz-Bruhat function on $\mathcal{L}ie(T) \simeq \mathbb{A}^3$. As $t \mapsto \int_{\mathbb{A}^\times} |a|^{3s} \tilde{\Phi}_{P_0}(at) d^\times a$ defines a function in $\mathcal{C}(T(\mathbb{A})^1, K \cap T(\mathbb{A}))$ for any s with $\Re s > 1$, the absolute convergence, and thus holomorphic continuation in this region follows from [FiLa11b]. □

Proposition 30. *If we insert f_s as the test function into the trace formula, each of the functions, which are defined by the terms in Theorem 1 (ii) – (v) is well-defined and holomorphic for $\Re s > 2$, and can be continued to a holomorphic function at least in the region $\Re s > \frac{2}{3}$.*

Proof. We study each of the functions separately, and denote them by $\mathcal{H}_{(k)}(s)$, $k \in \{ii, iii, iv, v\}$ according to the enumeration of Theorem 1.

$\mathcal{H}_{(ii)}$: According to the three summands in Theorem 1 (ii), $\mathcal{H}_{(ii)}(s)$ can be written as the sum of three functions $\mathcal{H}^{(1)}_{(ii)}(s) + \mathcal{H}^{(2)}_{(ii)}(s) + \mathcal{H}^{(3)}_{(ii)}(s)$, each being well-defined and holomorphic at least for $\Re s > 2$. We have

$$\mathcal{H}^{(3)}_{(ii)}(s) = \Lambda_S \left(\prod_{i>j} \Phi_{ij}(0) \right) \cdot \int_{U_0(\mathbb{A})} \int_{\mathbb{A}^\times} |a|^{3s} \left(\prod_{i=1}^{3} \Phi_{ii}(a) \right) \left(\prod_{i<j} \Phi_{ij}(u_{i+j-2}) \right) d^\times a du,$$

which therefore has a meromorphic continuation to the entire complex plane with only poles at $s = \frac{1}{3}$, and $s = 0$. If we make a similar change of variables for the functions $\mathcal{H}^{(1)}_{(ii)}$, and $\mathcal{H}^{(2)}_{(ii)}$, the weight functions

$\omega_{M_0}(t,u)$ and $\Omega_{M_0}(t,u)$ amount to polynomials in $\log|a|$ and $\log|u_i|$. Since this does not interfere with convergence issues with respect to a, the functions $\mathcal{H}_{(ii)}^{(1)}$ and $\mathcal{H}_{(ii)}^{(2)}$ are still given by an absolutely convergent expression for $\Re s > \frac{1}{3}$.

$\mathcal{H}_{(iii)}$: Write $\mathcal{H}_{(iii)}(s) = \mathcal{H}_{(iii)}^{(1)}(s) + \mathcal{H}_{(iii)}^{(2)}(s)$ corresponding to the two summands in Theorem 1 (iii). Then $\mathcal{H}_{(iii)}^{(2)}(s)$ equals

$$-3\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)}\left(\prod_{i>j}\Phi_{ij}(0)\right)\Phi_{12}(0)\Phi_{23}(0)$$
$$\int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times}|a|^{3s+1}\left(\prod_{i=1}^{3}\Phi_{ii}(a)\right)\Phi_{13}(x)|x|^2 d^\times a d^\times x,$$

which therefore has an analytic continuation to the entire $\mathbb{C}$ with only poles at $s=0$ and $s=-\frac{1}{3}$. $\mathcal{H}_{(iii)}^{(1)}(s)$ can also be continued holomorphically up to $\Re s > 0$ similar to the previous case.

$\mathcal{H}_{(iv)}$: Again write $\mathcal{H}_{(iv)}(s) = \mathcal{H}_{(iv)}^{(1)}(s) + \mathcal{H}_{(iv)}^{(2)}(s)$. Then as in the proof of Proposition 26 for $\Re s > 2$, $\mathcal{H}_{(iv)}^{(2)}(s)$ can be bounded by the product of

$$\nu(T)\lambda_S\left(\prod_{i>j}\Phi_{ij}(0)\right)\left(\prod_{i=1,2}\int_{\mathbb{A}}\Phi_{i3}(u)du\right)$$

with

$$\int_{N(\mathbb{A})}\Phi_{12}(n)dn\int_{\mathbb{A}^\times/F^\times}|a|^{3s}\sum_{\sigma\in Z^{M_1}(F)}\left(\prod_{i=1,2}\Phi_{ii}(a\sigma_1)\right)\Phi_{33}(a\sigma_2)d^\times a. \tag{22}$$

As $(t_1,t_2)\mapsto\left(\prod_{i=1,2}\Phi_{ii}(t_1)\right)\Phi_{33}(t_2)$ defines a Schwartz-Bruhat function on $\mathbb{A}^2$, we can apply the results of GL(2) for the regular hyperbolic contribution to conclude that (22) converges absolutely for $\Re s > \frac{2}{3}$, and thus $\mathcal{H}_{(iv)}^{(2)}$ is a holomorphic function in this region. Similarly we infer that $\mathcal{H}_{(iv)}^{(1)}$ is holomorphic for $\Re s > \frac{2}{3}$.

$\mathcal{H}_{(v)}$: As in the two previous cases it follows that the expression for $\mathcal{H}_{(v)}$ still converges absolutely at least up to $\Re s > \frac{2}{3}$,

□

IV The spectral side for $\mathrm{GL}(n)$

IV.i Notation and preliminaries

IV.i.i Notation and a first form of the spectral side

We continue to use the notation from the previous part, but now $G = \mathrm{GL}(n)$, $n \in \mathbb{N}$ arbitrary. In particular, $T \subseteq \mathrm{GL}(n)$ is the maximal split torus of diagonal matrices. For $P \in \mathcal{F}(T)$ write $P = MU$ with $M \in \mathcal{L}(T)$ the Levi component and U the unipotent radical of P. For $M \in \mathcal{L}(T)$, W_0^M is the Weyl group of M with respect to T, $W_0 = W_0^G$, and for $L \in \mathcal{L}(M)$ let $W^L(M)$ be the set of all $t \in W_0^L$ such that t induces an isomorphism $\mathfrak{a}_M \longrightarrow \mathfrak{a}_M$. Let $W^L(M)_{\mathrm{reg}} = \{t \in W^L(M) \mid \ker t = \mathfrak{a}_L\}$. Write Σ_P for the set of positive reduced roots of (P, A_P), and $\mathrm{cork}_G(P)$ for the co-rank of P in G, and similarly for P replaced by some Levi subgroup.
Let R denote the right regular representation of $G(\mathbb{A})$ on $L^2(A_G G(F)\backslash G(\mathbb{A}))$, and let R_{disc} be its restriction to $L^2_{\mathrm{disc}}(A_G G(F)\backslash G(\mathbb{A}))$, the subspace of $L^2(A_G G(F)\backslash G(\mathbb{A}))$ decomposing discretely under R. For any $M \in \mathcal{L}(T)$, let $\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ be the set of irreducible representations occuring in the decomposition of $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$, and let R_M be the right regular representation of $M(\mathbb{A})$ on $L^2(A_M M(F)\backslash M(\mathbb{A}))$. Let $\mathcal{A}^2(P)$ be the space of all $\varphi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$ such that $\varphi_x \in L^2(A_M M(F)\backslash M(\mathbb{A}))$, $\varphi_x(g) = \delta_P(g)^{-\frac{1}{2}}\varphi(gx)$, for all $x \in G(\mathbb{A})$, $g \in M(\mathbb{A})$, and let $\bar{\mathcal{A}}^2(P)$ be its Hilbert space completion. In particular, $\mathcal{A}^2(P)$ is the $(\mathbf{K}, \mathfrak{z})$-finite part of $\bar{\mathcal{A}}^2(P)$. For $\varphi_1, \varphi_2 \in \mathcal{A}^2(P)$ we have the inner product given by

$$< \varphi_1, \varphi_2 >= \int_{A_M M(F)\backslash M(\mathbb{A})} \int_{\mathbf{K}} \varphi_1(mk)\overline{\varphi_2(mk)}dkdm.$$

For $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ there is a representation $\rho(P, \lambda, \cdot)$ of $G(\mathbb{A})$ on $\mathcal{A}^2(P)$ given by

$$(\rho(P, \lambda, y)\varphi)(x) = \varphi(xy)e^{<\lambda+\rho_P, H_P(xy)>}e^{-<\lambda+\rho_P, H_P(x)>}$$

and it is isomorphic to $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\mathrm{R}_{\mathrm{M,disc}} \otimes \mathrm{e}^{<\lambda, \mathrm{H}_{\mathrm{P}}(\cdot)>})$ for $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\cdot)$ denoting the parabolically induced representation. (With this definition, $\rho(G, 0, g) = R_{\mathrm{disc}}(g)$.) Thus for sufficiently reasonable functions $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$, we get an operator $\rho(P, \lambda, f)$ on $\mathcal{A}^2(P)$. If $\pi \subseteq L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ is a subrepresentation, we denote the restriction of $\rho(P, \lambda, f)$ to the space of $\mathcal{A}^2_\pi(P)$ by $\rho_\pi(P, \lambda, f)$, where $\mathcal{A}^2_\pi(P) \subseteq \mathcal{A}^2(P)$ is the space of all φ such that φ_x is contained in the π-isotypical component of $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ for any $x \in G(\mathbb{A})$. Again fix an open-compact subgroup $K \subseteq G(\mathbb{A}_f)$ being hyperspecial at almost all places. Denote by $\mathcal{A}^2_\pi(P)^K$ the K-invariant subspace of $\mathcal{A}^2_\pi(P)$. This is equivalent to a finite direct sum of unitary $G(F_\infty)$-representations. For any infinite place v, $\widehat{\mathbf{K}_v}$ is the unitary dual of $\mathbf{K}_v$, and similarly for $\widehat{\mathbf{K}_\infty}$. For $\tau \in \widehat{\mathbf{K}_v}$ we denote by $||\tau||$ the norm of its highest weight vector. Note that if λ_τ denotes the Casimir eigenvalue of τ, then $\lambda_\tau = ||\tau - \rho||^2 - ||\rho||^2$ for ρ the half sum of all positive roots of GL(n) [Kn02, Proposition (5.28)]. If $\tau \in \widehat{\mathbf{K}_\infty}$, let $\mathcal{A}^2_\pi(P)^K_\tau$ be the τ-isotypical component of $\mathcal{A}^2_\pi(P)^K$ when considered as a representation of $\mathbf{K}_\infty$. This is a finite dimensional vector space by the admissibility of π. As f_s is invariant under $Z(\mathbb{A})$, the representations which will contribute non-trivially in the decomposition $\rho(P,\lambda,f_s) = \widehat{\bigoplus}_{\pi\in\Pi_{\text{disc}}(M(\mathbb{A})^1)}\rho_\pi(P,\lambda,f_s)$ must have trivial central character. Let $\mathcal{S}_0(\text{Mat}_{n\times n}(\mathbb{A}))$ be the space of all $\mathbf{K}$-finite functions in $\mathcal{S}(\text{Mat}_{n\times n}(\mathbb{A}))$, i.e. $\Phi \in \mathcal{S}(\text{Mat}_{n\times n}(\mathbb{A}))$ such that the set Φ_k, $k \in \mathbf{K}$, spans a finite dimensional subspace of $\mathcal{S}(\text{Mat}_{n\times n}(\mathbb{A}))$, where $\Phi_k(g) = \Phi(gk)$. Note that for $\Phi \in \mathcal{S}(\text{Mat}_{n\times n}(\mathbb{A}))$ to be $\mathbf{K}$-finite this, in fact, is only an obstruction for the infinite part so that it suffices to check that Φ_a is $\mathbf{K}_\infty$-finite for each fixed $a \in \text{Mat}_{n\times n}(\mathbb{A}_f)$. Equivalently, there is a finite index subgroup $K_\infty \subseteq \mathbf{K}_\infty$ under which Φ_∞ is bi-invariant. The notation $\mathcal{S}_0(\text{Mat}_{n\times n}(\mathbb{A}), K_\infty K)$ then has the obvious meaning. $\mathcal{C}(G(\mathbb{A})^1, K)$ denotes the same space as before and we define $\mathcal{C}(G(\mathbb{A})^1) = \varinjlim \mathcal{C}(G(\mathbb{A})^1, K)$ to be the direct limit over all $K \subseteq G(\mathbb{A}_f)$. For later purposes we choose a non-trivial character $\psi = \bigotimes_v \psi_v : \mathbb{A}^\times/F^\times \longrightarrow \mathbb{C}$.

The spectral side of the trace formula can be written as

$$J_{\text{spec}}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f)$$

for test functions $f \in C_c^\infty(G(F_S))$ where the sum is over the cuspidal automorpic data $\chi \in \mathfrak{X}$ (see [Ar05, §12] for a definition) and certain distributions J_χ. The distributions have a finer expansion valid for bi-$\mathbf{K}$-finite $f \in C_c^\infty(G(F_S))$ [Ar82, Theorem 8.2]. Such an expansion converges absolutely for test functions $f \in \mathcal{C}(G(\mathbb{A})^1)$, which was shown in [MuSp04] for $G = \text{GL}(n)$, and for general reductive groups in [FiLaMu11]. More precisely, it was shown there that the spectral side of the trace formula for GL(n) can be written as

$$\sum_{M\in\mathcal{L}}\sum_{L\in\mathcal{L}(M)}\sum_{P\in\mathcal{P}(M)}\sum_{t\in W^L(\mathfrak{a}_M)_{\text{reg}}} a(M,L,t) J^L_{M,P}(f,t) \tag{23}$$

with

$$J^L_{M,P}(f,t) = \sum_{\pi\in\Pi_{\text{disc}}(M(\mathbb{A})^1)} \int_{i(\mathfrak{a}^G_L)^*} \text{tr}\left(\mathcal{M}_L(P,\lambda) M_{P|P}(t,0)\rho_\pi(P,\lambda,f)\right) d\lambda \tag{24}$$

and this sum-integral is absolutely convergent with respect to the trace norm for each $f \in \mathcal{C}(G(\mathbb{A})^1)$. Here $a(M,L,t) = \frac{|W_0^M|}{|\mathcal{P}(M)||W_0|}|\det(t-1)_{|\mathfrak{a}^L_M}|^{-1}$. It is this expansion with which we shall start with. Here $\mathcal{M}_L$ is a certain operator which is build out of a (G,M)-family associated with intertwining operators the exact defintion of which will be recalled later.

Example 31. *(i) Consider the summand belonging to* $M = G$ *in* (23), *which means that also* $L = P = G$. *Then* $W^L(\mathfrak{a}_M)_{reg}$ *consists of the identity element only, but* $\mathfrak{a}_G^G$ *is empty so that* $a(G, G, \mathrm{id}) = 1$. *Thus we are left with*

$$\sum_{\pi \in \Pi_{disc}(G(\mathbb{A})^1)} \int_{i(\mathfrak{a}_G^G)^*} \mathrm{tr}\left(\mathcal{M}_G(G, \lambda) M_{G|G}(\mathrm{id}, 0)\rho_\pi(\mathrm{G}, \lambda, \mathrm{f})\right) d\lambda.$$

As $i(\mathfrak{a}_G^G)^* = 0$, *and the intertwining operators act trivially, this sum reduces to*

$$\sum_{\pi \in \Pi_{disc}(G(\mathbb{A})^1)} \mathrm{tr}\, \rho_\pi(G, 0, f)$$

which is exactly the discrete part of the spectral side. The absolute convergence of this sum is the so called trace class conjecture, which was settled for $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ *(a slightly smaller space than* $\mathcal{C}(G(\mathbb{A})^1)$*) by Müller in [Mu98].*

(ii) If, more generally, we assume that $L = G$ *but* $M \in \mathcal{L}$, $M \neq G$, *is arbitrary, the respective summand equals*

$$\sum_{P \in \mathcal{P}(M)} \sum_{t \in W^G(M)_{reg}} \sum_{\pi \in \Pi_{disc}(M(\mathbb{A})^1)} a(M, G, t)\, \mathrm{tr}\left(M_{P|P}(t, 0)\rho_\pi(P, 0, f)\right).$$

IV.i.ii Results for the spectral side

The following will be our main result.

Theorem 32. *Let* $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$, $L \in \mathcal{L}(M)$ *and* $t \in W^L(M)$. *Suppose that* $\mathrm{cork}_G\, L \leq 1$. *As a function of* s, $J_{M,P}^L(f_s, t)$ *can be meromorphically continued to all* $s \in \mathbb{C}$, *and is holomorphic at least in* $\Re s > \frac{n}{2}$ *except for a possible simple pole at* $s = \frac{n+1}{2}$. *The pole at* $s = \frac{n+1}{2}$ *occurs if and only if* $L = M = P = G$, $t = 1$, *and then has residue*

$$\mathrm{vol}(G(F) \backslash G(\mathbb{A})^1) \int_{\mathrm{Mat}_{n \times n}(\mathbb{A})} \Phi(x) dx = \zeta_F^*(n) \cdot \ldots \cdot \zeta_F^*(2) \operatorname*{res}_{s=1} \zeta_F^*(s) \int_{\mathrm{Mat}_{n \times n}(\mathbb{A})} \Phi(x) dx.$$

This in particular implies that every spectral term for GL(2) can be continued to a meromorphic function on the whole complex plane. We shall investigate this case in greater detail in IV.iv.

For $L \in \mathcal{L}(M)$ of arbitrary corank we at least have the following.

Theorem 33. *Suppose that* $\mathrm{cork}_G\, L = r \geq 1$. *Then the function* $J_{M,P}^L(f_s, t)$ *can be meromorphically continued at least in* $\Re s > \frac{n+1-r}{2}$ *and is holomorphic there.*

It is questionable whether there exists a meromorphic continuation of the distributions $J^L_{M,P}(f_s,t)$ for $\operatorname{cork}_G L \geq 2$ to a larger half plane. See also the example and discussion in IV.iii.iii.

We separate the proof of the two theorems in different parts, and do not consider them in chronologiacal order. First we treat the case $L = G$, and then the case $L \subsetneq G$ and finally specialise to $\operatorname{cork}_G L = 1$. The case $L = G$ as well would fit in the framwork of the general case, but as the situation there is not obscured by the presence of any intertwining operators, we treat it separatly. As we do not need the intertwining operators for the first part, we do not define them until IV.iii. In general, we need to study the analytic properties of intertwining operators and meromorphic functions of several variables.

Remark 34. *Note that the proofs in [MuSp04] and [FiLaMu11] in fact stay valid for test functions in $\mathcal{C}^N(G(\mathbb{A})^1, K)$ if N is sufficiently large. Hence we can use slightly more general functions Φ to construct our test function f_s and still get holomorphic functions on the spectral side at least for $\Re s > \frac{n+1}{2}$. More precisely, let $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}_f), K)$, and $\Phi_\infty \in \mathcal{S}(\mathrm{Mat}_{n\times n}(F_\infty))$. Let $\varphi : \mathrm{Mat}_{n\times n}(F_\infty) \longrightarrow \mathbb{C}$ be a function in $C^N(\mathrm{Mat}_{n\times n}(F_\infty))$ for some $N > 1$ such that $|X * \varphi|$ is bounded for any $X \in \mathcal{U}(\mathfrak{gl}_n)_{\leq N}$ (this is the case, for example, if φ has compact support). Put $\tilde{\Phi}(x) = \varphi(x_\infty)\Phi_\infty(x_\infty)\Phi_f(x_f)$, $x \in \mathrm{Mat}_{n\times n}(\mathbb{A})$, and let $\tilde{f}_s(g) = \int_{\mathbb{A}^\times} |\det(ag)|^{s+\frac{n-1}{2}} \tilde{\Phi}(ax) d^\times a$. This is still well-defined, and bi-K-invariant. It is not necessarily smooth any more, but by our assumption on the boundedness of φ and all its derivatives, the semi-norms $||X * \tilde{f}_s||_{L^1(K\backslash G(\mathbb{A})^1/K)}$ are still finite for all $X \in \mathcal{U}(\mathfrak{gl}_n)_{\leq N}$ so that at least $\tilde{f}_s \in \mathcal{C}^N(G(\mathbb{A})^1, K)$. Hence, if N is sufficiently large, $\tilde{f}_s$ may be used as a test function with all terms on the spectral side converging absolutely for $\Re s > \frac{n+1}{2}$. For $n = 3$ (and also $n = 2$) this is also true for the geometric terms, as remarked before. See also Remark 39.*

IV.ii The "discrete" parts (i.e. $L = G$)

As announced earlier, we first treat the "discrete" parts of Theorem 32. To begin with, we consider the case $M = G$ (which implies $L = P = G$). As discussed before, $J^G_{G,G}(f, \mathrm{id})$ then equals $\operatorname{tr} R_{\mathrm{disc}}(f_s)$. The general case with $L = G$ follows from this, see Corollary 36.

By [Mu98] the trace of the operator R restricted to the discrete subspace can be written for $\Re s > \frac{n+1}{2}$ as an absolutely convergent sum

$$\Theta(s) := \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A})^1)} \sum_{\tau \in \widehat{\mathbf{K}_\infty}} \operatorname{tr}(R(f_s)_{|\mathcal{A}^2_\pi(G)^K_\tau})$$

so that in particular it defines a holomorphic function in this region. We seperate the one-dimensional representations to write $\Theta(s) = \Theta_1(s) + \Theta_2(s)$ with $\Theta_1(s) = \sum_{\chi:\mathbb{A}^\times/F^\times \longrightarrow \mathbb{C}^\times} \operatorname{tr}(\chi \circ \det(f_s))$ and

$$\Theta_2(s) = \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A})),\ \dim \pi = \infty} \operatorname{tr} \pi(f_s).$$

Note that if $G = \mathrm{GL}(2)$, this last sum ranges only over all cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})/Z(\mathbb{A})$.

In this section we show the following.

Proposition 35. *The trace $\Theta(s) = \operatorname{tr} R_{disc}(f_s)$ is well-defined and holomorphic for $\Re s > \frac{n+1}{2}$. It can be continued to a meromorphic function on $\mathbb{C}$ with the first pole occuring at $s = \frac{n+1}{2}$, and subsequently at $s = \frac{n+1}{2} - i$, $i = 1, \dots, n$. The first and last poles are simple, and all others are of second order. For the residue at $s = \frac{n+1}{2}$ we obtain*

$$\operatorname*{res}_{s=\frac{n+1}{2}} \Theta(s) = \zeta^*_F(n) \cdot \ldots \cdot \zeta^*_F(2) \operatorname*{res}_{s=1} \zeta^*_F(s) \int_{\mathrm{Mat}_{n\times n}(\mathbb{A})} \Phi(x) dx.$$

The function $\Theta_1(s)$ is an entire function.

For general Levi subgroups M we then obtain more generally.

Corollary 36. *Let $M \simeq \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$ with $m = \max\{n_1, \dots, n_r\}$. Then $J^G_{M,P}(f_s, t)$ has a meromorphic continuation to all $s \in \mathbb{C}$ which is holomorphic at least in $\Re s > \frac{m+1}{2}$. In particular, if $M \subsetneq G$, then $J^G_{M,P}(f_s, t)$ is holomorphic at least for $\Re s > \frac{n}{2}$, and a pole at $s = \frac{n}{2}$ can only occur if* $\operatorname{cork}_G M = 1$ *and* $M \simeq \mathrm{GL}_{n-1} \times \mathrm{GL}_1$.

The rest of this section will be concerned with the proof of the proposition, which heavily depends on the strategy employed by Müller in [Mu98].

We first recall some well-known facts about zeta functions associated with automorphic representations. Let π be an irreducible automorphic representation of $G(\mathbb{A})$ in $L^2_{\text{disc}}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))$. For our purposes only such π having a vector fixed by K will matter. This is because by our stipulation on Φ and by definition of f_s, all other traces vanish. By the classification of the discrete automorphic representations by Moeglin and Waldspurger in [MoWa89] (cf. [ArGe91, p.17]) there is a bijective correspondence

$$\{\pi \text{ automorphic representations of } G(\mathbb{A}) \text{ in the discrete spectrum}\}$$

$$\updownarrow$$

$$\{(d,\sigma) \mid d \in \mathbb{N}, d|n, d \neq n, \sigma \text{ automorphic cuspidal repr.'s of } \mathrm{GL}_d(\mathbb{A})\},$$

and if π corresponds to the pair (d,σ), then π is the unique irreducible component of the representation induced from $\sigma[\frac{m-1}{2}]\otimes\sigma[\frac{m-3}{2}]\otimes\ldots\otimes\sigma[\frac{1-m}{2}]$, $m = \frac{n}{d}$, from the standard parabolic associated to $(k,\ldots,k)$ to G. Therefore, $L(s,\pi) = L(s+\frac{m-1}{2},\sigma)\cdot\ldots\cdot L(s+\frac{1-m}{2},\sigma)$. Note that it follows from this classification that for $G = \mathrm{GL}(3)$, Θ_2 consists of terms belonging to cuspidal representations only. $L(s,\pi)$ is entire unless $d = 1$. If $d = 1$, σ is a Hecke character so that $L(s,\pi)$ is again holomorphic unless σ is unramified, which means $\sigma = |\cdot|^{it}$ for some $t \in \mathbb{R}$ in which case $L(s,\pi) = \zeta_F^*(s+\frac{n-1}{2}-it)\cdot\ldots\cdot\zeta_F^*(s+\frac{1-n}{2}-it)$. The representations induced from a character σ in this way are not irreducible, but factor through $\sigma\circ\det$, and thus give rise to the one-dimensional representations. Thus for such σ has to satisfy the additional property $\sigma^n = 1$ (or more generally, its n-th power has to equal the central character). If π is not one-dimensional, the corresponding σ is an infinite dimensional representation so that in particular the corresponding L-functions are entire.

Let $\varphi =< \pi\varphi_1, \varphi_2 >$ be some matrix coefficient of π with $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(G)$ of norm 1. Let

$$Z(\Phi, s+\frac{n-1}{2}, \varphi) = \int_{G(\mathbb{A})} \Phi(x)|\det x|^{s+\frac{n-1}{2}}\varphi(x)dx$$

be the zeta function associated to Φ and φ which continues to a meromorphic function on all of $\mathbb{C}$. By defintion, the quotient $Z(\Phi, s+\frac{n-1}{2},\varphi)/L(s,\pi)$ is an entire function.

Lemma 37. *Let φ be a matrix coefficient for $\pi \in \Pi_{disc}(G(\mathbb{A}))$, π not 1-dimensional. Let $a \in (-\infty, -\frac{n}{2})$, $b \in (\frac{n}{2}+1, \infty)$ be two real numbers, and write $I_{a,b} = \{z \in \mathbb{C} \mid a \leq \Re z \leq b\}$ for the vertical strip bounded by a and b. Then there exists a constant $M(\Phi,a,b) \geq 0$ independent of π, φ, and s such that*

$$|Z(\Phi, s+\frac{n-1}{2}, \varphi)| \leq M(\Phi,a,b)$$

for all $s \in I_{a,b}$.

Proof. As π is assumed to be infinite dimensional, all occuring zeta functions are entire so that if $\varphi(x) =< \pi(x)\varphi_1, \varphi_2 >$, then

$$|Z(\Phi, s + \frac{n-1}{2}, \varphi)| \leq A(\Phi, s)||\varphi_1||||\varphi_2|| = A(\Phi, s)$$

for $\Re s > \frac{n+1}{2}$, where we can choose the constant $A(\Phi, s)$ inpendently of φ_1, φ_2, (cf. [GoJa72, p. 184] for the cuspidal case) namely

$$A(\Phi, s) = \int_{G(\mathbb{A})} |\Phi(x)|| \det x|^{\Re(s+\frac{n-1}{2})} dx$$

which converges for $\Re s > \frac{n+1}{2}$, and only depends on $\Re s$. In particular, this inequality is true on the line $\Re s = b$. Using the functional equation $Z(\widehat{\Phi}, 1 - s + \frac{n-1}{2}, \varphi^\vee) = Z(\Phi, s + \frac{n-1}{2}, \varphi)$ with $\varphi^\vee(g) = \varphi(g^{-1})$, we get a similar inequality

$$\begin{aligned} |Z(\Phi, s + \frac{n-1}{2}, \varphi)| &= |Z(\widehat{\Phi}, 1 - s + \frac{n-1}{2}, \varphi^\vee)| \\ &\leq B(\Phi, s)||\varphi_1^\vee||||\varphi_2^\vee|| = B(\Phi, s), \end{aligned}$$

which is valid for $\Re s < \frac{1-n}{2}$ with

$$B(\Phi, s) = \int_{G(\mathbb{A})} |\widehat{\Phi}(x)| \det x|^{\Re(1-s+\frac{n-1}{2})} dx.$$

Consequently, $Z(\Phi, s + \frac{n-1}{2}, \varphi)$ is bounded on the lines $\Re s = b$, $\Re s = a$, and, since we excluded 1-dimensional π, it is also bounded on $\Im s = 0, \Re s \in [-\frac{n}{2}, \frac{n}{2} + 1]$. From the form of the constants A and B one also sees that the zeta functions are bounded in each vertical strip of finite width, which is entirely contained in $\Re s > \frac{n+1}{2}$ or $\Re s < \frac{1-n}{2}$. We now first show that the zeta function is of finite order, i.e. the growth of $|Z(\Phi, s + \frac{n-1}{2}, \varphi)|$ is bounded by some constant multiple of $e^{|\Im s|^\alpha}$ for some $\alpha > 0$ as $|\Im s| \to \infty$. By the classification of the residual spectrum by Moeglin-Waldspurger, π is the unique irreducible quotient of the representation induced from $\sigma[\frac{m-1}{2}] \otimes \ldots \otimes \sigma[\frac{1-m}{2}]$ for some $m|n$, $m \neq n$, and $\sigma \in \Pi_{\text{cusp}}(\text{GL}_{\frac{n}{m}}(\mathbb{A}))$. Hence if Φ is $\mathbf{K}$-finite, there exists a finite set of Schwartz-Bruhat functions $\Psi_i \in \mathcal{S}(\text{Mat}_{\frac{n}{m} \times \frac{n}{m}}(\mathbb{A}))$, and a finite set of Matrix coefficients φ_i for σ such that $Z(\Phi, s + \frac{n-1}{2}, \varphi)$ is the sum over finitely many products of the form

$$\prod_{i=0}^{m} Z(\Psi_i, s + \frac{d-1}{2} - \frac{m-1}{2} - i, \varphi_i)$$

with the zeta function now defined as an integral over the group $\text{GL}_{\frac{n}{m}}(\mathbb{A})$. Such finite sets exist for each place by [Ja79, (2.3) Proposition,§4]. As for almost all places the zeta function coincides with $L_v(s, \pi) = L_v(s + \frac{m-1}{2}, \sigma) \cdot \ldots \cdot L_v(s - \frac{m-1}{2}, \sigma)$, only a finite number of places are relevant. By eventually "refining" the local Schwartz-Bruhat functions and matrix coefficients, we finally obtain appropriate global functions. By [GoJa72, §13] each of the cuspidal zeta functions is bounded in every vertical strip of finite width, so that consequently also $Z(\Phi, s+\frac{n-1}{2}, \varphi)$ is of finite order as well. Hence we may apply the Phragmen-Lindelöf theorem to conclude that they are also bounded on the strip $I_{a,b} = \{s \mid a \leq \Re s \leq b\}$ by a constant, which only depends on Φ, a, b. For not necessarily $\mathbf{K}$-finite Φ, the assertion follows from the $\mathbf{K}$-finite case by using that the space of $\mathbf{K}$-finite Φ is dense in $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$, and each sequence of $\mathbf{K}$-finite functions converging to some element in $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ yields a locally uniformly convergent sequence of the associated zeta functions by [Ja79, (4.5.2), (4.5.3)]. □

Let Ω_G and $\Omega_{\mathbf{K}_\infty}$ be the Casimir elements for $G(F_\infty)$ and $\mathbf{K}_\infty$, respectively (see [Kn86, Chapter 8.6]). Since they commute with the right regular representation, they both operate by scalars on $\mathcal{A}^2_\pi(G)^K_\tau$ (because of the irreducibility of π and τ), which we denote by λ_π and λ_τ, respectively (the eigenvalue of Ω_G only depends on π_∞, and the eigenvalue of $\Omega_{\mathbf{K}_\infty}$ only depends on τ, so that the notation is justified).Let $D = \mathrm{id} + \Omega^2_{\mathrm{G}} + \Omega^2_{\mathbf{K}_\infty}$. Let $\varphi =< \pi\varphi_1, \varphi_2 >$ be a matrix coefficient of π with $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(G)^K_\tau$ of norm 1. Then the operator $R(D)$ is essentially self-adjoint, and we have

$$\begin{aligned}
&\int_{G(\mathbb{A})} \Phi(x)|\det x|^{s+\frac{n-1}{2}}\varphi(x)dx =< R(f_s)\varphi_1, \varphi_2 > \\
&=< R(D^N f_s)\varphi_1, R(D^{-N})\varphi_2 >= \nu(\pi,\tau)^{-N} < R(D^N f_s)\varphi_1, \varphi_2 > \\
&\qquad\qquad = \nu(\pi,\tau)^{-N}\int_{G(\mathbb{A})} \Phi_N(x)|\det x|^{s+\frac{n-1}{2}}\varphi(x)dx,
\end{aligned}$$

where $\nu(\pi,\tau) = 1+\lambda^2_\pi+\lambda^2_\tau$, and $\Phi_N \in \mathcal{S}(M_n(\mathbb{A}), K)$ is a suitable Schwartz-Bruhat function depending only on N, but neither on π nor on τ, which exists by the following lemma.

Lemma 38. *Let $\psi \in \mathcal{S}(\mathrm{Mat}_{n\times n}(F_\infty))$ and put $g_t(x) = \psi(x)|\det(x)|^t$ for $t \in \mathbb{C}$, $\Re t > n$, and $x \in G(F_\infty)$. Let $X \in \mathcal{U}(\mathfrak{g}_{F_\infty \otimes \mathbb{C}})$. Then $X * g_t$ again has the form $(X*g_t)(x) = \tilde{\psi}(x)|\det(x)|^t$ for some function $\tilde{\psi} \in \mathcal{S}(\mathrm{Mat}_{n\times n}(F_\infty))$. If $\Psi \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ and $F_t(\Psi, g) = \Psi(g)|\det(g)|^t$, we can find $\tilde{\Psi} \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ such that $(X * F_t(\Phi, \cdot))(g) = F_t(\tilde{\Psi}, g)$.*

Proof. We have

$$\begin{aligned}
(X * g_t)(x) &= \frac{\partial}{\partial r}_{|r=0}\left(\psi(\exp(rX)x)|\det(\exp(rX)x)|^t\right) \\
&= \left(\frac{\partial}{\partial r}_{|r=0}\psi(\exp(rX)x) + \psi(x)\frac{\partial}{\partial r}_{|r=0}|\det(\exp(rX))|^t\right)|\det(x)|^t
\end{aligned}$$

from which our assertions are clear. □

Proof of Proposition 32. We first show that Θ_2 continues to an entire function. For $\Re s > \frac{n+1}{2}$ write $\Theta_2(s)$ as the absolute convergent sum

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A})^1), \dim \pi \neq 1} \sum_{\tau \in \widehat{\mathbf{K}_\infty}} \nu(\pi,\tau)^{-N} \sum_{\varphi \in \Phi_{\pi,\tau}} Z(\Phi_N, s + \frac{n-1}{2}, \varphi) \qquad (25)$$

for suitable sets of matrix coefficients $\Phi_{\pi,\tau}$ for the τ-isotypic component of the K-invariant part of π. Each of the zeta functions continues to an entire function on $s \in \mathbb{C}$, so that we are done if we show that (25) converges absolutely for all $s \in \mathbb{C}$. Let $a \ll 0$ and $b = \frac{n+2}{2}$. Then by Lemma 37, there exists $M(\Phi_N, a, b)$ such that $|Z(\Phi_N, s + \frac{n-1}{2}, \varphi)| \leq M(\Phi_N, a, b)$ for all φ, and all s with $a \leq \Re s \leq b$. One of the main ingredients of Müller's proof of the trace class conjecture was [Mu98, Corollary 0.3], whose adelic version as explained in [Mu02, §6] is that for any $M \in \mathcal{L}$ and any $P \in \mathcal{P}(M)$,

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \sum_{\tau \in \widehat{\mathbf{K}_\infty}} \dim(\mathcal{A}^2_\pi(P)^K_\tau)(1 + \lambda_\pi^2 + \lambda_\tau^2)^{-N} < \infty. \qquad (26)$$

Hence (25) converges absolutely by (26) so that we get a holomorphic continuation to $\Re s > a$, and thus to all of $\mathbb{C}$.

We are left with $\Theta_1(s)$, i.e. the trace of the one-dimensional representations so that $\Theta_1(s) = \sum_{\chi : \mathbb{A}^\times / F^\times \longrightarrow \mathbb{C}} \operatorname{tr} \chi \circ \det(f_s)$ with χ ranging over all Hecke characters which are invariant under the group $\det(\mathbf{K}_\infty K) \subseteq \mathbb{A}^1$ and such that $\chi^n = 1$. The group $\det(\mathbf{K}_\infty K)$ is of finite index in $\mathbb{A}^1$ so that the conductor of the possible characters is bounded. Since $\chi^n = 1$ implies that $\chi_{|\mathbb{R}_{>0}} = 1$, where $\mathbb{R}_{>0}$ is embedded in $\mathbb{A}^\times$ by putting the same entry at all infinite places, and 1 at all finite places, there are overall only finitely many possible characters. The trace can be computed to be $\operatorname{tr} \chi \circ \det(f_s) = \int_{G(\mathbb{A})} \Phi(g) |\det g|^{s + \frac{n-1}{2}} \chi(\det g) dg$. Hence the sum $\sum_{\chi,\ \chi \neq 1} \operatorname{tr} \chi \circ \det(f_s)$ yields an entire funtion. Thus only the trivial representation is left over, and it equals

$$Z(\Phi, s + \frac{n-1}{2}, 1) = Z(\Phi, s + \frac{n-1}{2}) = \int_{\mathrm{GL}_n(\mathbb{A})} |\det x|^{s + \frac{n-1}{2}} \Phi(x) dx.$$

Hence up to multiplication with and addition of an entire function $\Theta_1(s)$ equals

$$\prod_{i=0}^{n-1} \zeta_F^*(s + \frac{n-1}{2} - i)$$

a priori for $\Re s > \frac{n+1}{2}$, but the right hand side also gives the meromorphic continuation to all of $\mathbb{C}$, and from which we can readily read of the location and multiplicities of the poles. □

Proof of Corollary 36. Now suppose that $M \subsetneq G$ is a Levi subgroup and $P \in \mathcal{P}(M)$, $P = MU$. Then the proof above applies also to (24), since we have $M_{P|P}(t,0)\rho(P,0,f_s) = \rho(P,0,f_s)M_{P|P}(t,0)$, we can assume $t = \mathrm{id}$ for the question of analytic continuation. Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ and let $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(P)^K$ be of norm 1, and let $\varphi =< \rho(P,\pi,\cdot)\varphi_1, \varphi_2 >$ again be a matrix coefficient. Then

$$Z(\Phi, s+\frac{n+1}{2}, \varphi)$$
$$= \int_{G(\mathbb{A})}\int_{\mathbf{K}}\int_{A_M M(F)\backslash M(\mathbb{A})} |\det g|^{s+\frac{n-1}{2}}\Phi(g)\varphi_1(mkg)\overline{\varphi_2(mk)}dmdkdg$$

and changing variables, this equals

$$\int_{M(\mathbb{A})}\int_{U(\mathbb{A})}\int_{\mathbf{K}}\int_{\mathbf{K}}\int_{A_M M(F)\backslash M(\mathbb{A})} |\det m'|^{s+\frac{n-1}{2}}\Phi(k^{-1}m'uk')$$
$$\cdot \varphi_1(mm'k')\overline{\varphi_2(mk)}dkdmdk'dudm'. \quad (27)$$

Note that for any k, k' fixed,

$$m' \mapsto \int_{A_M M(F)\backslash M(\mathbb{A})} \varphi_1(mm'k')\overline{\varphi_2(mk)}dm$$

is a matrix coefficient for π, and by the $\mathbf{K}$-finiteness of φ_1, φ_2 (π is admissible), and if we further assume that Φ is $\mathbf{K}$-finite, (27) therefore is a finite sum over a finite set of matrix coefficients ψ for π and a finite set $\{\Psi\} \subseteq \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), \mathbf{K}_\infty K)$ of

$$\int_{M(\mathbb{A})}\int_{U(\mathbb{A})} |\det m'|^{s+\frac{n-1}{2}}\Psi(m'u)\psi(m')dudm'.$$

Put

$$\Psi_M(m) = e^{<\alpha_M, H_M(m)>}\int_{U(\mathbb{A})}\Psi(mu)du$$

with $\alpha_M = (n-n_1, n-n_1-n_2, \ldots, n_r, 0)$, which is a Schwartz-Bruhat function on $\mathrm{Mat}_{n_1\times n_1}(\mathbb{A}) \times \ldots \times \mathrm{Mat}_{n_r\times n_r}(\mathbb{A})$ invariant under $K \cap M(\mathbb{A})$. Note that $\rho_M - \alpha_M = -(\frac{n-n_1}{2}, \frac{n-n_2}{2}, \ldots, \frac{n-n_r}{2})$. Write $\pi = \pi_1 \otimes \ldots \otimes \pi_r$ with $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}_{n_i}(\mathbb{A})^1)$, and accordingly $\psi = \psi_1 \cdot \ldots \cdot \psi_r$. By eventually refining the sum over Ψ, we can also assume $\Psi_M(m) = \prod_{i=1}^r \Psi_{m,i}(m_i)$, $\Psi_{M,i} \in \mathcal{S}(\mathrm{Mat}_{n_i\times n_i}(\mathbb{A}))$. Then (27) equals the finite sum over ψ and Ψ of

$$\prod_{i=1}^r \int_{\mathrm{GL}_{n_i}(\mathbb{A})} |\det m_i|^{s+\frac{n_i-1}{2}}\Phi_{M,i}(m_i)\psi_i(m_i)dm_i$$

for which we can apply the previous results. If now Φ is not $\mathbf{K}$-finite, there is a sequence of $\mathbf{K}$-finite functions in $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ converging to Φ and such that the resulting zeta functions converge locally uniformerly to the zeta function associated with Φ (cf. [Ja79]) so that the assertion also follows for general Φ. □

Remark 39. *Consider one of the functions $\tilde{f}_s$ associated with $\tilde{\Phi} = \varphi\Phi_\infty\Phi_f$ as in Remark 34. As observed there, they yield holomorphic functions in $\Re s > \frac{n+1}{2}$ for the spectral terms. If $\pi \in \Pi_{disc}(G(\mathbb{A})^1)$ and f a matrix coefficient for π, we can still define a zeta function*

$$Z(\tilde{\Phi}, s + \frac{n-1}{2}, f) = \int_{G(\mathbb{A})} |\det g|^{s+\frac{n-1}{2}} \tilde{\Phi}(g) f(g) dg$$

which is a well-defined and holomorphic function for $\Re s > \frac{n+1}{2}$. Let $\hat{\tilde{\Phi}}(x) = \int_{\mathrm{Mat}_{n\times n}(\mathbb{A})} \tilde{\Phi}(y)\psi(\mathrm{tr}\, xy) dy$ be the Fourier transform. Using the standard identities for Fourier transform and keeping in mind that $X * \varphi$ is absolutely bounded for any $X \in \mathcal{U}(\mathfrak{gl}_n)_{\leq N}$, we see that there is $c > 0$ such that

$$|\hat{\tilde{\Phi}}(x)| \leq c(1 + ||x||)^{-N+1}$$

for any $x \in \mathrm{Mat}_{n\times n}(\mathbb{A})$. This in particular allows us to apply Poisson summation to sums of the form $\sum_{x\in \mathrm{Mat}_{n\times n}(F),\ \mathrm{rk}\, x = k} \Phi(ax)$, $a \in \mathbb{A}^\times$, $0 \leq k \leq n$. We may also consider the integral defining $Z(\hat{\tilde{\Phi}}, s + \frac{n-1}{2}, \hat{f})$, but the absolute convergence of this can now be guaranteed only for $\Re s \in (\frac{n+1}{2}, \frac{n+1}{2} + \frac{N-1}{n})$. Following along the lines of [GoJa72] for the proof of the meromorphic continuation and the functional equation of the zeta functions of cuspidal representations, we see that $Z(\tilde{\Phi}, s + \frac{n-1}{2}, f)$ does not necessarily has a continuation to all of $\mathbb{C}$, but at least in the half plane $\Re s > \frac{1-N}{n} + \frac{1-n}{2}$ if π is cuspidal, and it has poles at the usual points if $n = 1$. The analogue statements are similarly true for general $\pi \in \Pi_{disc}(G(\mathbb{A})^1)$. Moreover, in the region of continuation, it satisfies the usual functional equation, i.e.

$$Z(\tilde{\Phi}, s + \frac{n-1}{2}, f) = Z(\hat{\tilde{\Phi}}, 1 - s + \frac{n-1}{2}, \hat{f})$$

for $\Re s \in (\frac{1-N}{n} + \frac{1-n}{2}, \infty)$. Hence if N is sufficiently large, we can continue up to a region, where $Z(\hat{\tilde{\Phi}}, 1 - s + \frac{n-1}{2}, \hat{f})$ is again given by an absolutely convergent integral. Hence we can conclude that in the half plane of continuation, Lemma 37 stays valid, and we get a continuation of the spectral terms at least in some larger half plane $\Re s > \frac{n+1}{2} - \delta$ with a simple pole at $\Re s = \frac{n+1}{2}$.
It will be clear from the next sections that also any other spectral term can be continued to some half plane $\Re s > 2 - \delta$ for functions from $\mathcal{C}^N(G(\mathbb{A})^1, K)$.

IV.iii The non-discrete part

IV.iii.i Estimates for normalising factors and matrix coefficients of normalised intertwining operators

The purpose of this section is to introduce some well-known notation, and to find estimates for certain quantities, which will allow us later to adapt the convergence proofs of [MuSp04, FiLaMu11] to our situation. Let $(n_1, \ldots, n_r)$ be a partition of n, $r \geq 2$, and let M be the standard Levi subgroup associated to $(n_1, \ldots, n_r)$. Let $P \in \mathcal{P}(M)$, and U the unipotent radical of P. Let $\pi = \pi_1 \otimes \ldots \otimes \pi_r \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$, i.e. $\pi_i \in \Pi_{\text{disc}}(\mathrm{GL}_{n_i}(\mathbb{A})^1)$, $i = 1, \ldots, r$. Then $\pi = \bigotimes_v \pi_v = (\bigotimes_v \pi_{1,v}) \otimes \ldots \otimes (\bigotimes_v \pi_{r,v})$ with $\pi_{j,v}$ local admissible representations of $\mathrm{GL}_{n_j}(F_v)$. For $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, π_λ is the twisted representation given by $\pi_\lambda(m) = \pi(m)e^{\lambda(H_M(m))}$. Denote by $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\pi)$ the representation of $G(\mathbb{A})$ parabolically induced from π, and denote by $\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_\mathrm{v})$ the respective local version. If Q is another parabolic subgroup with Levi component M, there are intertwining operators

$$M_{Q|P}(\pi, \lambda) : \mathcal{A}^2_\pi(Q) \longrightarrow \mathcal{A}^2_\pi(P)$$

which are initially defined for λ with $\Re\lambda$ contained in a certain translate of the positive chamber associated with P by an integral over $(U_Q(\mathbb{A}) \cap U_P(\mathbb{A}))\backslash U_Q(\mathbb{A})$ [Ar05, (7.2), Lemma 7.1], and are defined by analytic continuation elsewhere. Here an operator, depending on complex variables is said to be holomorphic or has a meromorphic continuation if all its matrix coefficients have the respective property. It has a pole at some point if there exists a matrix coefficient having a pole at this point, and it has a zero if the inverse operator has a pole there. This definition also allows us to speak of traces, operator norms, etc. even at points, where the operator is only defined by means of analytic continuation.
More generally, if t is some Weyl group element, which maps $\mathfrak{a}_P$ isomorphically to $\mathfrak{a}_Q$, there is an intertwining operator

$$M_{Q|P}(\lambda, t) : \mathcal{A}^2(Q) \longrightarrow \mathcal{A}^2(P)$$

satisfying $M_{Q|P}(\pi, \lambda) = M_{Q|P}(\mathrm{id}, \lambda)_{|\mathcal{A}^2_\pi(\mathrm{Q})}$. Write $M_{Q|P}(\lambda) = M_{Q|P}(\lambda, 1)$ Similarly, there are local intertwining operators [Ar05, §21]

$$J_{Q|P}(\pi_v, \lambda) = J_{Q|P}(\pi_{v,\lambda}) : \mathrm{Ind}_{\mathrm{Q(F_v)}}^{\mathrm{G(F_v)}}(\pi_{\mathrm{v},\lambda}) \longrightarrow \mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_{\mathrm{v},\lambda})$$

which are initially defined by an integral over $(U_Q(F_v) \cap U_P(F_v))\backslash U_Q(F_v)$ for $\Re\lambda$ sufficiently regular, and are defined by analytic continuation elsewhere. There are local normalising factors $r_{Q|P}(\pi_v, \lambda)$ such that the normalised operators $R_{Q|P}(\pi_{v,\lambda}) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_{v,\lambda})$ satisfy certain conditions as in [Ar89, Theorem 2.1]. In particular, if $\pi_{v,\lambda}$ is a spherical representation, $R_{Q|P}(\pi_{v,\lambda})$ maps the spherical vector to itself so that $R_{Q|P}(\pi_{v,\lambda})$ acts as the identity at unramified places. We identify $(\mathfrak{a}^G_{M,\mathbb{C}})^*$ with the hyperplane $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$, $\sum \lambda_i = 0$. As explained in [ArCl89, Chapter 2.2] the local normalising factors are then given as a product $r_{Q|P}(\pi_v, \lambda) = \prod_{\alpha \in \Sigma(Q) \cap \Sigma(\bar{P})} r_\alpha(\pi_v, \lambda(\alpha^\vee))$ with $r_\alpha(\pi_v, x)$, $x \in \mathbb{C}$, meromorphic functions defined by

$$r_\alpha(\pi_v, x) = \frac{L_v(x, \alpha(\pi_v))}{L_v(1+x, \alpha(\pi_v)) \varepsilon_v(x, \alpha(\pi_v), \psi_v)}.$$

Here $\alpha(\pi_v)$ is defined as $\pi_{v,i} \times \tilde{\pi}_{v,j}$ if $\alpha = e_i - e_j$, $\{e_k\}_{k=1,\dots,r}$ the standard basis vector in $\mathbb{C}^r$, and the occuring L-functions are the local Rankin-Selberg L-functions with the corresponding ε-factor (see, e.g., [RuSa96]). By the properties of the local intertwining operators and normalising factors, their global products $J_{Q|P}(\pi_\lambda) = \bigotimes_v J_{Q|P}(\pi_{v,\lambda})$, and $r_{Q|P}(\pi, \lambda) = \bigotimes_v r_{Q|P}(\pi_{v,\lambda})$ are well-defined functions (cf. [Ar05, §21]). If we write $L(x, \sigma_1 \times \sigma_2)$ for the completed Rankin-Selberg L-function and use its functional equation, we obtain

$$r_{Q|P}(\pi, \lambda) = \prod_{\alpha \in \Sigma(Q) \cap \Sigma(\bar{P})} \frac{L(1 - \lambda(\alpha^\vee), \alpha(\tilde{\pi}))}{L(1 + \lambda(\alpha^\vee), \alpha(\pi))}.$$

As in [Ar05] let $R_{Q|P}(\lambda)$ be the operator on $\mathcal{A}^2(P)$ which equals the global normalised operator $r_{P|Q}(\pi, \lambda)^{-1} M_{P|Q}(\pi, \lambda)$ whenever restricted to $\mathcal{A}^2_\pi(P)$. In particular, this restriction is isomorphic to $m_{\text{disc}}(\pi)$ many copies of the operator $\otimes_v R_{Q|P}(\pi_{v,\lambda})$ by the isomorphism

$$\operatorname{Hom}(\pi, L^2(A_M M(F) \backslash M(\mathbb{A}))) \otimes \operatorname{Ind}^{\mathrm{G}(\mathbb{A})}_{\mathrm{P}(\mathbb{A})}(\pi) \longrightarrow \mathcal{A}^2_\pi(\mathrm{P})$$

(cf. [FiLaMu11]), and $m_{\text{disc}}(\pi)$ is the multiplicity of π in $L^2_{\text{disc}}(A_M M(F) \backslash M(\mathbb{A}))$ (which is 1 if π is cuspidal). We recall how the intertwining operators and normalising factors give rise to (G, M)-families [Ar05, §21]: For $Q, P \in \mathcal{P}(M)$, $\lambda, \Lambda \in \mathfrak{a}^*_M$, $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$ set

$$\mathcal{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda),$$

$$r_Q(\Lambda, \lambda, \pi, P) = r_{Q|P}(\pi, \lambda)^{-1} r_{Q|P}(\pi, \lambda + \Lambda),$$

and

$$\mathcal{R}_Q(\Lambda, \lambda, P) = R_{Q|P}(\lambda)^{-1} R_{Q|P}(\lambda + \Lambda).$$

These are all three (G, M)-families in the sense of [Ar81]. The operators and functions $\mathcal{M}_L(P, \lambda)$, $r^S_L(\pi, \lambda)$, $\mathcal{R}_S(\lambda)$, $L, S \in \mathcal{L}(M)$, $L \subseteq S$, are then associated with these (G, M)-families as explained in [Ar81]. By [Ar81, Corollary 6.5] one has for each $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$

$$\mathcal{M}_L(P, \lambda)_{|\mathcal{A}^2_\pi(P)} = \sum_{S \in \mathcal{L}(L)} r^S_L(\pi, \lambda) \mathcal{R}_S(\lambda, P)_{|\mathcal{A}^2_\pi(P)}, \tag{28}$$

where we used that r^R_L is independent of $R \in \mathcal{P}(L)$. Since $r_{Q|P}(\pi, \lambda)$, and hence also $r_Q(\Lambda, \lambda, \pi, P)$ are products over functions associated with roots as above, $r_Q^S(\pi,\lambda)$ equals by [Ar82, Proposition 7.5]

$$\sum_F \mathrm{vol}_{\mathfrak{a}_L^{L_1}}(F_L^\vee) \prod_{\alpha\in F} r_\alpha(\pi,\lambda(\alpha^\vee))^{-1} r'_\alpha(\pi,\lambda(\alpha^\vee)) \tag{29}$$

where $L_1 \in \mathcal{L}$ is such that $S \in \mathcal{P}(L_1)$, and F runs over all subsets of reduced roots of L_1 with respect to A_M such that F_L (the set of all roots of F restricted to $\mathfrak{a}_L$) is a basis of $\mathfrak{a}_L^{L_1}$. $\mathrm{vol}_{\mathfrak{a}_L^{L_1}}(F_L^\vee)$ is the volume of the lattice spanned by $F_L^\vee$ in $\mathfrak{a}_L^{L_1}$.

Before proceeding to proving the "continuous" parts of Theorem 32, we first collect some auxiliary results on the normalising factors and matrix coefficients of local intertwining operators.

Later on we shall need estimates on the growth of the matrix coefficients of the local intertwining operators $R_{P|Q}(\pi_v,\lambda)$ along subspaces for which $\Re\lambda(\alpha^\vee)$ is fixed and not necessarily 0. As the intertwining operators are factorisable into a product of operators belonging to adjacent parabolics, we only state those estimates for the adjacent case. In view of the results of [FiLaMu11] and Lemma 1 therein, it is sufficient to obtain estimates for the operator itself rather than all its derivatives as in [MuSp04] for $\lambda \in i(\mathfrak{a}_M^G)^*$. Results similar to the next lemma are contained in [MuSp04], in particular in the proof of Proposition 0.2 and Lemma A.1 therein, but as we need them in a slightly different version, we include a proof of our version. For a place v of F we denote by $\Pi_{\mathrm{disc}}(M(F_v))$ the set of all π_v, which occur as a local component of some $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$.

Lemma 40. *Let $M \in \mathcal{L}$, and $P, Q \in \mathcal{P}(M)$ adjacent along α. Suppose that $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$. Let v be a place of F, π_v the local component of π. In particular, $R_{Q|P}(\pi_v,\lambda)$ only depends on $\lambda(\alpha^\vee) \in \mathbb{C}$.*

(i) Suppose $v<\infty$. There is a finite set $X_0 \subseteq \mathbb{R}\backslash[-\frac{1}{1+n^2},\frac{1}{1+n^2}]$, which can be chosen independently of $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$ with $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}\pi_{\mathrm{v}})^{\mathrm{K_v}} \neq 0$, such that for all $x_0 \notin X_0$ no K_v-invariant matrix coefficient of $R_{Q|P}(\pi_v,\lambda)$ has poles or zeros on $\lambda(\alpha^\vee) \in x_0+i\mathbb{R}$. Let $\xi_2 = \sup_{x\in X_0}|x|$. Suppose that $x_0 \notin X_0$. Then there exists a constant $C = C_{x_0}$ not depending on π such that

$$|< R_{Q|P}(\pi_v,\lambda)\varphi_1,\varphi_2 >| \le C$$

for all $\lambda \in (\mathfrak{a}_M^G)^*$, $\lambda(\alpha^\vee) \in x_0+i\mathbb{R}$, and all $\varphi_1,\varphi_2 \in (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_{\mathrm{v}}))^{\mathrm{K_v}}$ of norm 1. For $|x_0| < \frac{1}{1+n^2}$ or $|x_2| > 1+\xi_2$, we can choose C_{x_0} independent of x_0.

(ii) Let $v|\infty$. There exists a finite set $\mathcal{R} \subseteq \mathbb{R}\backslash[-\frac{1}{1+n^2},\frac{1}{1+n^2}]$, which can be chosen independently of π, such that all poles and zeros of $R_{Q|P}(\pi_v,\lambda)$

are contained in $\lambda(\alpha^\vee) \in \mathcal{R} + \mathbb{Z}$. *Let* $\tau \in \widehat{\mathbf{K}_v}$. *There exists a constant* $c \geq 1$ *not depending on* π *or* τ, *such that for any* $x_0 \notin \mathcal{R} + \mathbb{Z}$,

$$| < R_{Q|P}(\pi_v, \lambda)\varphi_1, \varphi_2 > | \leq \left(c \frac{3 + ||\tau||}{\min_{z \in \mathcal{R}+\mathbb{Z}} |x_0 - z|} \right)^c$$

for all $\lambda(\alpha^\vee) \in x_0 + i\mathbb{R}$, *and all* $\varphi_1, \varphi_2 \in (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_\mathrm{v}))_\tau$ *of norm* 1. *Moreover, if* $r > 0$, *then there are only finitely many* $\tau \in \widehat{\mathbf{K}_v}$ *such that* $< R_{Q|P}(\pi_v, \lambda)\varphi_1, \varphi_2 >$ *has a pole in* $\lambda \in [-r, r]$ *for some* $\varphi_1, \varphi_2 \in (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_\mathrm{v}))_\tau$.

Note that by [MuSp04, Example, Remark A.4] at least the second part of the lemma does not remain true for general unitary local π_v so that we must assume π_v to be a local component of some unitary global representation occuring in the discrete spectrum.

Before starting to prove the lemma we recall some representation theory, which will use. First assume that v is a finite place and $\mathrm{cork}_G M = r$. Let $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}_{n_i}(\mathbb{A})^1)$, $i = 1, \ldots, r$. By the classification of the discrete spectrum of $\mathrm{GL}_{n_i}(\mathbb{A})$ by Moeglin-Waldspurger , and the classification of the v-adic representations by Silberger, there exist the following data:

- a partition $(m_1^i, \ldots, m_{r_i}^i)$ of n_i,
- a parabolic subgroup P_i' with Levi component M_i', which is of type $(m_1^i, \ldots, m_r^i)$,
- discrete series representations δ_j^i of $\mathrm{GL}_{m_j^i}(F_v)$,
- real parameters $s_1^i \geq \ldots \geq s_{r_i}^i$ with $|s_j^i| \leq \frac{n_i}{2}$ for all j

such that
$$\pi_{i,v} \simeq J_{P_i'(F_v)}^{\mathrm{GL}_{n_i}(F_v)}(\delta_1^i[s_1^i] \otimes \ldots \otimes \delta_{r_i}^i[s_{r_i}^i])$$

where $J_{P_i'(F_v)}^{\mathrm{GL}_{n_i}(F_v)}$ denotes the Langlands quotient, i.e. the unique irreducible quotient of the induced representation $\mathrm{Ind}_{\mathrm{P_i'(F_v)}}^{\mathrm{GL_{n_i}(F_v)}}(\delta_1^\mathrm{i}[\mathrm{s_1^i}] \otimes \ldots \otimes \delta_{\mathrm{r_i}}^\mathrm{i}[\mathrm{s_{r_i}^i}])$ (see [MuSp04, §3]). (There are better estimates for the Langlands parameters s_j^i available, see [MuSp04], but we do not need them at the moment and they would only complicate the description here.) Put $M' = M_1' \times \ldots \times M_r' \subseteq M$, $P' = P_1' \times \ldots \times P_r' \in \mathcal{P}^M(M')$, and for $P, Q \in \mathcal{P}(M)$ let $P(P') \in \mathcal{P}(M)$ be defined by $P(P') \cap M = P'$ and $P(P') \subseteq P$, and define $Q(P')$ similarly. Let $\pi = \pi_1 \otimes \ldots \otimes \pi_r$, and $\delta = \delta_1^1 \otimes \ldots \otimes \delta_{r_r}^r$. As explained in [Ar89, §2] (cf. also [Mu02, §7]) each K_v-invariant matrix coefficient of the local intertwining operator $R_{P|Q}(\pi_v, \lambda)$ is also a K_v-invariant matrix coefficient of $R_{P(P')|Q(P')}(\delta, \lambda + s)$ for $\lambda \in (\mathfrak{a}_{M,\mathbb{C}}^G)^*$, which is identified with its image under the embedding $(\mathfrak{a}_{M,\mathbb{C}}^G)^* \hookrightarrow (\mathfrak{a}_{M',\mathbb{C}}^G)^*$, and $s = (s_1^1, \ldots, s_{r_r}^r)$. Hence it suffices to consider the second operator. Now suppose that $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \pi_\mathrm{v})^{\mathrm{K_v}} \neq 0$. $i(\mathfrak{a}_{M'}^G)^*$ acts by $\mu \mapsto \delta[\mu]$ on the square-integrable representations, and if π_v belonging to δ has a K_v-invariant vector, the representation belonging to $\delta[\mu]$ as well has one. If we denote by $\mathcal{M}' \subseteq i(\mathfrak{a}_{M'}^G)^*$ the stabiliser of this action, $\mathcal{M}'$ is a cocompact lattice. By a theorem of Harish-Chandra (see [Mu02, §7]), there are only finitely many orbits of square-integrable representations on $M'(F_v)$, which yield some π_v with $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \pi_\mathrm{v})^{\mathrm{K_v}} \neq 0$. Moreover, $R_{P(P')|Q(P')}(\delta[\mu], \lambda + s) = R_{P(P')|Q(P')}(\delta, \lambda + s + \mu)$ so that it suffices to assume $(\delta_j^i)_{|A_{\mathrm{GL}_{m_j^i}}} \equiv 1$ for all i, j. Hence we get a map

$$\{\sigma_v \in \Pi_{\mathrm{disc}}(M(F_v)) \mid (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \sigma_\mathrm{v})^{\mathrm{K_v}} \neq 0\} \ni \pi_\mathrm{v} \longrightarrow (\mathrm{M}', \delta, \mu, \mathrm{s})$$

which associates to π_v the data constructed above. Note that the possibilities for M' and δ are finite, and that s and μ can only vary in a compact subset of $(\mathfrak{a}_{M,\mathbb{C}}^G)^*$. Hence as we assume that π occurs discretely, there are in fact overall only finitely many possibilities for the tuples (M', δ, μ, s), which yield $\pi_v \in \Pi_{\mathrm{disc}}(M(F_v))$ possessing a K_v-fixed vector.

Now assume that v is an archimedean place. Again using the classification by Moeglin and Waldspurger together with the Langlands classification, we can associate to π_v data M_i', P_i', δ_j^i, and Langlands parameter $s_j^i \in \mathbb{R}$, which satisfy the inequalities above, so that $\pi_{i,v}$ are the Langlands quotients as above. We again have a bijection between the matrix coefficients of $R_{P|Q}(\pi_v, \lambda)$ and of $R_{P(P')|Q(P')}(\delta, \lambda + s)$. If v is real, $\mathrm{GL}_m(\mathbb{R})$ has discrete series only for $m = 1, 2$. For $m = 1$, there are only two possibilities up to unramified characters: $\delta = 1$ or $\delta = \frac{x}{|x|}$, $x \in \mathbb{R}$. For $m = 2$ the discrete series is, again up to unramified characters, parametrised by integers (see for example [Bu97, §2.5]). If v is complex, $\mathrm{GL}_m(\mathbb{C})$ has discrete series only for $m = 1$, and it is parametrised by $l \in \mathbb{Z}$ up to an unramified character (see also [MuSp04, §3]). Twisting δ with unramified characters is the same as the action of $i(\mathfrak{a}_{M'}^G)^*$ on the representation, $\mu \mapsto \delta[\mu]$. Again $R_{P(P')|Q(P')}(\delta[\mu], \lambda + s) = R_{P(P')|Q(P')}(\delta, \lambda + s + \mu)$, so that we may assume that each δ_j^i is uniquely determined by the integer, which is associated with δ_j^i. Suppose that $\tau \in \widehat{\mathbf{K}_v}$ with $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_\mathrm{v}))_\tau \neq 0$. By [Vo86, (5.4), Proposition 5.17], the representations $\tau \in \widehat{\mathbf{K}_v}$ are parametrised by their highest weight vectors, which are tuples of certain half-integers. For τ to appear in the induced representation the tuple $\mu + s$ must therefore consists of half-integers. As μ is purely imaginary and s real, this means that $\mu = 0$, and s must consists of half-integers. This limits the possibilities of s to a finite number, as the modulus of the entries of s was bounded by $\frac{n}{2}$. By the parametrisation of the discrete series and the unitary representations of $\mathbf{K}_v$ we can therefore even find a finite set $\mathcal{S} \subseteq (\mathfrak{a}_M^G)^*$ such that the image of the map $\Pi_{\mathrm{disc}}(M(F_v)) \ni \pi_v \mapsto s \in (\mathfrak{a}_{M'}^G)^* \hookrightarrow (\mathfrak{a}_M^G)^*$ is contained in $\mathcal{S} + L$ for some lattice $L \subseteq (\mathfrak{a}_M^G)^*$ (in fact, L can be chosen to be the lattice spanned by the roots of P).

Proof of Lemma 40. (i) The existence of a finite set $X_0 \subseteq \mathbb{R}$ as assumed in the lemma follows from the considerations just before the proof and the functional equation $R_{Q|P}(\pi_v, \lambda) = R_{Q|P}(\pi_v, -\lambda)^{-1}$ for P, Q adjacent. By [MuSp04, Proposition 4.2] X_0 can be chosen such that $\min_{x\in X_0} |\Re x| \geq \frac{2}{1+n^2}$. Let $\xi_2 = \max_{x\in X_0} |x|$. Suppose now that $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ with $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \pi_{\mathrm{v}})^{\mathrm{K_V}} \neq 0$. Let $\varphi_1, \varphi_2 \in (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}}(\pi_{\mathrm{v}}))^{\mathrm{K_v}}$ be of norm 1, and write $f(s) =< R_{Q|P}(\pi_v, s\varpi_\alpha)\varphi_1, \varphi_2 >$ for $s \in \mathbb{C}$, and $\varpi_\alpha \in \mathfrak{a}_M^*$ such that $\varpi_\alpha(\beta^\vee) = \delta_{\alpha\beta}$ for all roots β. By [FiLaMu11, Lemma 2] $f(s)$ is a rational function in q_v^{-s} the degree of which is bounded by some $m \in \mathbb{N}$, which only depends on K_v, but not on φ_i or π. We now follow a similar strategy as in the proof of [MuSp04, Proposition 0.2] to find an upper bound for the matrix coefficients. Write $F(q_v^{-s}) = f(s)$ for some rational function F of degree at most m. Note that $f(s)$ is holomorphic in $\Re s > 0$. Since $R_{Q|P}(\pi_v, \lambda)$ is unitary on $\lambda(\alpha^\vee) \in i\mathbb{R}$, $|F(z)| \leq 1$ for all $z \in \mathbb{C}$, $|z| = 1$. F has at most a pole of order $m'' \leq m$ at $z = 0$ in $\{z \in \mathbb{C} \mid |z| \leq 1\}$, and at most m poles in $|z| > 1$, say $a_1, \ldots, a_{m'}$ with $m' \leq m$ and counted with multiplicities. Hence $H_1(z) = F(z)z^{m''}$ and $H_2(z) = F(z^{-1}) \prod_{i=1,\ldots,m'} (z^{-1} - a_i)$ are holomorphic functions for all of $z \in \mathbb{C}, |z| \leq 1$ with

$$\sup_{|z|=1} |H_1(z)| = \sup_{|z|=1} |F(z)| \leq 1$$

and

$$\sup_{|z|=1} |H_2(z)| \leq \sup_{|z|=1} |F(z^{-1})| \prod_{i=1,\ldots,m'} \sup_{|z|=1} |z - a_i| \leq \max\{2, 1 + \xi_2\}^m.$$

By the maximum principle, $|H_1(z)|$ is therefore bounded by 1, and $|H_2(z)|$ is bounded by $\max\{2, 1 + \xi_2\}^m$ for all $z \in \mathbb{C}, |z| \leq 1$. Now suppose that $x_0 \in \mathbb{R}$ such that $R_v(\lambda)$ is holomorphic on $\lambda(\alpha^\vee) \in x_0 + i\mathbb{R}$. This means that $F(z)$ is holomorphic on $|z| = q_v^{-x_0}$ so that

$$\sup_{t\in\mathbb{R}} |f(x_0 + it)| = \sup_{|z|=q_v^{-x_0}} |F(z)| \leq \sup_{|z|=q_v^{-x_0}} |H_1(z)| q^{x_0 m''} \leq q_v^{x_0 m''}$$

if $q_v^{-x_0} \leq 1$, and

$$\begin{aligned} \sup_{t\in\mathbb{R}} |f(x_0 + it)| &= \sup_{|z|=q_v^{-x_0}} |F(z)| \\ &\leq \sup_{|z|=q_v^{-x_0}} |H_2(z)| \sup_{|z|=q_v^{-x_0},\ i=1,\ldots m'} |z^{-1} - a_i|^{-m} \\ &\qquad \leq q_v^{x_0 m} \max\{2, 1 + \xi_2\}^m \sup_{|z|=q_v^{x_0},\ x\in X_0} |z - x|^{-m}, \end{aligned}$$

if $q_v^{-x_0} > 1$. Both last bounds depend only on x_0, but not on φ_i. If additionally $|x_0| \not\in [\frac{1}{1+n^2}, \xi_2 + 1]$,

$$\sup_{|z|=q_v^{x_0},\ x\in X_0} |z-x|^{-m} \leq \min\{1, \frac{1}{1+n^2}\}^{-m} = (1+n^2)^m,$$

so that we even obtain a bound, which is independent of x_0.

(ii) Again by the considerations just before the proof together with the functional equation of the intertwining operator, it is clear that there is a finite set $\mathcal{R} = \{\rho\} \subseteq \mathbb{R}$ such that $R_{P|Q}(\pi_v, \lambda)$ is pole- and zero-free outside of $\mathcal{R} + \mathbb{Z}$. We are left to show that for $x_0 \not\in \mathcal{R} + \mathbb{Z}$ there is a constant as asserted. But this follows from [MuSp04, Lemma A.1, Proposition A.2] and the proof of the lemma there.

□

Note that this lemma also yields bounds for the norm of the operators $R_{P|Q}(\pi_v, \lambda)^{K_v}$ (the restriction of $R_{P|Q}(\pi_v, \lambda)$ to the K_v-invariant space) in the non-archimedean, and $R_{P|Q}(\pi_v, \lambda)_\tau$ (the restriction of $R_{P|Q}(\pi_v, \lambda)$ to the τ-isotypical component) in the archimedean case: Under the hypotheses of the lemma we get

$$||R_{P|Q}(\pi_v, \lambda)^{K_v}|| \leq C \left(\frac{\dim \mathcal{A}_\pi^2(P)_\tau^K}{m_{\text{disc}}(\pi)}\right)^2, \tag{30}$$

and

$$||R_{P|Q}(\pi_v, \lambda)_\tau|| \leq \left(c \frac{3 + ||\tau||}{\min_{z\in\mathcal{R}+\mathbb{Z}} |x_0 - z|}\right)^c \left(\frac{\dim \mathcal{A}_\pi^2(P)_\tau^K}{m_{\text{disc}}(\pi)}\right)^2 \tag{31}$$

for the constants and λ as in the lemma.

For the normalising factors we have the following.

Lemma 41. *Let $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$ and α be as above. Then there exists a meromorphic function $\mathfrak{L}_\alpha(\cdot, \pi) : \mathbb{C} \longrightarrow \mathbb{C}$ such that*

$$\frac{r'_\alpha(\pi, \lambda(\alpha^\vee))}{r_\alpha(\pi, \lambda(\alpha^\vee))} = \mathfrak{L}_\alpha(\lambda(\alpha^\vee), \pi) + \mathfrak{L}_\alpha(-\lambda(\alpha^\vee), \tilde{\pi}),$$

and $\mathfrak{L}_\alpha(\lambda(\alpha^\vee), \pi)$ is holomorphic for $\Re\lambda(\alpha^\vee) \geq 0$. If π_i, π_j are both cuspidal, then

$$\mathfrak{L}_\alpha(\lambda, \pi) = \frac{L'(1 + \lambda(\alpha^\vee), \alpha(\pi))}{L(1 + \lambda(\alpha^\vee), \alpha(\pi))} - \lambda(\alpha^\vee)^{-1}\delta_{\pi,\alpha}$$

where

$$\delta_{\pi,\alpha} = \begin{cases} 1 & \text{if } n_i = n_j \text{ and } \pi_i \simeq \pi_j, \\ 0 & \text{else.} \end{cases}$$

Moreover, there exist constants $m \in \mathbb{N}$, $C_N > 0$ for $N \in \mathbb{N}$, such that for N sufficiently large

$$\int_{a+i\mathbb{R}} |\mathfrak{L}_\alpha(\pi, t)|(1+|t|^2)^{-N} dt \leq C_N(1+\Lambda_\pi^2)^m \tag{32}$$

for all $a \in \mathbb{R}_{\geq 0}$, and $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$ with $\mathcal{A}_\pi^2(P)^K \neq 0$. Here $\Lambda_\pi = \min_\tau(|\lambda_\pi|^2 + |\lambda_\tau|^2)^{\frac{1}{2}}$ with λ_π the Casimir eigenvalue of π_∞, and λ_τ the Casimir eigenvalue of τ, and τ runs over all minimal $\mathbf{K}_\infty$-types in π (see [Mu02, (4.5)]).

Proof. By the classification of the residual spectrum of $\mathrm{GL}(n)$ by Moeglin-Waldspurger, there are $d_k|n_k$, and cuspidal unitary representations σ_k of $\mathrm{GL}(n_k)$, $k = i, j$, such that π_k is isomorphic to the unique irreducible quotient of the induced representation $\mathrm{Ind}_{\mathrm{Q}_{(\mathrm{d_k},\ldots,\mathrm{d_k})}}^{\mathrm{GL}(\mathrm{n_k})}(\sigma_\mathrm{k}[\frac{\mathrm{m_k}-1}{2}]\otimes\ldots\otimes\sigma_\mathrm{k}[-\frac{\mathrm{m_k}-1}{2}])$ for $m_k = n_k/d_k$ and $Q_{(d_k,\ldots,d_k)} \subseteq \mathrm{GL}(n_k)$ the standard parabolic associated with the partititon $(d_k, \ldots, d_k)$ of n_k. Hence by the definition of the Rankin-Selberg L-function for general π (see [JPSS83] or [RuSa96, Appendix]),

$$\begin{aligned}
&L(s, \alpha(\pi)) = L(s, \pi_i \times \tilde{\pi}_j) \\
&= \prod_{\nu=0}^{m_i-1}\prod_{\mu=0}^{m_j-1} L(s + \frac{m_i-1}{2} - \nu - \frac{m_j-1}{2} + \mu, \sigma_i \times \tilde{\sigma}_j) \\
&\qquad = \prod_{\kappa=0}^{m_i+m_j-2} L(s + \frac{m_i+m_j-2}{2} - \kappa, \sigma_i \times \tilde{\sigma}_j).
\end{aligned}$$

This product is an entire function unless $n_i = n_j$ and $\sigma_i \simeq \sigma_j$, and all zeros are contained in $0 < \Re(s + \frac{m_i+m_j-2}{2} - \kappa) < 1$, $\kappa = 0, \ldots, m_i + m_j - 2$ by [Co07, Theorem 4.3]. Moreover, each of the L-functions is bounded in vertical strips of finite width away from its poles, see [Co07]. Hence, if we set

$$\begin{aligned}
-\tilde{\mathfrak{L}}_\alpha(\lambda(\alpha^\vee), \pi) = &\sum_{\kappa=0}^{\lfloor\frac{m_i+m_j-2}{2}\rfloor} \frac{L'(1+\lambda(\alpha^\vee) + \frac{m_i+m_j-2}{2} - \kappa, \sigma_i \times \tilde{\sigma}_j)}{L(1+\lambda(\alpha^\vee) + \frac{m_i+m_j-2}{2} - \kappa, \sigma_i \times \tilde{\sigma}_j)} \\
&+ \sum_{\kappa=0}^{\lfloor\frac{m_i+m_j-2}{2}-1\rfloor} \frac{L'(1-\lambda(\alpha^\vee) - \frac{m_i+m_j-2}{2} + \kappa, \tilde{\sigma}_i \times \sigma_j)}{L(1-\lambda(\alpha^\vee) - \frac{m_i+m_j-2}{2} + \kappa, \tilde{\sigma}_i \times \sigma_j)},
\end{aligned}$$

then $\frac{r'_\alpha(\pi,\lambda(\alpha^\vee))}{r_\alpha(\pi,\lambda(\alpha^\vee))} = \tilde{\mathfrak{L}}_\alpha(\lambda(\alpha^\vee), \pi) + \tilde{\mathfrak{L}}_\alpha(-\lambda(\alpha^\vee), \tilde{\pi})$, and $\tilde{\mathfrak{L}}_\alpha(\pi, \lambda(\alpha^\vee))$ is holomorphic on $\Re\lambda(\alpha^\vee) > 0$, since all zeros are contained in $\Re\lambda(\alpha^\vee) < 0$, but it might have a pole at $\lambda(\alpha^\vee) = 0$. This pole occurs if and only if $\sigma_i \simeq \sigma_j$, and is simple if it occurs. Thus the function

$$\mathfrak{L}_\alpha(\pi, \lambda(\alpha^\vee)) := \tilde{\mathfrak{L}}_\alpha(\pi, \lambda(\alpha^\vee)) - \lambda(\alpha^\vee)^{-1} \operatorname*{res}_{s=0} \tilde{\mathfrak{L}}_\alpha(\pi, s)$$

is holomorphic in $\Re\lambda(\alpha^\vee) \geq 0$. The expression for the case that π_i and π_j are both cuspidal is clear from our computations.

The estimate (32) follows from [Mu07, §4, §5] together with the discussion following the last lemma. $\square$

(32) is a substitute for [Mu02, Theorem 5.3] we need to justify the replacement of the logarithmic derivative of the normalising factors by a sum of products of the functions $\mathfrak{L}_\alpha(z,\pi)$ in the expansion of the trace formula. At first glance it might appear odd, why one would wish to replace $r_L^S(\pi,\lambda)$ by the functions $\mathfrak{L}_\alpha(\lambda,\pi)$. The reason will become obvious later: The definition of $\mathfrak{L}_\alpha(\lambda,\pi)$ ensures that they do not have poles in the right half plane, whereas $r_L^S(\pi,\lambda)$ has infinitely many poles in every direction, when moving away from the purely imaginary subspace $i(\mathfrak{a}_L^S)^* \subseteq (\mathfrak{a}_{L,\mathbb{C}}^S)^*$. This will be later used for the deformation of the contour of certain integrals.

IV.iii.ii An expansion of the spectral side and its absolute convergence

We first recapitulate some notation from [FiLaMu11], but in a slightly more general version, as we shall need it in a relative context, i.e. with respect to Levi subgroups and not just the group G itself. For standard Levi subgroups $S, L \in \mathcal{L}$, $L \subseteq S$, $\operatorname{cork}_S(L) = m$, and $Q \in \mathcal{P}^S(L)$, $\mathfrak{B}_{Q,L}^S$ is the set of all m-tuples $\beta = (\beta_1^\vee, \dots, \beta_m^\vee) \in \Sigma^{S,\vee}(Q)^m$ such that the set $\{\beta_1^\vee, \dots, \beta_m^\vee\}$ forms a basis of $\mathfrak{a}_L^S$ when restricted and projected to $\mathfrak{a}_L^S$. If $S = G$, we shall simply write $\mathfrak{B}_{Q,L}$. If $\beta \in \mathfrak{B}_{Q,L}^S$, $\operatorname{vol}_{\mathfrak{a}_L^S}(\beta)$ is the covolume of the lattice spanned by the restrictions/projections of the vectors $\beta_1^\vee, \dots, \beta_m^\vee$ in $\mathfrak{a}_L^S$. In particular, the splitting formula (29) for the normalising factors reads in this notation

$$r_L^S(\pi_\lambda) = \sum_{\beta \in \mathfrak{B}_{Q,L}^S} \operatorname{vol}_{\mathfrak{a}_L^S}(\beta) \prod_{i=1}^m r'_{\beta_i}(\pi, \lambda(\beta_i^\vee)) r_{\beta_i}(\pi, \lambda(\beta_i^\vee))^{-1}.$$

Denote by $\delta_L^S(\beta, \lambda, \pi)$ the summand corresponding to $\beta \in \mathfrak{B}_{Q,L}^S$. For $\beta \in \mathfrak{B}_{Q,L}^S$, let $\Xi_L^S(\beta) = \{(Q_1, \dots, Q_m) \in \mathcal{F}^S(L)^m \mid \operatorname{cork}_{M_{Q_i}}(L) = 1,\ \beta_i^\vee \in \mathfrak{a}_L^{Q_i}, i = 1, \dots, m\}$. Then for each $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L^S$ there are uniquely determined parabolics $P_1, P_1', \dots, P_m, P_m' \in \mathcal{P}^S(L)$ such that $Q_i = \overline{P_i P_i'}$ and $P_i|^{\beta_i} P_i'$ for $i = 1, \dots, m$ (see [FiLaMu11, §2.1]). We now want to define a non-commutative analogue of the expansion (29) for the normalised intertwining operators analogue to the expansion for the unnormalised intertwining operators in [FiLaMu11]. Let $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, $S \in \mathcal{L}(L)$, $P \in \mathcal{P}(M)$, with

$\mathrm{cork}_G(S) = k \geq 1$, and $\mathcal{X}_S(\beta) \in \Xi_S^G(\beta)$. Then set

$$\tilde{\Delta}_{\mathcal{X}_S(\beta)}(P,\lambda) = \frac{\mathrm{vol}_{\mathfrak{a}_S^G}(\beta)}{k!} R_{P_1|P}(\lambda)^{-1}\left(\frac{d}{d\lambda(\beta_1^\vee)} R_{P_1|P_1'}(\lambda)\right) R_{P_1'|P_2}(\lambda)\cdot$$
$$\cdot \ldots \cdot R_{P_{k-1}|P_k}(\lambda)\left(\frac{d}{d\lambda(\beta_k^\vee)} R_{P_k|P_k'}(\lambda)\right) R_{P_k'|P}(\lambda). \quad (33)$$

The operators $R_{P_i|P_i'}(\lambda)$ are in fact meromorphic and only depend on $\lambda(\beta_i^\vee)$ so that the partial derivatives are ordinary derivatives of meromorphic operators in one variable. From [FiLaMu11, Theorem 2] it follows that there is a non-commutative analogue of (29) for $\mathcal{R}_S(\lambda, P)$ given by

$$\mathcal{R}_S(\lambda,P) = \sum_{\beta\in\mathfrak{B}_{P,S}} \tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)$$

where $\mathcal{X}_{S,\mu}(\beta)$ are certain tuples in $\Xi_S^G(\beta)$ associated to β as in [FiLaMu11, §2.3] and μ is a vector in $(\mathfrak{a}_M^*)^k$ in general position. The vector μ is an auxiliary variable of which the expansion above is basically independent. If $\mu = (\mu_1, \ldots, \mu_k)$, then $\mathcal{X}_{S,\mu}(\beta) = (Q_1, \ldots, Q_k)$ is chosen such that $\tilde{\mu} - \mu_i \in \mathfrak{a}_{Q_i,+}^*$, $i = 1, \ldots, k$, for some uniquely determined $\tilde{\mu} \in \mathfrak{a}_S^*$.

We leave the Levi subgroups fixed and assume $\mathrm{cork}_G(L) = m \geq 1$, and $\mathrm{cork}_S(L) = \nu \in \{0, \ldots, m\}$ in the following. Hence for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ the operator $\mathcal{M}_L(P,\lambda)$ restricted to the space $\mathcal{A}_\pi^2(P)$ can be written as

$$\mathcal{M}_L(P,\lambda)_{|\mathcal{A}_\pi^2(P)} = \sum_{S\in\mathcal{L}(L)} \sum_{\beta\in\mathfrak{B}_{P,S}} \sum_{\alpha\in\mathfrak{B}_{P\cap S,L}^S} \delta_L^S(\alpha,\lambda,\pi)\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(\lambda,P)_{|\mathcal{A}_\pi^2(P)}$$

By [FiLaMu11, Corollary 1] the integral

$$\int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L,\mu}(\beta)}(P,L)M_P(t,0)\rho(P,\lambda,f))d\lambda$$

converges absolutely in the trace norm for all $f \in \mathcal{C}(G(\mathbb{A})^1)$. As the proof of the absolute convergence was reduced to the consideration of the normalising factors and the normalised intertwining operators, it was in fact shown there that

$$\sum_{\beta\in\mathfrak{B}_{P,S}} \sum_{\alpha\in\mathfrak{B}_{P\cap S,L}^S} \sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}^1))} \int_{i(\mathfrak{a}_L^G)^*}$$
$$\delta_L^S(\alpha,\lambda,\pi)\,\mathrm{tr}(\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(\lambda,P)M_P(t,0)\rho_\pi(P,\lambda,f))d\lambda$$

converges absolutely for all $f \in \mathcal{C}(G(\mathbb{A})^1)$. Note that the sums over α and β are both finite.

Each pair $\alpha \in \mathfrak{B}_{P,L}^S$ and $\beta \in \mathfrak{B}_{P,S}$ determines isomorphisms

$$\mathfrak{a}_L^S \oplus \mathfrak{a}_S^G \xrightarrow{\simeq} \mathfrak{a}_L^G, \quad \text{and} \qquad (\mathfrak{a}_L^S)^* \oplus (\mathfrak{a}_S^G)^* \xrightarrow{\simeq} (\mathfrak{a}_L^G)^*$$

with respect to which the integration over $i(\mathfrak{a}_L^G)^*$ decomposes into two parts. Under this isomorphism write $\lambda \in (\mathfrak{a}_L^G)^*$ as $\lambda = \lambda_\alpha + \lambda_\beta$ with $\lambda_\alpha = \sum_{i=1}^{m-k} \lambda(\alpha_i^\vee)\alpha_i \in (\mathfrak{a}_L^S)^*$ and $\lambda_\beta = \sum_{i=1}^{k} \lambda(\beta_i^\vee)\beta_i \in (\mathfrak{a}_S^G)^*$, where all (co-)roots are viewed as elements in $\mathfrak{a}_L^G$ or $(\mathfrak{a}_L^G)^*$. Then the function $\delta_L^S(\alpha, \lambda, \pi)$ only depends on λ_α, and $\tilde{\Delta}_{\mathcal{X}_{S,\mu(\beta)}}(\lambda, P)$ only on λ_β.

By Lemma 41, there are functions $\mathfrak{L}_{\alpha_i}(\cdot, \pi) : \mathbb{C} \longrightarrow \mathbb{C}$ (which implicitly depend on S and L now, but we shall suppress such indices), holomorphic in $\Re\lambda(\alpha_i^\vee) \geq 0$, such that

$$\begin{aligned} \delta_L^S(\alpha, \pi, \lambda) &= \mathrm{vol}_{\mathfrak{a}_L^S}(\alpha) \prod_{i=1,\ldots,\nu} (\mathfrak{L}_{\alpha_i}(\lambda(\alpha_i^\vee), \pi) + \mathfrak{L}_{\alpha_i}(-\lambda(\alpha_i^\vee), \tilde{\pi})) \\ &= \mathrm{vol}_{\mathfrak{a}_L^S}(\alpha) \sum_{\varepsilon \in \{\pm 1\}^\nu} \prod_{i=1,\ldots,\nu} \mathfrak{L}_{\alpha_i}(\varepsilon_i \lambda(\alpha_i^\vee), \varepsilon_i \pi) \end{aligned}$$

with $\varepsilon_i \pi = \pi$ if $\varepsilon_i = 1$, and $\varepsilon_i \pi = \tilde{\pi}$ if $\varepsilon_1 = -1$. For $\eta \in \{\pm 1\}$ put $\eta\mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0}$ if $\eta = 1$, and $\eta\mathbb{R}_{\geq 0} = \mathbb{R}_{\leq 0}$ if $\eta = -1$. Then for $\varepsilon \in \{\pm 1\}^\nu$, the product $\mathfrak{L}(\pi, \alpha, \varepsilon, \lambda) := \mathrm{vol}_{\mathfrak{a}_L^S}(\alpha) \prod_{i=1,\ldots,\nu} \mathfrak{L}_{\alpha_i}(\varepsilon_i \lambda(\alpha_i^\vee), \varepsilon_i \pi)$ is a holomorphic function for all $\lambda \in (\mathfrak{a}_{L,\mathbb{C}}^G)^*$ whose real part is contained in the chamber $\Re\lambda(\alpha_i^\vee) \in \varepsilon_i \mathbb{R}_{\geq 0}$ for $i = 1, \ldots, \nu$. Again, a priori $\mathfrak{L}(\pi, \alpha, \varepsilon, \lambda)$ also depends on the choice of L and S, but since the function in fact only depends on $\lambda(\alpha_i^\vee)$, specifying α and λ suffices.

Proposition 42. *Keep the notation introduced earlier. Denote by $\rho_\pi(P, \lambda, f_s)$ the meromorphic continuation of the operator to all $s \in \mathbb{C}$. Suppose $\eta \in (\mathfrak{a}_L^G)^*$ such that $\eta(\alpha_i^\vee) \in \varepsilon_i \mathbb{R}_{\geq 0}$ for all $i = 1, \ldots, \nu$. Further assume that for all $v \in S$, and all $P, Q \in \mathcal{P}(M)$, the operator $R_{Q|P}(\pi_v, \lambda)$ is holomorphic and zero-free for all $\lambda \in \eta + i(\mathfrak{a}_L^G)^*$, and all $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$. Suppose further that there are $a, b \in \mathbb{R}$ such that for $s \in \mathbb{C}$ with $\Re s \in (a, b)$, $\rho(P, \lambda, f_s)$ is holomorphic for all $\lambda \in \eta + i(\mathfrak{a}_L^G)^*$. Then the sum-integral*

$$\sum_{\pi \in \Pi_{disc}(M(\mathbb{A})^1)} \int_{\eta + i(\mathfrak{a}_L^G)^*} \mathfrak{L}(\pi, \alpha, \varepsilon, \lambda) \operatorname{tr}(\tilde{\Delta}_{\mathcal{X}_{S,\mu(\beta)}}(P, \lambda) \rho_\pi(P, \lambda, f_s)) d\lambda \quad (34)$$

converges absolutely in the trace norm for any $s \in \mathbb{C}$ with $\Re s \in (a, b)$ and uniformely in such s.

Since the local operators are all holomorphic and zero-free at least in the region $|\Re\eta(\gamma^\vee)| \leq \frac{1}{1+n^2}$ by the results of [MuSp04], there is a region given by the intersection of the complexification of a certain Weyl chamber with a small tube around $i(\mathfrak{a}_L^G)^* \subseteq (\mathfrak{a}_{L,\mathbb{C}}^G)^*$ with η satisfying at least the first requirements.

A priori it is not clear that this converges even for $\Re s \gg 0$ and $\eta = 0$, since the function $\mathfrak{L}(\pi, \alpha, \varepsilon, \lambda)$ may grow very fast if the zeros of the Rankin-Selberg L-functions come very close to the lines $1 + \lambda(\alpha_i^\vee) \in 1 + i\mathbb{R}$. The

construction of the original normalising factors took in some way care of this problem, since the zeros almost cancelled each other by the functional equation. (Of course, this as well needs more justification as was given in [Mu02].) It is this point where (32) becomes important.
The Proposition 42 therefore shows the necessary convergence, and yields also the analytic continuation to a slightly larger half plane. We formulated it in a more general context to make it suitable for later application. For the proof of the proposition we first state some simple facts about the matrix coefficients of the induced representation.

Let $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$ so that π is equivalent to $\pi_1 \otimes \ldots \otimes \pi_r$ with $\pi_i \in \Pi_{\text{disc}}(\text{GL}_{n_i}(\mathbb{A})^1)$. Let $\varphi(g) =< \pi(g)\varphi_1, \varphi_2 >$ be a matrix coefficient of the representation induced from π, $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(P)^K$, and put $\tau(s, \lambda, \varphi_1, \varphi_2) =< \rho_\pi(P, \lambda, f_s)\varphi_1, \varphi_2 >$.

We use coordinates on $(\mathfrak{a}_L^G)^*$ as follows: if $\text{cork}_G L = s$, we can identify $(\mathfrak{a}_L^G)^*$ in $\mathbb{R}^s$ with the hyperplane $\lambda_1, \ldots, \lambda_s = 0$, and a basis of $(\mathfrak{a}_L^G)^*$ is then given by the simple roots $e_1 - e_2, \ldots, e_{s-1} - e_s$ for $\{e_1, \ldots, e_s\}$ the standard euclidean basis of $\mathbb{R}^s$. As $M \subseteq L$, there is a canonical embedding $i(\mathfrak{a}_L^G)^* \hookrightarrow i(\mathfrak{a}_M^G)^*$, $\lambda \mapsto \lambda_M$, and λ_M can be given explicitely by $\lambda_M = (\frac{n_1}{m_1}\lambda_1, \ldots, \frac{n_{i_1}}{m_1}\lambda_1, \ldots, \frac{n_r}{m_s}\lambda_s)$ if $L = \text{GL}(m_1) \times \ldots \text{GL}(m_s)$, and $1 = i_0 < i_1 < \ldots < i_s = r$ are so that $n_{i_j+1} + \ldots + n_{i_{j+1}} = m_{j+1}$, $j = 0, \ldots, s-1$.

The following is some kind of multidimensional analogue of Lemma 37.

Lemma 43. *$\tau(s, \lambda, \varphi_1, \varphi_2)$ can be continued to a meromorphic function of $(s, \lambda) \in \mathfrak{a}^*_{L,\mathbb{C}}$, which for $(\Re s, \Re\lambda)$ varying in some compact set, is bounded away from its poles. More precisely, there exists an entire function $G(s, \lambda) : \mathfrak{a}^*_{L,\mathbb{C}} \longrightarrow \mathbb{C}$ such that*

$$\tau(s, \lambda, \varphi_1, \varphi_2) = G(s, \lambda) \prod_{i=1}^r L(s + \lambda_{M,i}, \pi_i),$$

and if $\mathcal{C} \subseteq \mathbb{R} \times (\mathfrak{a}_L^G)^* \simeq \mathfrak{a}_L^*$ is some compact set such that $\tau(s, \lambda, \varphi_1, \varphi_2)$ is holomorphic for all s, λ with $(\Re s, \Re\lambda) \in \mathcal{C}$, there exists a constant $C = C(\Phi, \mathcal{C})$ independent of π and φ such that

$$|\tau(s, \lambda, \varphi)| \leq C(\Phi, \mathcal{C})$$

for all $(s, \lambda) \in \mathfrak{a}^*_{L,\mathbb{C}}$ with $(\Re s, \Re\lambda) \in \mathcal{C}$. Moreover, if π_i is not 1-dimensional for all i, then $\tau(s, \lambda, \varphi_1, \varphi_2)$ continues to an entire function of s and λ. For $\lambda \in i(\mathfrak{a}_L^G)^*$ and $\text{cork}_G M = r \geq 1$, $\tau(s, \lambda, \varphi_1, \varphi_2)$ is holomorphic at least in $\Re s > \frac{n+1-r}{2}$.

Note that if we replace τ by $\tau_t(s, \lambda, \varphi_1, \varphi_2) =< M_{P|P}(t, 0)\rho_\pi(P, \lambda, f_s)\varphi_1, \varphi_2 >$ for some $t \in W^L(M)_{\text{reg}}$ the lemma stays valid up to permutation of the variables λ_{Mi}.

Proof. Note that $\rho_\pi(P,\lambda,f_s) = \rho_{\pi[\lambda_M]}(P,0,f_s)$. Hence a matrix coefficient of $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\pi[\lambda_\mathrm{M}])$ is obtained from one of $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\pi)$ by twisting with $e^{\lambda_M(H_M(\cdot))}$. Then $\tau(s,\lambda,\varphi_1,\varphi_2)$ equals

$$\int_{Z(\mathbb{A})\backslash G(\mathbb{A})}\int_{\mathbf{K}}\int_{A_M M(F)\backslash M(\mathbb{A})} f_s(x)\varphi_1(mkx)e^{(\lambda_M+\rho_M)(H_P(kx))} \cdot e^{-(\lambda_M+\rho_M)(H_P(k))}\overline{\varphi_2(mk)}dmdkdx.$$

The change of variables $x \mapsto k^{-1}x$ yields

$$\int_{Z(\mathbb{A})\backslash G(\mathbb{A})}\int_{\mathbf{K}}\int_{A_M M(F)\backslash M(\mathbb{A})} f_s(k^{-1}x)e^{(\lambda_M+\rho_M)(H_P(x))} \cdot \varphi_1(mx)\overline{\varphi_2(mk)}dmdkdx,$$

Inserting the definitions, this is

$$\int_{M(\mathbb{A})}\int_{U(\mathbb{A})}\int_{A_M M(F)\backslash M(\mathbb{A})}\int_{\mathbf{K}}\int_{\mathbf{K}} |\det m'|^{s+\frac{n-1}{2}}\Phi(k^{-1}m'uk') \cdot e^{(\lambda_M+\rho_M)(H_P(m'))}\varphi_1(mm'uk')\overline{\varphi_2(mk)}dkdk'dmdudm'.$$

As $M(\mathbb{A})$ normalises $U(\mathbb{A})$, $\varphi(muk') = \varphi(mk')$ so that we get

$$\int_{M(\mathbb{A})}\int_{U(\mathbb{A})}\int_{A_M M(F)\backslash M(\mathbb{A})} |\det m'|^{s+\frac{n-1}{2}}e^{(\lambda_M+\rho_M)(H_P(m'))} \int_{\mathbf{K}}\int_{\mathbf{K}} \Phi(k^{-1}m'uk')\varphi_1(mm'k')\overline{\varphi_2(mk)}dkdk'dmdudm'.$$

Assume now that $\Phi \in \mathcal{S}_0(\mathrm{Mat}_{n\times n}(\mathbb{A}), K_\infty K)$ for some finite index subgroup $K_\infty \subseteq \mathbf{K}_\infty$ (we may suppose $K_\infty = \mathbf{K}_\infty$. The general case then follows as the subspace $\mathcal{S}_0(\mathrm{Mat}_{n\times n}(\mathbb{A}), K_\infty K) \subseteq \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ is dense, and for any sequence of Schwartz-Bruhat functions in the $\mathbf{K}$-finite space converging to some arbitrary element in $\mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$, the sequence of associated zeta integrals converges locally uniformely to the zeta function associated with the limit function by [Ja79, (4.5.2), (4.5.3)].
For k, k' fixed the function

$$m' \mapsto \int_{A_M M(F)\backslash M(\mathbb{A})} \varphi_1(mm'k')\overline{\varphi_2(mk)}dm$$

defines a matrix coefficient of π so that by the $\mathbf{K}$-finiteness of π (π is admissible) and our stipulation on Φ_∞ to be $\mathbf{K}_\infty$-finite, the function

$$m \mapsto \int_{\mathbf{K}}\int_{\mathbf{K}} \Phi(k^{-1}muk')\varphi_1(mk')\overline{\varphi_2(k)}dkdk'dudm$$

is a finite sum over ψ, Ψ of

$$\int_{M(\mathbb{A})}\int_{U(\mathbb{A})} e^{(\lambda_M+\rho_M)(H_P(m))}|\det m|^{s+\frac{n-1}{2}}\int_{\mathbf{K}}\int_{\mathbf{K}}\Psi(kmuk')\psi(m)dkdk'dudm$$

for suitable matrix coefficients $\{\psi\}$ of π and Schwartz-Bruhat functions $\{\Psi\}$ on $\mathrm{Mat}_{n\times n}(\mathbb{A})$ as in [Ja79]. By eventually "refining" the sum over Ψ, we may assume that $\Psi(m)=\prod_{i=1}^r\Psi_i(m_i)$ for $m=\mathrm{diag}(m_1,\ldots,m_r)$, $m_i\in\mathrm{GL}_{n_i}(\mathbb{A})$, and suitable $\Psi_i\in\mathcal{S}(\mathrm{Mat}_{n_i\times n_i}(\mathbb{A}))$, and we can also write $\psi(m)=\prod_{i=1}^r\psi_i(m_i)$ accordingly with ψ_i matrix coefficients of π_i, Let

$$\Psi_M(m)=e^{\alpha_M(H_M(m))}\int_{\mathbf{K}}\int_{\mathbf{K}}\int_{U(\mathbb{A})}\Psi(kmuk')dkdk'du.$$

This is a Schwartz-Bruhat function on $\mathrm{Mat}_{n_1\times n_1}(\mathbb{A})\times\ldots\times\mathrm{Mat}_{n_r\times n_r}(\mathbb{A})$ with $\alpha_M=(n-n_1,n-n_1-n_2,\ldots,n_r,0)\in\mathfrak{a}_M^*$.

Hence we get that $\tau(s,\lambda,\varphi_1,\varphi_2)$ equals

$$\sum_{\psi,\Psi}\int_{M(\mathbb{A})}e^{-\alpha_M(H_M(m))}|\det m|^{s+\frac{n-1}{2}}\Psi_M(m)e^{(\lambda+\rho_M)(H_P(m))}\psi(m)dm$$

$$=\sum_{\psi,\Psi}\prod_{i=1}^r\int_{\mathrm{GL}_{n_i}(\mathbb{A})}|\det g_i|^{s+\frac{n_i-1}{2}+\lambda_{Mi}}\Psi_{M,i}(g_i)\psi_i(g_i)dg_i. \quad (35)$$

Integrating the variables $g_1,\ldots,g_r$ one by one, the first assertion follows. The boundednes for $\Re s$, $\Re\lambda$ varying in a compact set, now follows as in the proof of Lemma 37. Since $\mathrm{cork}_G M\geq 1$, we have $n_i\leq n-r$, and hence $\frac{n_i+1}{2}\leq\frac{n+1-r}{2}$. Hence the function is holomorphic at least in $\Re s>\frac{n+1-r}{2}$. □

Proof of Proposition 42. We can proceed along the lines of [FiLaMu11, §5], but have to be more careful, since we have to keep track of the variable s, and additionally are no longer on $i(\mathfrak{a}_L^G)^*$ so that the normalised intertwining operators are not necessarily unitary any more. For each $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ and $\tau\in\widehat{\mathbf{K}_\infty}$ let $d(\pi,\tau)=\dim\mathcal{A}^2_\pi(P)^K_\tau$. Let $\Delta=\mathrm{id}-\Omega_{\mathrm{G}}+2\Omega_{\mathbf{K}_\infty}$. Then the operator $\rho(P,\lambda,\Delta)$ acts on $\mathcal{A}^2_\pi(P)^K_\tau$ by the scalar $\mu(\lambda,\pi,\tau)=1+||\lambda||^2-\lambda_\pi+2\lambda_\tau$, which satisfies $|\mu(\lambda,\pi,\tau)|^2\geq\frac14(1+||\lambda||^2+\lambda_\pi^2+\lambda_\tau^2)$ for any $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ and any $\tau\in\widehat{\mathbf{K}_\infty}$ with $\mathcal{A}^2_\pi(P)^K_\tau\neq 0$ (see [Mu02, (6.2), (6.9)]). Let $\varphi_1,\varphi_2\in\mathcal{A}^2_\pi(P)^K_\tau$ be of norm 1. Then for any λ,s for which the matrix coefficient is holomorphic,

$$\begin{aligned}&|<\tilde\Delta_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)\rho_\pi(P,\lambda,f_s)\varphi_1,\varphi_2>|\\&\quad\leq|\mu(\lambda,\pi,\tau)|^{-M}||\tilde\Delta_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)^K_{\pi,\tau}|||<\rho_\pi(P,\lambda,\Delta^M*f_s)\varphi_1,\varphi_2>|\end{aligned}$$

for any $M\in\mathbb{N}_0$. Here $||\tilde\Delta_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)^K_{\pi,\tau}||$ is the operator norm of the operator $\tilde\Delta_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)$ restricted to $\mathcal{A}^2_\pi(P)^K_\tau$. By Lemma 38 there exists some function $\Phi_M \in \mathcal{S}(\mathrm{Mat}_{n\times n}(\mathbb{A}), K)$ such that $\Delta^M * f_s$ is the same as the function f_s obtained by replacing Φ with Φ_M. Write f_s^M for this function. By the definition of $\tilde{\Delta}_{\mathcal{X}_{S,\mu(\beta)}}(P,\lambda)_\pi$, we have

$$||\tilde{\Delta}_{\mathcal{X}_{S,\mu(\beta)}}(P,\lambda)^K_{\pi,\tau}|| \leq \frac{\mathrm{vol}_{\mathfrak{a}_S^G}(\beta)}{k!}||(R_{P_1|P}(\lambda)^K_{\pi,\tau})^{-1}||||R_{P'_k|P}(\lambda)^K_{\pi,\tau}|| \cdot \prod_{i=1}^{m-\nu-1}||R_{P'_i|P_{i+1}}(\lambda)^K_{\pi,\tau}||\prod_{i=1}^{m-\nu}||R_{P_i|P'_i}(\lambda)^K_{\pi,\tau}||||\varsigma_i(\pi,\lambda)||$$

with $R_{P|Q}(\lambda)^K_{\pi,\tau}$ the restriction of $R_{P|Q}(\lambda)$ to $\mathcal{A}^2_\pi(P)^K_\tau$ and

$$\varsigma_i(\pi,\lambda)=\varsigma_i(\pi,\lambda(\beta_i^\vee)) := (R_{P_i|P'_i}(\lambda)^K_{\pi,\tau})^{-1}\frac{d}{d\lambda(\beta_1^\vee)}R_{P_i|P'_i}(\lambda)^K_{\pi,\tau}$$

the logarithmic derivative. The operators $R_{P|Q}(\lambda)_\pi$ equal $m_{\mathrm{disc}}(\pi)$ copies the products of local operators $\prod_{v\in S} R_{P|Q}(\pi_v,\lambda)$ for any π with $(\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\pi)^{\mathrm{K}} \neq 0$, since for such π, π_v is unramified for $v\notin S$ so that the local intertwining operators act as the identity at places outside S [Ar89, Theorem 2.1 (R_8)]. By the transitivity property [Ar89, Theorem 2.1 (R_2)], each of the local (or global) normalised intertwining operators can be factorised into a product over intertwining operators belonging to adjacent parabolics. Hence by our assumption on η, Lemma 40 and the estimates (30) and (31) thereafter, there are constants $C, C_1 > 0$ such that

$$\frac{\mathrm{vol}_{\mathfrak{a}_S^G}(\beta)}{k!}||R_{P'_k|P}(\lambda)^K_{\pi,\tau}||\prod_{i=1}^{m-\nu-1}||R_{P'_i|P_{i+1}}(\lambda)^K_{\pi,\tau}||\prod_{i=1}^{m-\nu}||R_{P_i|P'_i}(\lambda)^K_{\pi,\tau}|| \leq C_1 d(\pi,\tau)^{N_1}(3+||\tau||)^C$$

for all $\lambda\in\eta+i(\mathfrak{a}_L^G)^*$. By the functional equation together with Lemma 40, there exist $C_2, N_2>0$ such that $||(R_{P_1|P}(\lambda)^K_{\pi,\tau})^{-1}|| \leq C_2(3+||\tau||)^C d(\pi,\tau)^{N_2}$ for all $\lambda\in\eta+i(\mathfrak{a}_L^G)^*$. Thus we have

$$\int_{\eta+i(\mathfrak{a}_L^G)^*}|\mathfrak{L}(\pi,\alpha,\varepsilon,\lambda)||<\tilde{\Delta}_{\mathcal{X}_{S,\mu(\beta)}}(P,\lambda)\rho_\pi(P,\lambda,f_s)\varphi_1,\varphi_2>|d\lambda$$

$$\leq C_1C_2(1+||\tau||)^{2C}d(\pi,\tau)^{N1+N_2}\int_{\eta+i(\mathfrak{a}_L^G)^*}|\mu(\lambda,\pi,\tau)|^{-M}$$

$$\cdot\prod_{\alpha_i\in\alpha}\mathfrak{L}_{\alpha_i}(\varepsilon_i\lambda(\alpha_i),\pi)\prod_{\beta_j\in\beta}||\varsigma_j(\pi,\lambda(\beta_j^\vee))||| <\rho_\pi(P,\lambda,f_s^M)\varphi_1,\varphi_2>|d\lambda.$$

The function $h(\lambda,s) := <\rho(P,\lambda,f_s^M)\varphi_1,\varphi_2>$ can be meromorphically continued to all $s\in\mathbb{C}$ and all $\lambda\in(\mathfrak{a}^G_{L,\mathbb{C}})^*$, and it is bounded for $\Re s, \Re\lambda$ varying in a compact set by Lemma 43 away from its poles. If the continuation of the matrix coefficient $<\rho(P,\lambda,f_s)\varphi_1,\varphi_2>$ is holomorphic at some point, so is $h(\lambda,s)$. Suppose that $a,b\in\mathbb{R}$ are such that $h(\lambda,s)$ is holomorphic for $\lambda \in \eta + i(\mathfrak{a}_L^G)^*$ and $\Re s \in (a,b)$. Hence by Lemma 43, there exists $M(\Phi^M, a, b)$ such that $|h(\lambda, s)| \leq M(\Phi^M, a, b)$ for all s with $\Re s \in (a,b)$, and all $\lambda \in \eta + i(\mathfrak{a}_L^G)^*$, and $M(\Phi^M, a, b)$ is independent of $\pi, \varphi_1, \varphi_2$. Hence the above integral can be bounded by the product of $C_1 C_2 M(\Phi^M, a, b)$ with

$$(1+||\tau||)^{2C} d(\pi,\tau)^{N_1+N_2} \int_{i\mathbb{R}^m} (1+||\eta||^2+||u||^2+\lambda_\pi^2+\lambda_\tau^2)^{-M/2} \cdot \prod_{\alpha_i\in\alpha} \mathfrak{L}_{\alpha_i}(\eta_i(\alpha_i^\vee)+u_i, \varepsilon_i\pi) \prod_{\beta_j\in\beta} ||\varsigma_j(\pi, \eta(\beta_j^\vee)+u_{\nu+j})|| du.$$

As the logarithmic derivative $\varsigma_j(\pi, \lambda(\beta_j^\vee))$ equals the sum of local logarithmic derivatives $\sum_{v\in S} R_{P_i|P_i'}(\pi_v, \lambda(\beta_j^\vee))_\tau^K)^{-1} \frac{d}{d\lambda(\beta_1^\vee)} R_{P_i|P_i'}(\pi_v, \lambda(\beta_j^\vee))_\tau^K$, and for all u we have $(1+||\eta||^2+||u||^2+\lambda_\pi^2+\lambda_\tau^2)^{-M/2} \leq (1+||u||^2+\lambda_\pi^2+\lambda_\tau^2)^{-M/2}$, it suffices to estimate

$$(3+||\tau||)^{2C} \int_{i\mathbb{R}^m} (1+||u||^2+\lambda_\pi^2+\lambda_\tau^2)^{-M/2} \prod_{\alpha_i\in\alpha} \mathfrak{L}_{\alpha_i}(\eta_i(\alpha_i^\vee)+u_i, \varepsilon_i\pi)$$

$$\cdot \prod_{\beta_j\in\beta} ||(R_{P_i|P_i'}(\pi_v, \eta(\beta_j^\vee)+u_{\nu+j})_\tau^K)^{-1} \frac{d}{d\lambda(\beta_1^\vee)} R_{P_i|P_i'}(\pi_v, \eta(\beta_j^\vee)+u_{\nu+j})_\tau^K|| du$$

for each $v \in S$. Again by the assumptions on η and Lemma 40, there exist $C_3, N_3 > 0$ such that $||(R_{P_i|P_i'}(\pi_v, \eta(\beta_j^\vee)+u_{\nu+j})_\tau^K)^{-1}|| \leq C_3 d(\pi,\tau)^{N_3}$, as well as $||R_{P_i|P_i'}(\pi_v, \eta(\beta_j^\vee)+u_{\nu+j})_\tau^K|| \leq C_3 d(\pi,\tau)^{N_3}$, for all $u_{\nu+j} \in i\mathbb{R}$ if v is non-archimedean. If v is an archimedean place, such inequalities stay true if we replace C_3 by $C_3(3+||\tau_v||)^C d(\pi,\tau)^{N_3}$. In particular, we may apply [FiLaMu11, Lemma 1] to the integral over a matrix coefficients of $\frac{d}{d\lambda(\beta_1^\vee)} R_{P_i|P_i'}(\pi_v, \eta(\beta_j^\vee)+u_{\nu+j})_\tau^K$, as they still are rational functions of degree bounded as asserted in [FiLaMu11, Lemma 2]. Hence the integration with respect to the second part $(u_{\nu+1}, \ldots, u_m)$ of the variable u can be bounded by [FiLaMu11, Lemma 1] by a constant multiple (which only depends on M and Φ) of the product of $(3+||\tau||)^{3C} d(\pi,\tau)^{N_1+N_2+N_3+2}$ with

$$\int_{i\mathbb{R}^\nu} (1+||w||^2+\lambda_\pi^2+\lambda_\tau^2)^{-M/2} \prod_{\alpha_i\in\alpha} \mathfrak{L}_{\alpha_i}(\eta_i(\alpha_i^\vee)+w_i, \varepsilon_i\pi) dw$$

for $v \in S$ archimedean as well as non-archimedean. Then the repeated application of (32) shows that this integral is bounded by some constant multiple (which again only depends on M and Φ) of

$$(1+\lambda_\pi^2+\lambda_\tau^2)^{-\frac{M-\nu}{2}} (1+\Lambda_\tau^2)^m (3+||\tau||)^{3C} d(\pi,\tau)^{N_1+N_2+N_3}.$$

The assertion of the lemma now follows from (26), when choosing M sufficiently large. □

Note that the strategy of the proof applies equally well if we replace the operators $\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}$ and scalar functions $\mathfrak{L}(\pi, \alpha, \varepsilon, \lambda)$ by operators and scalar functions satisfying similar factorisation and growth properties.

From the last proposition and [FiLaMu11] we get the following, which is a slight variant of [FiLaMu11, Corollary 1].

Corollary 44. *The spectral side of the trace formula for the test functions* $f \in \mathcal{C}(G(\mathbb{A})^1)$ *can be written as a sum over associate classes of parabolics* $[P] \subseteq \mathcal{F}(T)$, $t \in W(M_P)$, $S \in \mathcal{L}(L_t)$, $\beta \in \mathfrak{B}_{P,S}$, $\alpha \in \mathfrak{B}^S_{P\cap S,L_t}$, *and* $\varepsilon \in \{\pm 1\}^{\mathrm{cork}_S(L_t)}$ *of the product of* $\frac{|\det(1-t)_{\mathfrak{a}_M^{L_t}}|^{-1}}{|W(M_P)|}$ *with*

$$\sum_{\pi\in\Pi_{disc}(M_P(\mathbb{A})^1)} \int_{i(\mathfrak{a}^G_{L_t})^*} \mathfrak{L}(\pi,\alpha,\varepsilon,\lambda)\,\mathrm{tr}(\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(P,\lambda)\rho_\pi(P,\lambda,f))d\lambda$$

and this is absolutely convergent in the trace norm.Here the Levi subgroup $L_t \in \mathcal{L}(M)$ *is uniquely determined by* $\mathfrak{a}_{L_t} = \{X \in \mathfrak{a}_M \mid tX = X\}$.

Here two parabolics are called associated if their Levi component is conjugate. Actually, at first this follows only for the test functions f_s from the last proposition. But restricting our attention to $\eta = 0$, we see that the argument simplifies for general $f \in \mathcal{C}(G(\mathbb{A})^1)$ as in [FiLaMu11, §5]

Another direct consequence is the following corollary finishing the proof of Theorem 33 and the first part of Theorem 32.

Corollary 45. *Suppose* $\mathrm{cork}_G M = r \geq 1$. *Then as a function of* s *the sum-integral* (34) *with* $\eta = 0$ *is well-defined and holomorphic at least in* $\Re s > \frac{n+1-r}{2}$.

IV.iii.iii Meromorphic continuation for $\operatorname{cork}_G L = 1$

For any finite place v, $i(\mathfrak{a}_L^G)^*$ acts on $\Pi_{\mathrm{disc}}(M(F_v)^1)$ by $(\lambda, \pi_v) \mapsto \pi_v[\lambda]$ with stabiliser some cocompact lattice $\mathcal{M}_v \subseteq i(\mathfrak{a}_L^G)^*$, and the local intertwining operators $R_{P|Q}(\pi_v, \lambda)$, $\lambda \in (\mathfrak{a}_{L,\mathbb{C}}^G)^*$ are periodic with respect to $\mathcal{M}_v$. Hence if $R_{P|Q}(\pi_v, \lambda)$ has a pole at $\lambda = \lambda_0$, it also has poles at any $\lambda \in \lambda_0 + \mathcal{M}_v$ with the same residue as at λ_0. Therefore, by the considerations leading to the proof of Lemma 40 and the lemma itself, there is a constant $c > 0$ such that for any finite place $v \in S$, and any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ with $(\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \pi_{\mathrm{v}})^{\mathrm{K_v}} \neq 0$, the residues are bounded by c,

$$|\operatorname*{res}_{\lambda=\lambda_0} < R_{P|Q}(\pi_v, \lambda)\varphi_1, \varphi_2 > | \leq c \tag{36}$$

for all $\varphi_1, \varphi_2 \in (\mathrm{Ind}_{\mathrm{P(F_v)}}^{\mathrm{G(F_v)}} \pi_{\mathrm{v}})^{\mathrm{K_v}}$. (This notation makes sense only when $(\mathfrak{a}_{L,\mathbb{C}}^G)^* \simeq \mathbb{C}$. However, the statement remains true for L arbitrary by using the concept of multidimensional residues as in [Ar89].) Restricting $\Re\lambda$ to a compact subset, there is $c > 0$ such that

$$|\operatorname*{res}_{\lambda=\lambda_0} < R_{P|Q}(\lambda)\varphi_1, \varphi_2 > | \leq c(1 + ||\tau||)^c \tag{37}$$

for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$, $\tau \in \widehat{\mathbf{K}_\infty}$, and $\varphi_1, \varphi_2 \in \mathcal{A}_\pi^2(P)_\tau^K$ and any λ_0 with $\Re\lambda_0$ contained in this compact set, as there are only finitely many τ for which $R_{P|Q}(\pi_\infty, \lambda)_\tau$ has a pole at the respective point.

Let $\mathcal{M} = \bigcup_{v \in S,\ v < \infty} \mathcal{M}_v \subseteq i(\mathfrak{a}_L^G)^*$. Each $\gamma \in \Sigma(P)$ defines a map $i(\mathfrak{a}_L^G)^* \longrightarrow i\mathbb{R}$, $\lambda \mapsto \lambda(\gamma^\vee)$. Let $\gamma(\mathcal{M}) := \{\lambda(\gamma^\vee) \mid \lambda \in \mathcal{M}\} \subseteq i\mathbb{R}$, and $\Sigma_P\mathcal{M} = \bigcup_{\gamma \in \Sigma(P)} \gamma(\mathcal{M}) \subseteq i\mathbb{R}$, which are both discrete sets in $i\mathbb{R}$.

Lemma 46. *Suppose that* $\operatorname{cork}_G M \geq 1$.

(i) There exists a finite collection of functions $\Lambda_k : \mathfrak{a}_L^* \longrightarrow \mathbb{C}$, $k = 1, \dots, K$, $K \leq n$, *with*

$$\Lambda_k(s + \lambda) = \Lambda_k(s + \sum_{i=1}^{\nu} \lambda(\alpha_i^\vee)\varpi_{\alpha_i} + \sum_{j=1}^{m-\nu} \lambda(\beta_j^\vee)\varpi_{\beta_j}) = q_0^k + s + q_1^k \lambda(\varpi_k^\vee)$$

where $q_0^k \in \frac{1}{2}\mathbb{Z}, q_1^k \in \mathbb{Q}$, $|q_0^k| \leq \frac{n}{2}$, $0 < |q_1^k| \leq 1$, *and* ϖ_k *reduced weights for* P_L *such that the following holds: If* $\rho(P, \lambda, f_s)$ *has a pole at* $s + \lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$, *then there exists* $k \in \{1, \dots, K\}$ *with*

$$\Lambda_k(s + \lambda) = 0.$$

(ii) There exists a discrete set $\mathcal{S} \subseteq \mathbb{C}$, *which is finite modulo* $\Sigma_{P_S}\mathcal{M} \cup \mathbb{Z}$, *such that all poles of* $\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(P, \lambda)$ *are along the hyperplanes*

$$\lambda(\gamma^\vee) = x, \qquad \lambda \in (\mathfrak{a}_{L,\mathbb{C}}^G)^*,$$

for $x \in \mathcal{S}$, $\gamma \in \Sigma(P_S)$, *where* $P_S \in \mathcal{P}(S)$ *is such that* $P_S \cap P = P$.

Proof. (i) This is a direct consequence of Lemma 43.

(ii) This follows directly from the fact that each normalised intertwining operator can be written as a product of intertwining operators belonging to adjacent parabolics together with Lemma 40.

□

Suppose that $\operatorname{cork}_G L = 1$. We shall see later why we restrict ourselves to this case. Then all occuring integrals are 1-dimensional, and they are of the form

$$\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \int_{i(\mathfrak{a}_L^G)^*} r'_\beta(\pi,\lambda(\beta^\vee)) r_\beta(\pi,\lambda(\beta^\vee)) \operatorname{tr} M_{P|P}(t,0)\rho_\pi(P,\lambda,f_s)d\lambda \tag{38}$$

and

$$\int_{i(\mathfrak{a}_L^G)^*} \operatorname{tr}(R^{-1}_{P_1|P}(\lambda)\frac{d}{d\lambda(\beta^\vee)} R_{P_1|P_1'}(\lambda)R_{P_1'|P}(\lambda)M_{P|P}(t,0)\rho(P,\lambda,f_s))d\lambda \tag{39}$$

where β is one of the two roots in $(\mathfrak{a}_L^G)^*$ and P_1 and P_1' are adjacent along β. We can identify $(\mathfrak{a}_L^G)^*$ with $\mathbb{R}$ via β.

By Corollary 45, we know that such sum-integrals converges absolutely at least in $\Re s > \frac{n}{2}$.

Lemma 47. *The functions defined by* (38) *and* (39) *can be continued to meromorphic functions on all of* $\mathbb{C}$*.*

We first need a similar result as Lemma 37 for residues of matrix coefficients.

Lemma 48. *Let L and M be as before. Let $\pi \in \Pi_{disc}(M(\mathbb{A})^1)$, $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(P)^K$ of norm 1. Suppose that $h(\lambda,s) :=< M_{P|P}(t,0)\rho(P,\lambda,f_s)\varphi_1,\varphi_2 >$ has a pole at $\lambda = \rho(s) = as+b$, $a,b\in\mathbb{Q}$. Further suppose that $C \subseteq \mathbb{C}$ is a compact region such that $H(s) := \operatorname{res}_{\lambda=\rho(s)} h(\lambda,s)$ is holomorphic for all $s\in C$. Then there exists a constant $c>0$ such that $|H(s)|\le c$ for all $s\in C$ and c can be chosen independently of π and $\rho(s)$.*

Proof. This follows from the fact that a pole can occur only if $\pi_i = 1$ for some i, in which case the residue with respect to this representation is bounded by the integral over the respective part of function Φ or its Fourier transformation. The bound for the remaining part follows as in Lemma 43. □

Proof of Lemma 47. We begin with (38). Suppose $s \in (\frac{n}{2}, \frac{n+1}{2})$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$, $\varphi_1, \varphi_2 \in \mathcal{A}^2_\pi(P)^K$. We begin by splitting up the normalising factor as

$$r'_\beta(\pi,\lambda(\beta^\vee))r_\beta(\pi,\lambda(\beta^\vee)) = \mathfrak{L}_\beta(\lambda(\beta^\vee),\pi) + \mathfrak{L}_\beta(-\lambda(\beta^\vee),\tilde{\pi})$$

and consider the integral

$$\int_{i(\mathfrak{a}_L^G)^*} \mathfrak{L}_\beta(\lambda(\beta^\vee),\pi) < M_P(t,0)\rho(P,\lambda,f_s)\varphi_1,\varphi_2 > d\lambda,$$

which we are allowed to do by Proposition 42. The other integral is similar (we only move the contour in the opposite direction). We first make formal computations and verify later that we were allowed to perform each step. For $\Re\lambda(\beta^\vee) \in \mathbb{R}_{\geq 0}$, $\lambda \in (\mathfrak{a}_{L,\mathbb{C}}^G)^* \simeq \mathbb{C}$, the function $\mathfrak{L}_\beta(\lambda(\beta^\vee),\pi)$ is holomorphic. By Lemma 46, we can choose $\lambda_0 \in \mathbb{R}_{>0}$, sufficiently large such that for all λ with $\Re\lambda > \lambda_0$ and $\Re s \in (\frac{n}{2}, \frac{n+1}{2})$, the operator $\rho(P,\lambda,f_s)$ is holomorphic. We want to apply the residue theorem to deform the contour of integration from $i(\mathfrak{a}_L^G)^*$ to $\lambda_0 + i(\mathfrak{a}_L^G)^*$, and thereby pick up a finite number of residues, namely at the singularities of $\rho(P,\lambda,f_s)$ in the region crossed and such singularities depend linearly on s. As the integral

$$\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \int_{\lambda_0+i(\mathfrak{a}_L^G)^*} \mathfrak{L}_\beta(\lambda(\beta^\vee),\pi)\,\mathrm{tr}(M_P(t,0)\rho(P,\lambda,f_s))d\lambda$$

converges absolutely by Proposition 42 whenever the integrand is a holomorphic function of s, we may move the contour of integration to $\lambda_0 + i(\mathfrak{a}_L^G)^*$ whenever the integrand is holomorphic there. Given s_0, by Lemma 46, we can choose $\lambda_0 \gg 0$ such that the integrand above is holomorphic for all $\Re s \in (s_0, \frac{n+1}{2})$ and thus this integral defines a holomorphic function in this region. Let $\rho(s) \in \mathbb{C}$ be a residue of $\rho(P,\lambda,f_s)$, which is an affine linear function in s. If we can show that

$$\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \mathfrak{L}_\beta(\rho(s),\pi) \operatorname*{res}_{\lambda=\rho(s)} \mathrm{tr}(M_P(t,0)\rho_\pi(P,\lambda,f_s))$$

continues to a meromorphic function on all of $\mathbb{C}$ (or at least for $\Re s < \frac{n+1}{2}$), we are done with (38), since there are only finitely many $\rho(s)$. Each individual summand has a meromorphic continuation as the residue of the matrix coefficient is some zeta function in s as can be seen as in Lemma 43. By the considerations just before the proof of Lemma 40, there is a discrete set $\mathcal{R} \subseteq \mathbb{C}$ outside of which $\mathfrak{L}_\beta(\rho(s),\pi)\,\mathrm{res}_{\lambda=\rho(s)}\,\mathrm{tr}(M_P(t,0)\rho_\pi(P,\lambda,f_s))$ is holomorphic for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ for which $\mathcal{A}_\pi^2(P)^K \neq 0$. In fact, for the finite part of the Rankin-Selberg L-function $L_f(1 \pm \lambda(\beta), \beta(\pi))$ there are only finitely many possibilities, and for the infinite part the Langlands parameters are uniformely bounded. Let $C \subseteq \mathbb{C}\backslash\mathcal{R}$ be some compact set. Thus there exists a constant $C_1 > 0$ such that $|\mathfrak{L}_\beta(\rho(s),\pi)| \leq C_1$ for all $s \in C$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ with $\mathcal{A}_\pi^2(P)^K \neq 0$. Note that

$$\begin{aligned}\operatorname*{res}_{\lambda=\rho(s)} &\mathrm{tr}(M_P(t,0)\rho_\pi(P,\lambda,f_s)_{\mathcal{A}_\pi^2(P)_\tau^K}) \\ &= \mu(\pi,\tau,\rho(s))^{-M} \operatorname*{res}_{\lambda=\rho(s)} \mathrm{tr}(M_P(t,0)\rho_\pi(P,\lambda,f_s^M)_{\mathcal{A}_\pi^2(P)_\tau^K}) \quad (40)\end{aligned}$$

for any $M \in \mathbb{N}$, where we used the notation of the last section, and the function defined by the residue is uniformely bounded in any compact region away from its poles by Lemma 48, i.e. there exists $C_2 > 0$ such that $|\operatorname{res}_{\lambda=\rho(s)} < M_P(t,0)\rho_\pi(P,\lambda,f_s^M)\varphi_1,\varphi_2 > | \leq C_2$ for all $s \in C$, $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$ and $\varphi_1,\varphi_2 \in \mathcal{A}^2_\pi(P)^K$ of norm 1. Hence the above sum can be bounded by

$$\sum_{\pi\in\Pi_{\text{disc}}(M(\mathbb{A})^1)}\sum_{\tau\in\widehat{\mathbf{K}_\infty}} \mu(\pi,\tau,\rho(s))^{-M} C_1C_2 \dim\mathcal{A}^2_\pi(P)^K_\tau$$
$$\leq CC_2 \sum_{\pi\in\Pi_{\text{disc}}(M(\mathbb{A})^1)}\sum_{\tau\in\widehat{\mathbf{K}_\infty}} (1+\lambda_\pi^2+\lambda_\tau^2)^{-M}\dim\mathcal{A}^2_\pi(P)^K_\tau$$

for some other constant $\tilde{C} > 0$ only depending on C_1. This converges by (26) for M sufficiently large, and thus the above sum converges to a meromorphic function.

For (39) again first consider an individual matrix coefficient

$$\int_{i(\mathfrak{a}_L^G)^*} < R^{-1}_{P_1|P}(\lambda)\frac{d}{d\lambda(\beta^\vee)}R_{P_1|P_1'}(\lambda)R_{P_1'|P}(\lambda)M_P(t,0)\rho(P,\lambda,f_s)\varphi_1,\varphi_2 > d\lambda$$

for some $\varphi_1,\varphi_2 \in \mathcal{A}^2_\pi(P)^K_\tau$ of norm 1, π as before and $\tau \in \widehat{\mathbf{K}_\infty}$. Let $\Re s \in (\frac{n}{2},\frac{n}{2}+\delta_0)$ for some $\delta_0 > 0$ (to be determined later).

By Lemma 40, we can choose $\lambda_0 \gg 0$ such that none of the intertwining operators has a pole on $\lambda_0 + i(\mathfrak{a}_L^G)^*$. Given $s_0 < \frac{n+1}{2}$, we can choose by proposition 42 and Lemma 46 some $\lambda_0 \gg 0$ such that the resulting integral is holomorphic in $\Re s \in (s_0,\frac{n+1}{2})$. If we can show that the integral over $\lambda_0 + i(\mathfrak{a}_L^G)^*$ and the sum over all residues in the strip $0 < \Re\lambda < \lambda_0$ converge absolutely, we may apply the residue theorem. There are again only a finite number of poles arising from $\rho(P,\lambda,f_s)$, but there may be infinitely many from the intertwining operators at the finite places. Let $\mathcal{S} \subseteq \mathbb{C}$ be the set of all singularities of the intertwining operators occuring in the integral. This set is discrete and can be chosen independently of π with $\mathcal{A}^2_\pi(P)^K \neq 0$ by Lemma 40. By Lemma 40 $\{\Re z \mid z \in \mathcal{S}\} \cap [0,\lambda_0]$ is a finite set for all $\lambda_0 > 0$, and thus we can find $\delta_0 > 0$ such that the induced operator and the intertwining operators do not have a singularity at the same point $\lambda \in \mathbb{C}$, $\Re\lambda \in [0,\lambda_0]$ for any s with $\Re s \in (\frac{n}{2},\frac{n}{2}+\delta_0)$. Let $\mathcal{S}_{\lambda_0}$ be the set of all $z \in \mathcal{S}$ such that $\Re z \in (0,\lambda_0)$. Consider

$$\sum_{\pi\in\Pi_{\text{disc}}(M(\mathbb{A})^1)}\sum_{z\in\mathcal{S}_{\lambda_0}} \operatorname*{res}_{\lambda=z}\operatorname{tr}(R^{-1}_{P_1|P}(\lambda)\frac{d}{d\lambda(\beta^\vee)}R_{P_1|P_1'}(\lambda)R_{P_1'|P}(\lambda)$$
$$\cdot M_P(t,0)\rho_\pi(P,\lambda,f_s))$$

for which there is a discrete set of singularities outside of which any summand is holomorphic. Using (37) and the usual estimates, we see that this sum converges uniformely for s in any compact subset not intersecting the set of singularities, and hence defines a meromorphic function on $\mathbb{C}$. We are left with the finitely many poles at $\lambda = \rho(s)$ of $\rho(P, \lambda, f_s)$, i.e. a sum

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \operatorname*{res}_{\lambda=\rho(s)} \operatorname{tr}(R^{-1}_{P_1|P}(\lambda) \frac{d}{d\lambda(\beta^\vee)} R_{P_1|P_1'}(\lambda) R_{P_1'|P}(\lambda) \\ \cdot M_P(t, 0) \rho_\pi(P, \lambda, f_s)).$$

Each individual summand has a meromorphic continuation to all $\mathbb{C}$. By Lemma 40 there is a discrete set in $\mathbb{C}$ outside of which all contributing intertwining operators (i.e. those operating on spaces having a K-fixed vector) are pole- and zero-free. Using the usual estimates, it follows that this sum converges locally uniformely (away from its poles). Hence we obtain a meromorphic function also in this case. □

For general L, one might try to continue the distributions $J^L_{M,P}(f_s, t)$ following a similar argument by using multidimensional residues as used by Langlands for the classification of the continuous spectrum [La76]. However, the singular hyperplanes of the induced operator $\rho(P, \lambda, f_s)$ are in general not admissible in the sense of [MoWa95] as they are defined by weight equations $\lambda(\varpi^\vee) = c$ instead of root hyperplanes $\lambda(\beta^\vee) = c$ as the singular hyperplanes of the intertwining operator and normalising factors are. This might lead to serious difficulties.

Assume $\Re s > 1$. We consider only the example $G = \mathrm{GL}(3)$, $F = \mathbb{Q}$, $T = L = M$, Φ_f is the characteristic function of $\mathrm{Mat}_{3\times3}(\hat{\mathbb{Z}}) \subseteq \mathrm{Mat}_{3\times3}(\mathbb{A}_f)$, and $\Phi_\infty(x) = e^{-\pi \operatorname{tr} x_\infty^t x_\infty}$. Even in this simple case the above mentioned problem occurs. Moreover, $P = P_0$, $\alpha = \{\beta_1, \beta_2\}$ the set of simple roots, and $\varepsilon = (1, 1)$. Hence the only π contributing to the sum over $\pi \in \Pi_{\mathrm{disc}}(T(\mathbb{A})^1)$ is $\pi = 1 \otimes 1 \otimes 1$ and we may assume $S = \emptyset$. (Even though S was always supposed to contain the archimedean places, we may suppose $S = \emptyset$, as Φ_∞ is of a special form, i.e. in particular $\mathbf{K}_\infty$-biinvariant, and thus the local intertwining operator acts trivially.) Hence the operators $\tilde{\Delta}_{\mathcal{X}_{S,\mu}(\beta)}(\lambda, P)$ all vanish, and there is only $r^G_T(\pi, \lambda)$ left. The trace equals $\operatorname{tr} \rho_{1\otimes1\otimes1}(P, \lambda, f_s) = \zeta^*(s + \lambda_1)\zeta^*(s + \lambda_2 - \lambda_1)\zeta^*(s - \lambda_2)$ for $\lambda = \lambda_1\beta_1 + \lambda_2\beta_2$. Splitting $r^G_T(\pi, \lambda)$ up as usual, the function initially defined for $\Re s > 2$ by

$$\int_{i(\mathfrak{a}^G_T)^*} \mathfrak{L}(1 \otimes 1 \otimes 1, \alpha, \varepsilon, \lambda)\zeta^*(s + \lambda_1)\zeta^*(s + \lambda_2 - \lambda_1)\zeta^*(s - \lambda_2) d\lambda$$

seems to have no continuation to all $s \in \mathbb{C}$. Computing $\mathfrak{L}$ explicitely, we get the integral

$$\int_{i\mathbb{R}} \int_{i\mathbb{R}} (\frac{\zeta^{*\prime}(1 + \mu_1)}{\zeta^*(1 + \mu_1)} - \frac{1}{\mu_1})(\frac{\zeta^{*\prime}(1 + \mu_2)}{\zeta^*(1 + \mu_2)} - \frac{1}{\mu_2}) \\ \zeta^*(s + \frac{1}{3}(2\mu_1 + \mu_2))\zeta^*(s + \frac{1}{3}(\mu_2 - \mu_1))\zeta^*(s - \frac{1}{3}(\mu_1 + 2\mu_2)) d\mu_1 d\mu_2$$

where we changed variables to $\mu_1 = 2\lambda_1 - \lambda_2$, and $\mu_2 = 2\lambda_2 - \lambda_1$. Both factors involving the logarithmic derivatives are holomorphic for $\Re\mu_i \geq 0$. By Proposition 42 the integral is still convergent if we replace $i\mathbb{R}$ with $\eta_j + i\mathbb{R}$, $j = 1, 2$, such that the integrand is holomorphic on that subspace. Choose $\varepsilon > 0$ very small and irrational. We can consider the integral over $\mu_2 \in \varepsilon + i\mathbb{R}$, $\mu_1 \in i\mathbb{R}$ instead, as there are no singularities in the strip $0 \leq \Re\mu_2 \leq \varepsilon$, and the resulting integral still converges absolutely by proposition 42. Let $a_1 \gg 0$. We first deform the the contour $\mu_1 \in i\mathbb{R}$ to $\mu_1 \in a_1 + i\mathbb{R}$ and apply the residue theorem while $\mu_2 \in \varepsilon + i\mathbb{R}$ is temporarily fixed. The only poles in the strip crossed arise from the product of the zeta functions, but not from their logarithmic derivatives. We have $\Re(s + \frac{1}{3}(2\mu_1 + \mu_2)) > 0$ and thus for such μ_1, μ_2 and s, there are at most singularities at

$$s + \frac{1}{3}(\mu_1 + 2\mu_2) - \frac{1}{3}(2\mu_1 + \mu_2) = c \Leftrightarrow \mu_1 = 3s + \mu_2 - c$$

and at

$$s - \frac{1}{3}(\mu_1 + 2\mu_2) = c \Leftrightarrow \mu_1 = 3s - 2\mu_2 - 3c$$

with $c \in \{0, 1\}$. Hence we obtain the sum of

$$\int_{\varepsilon+i\mathbb{R}} \int_{a_1+i\mathbb{R}} (\frac{\zeta^{*\prime}(1+\mu_1)}{\zeta^*(1+\mu_1)} - \frac{1}{\mu_1})(\frac{\zeta^{*\prime}(1+\mu_2)}{\zeta^*(1+\mu_2)} - \frac{1}{\mu_2})$$
$$\zeta^*(s + \frac{1}{3}(2\mu_1 + \mu_2))\zeta^*(s + \frac{1}{3}(\mu_2 - \mu_1))\zeta^*(s - \frac{1}{3}(\mu_1 + 2\mu_2))d\mu_1 d\mu_2$$

with

$$\sum_{c=0,1} \int_{\varepsilon+i\mathbb{R}} (\frac{\zeta^{*\prime}(1+3s-2\mu_2-3c)}{\zeta^*(1+3s-2\mu_2-3c)} - \frac{1}{3s-2\mu_2-3c})(\frac{\zeta^{*\prime}(1+\mu_2)}{\zeta^*(1+\mu_2)} - \frac{1}{\mu_2})$$
$$\zeta^*(s + \frac{1}{3}(2(3s-2\mu_2-3c)+\mu_2))\zeta^*(2s - c - \frac{1}{3}(2(3s-2\mu_2-3c)+\mu_2))$$
$$\cdot \operatorname*{res}_{\sigma=s-c} \zeta^*(s-\sigma)d\mu_2 \quad (41)$$

and

$$\sum_{c=0,1} \int_{\varepsilon+i\mathbb{R}} (\frac{\zeta^{*\prime}(1+3s+\mu_2-c)}{\zeta^*(1+3s+\mu_2-c)} - \frac{1}{3s+\mu_2-c})(\frac{\zeta^{*\prime}(1+\mu_2)}{\zeta^*(1+\mu_2)} - \frac{1}{\mu_2})$$
$$\zeta^*(3s + \mu_2 - \frac{2}{3}c)) \operatorname*{res}_{\sigma=c-s} \zeta^*(s+\sigma)\zeta^*(\mu_2 - \frac{1}{3}c))d\mu_2 \quad (42)$$

By our assumption on ε there are no poles of the integrands on the line $\varepsilon + i\mathbb{R}$ so that all integrals are well-defined. Moving in the first summand the integration $\mu_2 \in \varepsilon + i\mathbb{R}$ to $\mu_2 \in a_2 + i\mathbb{R}$, $a_2 \gg 0$, yields a similar sum.

Choosing $a_1 \gg a_2 \gg 2$, the resulting double integral over $\mu_1 \in a_1 + i\mathbb{R}, \mu_2 \in a_2 + i\mathbb{R}$ is then by Proposition 42 holomorphic at least in $\Re s \in (-a+1, 2)$

with $a = \min\{a_1 - a_2, \frac{1}{3}(a_1 + 2a_2)\}$.
The zeta functions in (41) have singularities at most at

$$s + \frac{1}{3}(2(3s - 2\mu_2 - 3c) + \mu_2) = d \Leftrightarrow \mu_2 = 3s - 2c - d$$

and at

$$s + (s - c) - \frac{1}{3}(2(3s - 2\mu_2 - 3c) + \mu_2) = d \Leftrightarrow \mu_2 = d - c$$

for $d \in \{0, 1\}$. Let $a_2 \gg 0$. Moving the integral in (41) to $\mu_2 \in a_2 + i\mathbb{R}$ we get the sum of (provided all occuring sum-integrals do converge)

$$\int_{a_2+i\mathbb{R}} (\frac{\zeta^{*\prime}(1 + 3s - 2\mu_2 - 3c)}{\zeta^*(1 + 3s - 2\mu_2 - 3c)} - \frac{1}{3s - 2\mu_2 - 3c})(\frac{\zeta^{*\prime}(1 + \mu_2)}{\zeta^*(1 + \mu_2)} - \frac{1}{\mu_2})$$
$$\zeta^*(s + \frac{1}{3}(2(3s - 2\mu_2 - 3c) + \mu_2))\zeta^*(2s - c - \frac{1}{3}(2(3s - 2\mu_2 - 3c) + \mu_2))$$
$$\cdot \operatorname*{res}_{\sigma=s-c} \zeta^*(s - \sigma) d\mu_2,$$

(which converges absolutely for all $\Re s < 2$ by Proposition 42 if we choose a_2 sufficiently large) and

$$\sum_{d=0,1} (\frac{\zeta^{*\prime}(1 - 3s + c + 2d)}{\zeta^*(1 - 3s + c + 2d)} - \frac{1}{-3s + c + 2d})$$
$$\cdot (\frac{\zeta^{*\prime}(1 + 3s - 2c - d)}{\zeta^*(1 + 3s - 2c - d)} - \frac{1}{1 + 3s - 2c - d})$$
$$\cdot \operatorname*{res}_{\sigma=d} \zeta^*(\sigma) \operatorname*{res}_{\sigma=c} \zeta^*(\sigma)\zeta^*(3s - c - d),$$

which defines a meromorphic function on all of $\mathbb{C}$, and

$$\sum_{d=0,1} (\frac{\zeta^{*\prime}(1 + 3s - 2d - c)}{\zeta^*(1 + 3s - 2d - c)} - \frac{1}{3s - 2d - c})(\frac{\zeta^{*\prime}(1 + d - c)}{\zeta^*(1 + d - c)} - \frac{1}{d - c})$$
$$\cdot \operatorname*{res}_{\sigma=d} \zeta^*(\sigma) \operatorname*{res}_{\sigma=c} \zeta^*(\sigma)\zeta^*(3s - c - d),$$

which again defines a meromorphic function on $\mathbb{C}$ (not all summands occur for all d depending on the value of c), and finally the infinite sum

$$\sum_\rho m(\rho)(\frac{\zeta^{*\prime}(\frac{1}{2}(3 + 3s - 3c - \rho))}{\zeta^*(\frac{1}{2}(3 + 3s - 3c - \rho))} - \frac{1}{\frac{1}{2}(3 + 3s - 3c - \rho)})$$
$$\zeta^*(\frac{1}{2}(3s + \rho - c - 1))\zeta^*(\frac{1}{2}(3s + 1 - \rho - c)) \operatorname*{res}_{\sigma=s-c} \zeta^*(s - \sigma)$$

with ρ ranging over all zeros of ζ^* and $m(\rho)$ denotes the multiplicity of this zero. For $c = 1$ this is a well-defined sum of meromorphic functions, which converges to a meromorphic function on all of $\mathbb{C}$ (with infinitely many poles). If $c = 0$ however, we have

$$\sum_\rho m(\rho)\Big(\frac{\zeta^{*\prime}(\frac{1}{2}(3+3s-\rho))}{\zeta^*(\frac{1}{2}(3+3s-\rho))} - \frac{1}{\frac{1}{2}(3+3s-\rho)}\Big) \cdot \zeta^*(\frac{1}{2}(3s+\rho-1))\zeta^*(\frac{1}{2}(3s+1-\rho)) \operatorname*{res}_{\sigma=s} \zeta^*(s-\sigma)$$

and there is no cancelation occuring for the infinitely many zeros of the zeta function $\zeta^*(\frac{1}{2}(3+3s-\rho))$, and hence this sum has singularities at the points $s = \frac{1}{3}(2\rho' + \rho - 3)$ for ρ, ρ' zeros of ζ^* (such points satisfy $-1 < \Re s < 0$). But this set has accumulation points so that the sum does not converge to a meromorphic function. Therefore the original integral can not be continued to all of $\mathbb{C}$ using this method, but only to $\Re s > 0$.
Note that since we take the residue for $c = 0$, the residue is of the form

$$\int_{\mathbb{A}}\int_{\mathbb{A}}\int_{U_0(\mathbb{A})} \Phi(\mathrm{diag}(t_1, t_2, 0)u)|t_1|^{c_1}|t_2|^{c_2} dt_i du$$

for suitable c_1, c_2 with $\Re c_1, \Re c_2 > 2$, and hence is an integral over the singular matrices. We shall see in the next section that such distribution supported on the singular matrices already occur for GL(2).

The continuation of the term in (42), however, does not lead to any difficulties: Moving the integral in (42) to $a_2 + i\mathbb{R}$ yields a sum of residues at the finitely many points, where the product of zeta functions has poles, but the logarithmic derivatives again do not contribute. Hence for this term we get a meromorphic continuation to all of $\mathbb{C}$.

IV.iv Example: $G = \mathrm{GL}(2)$

The discussion simplifies insofar as the only non-trivial Levi subgroup L is the torus T so that apart from the discrete spectrum belonging to G and T, only the corank 1 case occurs. There are only the following terms (24): $J^G_{G,G}(f_s, 1)$, $J^G_{T,B}(f_s, 1)$, $J^G_{T,\overline{B}}(f_s, 1)$, $J^G_{T,B}(f_s, w)$, $J^G_{T,\overline{B}}(f_s, w)$, $J^T_{T,B}(f_s, 1)$, and $J^T_{T,\overline{B}}(f_s, 1)$, where $B \subseteq \mathrm{GL}(2)$ denotes the Borel subgroup of upper triangular matrices and $\overline{B}$ its opposite, and $w \in W^G$ is the unique non-trivial element. Note that $J^G_{T,B}(f_s, w) = J^G_{T,\overline{B}}(f_s, 1)$.

IV.iv.i The "discrete" spectrum

From the results on the discrete spectrum (and its generalisations to arbitrary Levis M) we can read off the following:

Corollary 49. *The functions $J^G_{G,G}(f_s, 1)$, $J^G_{T,B}(f_s, 1)$, and $J^G_{T,B}(f_s, w)$ have meromorphic continuations to all $s \in \mathbb{C}$. $J^G_{G,G}(f_s, 1)$ has simple poles at $s = \frac{3}{2}, -\frac{1}{2}$, a pole of second order at $s = \frac{1}{2}$, and is holomorphic elsewhere. The residue at $s = \frac{3}{2}$ is*

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1) \int_{\mathrm{Mat}_{2\times 2}(\mathbb{A})} \Phi(x) dx,$$

and at $s = -\frac{1}{2}$ it is

$$-\,\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)\Phi(0).$$

The functions $J^G_{T,B}(f_s, 1)$, $J^G_{T,B}(f_s, w)$ have poles of second order at $s = 1$ and $s = 0$, and are holomorphic elsewhere.

Proof. The meromorphic continuation, and the location and order of the poles follow from Corollary 35 and 36. Up to an entire functione, $J^G_{G,G}(f_s, 1)$ equals

$$Z(\Phi, s + \frac{2-1}{2}, 1) = \int_{\mathrm{GL}_2(\mathbb{A})} \Phi(x) |\det x|^{s+\frac{1}{2}} dx = Z(\Phi, s + \frac{1}{2})$$

with the second expression valid for $\Re s > \frac{3}{2}$. By the functional equation for the zeta function, the residues at the last pole follows. □

Note that the function $J^G_{T,B}(f_s, 1) + J^G_{T,B}(f_s, w)$ is given by [GeJa79, (6.37)] for $\Re s > \frac{3}{2}$.

IV.iv.ii The continuous spectrum

For $\Re s > \frac{3}{2}$, the function $a(T,T,\mathrm{id})(\mathrm{J}^{\mathrm{T}}_{\mathrm{T,B}}(\mathrm{f_s},1)+\mathrm{J}^{\mathrm{T}}_{\mathrm{T,\overline{B}}}(\mathrm{f_s},1))$ is the continuous spectrum given by [GeJa79, (6.36)]. Using the local expression [GeJa79, (7.13)], in which the sum reduces to $\chi = 1$ as we only consider the trivial central character, we get the sum of

$$\frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{r'(\sigma)}{r(\sigma)} \operatorname{tr} I(\sigma, f_s) d\sigma$$

and

$$\frac{1}{2\pi i}\sum_{u\in S}\int_{i\mathbb{R}} \operatorname{tr}(R_u(\sigma)^{-1}R_u'(\sigma)I_u(\sigma,f_s))(\prod_{v\neq u}\operatorname{tr} I_v(\sigma,f_s))d\sigma.$$

Here $r(\sigma)$ denotes the normalising factor for $\pi = 1\otimes 1$ so that we simply have $r(\sigma) = \frac{\zeta_F^*(1-2\sigma)}{\zeta_F^*(1+2\sigma)}$, and $R(\sigma)$ denotes the normalised intertwining operator $\mathcal{A}^2_{1\otimes 1}(B) \longrightarrow \mathcal{A}^2_{1\otimes 1}(B)$ with $R_u(\sigma)$ its local version. Moreover, we write $I(\sigma, f_s)$ for the induced operator $\rho_{1\otimes 1}(B,\sigma,f_s)$ and $I_v(\sigma,f_s)$ for the respective local operator. The space $(\mathfrak{a}^G_{T,\mathbb{C}})^*$ here is identified with $\mathbb{C}$ in the canonical way via the root $\alpha = (1,-1)$.

We know that each of the terms has a meromorphic continuation to all $s\in\mathbb{C}$, but we want to analyse the integral involving the logarithmic derivative of the normalising factor $r(\sigma)$ more closely to find a specific function as a part of it. This function will later also occur on the geometric side, and plays a special role there.

Let $s_0\in\mathbb{R}$, $\sigma_0 \gg 0$. The poles of $\operatorname{tr} I(\sigma, f_s)$ lie at $\sigma = \pm s$ and $\sigma = \pm(s-1)$ so that for $\Re s > \frac{3}{2}$

$$\begin{aligned}\frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{r'(\sigma)}{r(\sigma)} \operatorname{tr} I(\sigma, f_s) d\sigma &= \frac{1}{\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \mathfrak{L}(1+2\sigma)\operatorname{tr} I(\sigma,f_s)d\sigma \\ -2\left(\frac{\zeta_F^{*\prime}(2s-1)}{\zeta_F^*(2s-1)} + \frac{1}{2(s-1)}\right)&\operatorname*{res}_{\sigma=s-1}\operatorname{tr} I(\sigma,f_s) \\ &-2\left(\frac{\zeta_F^{*\prime}(2s+1)}{\zeta_F^*(2s+1)}+\frac{1}{2s}\right)\operatorname*{res}_{\sigma=s}\operatorname{tr} I(\sigma,f_s) \qquad (43)\end{aligned}$$

with $\mathfrak{L}(1+2\sigma) = \frac{\zeta_F^{*\prime}(2\sigma+1)}{\zeta_F^*(2\sigma+1)} + \frac{1}{2\sigma}$. The first integral on the right hand side now defines a holomorphic function for $\Re s > 1-\sigma_0$. It even defines an entire function since we may shift the contour of integration further right without picking up any additional residues (since there are no more). For $\Re s \gg 0$ the residues can be computed to be as follows:

$$\operatorname*{res}_{\sigma=s-1}\operatorname{tr} I(\sigma,f_s) = \operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^2}\int_{\mathbb{A}^\times}\Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & x \\ & t_2\end{pmatrix})|t_1|^{2s-1}d^\times t_1 dt_2 dx$$

$$\operatorname*{res}_{\sigma=s}\operatorname{tr} I(\sigma,f_s) = -\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}\Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & x \\ & 0\end{pmatrix})|t_1|^{2s}d^\times t_1 dx.$$

Note that for any finite (or empty) set of places P

$$\lim_{s\to 0}\left[\frac{\zeta_F^{P\prime}(1+s)}{\zeta_F^P(1+s)}+\frac{1}{s}\right]=\lim_{s\to 0}\frac{s\zeta_F^{P\prime}(1+s)+\zeta_F^P(1+s)}{s\zeta_F^P(1+s)}=\lambda_P$$

with $\lambda_P=\lambda_{1,P}$ the constant defined in [FiLa11a] in order to modify the geometric terms. In particular, $\lim_{s\to 0}\mathfrak{L}(1+2s)=\frac{\lambda_0}{\lambda_{-1}}$ for $\frac{\lambda_0}{\lambda_{-1}}=\frac{[(s-1)\zeta_F^*(s)]'_{s=1}}{\operatorname{res}_{s=1}\zeta_F^*(s)}$ and $\zeta_F^*(s)=(s-1)^{-1}\lambda_{-1}+\lambda_0+(s-1)\lambda_1+\dots$.

We denote by T a certain distribution on the space of Schwartz-Bruhat functions in two variables, which is defined in [Yu92, Da96], see (48). It defines a meromorphic function on all of $\mathbb{C}$, but has infinitely many poles. This T appears on the geometric side as the term leftover from the hyperbolic contribution, and it is responsible for the ifninitely many poles there.

Proposition 50. *The integral* $\varphi(s)=\frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{r'(\sigma)}{r(\sigma)}\operatorname{tr}I(\sigma,f_s)d\sigma$ *defines a holomorphic function for* $\Re s>1$. $\varphi(s)$ *can be meromorphically continued to a function on* $\mathbb{C}$ *such that* $\varphi(s)-T(2s+1,\Phi((\begin{smallmatrix}*&*\\0&0\end{smallmatrix}))$ *can be written as the sum of a meromorphic function, which is the analytic continuation of*

$$-\int_{\mathbb{A}}\int_{\mathbb{A}^\times}\Phi_{\mathbf{K}}(\begin{pmatrix}t&tx\\0&0\end{pmatrix})|t|^{2s+1}(\log||(1,x)||_S \\ -\frac{\zeta_{F,S}'(1-2s)}{\zeta_{F,S}(1-2s)}-\frac{\zeta_{F,S}'(1+2s)}{\zeta_{F,S}(1+2s)})d^\times t dx, \quad (44)$$

and a second meromorphic function, which has a pole of second order at $s=\frac{1}{2}$*, simple poles at* $s=1,0$*, and is holomorphic elsewhere. The residues at the simple poles of this function are*

$$-\operatorname{vol}(T(F)\backslash T(\mathbb{A})^1)\frac{\lambda_0}{\lambda_{-1}}\int_{\mathbb{A}^3}\Phi_{\mathbf{K}}((\begin{smallmatrix}x&y\\0&z\end{smallmatrix}))dx$$

at $s=1$*, and*

$$-\operatorname{vol}(T(F)\backslash T(\mathbb{A})^1)\frac{\lambda_0}{\lambda_{-1}}\int_{\mathbb{A}}\Phi_{\mathbf{K}}((\begin{smallmatrix}0&x\\0&0\end{smallmatrix}))dx$$

at $s=0$*, and the main part of the Laurent expansion at* $s=\frac{1}{2}$ *equals*

$$-(s-\frac{1}{2})^{-2}\operatorname{vol}(T(F)\backslash T(\mathbb{A})^1)\int_{\operatorname{Mat}^0_{2\times 2}(\mathbb{A})}\Phi_{\mathbf{K}}(x)dx \\ -(s-\frac{1}{2})^{-1}\operatorname{vol}(T(F)\backslash T(\mathbb{A})^1) \\ \cdot\int_{\operatorname{Mat}^0_{2\times 2}(\mathbb{A})}(\widehat{\Phi_{\mathbf{K}}}(x)+\Phi_{\mathbf{K}}(x))(\log|\operatorname{tr}x|_S+2\lambda_{1,S})dx \quad (45)$$

with $\operatorname{Mat}^0_{2\times 2}(\mathbb{A})=\{x\in\operatorname{Mat}_{2\times 2}(\mathbb{A})|\det x=0\}$.

The measure on $\mathrm{Mat}^0_{2\times 2}(\mathbb{A})$ is chosen as follows: for $A \in \mathrm{Mat}^0_{2\times 2}(\mathbb{A})\backslash\{0\}$ there exists by the polar decomposition a unique (up to units in $\prod_{v\leq\infty} \mathcal{O}_v$) $k \in \mathbf{K}$ such that $kA = \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ for unique (again up to units) $(x, y) \in \mathbb{A}^2\backslash\{(0,0)\}$. Then $kAk^{-1} \in \{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$. On the other hand, $k^{-1}\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)k \in \mathrm{Mat}^0_{2\times 2}(\mathbb{A})$ for any x, y, k. We then take the measure induced from $\mathbf{K} \times \mathbb{A}^2 \longrightarrow \mathrm{Mat}^0_{2\times 2}(\mathbb{A})$, cf. also [Co83, §9] for a similar construction.

Proof. We ignore the common factor $\mathrm{vol}(F^\times\backslash\mathbb{A}^1)$ during the proof. The last term of (43) equals

$$2\left(\frac{\zeta_F^{*\prime}(2s+1)}{\zeta_F^*(2s+1)} + \frac{\zeta_F^{*\prime}(1-2s)}{\zeta_F^*(1-2s)}\right)\int_{\mathbb{A}}\int_{\mathbb{A}^\times} \Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & t_1 x \\ 0 & 0 \end{pmatrix})|t_1|^{2s+1} d^\times t_1 dx$$
$$+ 2\left(\frac{1}{2s} - \frac{\zeta_F^{*\prime}(1-2s)}{\zeta_F^*(1-2s)}\right) Z(\Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & x \\ 0 & 0 \end{smallmatrix}\right)), 2s),$$

with the global zeta function $Z(\Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & x \\ 0 & 0 \end{smallmatrix}\right), 2s) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}} \Phi_{\mathbf{K}}(\left(\begin{smallmatrix} t & x \\ 0 & 0 \end{smallmatrix}\right))|t|^{2s} d^\times t dx$. The first summand can be written as

$$2\left(\frac{\zeta_F^{*\prime}(2s+1)}{\zeta_F^*(2s+1)} + \frac{\zeta_F^{*\prime}(1-2s)}{\zeta_F^*(1-2s)}\right) T(2s+1, 0, \Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)).$$

Using [Yu92, Proposition (2.12) (2)] this is

$$T(2s+1, \Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right))) - \zeta_F^S(2s) T_S(2s+1, \Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)))$$
$$- \zeta_F^S(2s) T_S(2s+1, 0, \Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)))\left(-\frac{\zeta_{F,S}'(1-2s)}{\zeta_{F,S}(1-2s)} - \frac{\zeta_{F,S}'(1+2s)}{\zeta_{F,S}(1+2s)}\right)$$

for S a sufficiently large finite set of places of F. The different distributions are again defined in [Yu92, Definition (2.7)]. The last two summands can be written as the product of $-\zeta_F^S(2s)$ with

$$\int_{\mathbb{A}_S}\int_{\mathbb{A}_S^\times} \Phi_{S,\mathbf{K}_S}(\begin{pmatrix} t & tx \\ 0 & 0 \end{pmatrix})|t|_S^{2s+1}(\log ||(1,x)||_S$$
$$- \frac{\zeta_{F,S}'(1-2s)}{\zeta_{F,S}(1-2s)} - \frac{\zeta_{F,S}'(1+2s)}{\zeta_{F,S}(1+2s)}) d^\times t dx$$

and we have $\frac{\zeta_{F,S}'(1-2s)}{\zeta_{F,S}(1-2s)} + \frac{\zeta_{F,S}'(1+2s)}{\zeta_{F,S}(1+2s)} = \frac{1}{2}\sum_{v\in S} \frac{r_v'(\sigma)}{r_v(\sigma)} = \frac{1}{2}\frac{r_S'(\sigma)}{r_S(\sigma)}$. Note that

$$2\left(\frac{1}{2s} - \frac{\zeta_F^{*\prime}(1-2s)}{\zeta_F^*(1-2s)}\right) Z(\Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & x \\ 0 & 0 \end{smallmatrix}\right)), 2s)$$
$$= 2\left(\frac{1}{2s} + \frac{\zeta_F^{*\prime}(2s)}{\zeta_F^*(2s)}\right) Z(\Phi_{\mathbf{K}}(\left(\begin{smallmatrix} * & x \\ 0 & 0 \end{smallmatrix}\right)), 2s)$$

and $Z(\Phi_{\mathbf{K}}((\begin{smallmatrix} * & x \\ 0 & 0 \end{smallmatrix})), 2s)/\zeta_F^*(2s)$ is an entire function. Hence this function only has a pole of second order at $s = \frac{1}{2}$, and a simple one at $s = 0$. The main part of the Laurent expansion at $s = \frac{1}{2}$ is the product of $\mathrm{vol}(F^\times\backslash\mathbb{A}^1)$ with

$$-(s-\frac{1}{2})^{-2}\frac{1}{2}\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} t & x \\ 0 & 0 \end{pmatrix})dtdx$$

$$+(s-\frac{1}{2})^{-1}(2+\frac{[\zeta_F^*(s)(s-1)]'_{s=1}+[\zeta_F^*(s)(s-1)]''_{s=1}}{\mathrm{res}_{s=1}\,\zeta_F^*(s)})\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} t & x \\ 0 & 0 \end{pmatrix})dtdx$$

$$-(s-\frac{1}{2})^{-1}\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} t & x \\ 0 & 0 \end{pmatrix})(\log|t|_S+\lambda_{1,S})dtdx, \tag{46}$$

and the residue at $s = 0$ is

$$-\frac{\lambda_0}{\lambda_{-1}}\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\Phi_{\mathbf{K}}(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})dx.$$

For the last residual term note that again $Z(\Phi_{\mathbf{K}}((\begin{smallmatrix} * & x \\ 0 & t_2 \end{smallmatrix})), 2s-1)/\zeta_F^*(2s-1)$ is an entire function with

$$Z(\Phi_{\mathbf{K}}((\begin{smallmatrix} * & x \\ 0 & t_2 \end{smallmatrix})), 2s-1) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix})|t_1|^{2s-1}dxdt_2d^\times t_1.$$

Thus the function

$$-2\left(\frac{\zeta_F^{*\prime}(2s-1)}{\zeta_F^*(2s-1)}+\frac{1}{2(s-1)}\right)Z(\Phi_{\mathbf{K}}((\begin{smallmatrix} * & x \\ & t_2 \end{smallmatrix})), 2s-1)$$

has its only poles at $s = 1$, which is simple, and at $s = \frac{1}{2}$, which is a pole of second order. The residue at $s = 1$ is

$$-\frac{\lambda_0}{\lambda_{-1}}\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^3}\Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix})dt_1dt_2dx.$$

The main part of the Laurent expansion at $s = \frac{1}{2}$ is given by the product of $\mathrm{vol}(F^\times\backslash\mathbb{A}^1)$ with

$$-(s-\frac{1}{2})^{-2}\frac{1}{2}\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} 0 & x \\ 0 & t \end{pmatrix})dtdx$$

$$-(s-\frac{1}{2})^{-1}(2+\frac{[\zeta_F^*(s)(s-1)]'_{s=1}+[\zeta_F^*(s)(s-1)]''_{s=1}}{\mathrm{res}_{s=1}\,\zeta_F^*(s)})\int_{\mathbb{A}^2}\Phi_{\mathbf{K}}(\begin{pmatrix} 0 & x \\ 0 & t \end{pmatrix})dtdx$$

$$-(s-\frac{1}{2})^{-1}\int_{\mathbb{A}^3}\mathcal{F}\Phi_{\mathbf{K}}(\begin{pmatrix} t_1 & x \\ 0 & 0 \end{pmatrix})(\log|t_1|_S+\lambda_{1,S})dt_1dt_2dx \tag{47}$$

with $\mathcal{F}\Phi_{\mathbf{K}}((\begin{smallmatrix} t_1 & x \\ 0 & t_2 \end{smallmatrix})) = \int_{\mathbb{A}^2}\Phi_{\mathbf{K}}((\begin{smallmatrix} y_1 & x \\ 0 & y_2 \end{smallmatrix}))\psi(y_1t_1+y_2t_2)dy_1dy_2$ the Fourier transform in the "diagonal" variables. Note that we have $\int_{\mathbb{A}}\mathcal{F}\Phi_{\mathbf{K}}((\begin{smallmatrix} t_1 & x \\ 0 & 0 \end{smallmatrix}))dx = \int_{\mathbb{A}}\widehat{\Phi_{\mathbf{K}}}((\begin{smallmatrix} t_1 & 0 \\ y & 0 \end{smallmatrix}))dy$. If we add the Laurent expansions (46) and (47) we therefore obtain (45). □

Hence up to an entire function we obtain on the spectral side the following:

- a meromorphic function on $\mathbb{C}$ with only finitely many poles,
- the meromorphic function $T(2s+1, \Phi((\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}))), s \in \mathbb{C}$,
- the meromorphic continuation of (44),
- and the function defined by

$$\frac{1}{2\pi i} \sum_{u \in S} \int_{i\mathbb{R}} \operatorname{tr}(R_u(\sigma)^{-1} R'_u(\sigma) I_u(\sigma, f_s)) (\prod_{v \neq u} \operatorname{tr} I_v(\sigma, f_s)) d\sigma.$$

  which is known to have a meromorphic continuation to all $s \in \mathbb{C}$.

The proof that this last function can be continued to all $\mathbb{C}$ does not tell us much about the location of possible poles (even though this could be made more explicit for GL(2)). However, from the analysis of the geometric side we shall see that the last two terms together as well can only have finitely many poles (which additionally must be contained in in the set $\{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$). As the terms of the spectral side are all subject to certain functional equations, there should be no pole at $s = -\frac{1}{2}$.
Note that $T(2s+1, \Phi((\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix})))$ is a distribution defined on the singular matrices as already mentioned in the last section. Hence one could try to use the trace formula for the Lie algebra $\mathfrak{gl}_2(\mathbb{A})$ [Ch02] to get rid of such singular contributions, see Remark 56.

V The elliptic contribution for GL(2) and GL(3)

V.i The geometric side for GL(2): A model for the higher rank case

The convergence of the geometric side of the trace formula for $G = \mathrm{GL}(2)$ for test functions $f \in \mathcal{C}(\mathrm{GL}_2(\mathbb{A})^1, K)$ was shown in [FiLa11a] (although the space of test functions was slightly smaller there, see Remark 51). In this case we now can give a complete analysis of the geometric side showing that each of the contributions has a meromorphic continuation to all $s \in \mathbb{C}$. The individual terms are explicitly given by [FiLa11a, Theorem 1], and we will use this expansion. The regular elliptic terms are of special interest: They amount for the first pole, and essentially yield the adelic Shintani zeta function from [Yu92]. This zeta function is a an example of a zeta function associated to a prehomogeneous vector space. The Shintani zeta function itself has infinitely many poles, but there is a regularised version having only finitely many poles and satisfying a functional equation. It will turn out that this regularisation can be found as well on the geometric side in terms of the summands belonging to the regular hyperbolic elements.

The geometric side now is of simpler form than before: The polynomials $\chi_{\mathfrak{o}}$ parametrising the equivalence classes $\mathfrak{o} \in \mathcal{O}$ are of the following form: Either $\chi_{\mathfrak{o}}(T) \in F[T]$ is irreducible over F, or $\chi_{\mathfrak{o}}(T) = (T - t_1)(T - t_2)$ with $t_1, t_2 \in F^\times$. In the former case, the class $\mathfrak{o}$ consists entirely of regular elliptic terms forming one single orbit. In the latter case, $\mathfrak{o}$ consists entirely of semi-simple elements if $t_1 \neq t_2$, but if $t_1 = t_2$, $\mathfrak{o} = \mathfrak{b}_1 \sqcup \mathfrak{b}_2$ with $\mathfrak{b}_1 = \{t_1 \mathbf{1}_2\}$ the trivial unipotent orbit (i.e. the singular elliptic elements), and $\mathfrak{b}_2$ the orbit belonging to $t_1 \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ (i.e. the singular hyperbolic elements). We will denote the decomposition as

$$\mathcal{O} = \mathcal{O}_{\mathrm{ell,reg}} \sqcup \mathcal{O}_{\mathrm{hyp,reg}} \sqcup \mathcal{O}_{\mathrm{sing}}.$$

We first study the elliptic and regular hyperbolic terms. The singular hyperbolic terms will be dealt with separately later on.

V.i.i Binary quadratic forms and the Shintani-ζ-function

In this section we shortly review some notation and results from [Da93] and [Yu92].
Write $G = \mathrm{GL}(2)$. We want to study the action of G on the three dimensional space of binary quadratic forms and Schwartz-Bruhat functions on this space over the field F. The affine algebraic group of such forms defined over F will be denoted by V, its F-points by V_F, and its adelic points by $V_{\mathbb{A}}$. Let $\mathcal{S}(V_{\mathbb{A}})$

be the space of Schwartz-Bruhat functions on $V_{\mathbb{A}}$. If $X = (X_1, X_2, X_3) \in V_{\mathbb{A}}$ is a binary quadratic form, $X(u,v) = X_1u^2 + X_2uv + X_3v^2$, the action of G is given by $g \cdot X(u,v) = X((u,v)g^t)$, which is the linear transformation

$$X \mapsto \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad+bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{A})$. Equipped with this action (G,V) is a prehomogeneous vector space. GL(1) acts by multiplication on the coefficients of V. If we then define H to be $\mathrm{GL}(1) \times \mathrm{GL}(2)$, we get an action of H on V which we denote by $h \cdot X$. We view H as embedded in $\mathrm{GL}(V) \simeq \mathrm{GL}(3)$ so that we can write $\det(h)$ for $h = (a,g) \in H$ and it is equal to $a \det(g)$.

Note that there is an isomorphism

$$\mathrm{Mat}_{2\times 2} \simeq V \oplus A^1$$

with A^1 the one-dimensional affine space such that under this isomorphism the adjoint action of G on $\mathrm{Mat}_{2\times 2}$ splits into the action of the subgroup $H_G = \{(\det g^{-1}, g) \in H \mid g \in G\} \subseteq H$ on V plus the identity on A^1. In particular, for the fibration given by

$$\mathrm{Mat}_{2\times 2} \longrightarrow A^1, \ g \mapsto \mathrm{tr}\, g,$$

each fibre is isomorphic to V and is invariant under the action of G. For $X \in V$ let $\gamma_X \in \mathrm{Mat}_{2\times 2}$ be the unique element in the fibre above $0 \in A^1$ defined by the above isomorphism. The measure on $V_{\mathbb{A}}$ is the natural one obtained from the identification $V_{\mathbb{A}} \simeq \mathbb{A}^3$. For the inner form $[\cdot,\cdot]$ on $V_{\mathbb{A}}$ we adopt the convention from [Da93] by defining $[X,Y] = X_1Y_3 - \frac{1}{2}X_2Y_2 + X_3Y_1$. Let $\psi = \bigotimes_v \psi_v : \mathbb{A} \longrightarrow \mathbb{C}^\times$ be a non-trivial character. Then $\widehat{\Psi}(Y) = \int_{V_{\mathbb{A}}} \Psi(X)\psi([X,Y])dX$ denotes the Fourier-transform with respect to ψ. If $\Phi \in \mathcal{S}(\mathrm{Mat}_{2\times 2}(\mathbb{A}))$, we use the same character to define the Fourier transform of Φ on the space of all 2×2 matrices by $\hat{\Phi}(x) = \int_{\mathrm{Mat}_{2\times 2}(\mathbb{A})} \Phi(y)\psi(\mathrm{tr}(xy))dy$. Note that if $a \in \mathbb{A}$, $X \in V_{\mathbb{A}}$, then

$$\begin{aligned} \hat{\Phi}(a + \gamma_X) &= \int_{\mathrm{Mat}_{2\times 2}(\mathbb{A})} \Phi(y)\psi(\mathrm{tr}((a+\gamma_X)y))dy \\ &= -\int_{\mathbb{A}} \int_{V_{\mathbb{A}}} \Phi(b+\gamma_Y)\psi(2ab)\psi([X,Y])dY\,db \end{aligned}$$

For a binary quadratic form $X \in V_F$ we denote the splitting field of X over F by $F(X)$, and write $P(X) = X_2^2 - 4X_1X_3$ for the discriminant of the form X. Clearly, $[F(X) : F] \leq 2$ and $[F(X) : F] = 2$ if and only if $P(X)$ is not a square in F. Let $V_F'' = \{X \in V_F | [F(X) : F] = 2\}$. Then V with the above action of H and the polynomial P forms a prehomogeneous vector space. An important consequence of this is that the action of $H(F_\infty)$ on V_{F_∞} has only finitely many orbits. This is no longer the case for the analogue situation of

GL(3) we are considering in the next section.
For $\Psi \in \mathcal{S}(V_{\mathbb{A}})$ and $s \in \mathbb{C}$, $\Re s > \frac{3}{2}$, the Shintani zeta-function (with trivial central character) is defined by [Da96, Yu92]

$$Z(\Psi, s) = \int_{H(F)\backslash H(\mathbb{A})} |\det(h)|^{2s} \sum_{X \in V_F''} \Psi(h \cdot X) dh.$$

This is a special case of a zeta function associated with a prehomogenous vector space. It can be shown (see [Da93]) that this zeta functions has a meromorphic continuation to the whole complex plane. In order to get a functional equation, the adjusted Shintani zeta function is defined which is a slight modification of the function above. It is this adjusted function which will occur naturally as a part of the geometric side of the trace formula.
For $u = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \in U(\mathbb{A})$ we define as in [Da96, (1.3)]

$$\alpha_v(u) = \begin{cases} \max\{1, |x_v|_v\} & \text{if } v < \infty \\ \sqrt{1 + |x_v|_v^2} & \text{if } v|\infty \text{ is real} \\ 1 + |x_v|_v & \text{if } v|\infty \text{ is complex} \end{cases}$$

and $\alpha(u) = \prod \alpha_v(u)$. If $x \in \mathbb{A}$ we also write $\alpha(x) = \alpha(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))$. For a Schwartz-Bruhat function $\Xi \in \mathcal{S}(\mathbb{A}^2)$ we put

$$T(\Xi, w, s) = \operatorname{vol}(T(F)\backslash T(\mathbb{A})^1) \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \Xi(t, tx) |t|^s \alpha(x)^w dx d^\times t \tag{48}$$

(note that the difference to the definition in [Da93, §3.7] is due to our different choice of normalisation of measures). Define the adjusted Shintani zeta function Z_{ad} to be

$$Z_{\mathrm{ad}}(\Psi, s) = Z(\Psi, s) + \frac{\partial}{\partial w}_{|w=0} T(\Psi(0, \cdot, \cdot), w, s)$$

with $\Psi \in \mathcal{S}(V_{\mathbb{A}})$ and viewing $\Psi(0, \cdot, \cdot)$ as a function in $\mathcal{S}(\mathbb{A}^2)$. We shall write $T(\Xi, s) = \frac{\partial}{\partial w}_{|w=0} T(\Xi, w, s)$. By [Yu92, Corollary 4.3] Z_{ad} can be meromorphically continued to all $\mathbb{C}$ with known poles and satisfies the functional equation

$$Z_{\mathrm{ad}}(\Psi, s) = Z_{\mathrm{ad}}(\widehat{\Psi}, 3 - s).$$

V.i.ii The elliptic contribution

We shall use the explicit form of the trace formula for $G = \mathrm{GL}(2)$ given in [FiLa11a, Theorem 1].

Remark 51. *The form of the geometric side from [FiLa11a] was shown there to be valid for all smooth functions $f : K\backslash \mathrm{GL}_2(\mathbb{A})^1/K \longrightarrow \mathbb{C}$ such that all seminorms $||X * f * Y||_{L^1(K\backslash \mathrm{GL}_2(\mathbb{A})^1/K)}$ are finite for all $X, Y \in \mathcal{U}(\mathfrak{gl}_{2,\mathbb{C}})$. Although this is sufficient for our purposes, we remark that it stays valid* *for the larger class of functions which are only subject to the condition that all seminorms* $||f * X||_{L^1(K\backslash \mathrm{GL}_2(\mathbb{A})^1/K)}$ *are finite. (This larger space of test functions is the space we use and was also considered in [FiLa11b].) The semisimple part converges for such test functions by [FiLa11b] so that we only need to show the claim for the non-semisimple part, i.e. the singular hyperbolic contribution. But examining the proof of [FiLa11a, Theorem 1] it suffices to note that for central* $t \in Z(\mathbb{R})$ *and* $\mathrm{O}(2)$*-central* $f \in \mathcal{C}(\mathrm{GL}_2(\mathbb{R})^1)$,

$$\sup_{t\in Z(\mathbb{R}),k\in \mathrm{O}(2)} \int_{U(\mathbb{R})} |f(tuk)|du = \sup_{t\in Z(\mathbb{R}),k\in \mathrm{O}(2)} \int_{U(\mathbb{R})} |f(utk)|du \leq \sum_{X\in\mathcal{B}_{\mathrm{O}(2)},Y\in\mathcal{B}_T} ||f * X * Y||_{L^1(K\backslash \mathrm{GL}_2(\mathbb{A})^1/K)},$$

and also

$$\sup_{k_1,k_2\in \mathrm{O}(2)} |f(k_1ak_2)| \leq \sum_{X\in\mathcal{B}_{\mathrm{O}(2)}} ||f * X||_{L^1(K\backslash \mathrm{GL}_2(\mathbb{A})^1/K)},$$

where we used the $\mathrm{O}(2)$*-centrality of* f.

Fix a function $\Phi \in \mathcal{S}(M_2(\mathbb{A}), K)$. Write $G(F)_{\mathrm{ell}}$ for the set of all elliptic elements in $G(F)$, i.e. those elements having an irreducible minimal polynomial. We have $G(F)_{\mathrm{ell}} = Z(F) \sqcup G(F)_{\mathrm{ell,reg}}$, where $G(F)_{\mathrm{ell,reg}} = \bigsqcup_{\mathfrak{o}\in\mathcal{O}_{\mathrm{ell,reg}}} \mathfrak{o}$ is the set of regular elliptic elements. The contribution from the singular elliptic elements reduces to a Tate integral:

Proposition 52. *The central contribution* $\sum_{\mathfrak{o}\in\mathcal{O}_{sing}} J_{\mathfrak{o}}^{\mathfrak{b}_1}(f_s)$ *is given by*

$$\mathrm{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A})) \int_{\mathbb{A}^\times} \Phi(z\mathbf{1}_2)|z|^{2s+1}d^\times z$$

for $\Re s > \frac{3}{2}$. *It has a meromorphic continuation to all* $s \in \mathbb{C}$ *with simple poles at* $s = 0$ *and* $-\frac{1}{2}$, *and is holomorphic elsewhere. Its residues are given by*

$$\frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)}{2} \int_{\mathbb{A}} \Phi(z\mathbf{1}_2)dz$$

at $s = 0$, *and at* $s = -\frac{1}{2}$ *by*

$$-\frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)}{2}\Phi(0).$$

Proof. See [We67, VII, §5, Theorem 2]. □

For the regular elliptic terms we have the following.

Theorem 53. *The regular elliptic contribution* $\sum_{\mathfrak{o}\in\mathcal{O}_{ell,reg}} J_{\mathfrak{o}}(f_s)$ *defines a holomorphic function for* $\Re s > \frac{3}{2}$ *and can be analytically continued to a meromorphic function to all* $s \in \mathbb{C}$. *Up to an entire function, it equals the Shintani zeta function* $Z(\Psi, 2s)$ *for*

$$\Psi(X) = \int_{\mathbb{A}} \Phi(a + \gamma_X)da, \quad X \in V_{\mathbb{A}}.$$

The first assertion is contained in [FiLa11a, Theorem 1], and the second one follows from the identification with the Shintani zeta function. Hence we shall only show the last assertion. In the proof we shall use the fibration $\mathrm{Mat}_{2\times 2}(F) \longrightarrow F$ from above. The intersection of each fibre with $G(F)_{\mathrm{ell,reg}}$ is isomorphic to V''_F.

Proof. Let $\Re s > \frac{3}{2}$. The sum $\sum_{\mathfrak{o}\in\mathcal{O}_{\mathrm{ell,reg}}} J_{\mathfrak{o}}(f_s)$ equals

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{Z(F)\backslash Z(\mathbb{A})} \sum_{X\in V''_F} \sum_{q\in F} \Phi(zq\mathbf{1}_2 + zg^{-1}\gamma_X g)|z|^{2s+1} dz dg.$$

We now use the isomorphism $Z(F)\backslash Z(\mathbb{A}) \simeq F^\times\backslash\mathbb{A}^1 \times \mathbb{R}_{>0}$, and split the integral over $\mathbb{R}_{>0}$ in one over $(0,1]$ and one over $[1,\infty)$. Since Φ is a Schwartz-Bruhat function, the integral

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{F^\times\backslash\mathbb{A}^1} \int_1^\infty \sum_{X\in V''_F} \sum_{q\in F} \lambda^{2s+1}\Phi(\lambda zq\mathbf{1}_2 + \lambda zg^{-1}\gamma_X g) d^\times\lambda d^\times z dg$$

converges absolutely for all $s \in \mathbb{C}$, i. e. defines a holomorphic function on $\mathbb{C}$.

The remaining part of the integral is

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{F^\times\backslash\mathbb{A}^1} \int_0^1 \lambda^{2s+1} \sum_{X\in V''_F} \sum_{q\in F} \Phi(\lambda zq\mathbf{1}_2 + \lambda zg^{-1}\gamma_X g) d^\times\lambda d^\times z dg.$$

We apply the Poisson summation formula to the inner sum over q to get

$$\sum_{q\in F} \Phi(\lambda zq\mathbf{1}_2 + \lambda zg^{-1}\gamma_X g) = \frac{1}{\lambda} \sum_{a\in F} \mathcal{F}_1\Phi(\frac{a}{\lambda z} + \lambda zg^{-1}\gamma_X g),$$

where

$$\mathcal{F}_1\Phi(y + \lambda zg^{-1}\gamma_X g) = \int_{\mathbb{A}} \Phi(q + \lambda zg^{-1}\gamma_X g)\psi(qy)dq$$

is the Fourier-transform in the "central" variable, which is again a Schwartz-Bruhat function on $\mathbb{A} \oplus V_{\mathbb{A}} \simeq \mathrm{Mat}_{2\times 2}(\mathbb{A})$. Using this, the integral for $(0,1]$ equals

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{F^\times\backslash\mathbb{A}^1} \int_0^1 \sum_{X\in V''_F} \lambda^{2s} \sum_{a\in F^\times} \mathcal{F}_1\Phi(\frac{a}{\lambda z} + \lambda zg^{-1}\gamma_X g) d^\times\lambda d^\times z dg$$

$$+\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^1}\int_0^1 \lambda^{2s}\sum_{X\in V''_F}\mathcal{F}_1\Phi(\lambda z g^{-1}\gamma_X g)d^\times\lambda d^\times z dg.$$

Changing the variables z to z^{-1} and λ to λ^{-1} in the first integral we get

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^1}\int_1^\infty \lambda^{-2s-1}\sum_{X\in V''_F}\sum_{a\in F^\times}\mathcal{F}_1\Phi(\lambda z a+\lambda^{-1}z^{-1}g^{-1}\gamma_X g)d^\times\lambda d^\times z dg,$$

which again converges absolutely for all $s\in\mathbb{C}$. So the analytic behaviour of the regular elliptic contribution is completely determined by

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^1}\int_0^1 \lambda^{2s}\sum_{X\in V''_F}\mathcal{F}_1\Phi(\lambda z g^{-1}\gamma_X g)d^\times\lambda d^\times z dg,$$

and we change nothing of its analytic properties if instead we consider

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^\times}|z|^{2s}\sum_{X\in V''_F}\mathcal{F}_1\Phi(z g^{-1}\gamma_X g)d^\times z dg,$$

which is exactly the Shintani zeta function $Z(\Psi,2s)$ for $\Psi(X)=\mathcal{F}_1\Phi(\gamma_X)$, $X\in V_{\mathbb{A}}$. □

V.i.iii Example: $F=\mathbb{Q}$, Φ of special form

In this section we want to see explicitely how one can recover Shintani's original definition of the Shintani zeta function from the regular elliptic terms. Suppose that we work over the field $F=\mathbb{Q}$ and take Φ to be of the following special form: for a finite prime p let Φ_p be the characteristic function of $\mathrm{Mat}_{2\times2}(\mathbb{Z}_p)\subseteq\mathrm{Mat}_{2\times2}(\mathbb{Q}_p)$, and at the real place let $\Phi_\infty(x)=e^{-\pi\operatorname{tr}(x^tx)}$, $x\in\mathrm{Mat}_{2\times2}(\mathbb{R})$, be the Gauss function. Consider $f_s(g)=\int_{\mathbb{R}_{>0}}|\det ag|^{s+\frac12}\Phi(ag)d^\times a$ as a function on $\mathrm{GL}_2(\mathbb{R})$. The conjugacy classes in $\mathrm{GL}_2(\mathbb{Q})_{\text{ell, reg}}$ are parametrised by monic irreducible quadratic polynomials with rational coefficients. Hence there is a bijective correspondence

$$\{[\gamma]\subseteq\mathrm{GL}_2(\mathbb{Q})_{\text{ell, reg}}\}\longleftrightarrow\{E\text{ quadratic field}\},$$

and for $\gamma\in\mathrm{GL}_2(\mathbb{Q})_{\text{ell,reg}}$, let $E=\mathbb{Q}(\gamma)$ be the respective quadratic field and let d_E its discriminant. Let $d(\gamma)$ be the discriminant of γ, and write $d(\gamma)=a_\gamma^2D_E$ for $D_E\in\mathbb{Z}$ squarefree so that $E=\mathbb{Q}(\sqrt{D_E})$. If $D_E\equiv1\mod 4$, $d_E=D_E$, but if $D_E\equiv2,3\mod 4$, $d_E=4D_E$. Let $\gamma_1,\gamma_2\in E$ be the roots of the characteristic polynomial of γ, i.e. $\gamma_1,\gamma_2=\pm a_\gamma\sqrt{D_E}$. The ring of integers of E has the form $\mathbb{Z}[\theta]$ with

$$\theta=\begin{cases}\sqrt{D_E} & \text{if } D_E\equiv2,3\mod 4\\ \frac{1+\sqrt{D_E}}{2} & \text{if } D_E\equiv1\mod 4.\end{cases}$$

Using this integral basis, there is a natural two-sheeted surjection (cf. also Lemma 64)

$$\mathcal{O}_E\backslash\mathbb{Z}$$
$$\downarrow$$
$$\{\gamma\in \mathrm{GL}_2(\mathbb{Q})_{\text{ell,reg}}\mid \mathbb{Q}(\gamma)=E,\ \int_{\mathrm{GL}_{2,\gamma}(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)}\Phi_f(g^{-1}\gamma g)dg\neq 0\}$$

sending $a+b\theta$ to the unique $\gamma\in \mathrm{GL}_2(\mathbb{Q})_{\text{ell,reg}}$ having eigenvalues $\gamma_1 = a+b\theta, \gamma_2=a+b\bar{\theta}$ with $\bar{\theta}$ denoting the image of θ under the action of the non-trivial element of the Galois group of $E/\mathbb{Q}$. For such γ, $\mathbb{Z}[\gamma_1]=\mathbb{Z}[b\theta]$ and $d(\gamma)=b^2D_E$. If p is a prime, we have

$$\mathcal{O}_{E\otimes\mathbb{Q}_p}=\mathcal{O}_E\otimes\mathbb{Z}_p=\begin{cases}\mathbb{Z}_p^2 & \text{if } \left(\frac{D_E}{p}\right)=1\\ \mathbb{Z}_p[\sqrt{D_E}] & \text{if } p\neq 2,\ \left(\frac{D_E}{p}\right)\neq 1, \text{ or } p=2,\\ & \left(\frac{D_E}{2}\right)\neq 1, D_E\not\equiv 1 \mod 4\\ \mathbb{Z}_2[\frac{1+\sqrt{D_E}}{2}] & \text{if } p=2,\ D_E\equiv -3 \mod 8,\end{cases}$$

and in the second case also $\mathbb{Z}[\gamma_1]\otimes\mathbb{Z}_p=\mathbb{Z}_p[b\sqrt{D_E}]$.

The local orbital integrals $\int_{\mathrm{GL}_{2,\gamma}(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)}\Phi_p(g^{-1}\gamma g)dg$ can now be computed by counting lattices (up to principal ideals) in $E\otimes\mathbb{Q}_p$ which have γ in their multiplier ring as follows.

Lemma 54. *Let $\gamma\in \mathrm{GL}_2(\mathbb{Q})_{ell,reg}$ correspond to a pair $a+b\theta, a+b\bar{\theta}\in\mathcal{O}_E$ as above. We have*

$$\int_{\mathrm{GL}_{2,\gamma}(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)}\Phi_p(g^{-1}\gamma g)dg=\begin{cases}p^\kappa & \textit{if } \left(\frac{D_E}{p}\right)=1\\ \frac{p^{\kappa+1}+p^\kappa-2}{p-1} & \textit{if } \left(\frac{D_E}{p}\right)=-1\\ \frac{p^{\kappa+1}-1}{p-1} & \textit{if } \left(\frac{D_E}{p}\right)=0\end{cases}$$

for $\kappa=\frac{1}{2}(\mathrm{val}_p(d(\gamma))-\mathrm{val}_p(D_E))=\mathrm{val}_p(b)$ with $d(\gamma)=(\mathrm{tr}\,\gamma)^2-4\det\gamma$ the discriminant of γ. Note that in any case, the right hand side can be written as

$$p^\kappa(1+(1-\left(\frac{D_E}{p}\right))\frac{1-p^{-\kappa}}{p-1}).$$

See also [Fl06, §II.1, Proposition 5], where this was computed by counting orders. We include another calculation, which is based on counting lattices with certain multiplier rings and is slightly longer than the computation with orders. However, it seems more promising count lattices for the respective p-adic integrals for GL(3), as one then do not need to compute any stabilisers, even if we have not yet succeeded in doing so.

Proof. Assume first that $p\neq 2$.

- p **split:** If p splits in E, i.e. $E \otimes \mathbb{Q}_p \simeq \mathbb{Q}_p^2$ as fields, γ is conjugate over $\mathrm{GL}_2(\mathbb{Q}_p)$ to a diagonal matrix with entries $\gamma_1, \gamma_2 \in \mathbb{Q}_p$. Hence there are $p^{|\operatorname{val}_p(\gamma_1) - \operatorname{val}_p(\gamma_2)|}$ many such lattices.

- p **inert:** We have $\mathrm{GL}_{2,\gamma}(\mathbb{Q}_p) = Z(\mathbb{Q}_p)$. For each $\mathbb{Z}_p$-order $\mathfrak{o} \subseteq \mathcal{O}_{E\otimes\mathbb{Q}_p} = \mathbb{Z}_p[\sqrt{D_E}]$, $\mathbb{Z}_p[\gamma_1] = \mathbb{Z}_p[p^\kappa \sqrt{D_E}] \subseteq \mathfrak{o}$ there is $l \in \{0, \dots, \kappa\}$ such that $\mathfrak{o} = \mathbb{Z}[\xi]$, $\xi = p^{-l}\gamma_1$. Let $k = \kappa - l$. Counting lattices as above amounts to the same as counting $\mathfrak{o}$-lattices (up to principal ideals) which have multiplier exactly equal to $\mathfrak{o}$. If $k = 0$ there is exactly one such ideal. Hence we may assume $k \geq 1$. Let $\mathfrak{a} \subseteq \mathfrak{o}$ be an $\mathfrak{o}$-ideal for $\mathfrak{o} = \mathbb{Z}_p[\xi]$ such that the multiplier of $\mathfrak{a}$ equals $\mathfrak{o}$. Then up to multiplication by elements of $\mathbb{Z}_p$, $\mathfrak{a}$ has a $\mathbb{Z}_p$-basis as a $\mathbb{Z}_p$-lattice of the form $p^{m_1}, \xi + \alpha p^{l_2}$ for $m_1, l_2 \in \mathbb{N}_0$, $l_2 < m_1$ or $l_2 = \infty$, and $\alpha = \varepsilon_0 + \varepsilon_1 p + \ldots + \varepsilon_{m_1-1-l_2} p^{m_1-1-l_2} \in \mathbb{Z}_p$, $\varepsilon_i \in \{0, \dots, p-1\}$, $\varepsilon_0 \neq 0$. For $\mathfrak{a}$ to be an $\mathfrak{o}$-ideal, the following systems must be soluble for $a, b, c, d \in \mathbb{Z}_p$:
$$\begin{pmatrix} 0 \\ p^{m_1} \end{pmatrix} = \begin{pmatrix} p^{m_1} & \alpha p^{l_2} \\ & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a p^{m_1} + b\alpha p^{l_2} \\ b \end{pmatrix}$$
and
$$\begin{pmatrix} p^{2k} D_E \\ \alpha p^{l_2} \end{pmatrix} = \begin{pmatrix} p^{m_1} & \alpha p^{l_2} \\ & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c p^{m_1} + d\alpha p^{l_2} \\ d \end{pmatrix}.$$
Solving these equations, we get
$$a = -\alpha p^{l_2}, \quad b = p^{m_1}, \quad c = D_E p^{2k-m_1} - \alpha^2 p^{2l_2 - m_1}, \quad d = \alpha p^{l_2}. \tag{49}$$
Since $\alpha^2 \neq D_E$, the solvability in $\mathbb{Z}_p$ is equivalent to $m_1 \leq 2l_2$ and $m_1 \leq 2k$ (if $l_2 \neq \infty$). Since we also assumed that the multiplier of $\mathfrak{a}$ is $\mathfrak{o}$, we moreover have $m_1 > 2k-2$. Thus $m_1 = 2k$ or $m_1 = 2k-1$. In the first case $l_2 \in \{k, \dots, 2k-1\}$ and in the second case $l_2 \in \{k, \dots 2k-2\}$. Counting these possibilities together, we get
$$\sum_{l_2=k}^{2k-1} (p^{2k-l_2} - p^{2k-l_2-1}) + \sum_{l_2=k}^{2k-2} (p^{2k-1-l_2} - p^{2k-1-l_2-1}) + 2 = p^k + p^{k-1}$$
and summation over k yields the assertion.

- p **ramified:** Consider the integral $\int_{Z(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)} \Phi_p(g^{-1}\gamma g) dg$ first. Then almost all the consideration from the unramified case apply. We only have to take into account that $p | D_E$ when counting the solutions of (49). Write $D_E = \delta p$ with $(\delta, p) = 1$. Hence (49) takes the form
$$a = -\alpha p^{l_2}, \quad b = p^{m_1}, \quad c = \delta p^{2k+1-m_1} - \alpha^2 p^{2l_2-m_1}, \quad d = \alpha p^{l_2}.$$
If $\delta \notin \mathbb{Z}_p^2$, we can argue as above that $m_1 \leq \min\{2k+1, 2l_2\}$, and $\mathfrak{o}$ being the multiplier of $\mathfrak{a}$ implies $m_1 > 2k-1$. Hence we get
$$\sum_{l_2=k+1}^{2k} (p^{2k+1-l_2} - p^{2k+1-l_2-1}) + \sum_{l_2=k}^{2k-2} (p^{2k-1-l_2} - p^{2k-1-l_2-1}) + 2 = 2p^k$$

solutions. If $\delta \in \mathbb{Z}_p^2$ there are again those p^{2k} solutions, but there could be more if $\alpha^2 \equiv \delta \mod p$. There exists an additional solution if and only if $2k+1 = 2l_2$, but this is not possible as $k, l_2 \in \mathbb{Z}$. Thus also in this case there are $2p^k$ solutions for $k \geq 1$. Note that $\mathrm{vol}(Z(\mathbb{Q}_p)\backslash \mathrm{GL}_{2,\gamma}(\mathbb{Q}_p))$ is 2 so that the assertion follows.

If $p = 2$, everything remains true except in the case $D_E \equiv -3 \mod 8$, i.e. 2 is inert and $\mathcal{O}_{E\otimes\mathbb{Z}_2} = \mathbb{Z}_2[\frac{1+\sqrt{D_E}}{2}]$. Similar considerations as before then yield analogous to (49) the following equations

$$a = -\alpha 2^{l_2},\ b = 2^{m_1},\ c = 2^{2k-m_1-1}(1-D_E) - \alpha^2 2^{2l_2-m_1} - 2^{2k-m_1},$$
$$d = \alpha 2^{l_2} + 2^{2k}.$$

As $D_E \equiv -3 \mod 8$, $1 - D_E = 4\delta$ with $(d,2) = 1$, so that $c = 2^{2k-m_1}(2\delta - 1) - \alpha^2 2^{2l_2-m_1}$. Hence we get $2^k + 2^{k-1}$ solutions as above for each $k \geq 1$. □

If E is totally real, the archimedean orbital integral can be computed to equal

$$\int_{\mathrm{GL}_{2,\gamma}(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})^1} \int_0^\infty a^{2s+1}\Phi_\infty(ag^{-1}\gamma g)d^\times a dg = \frac{\Gamma(s)}{2\pi^s\sqrt{d(\gamma)}}(\mathrm{tr}\,\gamma^2)^{-s}.$$

The sum over γ generating totally real quadratic extensions is

$$\sum_{\gamma\in\mathrm{GL}_2(\mathbb{Q})_{\mathrm{ell,reg}},\mathbb{Q}(\gamma)\text{ tot. real}} \nu(\gamma)\int_{\mathrm{GL}_{2,\gamma}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f_s(g^{-1}\gamma g)dg$$

with $\nu(\gamma) = \mathrm{vol}(\mathrm{GL}_{2,\gamma}(\mathbb{Q})\backslash \mathrm{GL}_{2,\gamma}(\mathbb{A})^1)$. Since

$$\mathrm{vol}(\mathrm{GL}_{2,\gamma}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})) = D_E^{\frac{1}{2}} \operatorname*{res}_{s=1}\zeta_E(s) = 2h_E\log\varepsilon_E$$

with ε_E a fundamental unit in $\mathcal{O}_{E,>0}$, this equals

$$\sum_{E/\mathbb{Q},[E:\mathbb{Q}]=2,\text{ tot. real}} \frac{h_E\log\varepsilon_E\Gamma(s)}{2\pi^s\sqrt{D_E}}$$
$$\sum_{a+b\theta\in\mathcal{O}_E\backslash\mathbb{Z}} \mathcal{N}_{E/\mathbb{Q}}(a+b\theta)^{-s}\prod_{p|b}(1+(1-\left(\frac{D_E}{p}\right))\frac{1-|b|_p}{p-1}).$$

Poisson summation over a yields as the main term

$$\sum_{E/\mathbb{Q},[E:\mathbb{Q}]=2,\text{ tot. real}} \frac{h_E\log\varepsilon_E\Gamma(s-\frac{1}{2})}{\pi^{s-\frac{1}{2}}}D_E^{-s}$$
$$\sum_{b\in\mathbb{N}} b^{-2s+1}\prod_{p|b}(1+(1-\left(\frac{D_E}{p}\right))\frac{1-|b|_p}{p-1}).$$

The sum over b can be computed to equal

$$\frac{\zeta(2s-1)\zeta(2s)}{\zeta_E(2s-1)},$$

and thus by [Da93, Theorem 0.2]

$$\sum_{E/\mathbb{Q},[E:\mathbb{Q}]=2,\text{ tot. real}} \frac{h_E \log \varepsilon_E \Gamma(s-\frac{1}{2})}{\pi^{s-\frac{1}{2}}} D_E^{-s} \frac{\zeta(2s-1)\zeta(2s)}{\zeta_E(2s-1)}$$
$$= \Gamma(s-\frac{1}{2})\pi^{-s+\frac{1}{2}} Z_{\text{Shin},+}(s)$$

with $Z_{\text{Shin},+} = \sum_{d=1}^{\infty} h_d \log \varepsilon_d d^{-s}$ the Shintani zeta function associated to the positive definite binary forms introduced by Shintani in [Sh75]. Here h_d is the class number of positive definite binary quadratic forms of discriminant d, $h_d \log \varepsilon_d$ is defined to be 0 if d is a square, and otherwise $\varepsilon_d = t + u\sqrt{d}$ is the minimal solution of $(t,u) \in \mathbb{N}^2$ of $t^2 - u^2 d = 4$. A similar computation is valid for the imaginary quadratic number fields, and one obtains

$$\sum_{\gamma\in \mathrm{GL}_2(\mathbb{Q})_{\text{ell,reg}},\mathbb{Q}(\gamma)\text{ imaginary}} \nu(\gamma) \int_{\mathrm{GL}_{2,\gamma}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f_s(g^{-1}\gamma g) dg$$
$$= 8\sqrt{2}\pi^{-s+1}\Gamma(s) I(s) Z_{\text{Shin},-}(s) + \text{entire fct.}$$

with $I(s) = \int_1^\infty (-\frac{1}{2}+\tau^2)^{-s} d\tau$, which is absolutely convergent and non-zero at least for $\Re s > 1$, and $Z_{\text{Shin},-}(s) = \sum_{-d=1}^{\infty} \frac{h_d}{w_d}(-d)^{-s}$ the Shintani zeta function associated with indefinite binary quadratic forms. Here again h_d is the class number of indefinite quadratic forms of discriminant d, and w_d is the order of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}^\times$, i.e. $w_d = 2$ unless $d = -3, -4$, in which case $w_{-3} = 6$ and $w_{-4} = 4$.

V.i.iv The regular hyperbolic contribution

In this section we analyse the contribution from the regular hyperbolic orbits $\sum_{\mathfrak{o}\in\mathcal{O}_{\text{hyp,reg}}} J_{\mathfrak{o}}(f_s)$. As explained before there is a bijection from $\mathcal{O}_{\text{hyp,reg}}$ to $T(F)\backslash Z(F)$, and each equivalence class is an actual conjugacy class.

Let $t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T(F)\backslash Z(F)$ be fixed for the moment. We can choose a finite set of places $\hat{S}_t$ (including all the archimedean ones) such that $|t_1|_v = |t_2|_v = |1-\frac{t_2}{t_1}|_v = 1$ for all $v \notin \hat{S}_t$. Let $S_0 \supseteq S_\infty$ be some fixed set of places such that Φ_v is the characteristic function of $\mathrm{Mat}_{2\times 2}(\mathcal{O}_v)$ for $v \notin S_0$. If $v \notin \hat{S}_t \cup S_0$, then $\Phi_v(t \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \neq 0$, $x \in F_v$, implies $|x|_v \leq 1$. Set $S_t = \hat{S}_t \cup S_0$. Then for any $v \notin S_t$ we have

$$\omega_v^{S_t}(t,u) = \log\max\{|x|_v, 1\},$$

but if $|x|_v > 1$, we have $\Phi_v(tu) = 0$ by our choice of S_t. Thus

$$\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})} |\det z|^{s+\frac{1}{2}}\Phi(ztu)\omega^{S_t}(t,u)dzdu$$

equals

$$\sum_{v\in S_t}\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})} |\det z|^{s+\frac{1}{2}}\Phi(ztu)\omega_v^{S_t}(t,u)dzdu$$

in which case the weights $\omega^S = \omega^{\mathrm{GL}(2),S}$ are the usual ones for GL(2) as in [FiLa11a]. Here we have $\lambda_{t,S_t} = 0$.

Given a finite set of places S including the archimedean places, we can collect all $t \in T(F)\backslash Z(F)$ with $S_t \subseteq S$ in a set W_S. If we consider the partial ordered set of all such finite sets S, every $t \in T(F)\backslash Z(F)$ lies in one of the W_S and is then also contained in all other $W_{\tilde{S}}$ with $\tilde{S} \supseteq S$. If for $t_1, t_2 \in W_S$ there is some $a \in F^\times$ with $t_1 = at_2$, we write $t_1 \sim t_2$. We consider now the limit over the net of finite sets of places S partially ordered by inclusion

$$\lim_S \sum_{t\in W_S/\sim}\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})} |\det z|^{s+\frac{1}{2}}\Phi(ztu)\omega^{S}(t,u)dzdu.$$

For each set S the sum-integrals converges absolutely for $\Re s > \frac{3}{2}$, and if $\{S_i\}_{i\in\mathbb{N}}$ is a chain of sets with $S_i \subseteq S_{i+1}, i \in \mathbb{N}$, the functions

$$\sum_{t\in W_{S_i}/\sim}\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})} |\det z|^{s+\frac{1}{2}}|\Phi(ztu)||\omega^{S_i}(t,u)|dzdu$$

form a non-decreasing sequence as $i \to \infty$ for $\Re s > \frac{3}{2}$. Thus if the above limit exists, it has to equal

$$\sum_{t\in(T(F)\backslash Z(F))/Z(F)}\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})/Z(F)} |\det z|^{s+\frac{1}{2}}\Phi(ztu)\omega(t,u)dzdu. \tag{50}$$

Up to the volume factor $\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)$ it is then the contribution from the regular hyperbolic terms. To show the existence of the limit it suffices to show the absolute convergence of this last sum-integral. Here

$$\omega_v(t,u) = \begin{cases} \log\max\{|1-\frac{t_2}{t_1}|_v, |x_v|_v\} & \text{if } v<\infty \\ \frac{1}{2}\log\left(|1-\frac{t_2}{t_1}|_v^2 + |x_v|_v^2\right) & \text{if } v|\infty \text{ is real} \\ \log\left(|1-\frac{t_2}{t_1}|_v + |x_v|_v\right) & \text{if } v|\infty \text{ is complex}\end{cases}$$

$$= \log|1-\frac{t_2}{t_1}|_v + \log\alpha_v\left(\frac{u_v}{1-\frac{t_2}{t_1}}\right),$$

and summation over all places v gives

$$\omega(t,u) = \log\alpha\left(\frac{u}{1-\frac{t_2}{t_1}}\right) = \frac{\partial}{\partial w}_{|w=0}\alpha\left(\frac{u}{1-\frac{t_2}{t_1}}\right)^w.$$

Lemma 55. *The sum-integral given by* (50) *converges absolutely for* $\Re s > \frac{3}{2}$.

Proof. The regular hyperbolic equivalence classes $\mathfrak{o} \in \mathcal{O}_{\text{hyp,reg}}$ correspond bijectively to $t \in T(F)\backslash Z(F)$, and each class contains only semi-simple elements, namely all $g^{-1}tg$, $g \in G(F)$. By [FiLa11b] the sum-integral

$$\sum_{t \in (T(F)\backslash Z(F))/Z(F)} \int_{T(\mathbb{A})\backslash G(\mathbb{A})} f_s(g^{-1}tg) v_T(g) dg$$

converges absolutely for all $\Re s > \frac{3}{2}$, where $v_T(g)$ is the volume of the convex hull of the points $\{-H_B(g), H_{\bar{B}}(g)\}$ in $\mathfrak{a}_T^G$. As $T(\mathbb{A})\backslash G(\mathbb{A}) \simeq U(\mathbb{A})(T(\mathbb{A}) \cap \mathbf{K})\backslash\mathbf{K}$, and v_T is right-$\mathbf{K}$-invariant, we have $v_T(uk) = \log ||(1,x)||$ with $u = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \in U(\mathbb{A}), k \in \mathbf{K}$. The change of variables $u \mapsto t^{-1}u^{-1}tu$ then shows that the above sum-integral equals (50), hence the lemma. □

Now write $\tau = \frac{t_1+t_2}{2}$ and $\sigma = \frac{t_1-t_2}{2} \neq 0$. Then we get with $u = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$

$$tu = \tau \mathbf{1}_2 + \begin{pmatrix} 1 & 2\frac{x}{1-\frac{t_2}{t_1}} \\ & -1 \end{pmatrix}.$$

Instead of taking the sum over all $t \in T(F)\backslash Z(F)$, we can sum over all $\sigma \in F^\times$ and $\tau \in F$, $\tau \neq \pm\sigma$,

$$\sum_{t \in T(F)\backslash Z(F)} \Phi(ztu) = \sum_{\sigma \in F^\times} \sum_{\tau \in F, \tau \neq \pm\sigma} \Phi(z\tau\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 2\frac{x}{1-\frac{t_2}{t_1}} \\ & -1 \end{pmatrix}),$$

and using $\frac{2}{1-\frac{t_2}{t_1}} = (1 + \frac{\tau}{\sigma})$ this is

$$\begin{aligned} &= \sum_{\sigma \in F^\times} \sum_{\tau \in F} \Phi(z\tau\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & x(1+\frac{\tau}{\sigma}) \\ & -1 \end{pmatrix}) \\ &\quad - \sum_{\sigma \in F^\times} \Phi(z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 2x \\ & -1 \end{pmatrix}) - \sum_{\sigma \in F^\times} \Phi(-z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}). \end{aligned}$$

Applying Poisson summation to the sum over $\tau \in F$, this equals

$$\begin{aligned} &\sum_{\sigma \in F^\times} \sum_{\tau' \in F} |z|^{-1} \mathcal{F}_1\Phi\left(\frac{\tau'}{z}, z\sigma \begin{pmatrix} 0 \\ 2 \\ x(1+\frac{\tau}{\sigma}) \end{pmatrix}\right) \\ &\quad - \sum_{\sigma \in F^\times} \Phi(z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 2x \\ & -1 \end{pmatrix}) - \sum_{\sigma \in F^\times} \Phi(-z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}) \end{aligned}$$

Changing $x(1+\frac{\tau}{\sigma})$ to x in the above integral (50), and using the notation as in the last section we thus get

$$\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times_{\geq 1}/F^\times} |z|^{2s+1} \sum_{\sigma \in F^\times} \sum_{\tau \in F} \Phi(z\tau + z\sigma\gamma_{(0,2,x)})\alpha(x)^w d^\times z dx$$

$$+\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times_{\leq 1}/F^\times} |z|^{2s+1} \sum_{\sigma \in F^\times} \sum_{\tau \in F} \Phi(z\tau + z\sigma\gamma_{(0,2,x)})\alpha(x)^w d^\times z dx \quad (51)$$

$$-\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times/F^\times} |z|^{2s+1} \sum_{\sigma \in F^\times} \Phi(z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 2x \\ & -1 \end{pmatrix})\alpha(2x)^w d^\times z dx \quad (52)$$

$$-\int_{\mathbb{A}} \int_{\mathbb{A}^\times/F^\times} |z|^{2s+1} \sum_{\sigma \in F^\times} \Phi(-z\sigma\mathbf{1}_2 + z\sigma \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}) \log\alpha(0) d^\times z dx$$

where we may put the derivation in the front of the first three integrals, since all occuring integrals converge absolutely. The last integral vanishes, since $\log\alpha(0) = 0$.

The first summand is again absolutely convergent for all $s \in \mathbb{C}$ since Φ is a Schwartz-Bruhat function and since $\alpha(\frac{x}{z})$ is bounded by some polynomial in $\log|x|$ and $\log|z|$. To the second summand we apply the Poisson summation formula to the inner sum over τ so that it reads

$$\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times_{\geq 1}/F^\times} |z|^{-2s} \sum_{\tau' \in F^\times} \sum_{\sigma \in F^\times} \mathcal{F}_1\Phi(\tau' z + z^{-1}\sigma\gamma_{(0,1,\frac{x}{2})})\alpha(x)^w d^\times z dx$$

$$+\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times_{\leq 1}/F^\times} |z|^{2s} \sum_{\sigma \in F^\times} \mathcal{F}_1\Phi(z\sigma\gamma_{(0,1,\frac{x}{2})})\alpha(x)^w d^\times z dx$$

after changing z to z^{-1} in the first part. So the sum over $\tau' \in F^\times$ is again a holomorphic function on the whole complex plane, but the summand corresponding to $\tau' = 0$ may give some non-holomorphic function. However, we may add the integral over $|z| > 1$ to this last term without changing its analytic behaviour to get

$$\frac{\partial}{\partial w}_{|w=0} \int_{\mathbb{A}} \int_{\mathbb{A}^\times} |z|^{2s} \mathcal{F}_1\Phi(z\gamma_{(0,1,x)})\alpha(x)^w d^\times z dx,$$

which equals up to the missing factor $\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)$ exactly the function $\frac{\partial}{\partial w}_{|w=0} T(2s, w, \Psi) = T(2s, \Psi)$ with Ψ and T defined as before.

The second last integral (52) equals

$$\int_{\mathbb{A}} \int_{\mathbb{A}^\times} |z|^{2s+1} \Phi(z \begin{pmatrix} 1 & x \\ & 0 \end{pmatrix}) \log\alpha(x) d^\times z dx$$
$$= \mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)^{-1} T(2s+1, \Phi^0)$$

with $\Phi^0 \in \mathcal{S}(\mathbb{A}^2)$, $\Phi^0(a,b) = \Phi(\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right))$.

Thus if we combine the regular elliptic terms with the first part of the hyperbolic terms obtained from (51), we get the adjusted Shintani zeta function of [Yu92] (cf. [Da93]) and some additional distribution. The poles of the adjusted Shintani zeta function are given by [Da93, Theorem 3.1]. By [Yu92, Proposition (2.12)] the distribution $T(2s+1, \Phi^0)$ has a meromorphic continuation to all $\mathbb{C}$, which is holomorphic in $\Re s > \frac{1}{2}$. However, this function has infinitely many poles (determined by the zeros of the Dedekind zeta function).

Remark 56. *Note that $T(2s+1, \Phi^0)$ can also be written as*

$$\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1) \int_{U(\mathbb{A})} F_s(u^{-1}\gamma_0 u) v_T(u) du$$

with $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $F_s : \mathrm{Mat}_{2\times 2}(\mathbb{A}) \to \mathbb{C}$, $F_s(g) = \int_{\mathbb{A}^\times} |z|^{2s+1}\Phi(zg) d^\times z$. This form of $T(2s+1, \Phi^0)$ as an orbital integral over a singular orbit suggests that it might be more suitable to define the test function f_s on the whole Lie algebra $\mathrm{Mat}_{2\times 2}(\mathbb{A})$, and use the trace formula for Lie algebras as developed in [Ch02]. In general, it then might be possible that the infinitely many poles occuring on the geometric and spectral side, and which prevented us from continuing the spectral terms further, cancel.

Proposition 57. *The function $\mathcal{G}$ given by*

$$\mathcal{G}(s) = \sum_{\mathfrak{o} \in \mathcal{O}_{ell,reg} \cup \mathcal{O}_{hyp,reg}} J_{\mathfrak{o}}(f_s)$$

for $\Re s > \frac{3}{2}$ has a meromorphic continuation to all $s \in \mathbb{C}$. $\mathcal{G}(s) - T(\Phi^0, 2s+1)$ equals the adjusted Shintani zeta function $Z_{ad}(\Psi, 2s)$ so that the poles and main parts of the Laurent expansions of $\mathcal{G}(s) - T(\Phi^0, 2s+1)$ are as follows:

(i) at $s = \frac{3}{2}$:

$$(s - \frac{3}{2})^{-1} \frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)}{2} \int_{\mathrm{Mat}_{2\times 2}(\mathbb{A})} \Phi(x) dx$$

(ii) at $s = 1$:

$$-(s-1)^{-2} \frac{\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)}{4} \int_{\mathbb{A}^3} \Phi(\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix}) dt_1 dt_2 dx$$

$$+(s-1)^{-1} \frac{\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)}{2} \int_{\mathbb{A}} \widehat{\Phi}(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})(\log |x|_S + \lambda_{1,S}) dt$$

with $\widehat{\Phi}$ the Fourier-transform in all variabels of Φ as a function on $M_2(\mathbb{A})$.

(iii) at $s = \frac{1}{2}$:

$$-(s-\frac{1}{2})^{-2}\frac{\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)}{4}\int_{\mathbb{A}}\int_{\mathbb{A}}\Phi(\begin{pmatrix} z & y \\ & z\end{pmatrix})dzdy$$

$$-(s-\frac{1}{2})^{-1}\frac{\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1)}{2}\int_{\mathbb{A}}\int_{\mathbb{A}}\Phi(\begin{pmatrix} z & y \\ & z\end{pmatrix})(\log|y|_S+\lambda_{1,S})dzdy$$

(iv) at $s = 0$:

$$-s^{-1}\frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)}{2}\int_{\mathbb{A}}\Phi(a\mathbf{1}_2)da$$

Moreover, $T(\Phi^0, 2s+1)$ is holomorphic for $\Re s > \frac{1}{2}$ so that the only poles of $\mathcal{G}(s)$ in $\Re s > \frac{1}{2}$ are at $s = \frac{3}{2}$ and $s = 1$, and they are given as above.

All the assertion about Z_{ad} are given in [Da93, Theorem 3.1].

V.i.v The singular hyperbolic contribution

We are left with the sums over the singular hyperbolic orbits $\sum_{\mathfrak{o}\in\mathcal{O}_{\mathrm{hyp,sing}}} J_{\mathfrak{o}}^{\mathfrak{b}_2}(f_s)$. According to [FiLa11a, Theorem 1] this equals the sum of

$$\begin{aligned}\mathrm{vol}(Z(\mathbb{A}^1)T(F)\backslash T(\mathbb{A})^1)\lambda_S\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})}F_s(zu)dzdu \\ = \mathrm{vol}(F^\times\backslash\mathbb{A}^1)\lambda_S\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})d^\times zdx \quad (53)\end{aligned}$$

and

$$\begin{aligned}\mathrm{vol}(Z(\mathbb{A}^1)T(F)\backslash T(\mathbb{A})^1)\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})}F_s(zu)\omega^S(z,x)dzdu \\ = \mathrm{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s+1}\Phi(\begin{pmatrix} z & zx \\ & z\end{pmatrix})\omega^S(z,x)d^\times zdx \quad (54)\end{aligned}$$

and now $\omega^S(z,x)$ is independent of z.

The first integral (53) is again just a Tate integral in the variable z. Hence the analytic behaviour of this part is as follows:

Proposition 58. *The integral*

$$\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\lambda_S\int_{U(\mathbb{A})}\int_{Z(\mathbb{A})}F_s(zu)dzdu$$

defines a holomorphic function for $\Re s > \frac{1}{2}$ and can be meromorphically continued to the whole complex plane. Its only poles are at $s = \frac{1}{2}$ and $s = 0$ and they are simple poles with residues

$$\frac{1}{2}\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\lambda_S\int_{\mathbb{A}}\int_{\mathbb{A}}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})dzdx$$

at $s=\frac{1}{2}$ *and*

$$-\frac{1}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\lambda_S\int_{\mathbb{A}}\Phi(\begin{pmatrix}0 & x\\ & 0\end{pmatrix})dx$$

at $s=0$.

We now analyse the second integral (54), which is

$$\sum_v \operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s+1}\Phi(\begin{pmatrix}z & zx\\ & z\end{pmatrix})\omega_v^S(z,x)d^\times z dx. \tag{55}$$

For a place v outside of our finite set S the function Φ_v coincides with the characteristic function of $\operatorname{Mat}_{2\times 2}(\mathcal{O}_v)$ so that we can compute

$$\int_{F_v}\int_{F_v^\times}|z|_v^{2s+1}\Phi_v(\begin{pmatrix}z & zx\\ & z\end{pmatrix})\omega_v^S(z,x)d^\times z dx = -\zeta_{F,v}'(2s)$$

and

$$\int_{F_v}\int_{F_v^\times}|z|_v^{2s+1}\Phi_v(\begin{pmatrix}z_v & z_v x_v\\ & z_v\end{pmatrix})d^\times z_v dx_v = \zeta_{F,v}(2s).$$

Thus (55) equals

$$\begin{aligned}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)&\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s+1}\Phi(\begin{pmatrix}z & zx\\ & z\end{pmatrix})\sum_{v\in S}\omega_v^S(z,x)d^\times z dx\\ &-\frac{\zeta_F^{S\prime}(2s)}{\zeta_F^S(2s)}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi(\begin{pmatrix}z & x\\ & z\end{pmatrix})d^\times z_v dx_v,\end{aligned}$$

which is

$$\begin{aligned}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)&\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi(\begin{pmatrix}z & x\\ & z\end{pmatrix})(\log|x|_S-\log|z|_S+\frac{\zeta_{F,S}'(2s)}{\zeta_{F,S}(2s)})d^\times z dx\\ &-\frac{\zeta_F^{*\prime}(2s)}{\zeta_F^*(2s)}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi(\begin{pmatrix}z & x\\ & z\end{pmatrix})d^\times z dx\end{aligned}$$

The integral $\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi((\begin{smallmatrix}z & x\\ & z\end{smallmatrix}))(-\log|z|_S+\frac{\zeta_{F,S}^{*\prime}(2s)}{\zeta_{F,S}(2s)})d^\times z dx$ can be written as $-z'(\Phi,2s)+\frac{\zeta_F^{*\prime}(2s)}{\zeta_F^*(2s)}z(\Phi,2s)$ with

$$z(\Phi,t)=\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^t\Phi(\begin{pmatrix}z & x\\ 0 & z\end{pmatrix})d^\times z dx.$$

Hence (55) is the same as

$$-\operatorname{vol}(F^\times\backslash\mathbb{A}^1)z'(\Phi,2s)+\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|z|^{2s}\Phi(\begin{pmatrix}z & x\\ & z\end{pmatrix})\log|x|_S d^\times z dx$$

for $\Re s>\frac{3}{2}$. As $z(\Phi,t)$ as well as the second integral are Tate integrals with well-known analytic properties, the following is immediate:

Proposition 59. *The function defined by* (54) *for* $\Re s > \frac{3}{2}$ *has a meromorphic continuation to the complex plane with poles of second order at* $s = \frac{1}{2}$ *and* $s = 0$*, and is holomorphic elsewhere. The principal part of the Laurent expansion at* $s = \frac{1}{2}$ *is*

$$(s-\frac{1}{2})^{-2}\frac{1}{4}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^2}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})dxdz \\ +(s-\frac{1}{2})^{-1}\frac{1}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^2}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})\log|x|_S dxdz,$$

and at $s = 0$

$$-s^{-2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\frac{1}{4}\int_{\mathbb{A}}\Phi(\begin{pmatrix} 0 & x \\ & 0\end{pmatrix})dx \\ -s^{-1}\frac{1}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\Phi(\begin{pmatrix} 0 & x \\ & 0\end{pmatrix})\log|x|_S dx.$$

For convenience we summarise the last two propositions to get

Corollary 60. *The contribution from the singular hyperbolic orbits converges absolutely for* $\Re s > \frac{1}{2}$*. It has a meromorphic continuation to all* $s \in \mathbb{C}$ *with poles of second order at* $s = \frac{1}{2}$ *and* $s = 0$*. The main part of the Laurent-expansion at* $s = \frac{1}{2}$ *is*

$$(s-\frac{1}{2})^{-2}\frac{1}{4}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^2}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})dxdz \\ +(s-\frac{1}{2})^{-1}\frac{1}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}^2}\Phi(\begin{pmatrix} z & x \\ & z\end{pmatrix})(\log|x|_S+\lambda_S)dxdz,$$

and at $s = 0$

$$-s^{-2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\frac{1}{4}\int_{\mathbb{A}}\Phi(\begin{pmatrix} 0 & x \\ & 0\end{pmatrix})dx \\ -s^{-1}\frac{1}{2}\operatorname{vol}(F^\times\backslash\mathbb{A}^1)\int_{\mathbb{A}}\Phi(\begin{pmatrix} 0 & x \\ & 0\end{pmatrix})(\log|x|_S+\lambda_S)dx.$$

Note that the last integrals involving the term $(\log|x|_S + \lambda_S)$ are invariant under enlarging S.

V.i.vi Summary for the geometric side for GL(2)

The results from the last sections can be summarised as follows. (Recall that Φ is assumed to be $\mathbf{K}$-central. Otherwise Φ has to be replaced by $\int_{\mathbf{K}}\Phi(k^{-1}\cdot k)dk$ in the following.)

Theorem 61. *The geometric side of the trace formula for* $\mathrm{GL}(2)$ *with test function* f_s *has, as a function of* s*, a meromorphic continuation to all* $s \in \mathbb{C}$ *(as a whole, but also each single contribution). Up to an entire function it equals the sum of*

- *the adjusted Shintani zeta function*

$$Z_{ad}(\Psi, 2s)$$

for $\Psi \in \mathcal{S}(V_{\mathbb{A}}), \quad \Psi(X) = \int_{\mathbb{A}} \Phi(a + \gamma_X) da,$

- *the unstable distribution*

$$T(2s + 1, \Phi^0),$$

with $\Phi^0 \in \mathcal{S}(\mathbb{A}^2), \quad \Phi^0(a, b) = \Phi(\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)),$

- *and the contribution from the singular classes*

$$\begin{aligned} \mathrm{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A})) z_0(\Phi, 2s+1) - \mathrm{vol}(F^\times \backslash \mathbb{A}^1) z'(\Phi, 2s) \\ + \lambda_S \, \mathrm{vol}(F^\times \backslash \mathbb{A}^1) z(\Phi, 2s) + \mathrm{vol}(F^\times \backslash \mathbb{A}^1) z_S(\Phi, 2s) \end{aligned}$$

where the zeta functions are defined for $\Re \sigma > 1$ *by*

$$z_0(\Phi, \sigma) = \int_{Z(\mathbb{A})} |\det z|^{\frac{\sigma}{2}} \Phi(z) dz,$$

$$z(\Phi, \sigma) = \int_{Z(\mathbb{A})} \int_{U(\mathbb{A})} |\det z|^{\frac{\sigma+1}{2}} \Phi(zu) dx dz,$$

and

$$z_S(\Phi, \sigma) = \int_{\mathbb{A}^\times} \int_{\mathbb{A}} |z|^\sigma \Phi(\left(\begin{smallmatrix} z & x \\ & z \end{smallmatrix}\right)) \log |x|_S dx d^\times z.$$

Note that the last two summands can be written as the derivative of a two-dimensional zeta function: If we define

$$\tilde{z}(\Phi, \sigma, \tau) = \int_{(\mathbb{A}^\times)^2} |x|^\tau |z|^\sigma \Phi(\begin{pmatrix} z & x \\ & z \end{pmatrix}) d^\times x d^\times z,$$

then

$$\begin{aligned} \lambda_S \, \mathrm{vol}(F^\times \backslash \mathbb{A}^1) z(\Phi, 2s) + \mathrm{vol}(F^\times \backslash \mathbb{A}^1) z_S(\Phi, 2s) \\ = \mathrm{vol}(F^\times \backslash \mathbb{A}^1) \frac{\partial}{\partial \tau} \left[(\tau - 1) \tilde{z}(\Phi, \tau, 2s) \right]_{|\tau=1}, \end{aligned}$$

which yields an expression apparently not depending on the chosen set S.

V.ii The regular elliptic terms for GL(3)

As we saw for the case of GL(2) the regular elliptic contribution yields a meromorphic function which contains information on some arithmetic quantities. We now want to study the analogous problem for $G = \mathrm{GL}(3)$. We assume our ground field to be $F = \mathbb{Q}$. Recall that for $\Re s > 2$ we defined $\mathcal{E}(s)$ to be the regular elliptic contribution. If we assume that Φ_f is trivial on $\hat{\mathbb{Z}}$, this is the same as

$$\begin{aligned}\mathcal{E}(s) &= \sum_{[\gamma]\subseteq \mathrm{GL}_3(\mathbb{Q})_{\mathrm{ell,reg}}} \nu(\gamma) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f_s(g^{-1}\gamma g)dg \\ &= \sum_{[\gamma]\subseteq \mathrm{GL}_3(\mathbb{Q})_{\mathrm{ell,reg}}} \nu(\gamma) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \int_0^\infty \lambda^{3s+3}\Phi(\lambda g^{-1}\gamma g)d^\times\lambda dg\end{aligned}$$

with $\nu(\gamma) = \mathrm{vol}(\mathbb{R}_{>0}G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))$. From the results for the spectral side, i.e. Theorems 32 and 33, and the fact that almost all of the geometric terms for GL(3) can be continued to some larger half plane by Propositions 29 and 30, we get the following.

Proposition 62. *The function $\mathcal{E}(s)$ is holomorphic for $\Re s > 2$ and has a meromorphic continuation at least up to $\Re s > \frac{3}{2}$. The only pole in this region is at $s = 2$, which is simple with residue*

$$\zeta^*(3)\zeta^*(2) \operatorname*{res}_{s=1} \zeta^*(s) \int_{\mathrm{Mat}_{3\times 3}(\mathbb{A})} \Phi(x)dx = \frac{\zeta(3)}{12} \int_{\mathrm{Mat}_{3\times 3}(\mathbb{A})} \Phi(x)dx.$$

In generalisation of the Remarks 34 and 39 in the last chapters, this stays true even if we replace f_s by one of the more general functions $\tilde{f}_s \in \mathcal{C}^N(\mathrm{GL}_3(\mathbb{A})^1, K)$ from Remark 34 and Φ by $\tilde{\Phi}$ provided that N is sufficiently large.

V.ii.i Cubic fields and elliptic elements

Suppose that E is a cubic extension over $\mathbb{Q}$, and let $\mathcal{O}_E$ be the ring of integers of E. Let $\mathcal{N}_{E/\mathbb{Q}} : E \longrightarrow \mathbb{Q}$ and $\mathrm{tr}_{E/\mathbb{Q}} : E \longrightarrow \mathbb{Q}$ be the norm and trace map of $E/\mathbb{Q}$. There is a simple corresondence between cubic field extensions and the regular elliptic elements. Here the conjugacy class of a number field $E/\mathbb{Q}$ consists of all subfields of the Galois closure E^{Gal} of $E/\mathbb{Q}$, which are conjugate to E via some element of the Galois group $\mathrm{Gal}(E^{\mathrm{Gal}}/\mathbb{Q})$.

Lemma 63. *(i) There is a well-defined, surjective map*

$$G(\mathbb{Q})_{\mathit{ell,reg}} \longrightarrow \{[E] \text{ conjugacy class of cubic extension } E/\mathbb{Q}\}$$

taking $\gamma \in G(\mathbb{Q})_{\mathit{ell,reg}}$ to $[\mathbb{Q}(\xi)]$ for $\xi \in \mathbb{C}$ some eigenvalue of γ. This map preserves the discriminant. If E is a cubic field, denote by Γ_E the fibre over $[E]$. Then Γ_E is closed under taking conjugacy classes in $G(\mathbb{Q})$, i.e., if $\gamma \in \Gamma_E$, then $[\gamma] \subseteq \Gamma_E$.

(ii) Let $E/\mathbb{Q}$ be a cubic extension. There is a surjection

$$E\backslash\mathbb{Q} \longrightarrow \Gamma_E/\textit{conjugacy}, \quad \xi \mapsto [\gamma_\xi]$$

such that the characteristic polynomials of ξ and γ_ξ coincide. In particular,

$$\det \gamma_\xi = \mathcal{N}_{E/\mathbb{Q}}(\xi) \text{ and } \operatorname{tr}\gamma_\xi = \operatorname{tr}_{E/\mathbb{Q}}\xi$$

and the characteristic polynomial of γ_ξ has integer coefficients if and only if $\xi \in \mathcal{O}_E\backslash\mathbb{Z}$. If $E/\mathbb{Q}$ is not Galois, the above map is a bijection, but if $E/\mathbb{Q}$ is Galois, it is a 3-to-1 covering.

Proof. (i) Let $\gamma \in G(\mathbb{Q})_{\text{ell,reg}}$ and let χ be its characteristic polynomial. Let $\gamma_i \in \mathbb{C}$, $i = 1, 2, 3$ be the roots of χ. Let $E_i = \mathbb{Q}(\gamma_i)$ which is a cubic extension over $\mathbb{Q}$ as χ is irreducible. For the above map to be well-defined, we have to show that $[E_i] = [E_j]$, $i, j = 1, 2, 3$. But this is clear, since E^{Gal} is the splitting field of χ over $\mathbb{Q}$ so that $\text{Gal}(E^{\text{Gal}}/\mathbb{Q})$ acts transitively on $\{E_1, E_2, E_3\}$.

(ii) Let $\xi \in E\backslash\mathbb{Q}$ and let γ_ξ be the companion matrix of the characteristic polynomial of ξ over $\mathbb{Q}$. Since $\mathbb{Q}(\xi) = E$, the characteristic polynomial of ξ is irreducible, and thus $\gamma_\xi \in G(\mathbb{Q})_{\text{ell, reg}}$. $\xi, \xi' \in E$ have the same image exactly when their characteristic polynomials coincide, i.e. if and only if there is some $\sigma \in \text{Gal}(E^{\text{Gal}}/\mathbb{Q})$ such that $\sigma(\xi) = \xi'$. If $E/\mathbb{Q}$ is Galois, the above map therefore is 3-to-1 if E, as the Galois group operates fixed point free on $E\backslash\mathbb{Q}$. If $E/\mathbb{Q}$ is not Galois, ξ can not be mapped into E by any non-trivial subgroup of $\text{Gal}(E^{\text{Gal}}/\mathbb{Q})$, as this would either imply that the extension $\mathbb{Q}(\xi)$ is quadratic or that $E/\mathbb{Q}$ is Galois. Hence in this case the map is $1 - 1$.

□

Note that if γ and $[E]$ are associated as in the lemma, then $\mathbb{Q}[\gamma] \simeq E$ as $\mathbb{Q}$-algebras.

For a cubic extension $E/\mathbb{Q}$ define

$$\eta_E(s) = \sum_{\xi \in E\backslash\mathbb{Q}} \int_{G_{\gamma_\xi}(\mathbb{A})\backslash G(\mathbb{A})} \int_0^\infty \lambda^{3s+3}\Phi(\lambda g^{-1}\gamma_\xi g) d^\times\lambda dg,$$

and further let $\nu(E) = \operatorname{res}_{s=1}\zeta_E(s)|D_E|^{\frac{1}{2}}$, which is the same as the volume of $\mathbb{R}_{>0}G_{\gamma_\xi}(\mathbb{Q})\backslash G_{\gamma_\xi}(\mathbb{A}) \simeq E^\times\backslash\mathbb{A}_E^1$. Hence by Lemma 63

$$\mathcal{E}(s) = \sum_{E/\mathbb{Q} \text{ cubic}} \frac{\nu(E)}{|\operatorname{Aut}(E/\mathbb{Q})|}\eta_E(s). \tag{56}$$

One would like to filter certain partial sums belonging to fields with prescribed splitting behaviour at finitely many places by choosing appropriate Φ as in the case GL(2), see [Da93]. We shall see that even in the case that one wishes to pick out the totally real cubic fields, one is meeting problems.

V.ii.ii Cubic orders and the non-archimedean integrals

Let E, $\xi \in E\backslash\mathbb{Q}$, $\gamma = \gamma_\xi \in G(\mathbb{Q})_{\mathrm{ell,reg}}$ be as before. In this section $\Re s > 2$, and we only consider $\Phi \in \mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{A}))$ such that $\Phi = \Phi_\infty\Phi_f$ with Φ_f the characteristic function of $\mathrm{Mat}_{3\times3}(\hat{\mathbb{Z}})$.

Lemma 64. *Let $\Phi \in \mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{A}))$, and assume that Φ_f is the characteristic function of $\mathrm{Mat}_{3\times3}(\hat{\mathbb{Z}}) \subseteq \mathrm{Mat}_{3\times3}(\mathbb{A}_f)$. Then the orbital integral $\int_{G_{\gamma_\xi}(\mathbb{A})\backslash G(\mathbb{A})}\int_0^\infty \lambda^{3s+3}\Phi(\lambda g^{-1}\gamma_\xi g)d^\times\lambda dg$ vanishes unless $\xi \in \mathcal{O}_E$.*

Accordingly, we shall henceforth assume $\xi \in \mathcal{O}_E$.

Proof. Note that the integral equals the product

$$\int_{G_{\gamma_\xi}(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^{3s+3}\Phi_\infty(\lambda g^{-1}\gamma_\xi g)d^\times\lambda dg \cdot \prod_p \int_{G_{\gamma_\xi}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \Phi_p(g^{-1}\gamma_\xi g)\lambda dg.$$

A necessary condition for the local integral $\int_{G_{\gamma_\xi}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \Phi_p(g^{-1}\gamma_\xi g)dg$ not to vanish is that the coefficients of the characteristic polynomial of γ_ξ are in $\mathbb{Z}_p$. By a local-global argument it follows that the coefficients have to be in $\mathbb{Z}$, i.e. $\xi \in \mathcal{O}_E$, in order that the above integral has a chance to be non-zero. □

For a prime p let $E_p = E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_{E,p} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$. As $E/\mathbb{Q}$ is separable, [Ne99, Chapter II, Proposition (8.3)] asserts that $E_p = \prod_{\mathfrak{p}|p} E_\mathfrak{p}$ where the product is over all prime ideals $\mathfrak{p} \subseteq \mathcal{O}_E$ above p, and $E_\mathfrak{p}$ denotes the completion of E at $\mathfrak{p}$. If $\chi(T) \in \mathbb{Z}[T]$ is the characteristic polynomial of ξ, the splitting behaviour of the prime p in $\mathcal{O}_E$ is determined by the factorisation of $\chi(T)$ into irreducible components over $\mathbb{Q}_p$. Write $\chi(T) = \prod_i \chi_i(T)$ with $\chi_i(T) \in \mathbb{Q}_p[T]$ irreducible over $\mathbb{Q}_p$. As $\operatorname{char}\mathbb{Q}_p = \operatorname{char}\mathbb{Q} = 0$, and χ was irreducible over $\mathbb{Q}$, $(\chi_i, \chi_j) = 1$ for all $i \neq j$. Then $E_p \simeq \prod_i K_i$ with $K_i = \mathbb{Q}_p[T]/(\chi_i(T))$ and between the sets $\{\chi_i\}$ and $\{\mathfrak{p}|p\}$ there is a bijection such that $E_\mathfrak{p} \simeq K_i$ if $\mathfrak{p}$ and i correspond to each other. For any field extension $E/\mathbb{Q}$ denote by $d : E \longrightarrow \mathbb{Q}$ the discriminant map. The discriminant map $\mathrm{Mat}_{n\times n}(\mathbb{Q}) \longrightarrow \mathbb{Q}$ as well is denoted by d.

We reduce the computation of the non-archimedean orbital integral to the irreducible case using the following lemma.

Lemma 65. *Let $n \geq 1$, $\chi(T) \in \mathbb{Z}[T]$ an irreducible polynomial (over $\mathbb{Q}$) of degree n. Let $\theta \in \overline{\mathbb{Q}}$ be some root of χ, and suppose that we have a* *factorisation* $\chi(T) = \prod_{i=1}^{r} \chi_i(T)$ *over* $\mathbb{Q}_p$ *with* $\chi_i(T) \in \mathbb{Q}_p[T]$ *irreducible over* $\mathbb{Q}_p$ *of degree* n_i. *Let* $\theta_i \in \overline{\mathbb{Q}_p}$ *be a root of* χ_i. *Put* $K = \mathbb{Q}[\theta]$, *and* $K_i = \mathbb{Q}_p[\theta_i]$. *Then* $K_p \simeq \prod_i K_i$ *and*

$$\int_{\mathrm{GL}_{n,\gamma_\theta}(\mathbb{Q}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)} \Phi_p^n(g^{-1}\gamma_\theta g)dg = \prod_{i<j} |\Delta(\theta_i,\theta_j)|_p^{-1} \prod_i \int_{\mathrm{GL}_{n_i,\gamma_{\theta_i}}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n_i}(\mathbb{Q}_p)} \Phi_p^{n_i}(g^{-1}\gamma_{\theta_i} g)dg$$

where Φ_p^k *is the characteristic function of* $\mathrm{Mat}_{k\times k}(\mathbb{Z}_p) \subseteq \mathrm{Mat}_{k\times k}(\mathbb{Q}_p)$, *and* $\Delta(\theta_i,\theta_j)$ *is the determinant of the linear map*

$$\varphi_{ij} : \mathrm{Mat}_{n_i\times n_j}(\mathbb{Q}_p) \longrightarrow \mathrm{Mat}_{n_i\times n_j}(\mathbb{Q}_p), \quad A \mapsto \gamma_{\theta_i}A - A\gamma_{\theta_j},$$

or more explicitly,

$$\Delta(\theta_i,\theta_j) = \prod_{\alpha,\beta} (\alpha - \beta)$$

where $\alpha \in \overline{\mathbb{Q}_p}$ *runs over all eigenvalues of* γ_{θ_i} *and* $\beta \in \overline{\mathbb{Q}_p}$ *over all eigenvalues of* γ_{θ_j}.

Note that by the definition of the discriminant $d(\theta)$ we have $|d(\theta)|_p = \prod_{i<j} |\Delta(\theta_i,\theta_j)^2|_p \prod_i |d(\theta_i)|_p$.

Proof. Let $P \subseteq \mathrm{GL}_n$ be the standard parabolic subgroup associated to the partition $(n_1,\dots,n_r)$ of n, $M \simeq \mathrm{GL}(n_1)\times\ldots\times\mathrm{GL}(n_r)$ its Levi component and U its unipotent radical. γ_θ is conjugate to $\mathrm{diag}(\gamma_{\theta_1},\dots,\gamma_{\theta_r}) \in M(\mathbb{Q}_p)$ in $\mathrm{GL}_n(\mathbb{Q}_p)$, and $\mathrm{GL}_{n,\gamma_\theta}(\mathbb{Q}_p) \simeq \mathrm{GL}_{n_1,\theta_1}(\mathbb{Q}_p)\times\ldots\times\mathrm{GL}_{n_r,\theta_r}(\mathbb{Q}_p) \subseteq M(\mathbb{Q}_p)$ (we write $\mathrm{GL}_{n_1,\theta_1}(\mathbb{Q}_p) = \mathrm{GL}_{n_1,\gamma_{\theta_1}}(\mathbb{Q}_p)$). As Φ_p^n is invariant under $\mathrm{GL}_n(\mathbb{Z}_p)$, using Iwasawa decomposition the above integral equals

$$\int_{(\mathrm{GL}_{n_1,\theta_1}(\mathbb{Q}_p)\times\ldots\times\mathrm{GL}_{n_r,\theta_r}(\mathbb{Q}_p))\backslash M(\mathbb{Q}_p)} \int_{U(\mathbb{Q}_p)} \Phi_p^n(u^{-1}\,\mathrm{diag}(m_1^{-1}\gamma_{\theta_1}m_1,\dots,m_r^{-1}\gamma_{\theta_r}m_r)u)dudm$$

where we wrote $m = \mathrm{diag}(m_1,\dots,m_r)$. Since the eigenvalues (in $\overline{\mathbb{Q}_p}$) of the γ_{θ_i}'s are pairwise distinct, the maps φ_{ij} are isomorphisms. Hence a change of variables shows the first assertion. Hence it remains to show the explicit form of $\Delta(\theta_i,\theta_j)$. Let $B_i \in \mathrm{GL}_{n_i}(\overline{\mathbb{Q}_p})$, $B_j \in \mathrm{GL}_{n_j}(\overline{\mathbb{Q}_p})$ such that $B_i^{-1}\gamma_{\theta_i}B_i = \mathrm{diag}(\alpha_1,\dots,\alpha_{n_i})$, $B_j^{-1}\gamma_{\theta_j}B_j = \mathrm{diag}(\beta_1,\dots,\beta_{n_j})$ (such B_i, B_j exist, since χ has no multiple roots). Then

$$\varphi_{ij}(A) = B_i((B_i^{-1}\gamma_{\theta_i}B_i)(B_i^{-1}AB_j) - (B_i^{-1}AB_j)(B_j^{-1}\gamma_{\theta_j}B_j))B_j^{-1},$$

i.e. the map φ_{ij} is the same as $\psi_{ij}^{-1} \circ \tilde{\varphi}_{ij} \circ \psi_{ij}$ restricted to $\mathrm{Mat}_{n_i \times n_j}(\mathbb{Q}_p)$, where $\psi_{ij}, \tilde{\varphi}_{ij} : \mathrm{Mat}_{n_i \times n_j}(\overline{\mathbb{Q}_p}) \longrightarrow \mathrm{Mat}_{n_i \times n_j}(\overline{\mathbb{Q}_p})$, $\psi_{ij}(A) = B_i^{-1} A B_j$, and $\tilde{\varphi}_{ij}(A) = \mathrm{diag}(\alpha_1, \ldots, \alpha_{n_i})A - A\,\mathrm{diag}(\beta_1, \ldots, \beta_{n_j})$. Hence the determinant of φ_{ij} is the same as the determinant of $\tilde{\varphi}_{ij}$, which is easily seen to equal the product given above. □

Let A be some finite-dimensional semi-simple $\mathbb{Q}$-algebra, and $R \subseteq A$ a $\mathbb{Z}$-order. We denote by $\mathrm{Frac}(R)$ the fractional ideals of R in A, i.e. the set of all full rank $\mathbb{Z}$-lattices $\mathfrak{a} \subseteq A$ such that $R\mathfrak{a} \subseteq \mathfrak{a}$. Let $\mathrm{Inv}(R) \subseteq \mathrm{Frac}(R)$ be the subset of invertibel R-ideals in A, and $P(R) = \{aR \mid a \in A^\times\}$ be the set of all principal ideals in R. If $\mathfrak{a} \subseteq A$ is a full rank lattice, let $\mathcal{M}(\mathfrak{a}) = \{a \in A \mid a\mathfrak{a} \subseteq \mathfrak{a}\}$ be the multiplier of $\mathfrak{a}$. This is a $\mathbb{Z}$-order in A (cf. [Ne99, Chapter I., §12]) so that $\mathfrak{a} \in \mathrm{Frac}(\mathcal{M}(\mathfrak{a}))$. Let $\mathrm{Frac}^0(R) = \{\mathfrak{a} \in \mathrm{Frac}(R) \mid \mathcal{M}(\mathfrak{a}) = R\}$. We make the same definitions if we replace $\mathbb{Q}$ by one of the local fields $\mathbb{Q}_p$ and $\mathbb{Z}$ by $\mathbb{Z}_p$. In general, neither $\mathrm{Frac}(R)$ nor $\mathrm{Frac}^0(R)$ are groups, but they are acted on by $P(R)$ so that we may build the quotients $\mathrm{Frac}(R)/P(R)$ and $\mathrm{Frac}^0(R)/P(R)$, which are both finite. (The finiteness follows along the same lines as the finiteness of the class group is deduced.)

Lemma 66. *Let $n \geq 1$, p a prime, and $K = \mathbb{Q}_p(\theta)$ a field extension of degree n generated by some element $\theta \in \overline{\mathbb{Q}_p}$, which has characteristic polynomial $\chi(T) \in \mathbb{Z}[T]$. Let Φ_p be the characteristic function of* $\mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \subseteq \mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$. *Then*

$$\int_{\mathbb{Q}_p^\times \backslash \mathrm{GL}_n(\mathbb{Q}_p)} \Phi_p(g^{-1}\gamma_\theta g)dg$$
$$= [K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] \sum_{\mathfrak{o} \subseteq \mathcal{O}_K,\ \theta \in \mathfrak{o}} |\mathrm{Frac}^0(\mathfrak{o})/P(\mathfrak{o})| [\mathcal{O}_K^\times : \mathfrak{o}^\times]$$

where $\mathfrak{o}$ runs over all $\mathbb{Z}_p$-orders in $\mathcal{O}_K$ containing θ.

Note that if $\theta \notin \mathcal{O}_K$, then the right hand side is 0, and if $\mathbb{Z}_p[\theta] = \mathcal{O}_K$, the right hand side equals

$$[K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] |\mathrm{Frac}^0(\mathcal{O}_K)/K^\times| = [K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] |\mathrm{Inv}(\mathcal{O}_K)/P(\mathcal{O}_K)|$$
$$= [K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)],$$

since $\mathcal{O}_K$ is a local ring. If the extension $K/\mathbb{Q}_p$ is unramified, $[K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] = 1$. In general, $[K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] = [K : \mathcal{O}_K^\times \mathbb{Q}_p]$, so that this index equals the ramification index, and we therefore have $[K^\times : (\mathcal{O}_K^\times \mathbb{Q}_p^\times)] = \mathrm{vol}(\mathbb{Q}_p^\times \backslash \mathrm{GL}_{n,\theta}(\mathbb{Q}_p))$. Thus in any case,

$$I_{\mathrm{orb},p}(\Phi_p, \gamma_\theta) := \int_{\mathrm{GL}_{n,\theta}(\mathbb{Q}_p) \backslash \mathrm{GL}_n(\mathbb{Q}_p)} \Phi_p(g^{-1}\gamma_\theta g)dg$$
$$= \sum_{\mathfrak{o} \subseteq \mathcal{O}_K,\ \theta \in \mathfrak{o}} |\mathrm{Frac}^0(\mathfrak{o})/P(\mathfrak{o})| [\mathcal{O}_K^\times : \mathfrak{o}^\times].$$

Proof of Lemma 66. The set $\{1, \theta, \dots, \theta^{n-1}\}$ forms a basis of K relative to which the matrix γ_θ corresponds to the endomorphism $K \longrightarrow K$ given by multiplication with θ. Moreover, this basis defines a map

$$\Psi : \mathrm{GL}_n(\mathbb{Q}_p) \longrightarrow \mathcal{L}_p = \{L \subseteq K \mid L \text{ is } \mathbb{Z}_p\text{-lattice of full rank}\}.$$

Hence $\Phi_p(g^{-1}\gamma_\theta g) \neq 0$ if and only if θ maps the lattice $L_g \subseteq K$ defined by g into itself, i.e. $\theta L_g \subseteq L_g$, or equivalently $\theta \in \mathcal{M}(L_g)$. Hence the integral equals

$$\sum_{\mathfrak{o} \subseteq \mathcal{O}_K,\ \theta \in \mathfrak{o}} \ \sum_{\mathfrak{a} \in \mathrm{Frac}^0(\mathfrak{o})/\mathbb{Q}_p^\times} \int_{\Psi^{-1}(\mathfrak{a})} dg.$$

Hence we have to compute the volume of $\Psi^{-1}(\mathfrak{a})$ as a subset of $\mathrm{GL}_n(\mathbb{Q}_p)$. Two elements $g_1, g_2 \in \mathrm{GL}_n(\mathbb{Q}_p)$ define the same $\mathbb{Z}_p$-lattice if and only if there is some $k \in \mathrm{GL}_n(\mathbb{Z}_p)$ with $g_2 = g_1 k$. Thus $\int_{\Psi^{-1}(\mathfrak{a})} dg = 1$. Since $|\mathrm{Frac}^0(\mathfrak{o})/\mathbb{Q}_p^\times| = |\mathrm{Frac}^0(\mathfrak{o})/(\mathfrak{o}^\times\mathbb{Q}_p^\times)| = |\mathrm{Frac}^0(\mathfrak{o})/(\mathcal{O}_K^\times\mathbb{Q}_p^\times)|[\mathcal{O}_K^\times : \mathfrak{o}^\times]$ the assertion follows. □

As a direct consequence of the last two lemmas we get

Corollary 67. *For any $\gamma \in \mathrm{GL}_n(\mathbb{Q})_{ell,reg}$ we have*

$$[\mathcal{O}_{\mathbb{Q}_p[\gamma]} : \mathbb{Z}_p[\gamma]]^{-1} I_{orb,p}(\Phi_p, \gamma) \geq 1.$$

Proof. By Lemma 66 the integral $[\mathcal{O}_{\mathbb{Q}_p[\gamma]} : \mathbb{Z}_p[\gamma]]^{-1} I_{\mathrm{orb},p}(\Phi_p, \gamma)$ equals a product of finitely many terms of the form

$$\frac{[\mathcal{O}_E^\times : \mathbb{Z}_p[\theta]^\times]}{[\mathcal{O}_E : \mathbb{Z}_p[\theta]]} |\mathrm{Frac}^0(\mathbb{Z}_p[\theta])/P(\mathbb{Z}_p[\theta])| + \frac{1}{[\mathcal{O}_E : \mathbb{Z}_p[\theta]]} \sum_{\mathbb{Z}_p[\theta] \subsetneq \mathfrak{o} \subseteq \mathcal{O}_E} |\mathrm{Frac}^0(\mathfrak{o})/P(\mathfrak{o})|[\mathcal{O}_E^\times : \mathfrak{o}^\times].$$

for $E/\mathbb{Q}_p$ a finite extension generated by $\theta \in E$ with maximal ideal $\mathfrak{p} \subseteq \mathcal{O}_E$ of norm q. Hence it certainly suffices to show $\frac{[\mathcal{O}_E^\times : \mathbb{Z}_p[\theta]^\times]}{[\mathcal{O}_E : \mathbb{Z}_p[\theta]]} \geq 1$, since $|\mathrm{Frac}^0(\mathbb{Z}_p[\theta])/P(\mathbb{Z}_p[\theta])| \geq 1$ and the rest of the sum is ≥ 0. Let $\mathfrak{f} \subseteq \mathbb{Z}_p[\theta]$ be the conductor of $\mathbb{Z}_p[\theta]$. For any order $\mathcal{O}$, $\mathfrak{f} \subseteq \mathcal{O} \subseteq \mathcal{O}_E$ there is an isomorphism $\mathcal{O}^\times/\mathfrak{f} \longrightarrow (\mathcal{O}/\mathfrak{f})^\times$. Moreover, $\mathfrak{p}/\mathfrak{f} \subseteq \mathcal{O}_E/\mathfrak{f}$ is the unique maximal ideal so that $(\mathfrak{p} \cap \mathbb{Z}_p[\theta])/\mathfrak{f}$ is the unique maximal ideal in $\mathbb{Z}_p[\theta]/\mathfrak{f}$. Hence

$$\#(\mathcal{O}_E/\mathfrak{f})^\times = \#(\mathcal{O}_E/\mathfrak{f}) - \#(\mathfrak{p}/\mathfrak{f}) = \#(\mathcal{O}_E/\mathfrak{f})(1 - q^{-1})$$

and

$$\#(\mathbb{Z}_p[\theta]/\mathfrak{f})^\times = \#(\mathbb{Z}_p[\theta]/\mathfrak{f})(1 - (\#(\mathbb{Z}_p[\theta]/(\mathbb{Z}_p[\theta] \cap \mathfrak{p})))^{-1}).$$

But since $\mathbb{Z}_p[\theta]/(\mathbb{Z}_p[\theta] \cap \mathfrak{p}) \hookrightarrow \mathcal{O}_E/\mathfrak{p}$ is injective, we altogether get

$$\frac{[\mathcal{O}_E^\times : \mathbb{Z}_p[\theta]^\times]}{[\mathcal{O}_E : \mathbb{Z}_p[\theta]]} = \frac{1 - q^{-1}}{1 - (\#(\mathbb{Z}_p[\theta]/(\mathbb{Z}_p[\theta] \cap \mathfrak{p})))^{-1}} \geq 1.$$

□

If $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{A}_f), \mathbf{K}_f)$ is the characteristic function of $\mathrm{Mat}_{3\times 3}(\hat{\mathbb{Z}})$, we define

$$c_0(\gamma) = \frac{1}{[\mathcal{O}_{\mathbb{Q}[\gamma]} : \mathbb{Z}[\gamma]]} \int_{\mathrm{GL}_{3,\gamma}(\mathbb{A}_f)\backslash \mathrm{GL}_3(\mathbb{A}_f)} \Phi_f(x^{-1}\gamma x)dx$$

and if $E/\mathbb{Q}$ is a cubic field, $c_0(\xi) = c_0(\gamma_\xi)$ for any $\xi \in E\backslash\mathbb{Q}$. In particular, $c_0(\xi) = 0$ unless $\xi \in \mathcal{O}_E\backslash\mathbb{Z}$. Interpreting $c_0(\xi)$ as in the last lemma, we see that $c_0(\xi + a) = c_0(\xi)$ for any $a \in \mathcal{O}_E\backslash\mathbb{Z}$ and $a \in \mathbb{Z}$ so that c_0 is a well-defined function on $(\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}$. If, more generally, $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{A}_f), K)$ is arbitrary, put

$$c(\Phi_f, \gamma) = \frac{1}{[\mathcal{O}_{\mathbb{Q}[\gamma]} : \mathbb{Z}[\gamma]]} \int_{\mathrm{GL}_{3,\gamma}(\mathbb{A}_f)\backslash \mathrm{GL}_3(\mathbb{A}_f)} \Phi_f(x^{-1}\gamma x)dx,$$

and similarly define $c(\Phi_f, \xi)$.

V.ii.iii A first asymptotic

The aim of this section is to prove an asymptotic of sums of certain orbital integrals, see Proposition 68. We first need some notation on quadratic forms on lattices and their successive minima. Suppose $\Lambda \subseteq \mathbb{R}^2$ is a lattice of full rank, and $Q : \Lambda \longrightarrow \mathbb{R}$ a positive definite quadratic form. Let $m_1(Q) < m_2(Q)$ be the successive minima of Q. Let $d(Q)$ be the discriminant of Q. For a totally real cubic field let $Q_E : (\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z} \longrightarrow \mathbb{R}$ be the positive definite quadratic form $Q_E(\xi) = \mathrm{tr}\,\xi^2 - \frac{1}{3}(\mathrm{tr}\,\xi)^2$ whose successive minima we denote by $m_1(E) < m_2(E)$. Note that $Q_E(\xi) = ||\xi||^2_{\mathbb{R}^2}$, if we view $\mathcal{O}_E/\mathbb{Z}$ as embedded in $\mathbb{R}^2$ so that the $m_i(E)$ are exactly the squares of the successive minima associated with the lattice $\mathcal{O}_E/\mathbb{Z} \subseteq \mathbb{R}^2$ in the sense of Minkowski. Moreover, $3d(Q_E) = D_E$.

Our first aim this section is to prove the following.

Proposition 68. *Let $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{A}_f), K)$ be supported in $\mathrm{Mat}_3(\widehat{\mathbb{Z}})$, $\Phi_f \neq 0$, and suppose that $c(\Phi_f, \gamma + a) = c(\Phi_f, \gamma)$ for all $\gamma \in \mathrm{GL}_3(\mathbb{Q})$ and $a \in \mathbb{Z}$. Then with $\rho_E = \mathrm{res}_{s=1}\,\zeta_E(s)$*

$$\sum_{E \text{ tot.real, } [E:\mathbb{Q}]=3} \frac{\rho_E}{|\mathrm{Aut}(E/\mathbb{Q})|} \sum_{\xi \in (\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}, Q_E(\xi)\leq X} c(\Phi_f, \xi) = \beta_0 X^{\frac{5}{2}} + o(X^{\frac{5}{2}})$$

for $X \to \infty$ and β_0 is given by

$$\frac{3}{5}\frac{2\sqrt{\pi}\zeta(3)}{12\sqrt{3}\pi^{-\frac{5}{2}}\Gamma(\frac{5}{2})} \int_{x\in \mathrm{Mat}_{3\times 3}(\mathbb{A}), d(x_\infty)>0} e^{-\pi\,\mathrm{tr}\, x_\infty^t x_\infty} \Phi_f(x_f) x_f$$
$$= \frac{2\zeta(3)\pi^{\frac{5}{2}}}{15\sqrt{3}} \int_{y\in \mathrm{Mat}_{3\times 3}(\mathbb{R}), d(y)>0} e^{-\pi\,\mathrm{tr}\, y^t y} dy \int_{\mathrm{Mat}_{3\times 3}(\mathbb{A}_f)} \Phi_f(x_f) x_f.$$

For the proof of this proposition we obviously have to separate the field extensions according to their signature at ∞. This is more complicated than in the case of GL(2) seemingly due to the absence of a prehomogoneous vector space structure, and hence the presence of infinitely many orbits under the action of $\mathbb{R}_{>0} \times \mathrm{GL}_3(\mathbb{A})$.

Let $\psi_\varepsilon^\pm : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be smooth non-negative functions satisfying

$$\psi_\varepsilon^+(x) \begin{cases} = 0 & \text{if } x < \frac{\varepsilon}{2} \\ \in [0,1] & \text{if } x \in [\frac{\varepsilon}{2}, \varepsilon] \\ 1 & \text{if } x > \varepsilon \end{cases}$$

and

$$\psi_\varepsilon^-(x) \begin{cases} = 0 & \text{if } |x| > \varepsilon \\ \in [0,1] & \text{if } |x| \leq \varepsilon \end{cases}$$

such that $\psi_\varepsilon^+(x) + \psi_\varepsilon^-(x) \in [1,2]$ for all $x > 0$. Let $\Psi_\varepsilon^\pm : \mathrm{Mat}_{3\times 3}(\mathbb{R}) \longrightarrow \mathbb{R}$ be defined by

$$\Psi_\varepsilon^\pm(x) = \psi_\varepsilon^\pm \left(\frac{d(x - \frac{1}{3}\operatorname{tr} x)}{|\operatorname{tr} x^2 - \frac{1}{3}(\operatorname{tr} x)^2|^3} \right).$$

This is can be thought of as a map on $\mathbb{R}^2$: Let $\mathrm{Mat}_{3\times 3}(\mathbb{R})^0$ be the set of trace-0-matrices. Then

$$\mathrm{Mat}_{3\times 3}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{3\times 3}(\mathbb{R})^0 \longrightarrow \mathbb{R}^2$$

with $x \mapsto x - \frac{1}{3}\operatorname{tr} x =: x_0$, and $x_0 \mapsto (\operatorname{tr}\mathrm{Sym}^2 x_0, \det x_0) =: (a,b)$ we have

$$\Psi_\varepsilon^\pm(x) = \psi_\varepsilon^\pm(\frac{-4a^3 - 27b^2}{|a/2|^3}).$$

This is well-defined: Consider Ψ_ε^+. Then $a \to 0$ with $\Psi_\varepsilon^+(x) \neq 0$ implies that $27b^2 < -4a^3 \to 0$, and in particular, $a \neq 0$ for any x in the support of Ψ_ε^+. Now consider Ψ_ε^-. If x is such that $a \to 0$, $-4a^3 - 27b^2$ stays bounded, so that either $\frac{-4a^3-27b^2}{|a/2|^3}$ tends to a finite number, or the argument in ψ_ε^- tends to ∞ and hence lies outside the support of Ψ_ε^- well in advance of any singularity.

For $x \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ put

$$\Phi_\infty^{\varepsilon,+}(x) = \psi_\varepsilon^+ \left(\frac{d(x - \frac{1}{3}\operatorname{tr} x)}{|\operatorname{tr} x^2 - \frac{1}{3}(\operatorname{tr} x)^2|^3} \right) \varphi(\operatorname{tr} x^2 - \frac{1}{3}(\operatorname{tr} x)^2) e^{-\pi \operatorname{tr} x^t x}$$

for $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$, $\varphi(0) = 0$, a suitable smooth function, which is not identically vanishing, such that $\Phi_\infty^{\varepsilon,+} \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{R}))$. For ψ_ε^- set

$$\Phi_\infty^{\varepsilon,-,l}(x) = \psi_\varepsilon^- \left(\frac{d(x - \frac{1}{3}\operatorname{tr} x)}{|\operatorname{tr} x^2 - \frac{1}{3}(\operatorname{tr} x)^2|^3} \right) (\operatorname{tr} x^2 - \frac{1}{3}(\operatorname{tr} x)^2)^l e^{-\pi \operatorname{tr} x^t x}$$

for $l \gg 0$, and for given $N \in \mathbb{N}$, we can choose l so large that $\Phi_\infty^{\varepsilon,-,l} \in C^N(\mathrm{Mat}_{3\times 3}(\mathbb{R}))$. The properties of $\Psi_\varepsilon^\pm$ can be summarised as follows.

Lemma 69. *For all $g, x \in \mathrm{GL}_3(\mathbb{R})$, $\lambda \in \mathbb{R}_{>0}$,*

(i) $\Psi_\varepsilon^\pm(x^{-1}gx) = \Psi_\varepsilon^\pm(g)$,

(ii) $\Psi_\varepsilon^\pm(\lambda g) = \Psi_\varepsilon^\pm(g)$,

(iii) $\Psi_\varepsilon^\pm(g+\lambda) = \Psi_\varepsilon^\pm(g)$,

(iv) $\Psi_\varepsilon^+(\lambda x^{-1}\gamma x) = 0$, *if $\gamma \in \mathrm{GL}_3(\mathbb{Q})_{ell,\ reg}$ corresponds to a field having a complex place.*

Let $\Phi^{\varepsilon,+} = \Phi_\infty^{\varepsilon,+}\Phi_f$ and $\Phi^{\varepsilon,-,l} = \Phi_\infty^{\varepsilon,-,l}\Phi_f$, with $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{A}_f), \mathbf{K}_f)$. Let $f_s^{\varepsilon,+}$ and $f_s^{\varepsilon,-,l}$ be defined in the usual fashion for $\Re s > 2$. Then $f_s^{\varepsilon,+} \in \mathcal{C}(G(\mathbb{A})^1, K)$, but $f_s^{\varepsilon,-,l} \in \mathcal{C}^N(G(\mathbb{A})^1, K)$ for N and l as above. However, as remarked before, we may use $f_s^{\varepsilon,-,l}$ as a test function for the trace formula for GL(3) provided that N, l are sufficiently large, and all parts of the trace formula are still absolutely convergent for $\Re s > 2$ and can be continued to some larger half plane $\Re s > 2-\delta$ with poles at the usual places.

By Lemma 69 (*iv*), the orbital integral for $\Phi^{\varepsilon,+}$ at infinity vanishes,

$$\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} \int_0^\infty \lambda^t \Phi_\infty^{\varepsilon,+}(\lambda x^{-1}\gamma x) d^\times\lambda dx = 0,$$

if $\gamma \in \mathrm{GL}_3(\mathbb{Q})_{\mathrm{ell,\ reg}}$ is not diagonisable over $\mathrm{GL}_3(\mathbb{R})$. Hence the contribution from the regular elliptic terms for the test function $f_s^{\varepsilon,+}$ is

$$\begin{aligned}
\mathcal{E}_\varepsilon^+(s) := \mathcal{E}(s) &= \sum_{E/\mathbb{Q}\text{ tot. real, }[E:\mathbb{Q}]=3} \frac{\nu(E)}{|\operatorname{Aut}(E/\mathbb{Q})|}\eta_E(s) \\
&= \sum_{E/\mathbb{Q}\text{ tot. real, }[E:\mathbb{Q}]=3} \frac{\nu(E)}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi\in\mathcal{O}_E\backslash\mathbb{Z}} [\mathcal{O}_E:\mathbb{Z}[\xi]] c(\Phi_f,\xi)\Psi_\varepsilon^+(\xi) \\
&\quad\cdot \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^t \varphi(\lambda^2 Q_E(\xi)) e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_\xi x)^t(x^{-1}\gamma_\xi x)} d^\times\lambda dx. \qquad (57)
\end{aligned}$$

Similarly, denote by $\mathcal{E}_\varepsilon^{-,l}(s)$ the regular elliptic terms obtained from the test function $f_s^{\varepsilon,-,l}$. In the following, let

$$\begin{aligned}
&I_{\mathrm{orb}}(\Phi_\infty^{\varepsilon,+},\xi,t) \\
&\qquad = \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^t \varphi(\lambda^2 Q_E(\xi)) e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_\xi x)^t(x^{-1}\gamma_\xi x)} d^\times\lambda dx
\end{aligned}$$

and similarly

$$\begin{aligned}
&I_{\mathrm{orb}}(\Phi_\infty^{\varepsilon,-,l},\xi,t) \\
&= \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^t (\lambda^2 Q_E(\xi))^l e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_\xi x)^t(x^{-1}\gamma_\xi x)} d^\times\lambda dx \\
&\qquad = Q_E(\xi)^l \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^{t+2l} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_\xi x)^t(x^{-1}\gamma_\xi x)} d^\times\lambda dx.
\end{aligned}$$

Lemma 70. *$\mathcal{E}_\varepsilon^+(s)$ converges absolutely for $\Re s > 2$, and has a holomorphic continuation at least in $\Re s > \frac{3}{2}$ except for a simple pole at $s = 2$. Suppose that $\Phi_f \in \mathcal{S}(\mathrm{Mat}_{3\times 3}(\mathbb{A}_f), K)$ is supported in $\mathrm{Mat}_3(\widehat{\mathbb{Z}})$ and is such that $c(\Phi_f, \gamma + a) = c(\Phi_f, \gamma)$ for all $\gamma \in \mathrm{GL}_3(\mathbb{Q}), a \in \mathbb{Z}$. Then for $\Re s > 2$, $\mathcal{E}_\varepsilon^+(s)$ equals up to an entire function the product of $\sqrt{3}\pi^{-\frac{1}{2}} I(s)$ with*

$$\sum_{E/\mathbb{Q} \text{ tot. real},\ [E:\mathbb{Q}]=3} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^+(\xi) Q_E(\xi)^{\frac{3s-1}{2}} \tag{58}$$

for

$$I(s) = \int_0^\infty \lambda^{3s-1} \varphi(\lambda^2) e^{-\pi\lambda^2} d^\times \lambda$$

and $\rho_E = \operatorname{res}_{s=1} \zeta_E(s)$. The function $I(s)$ converges absolutely at least for $\Re s > 1$.

Proof. The first assertion is just a restatement of Proposition 62. Consider the map $\mathcal{O}_E \longrightarrow \mathbb{Z} \oplus \mathcal{O}_E/\mathbb{Z}$, $\xi \mapsto (\operatorname{tr}\xi, [\xi])$, which is an additive group isomorphism. As the $c(\Phi_f, \cdot)$'s, φ and Ψ_ε^+'s are well-defined maps on $\mathcal{O}_E/\mathbb{Z}$ as mentioned before, the inner sum in (57) equals

$$\begin{aligned}
&\sum_{\xi \in \mathcal{O}_E \setminus \mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^+(\xi) I_{\mathrm{orb}}(\Phi_\infty^{\varepsilon,+}, \xi, 3s+3) \\
&= \sum_{\xi_0 \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} [\mathcal{O}_E : \mathbb{Z}[\xi_0]] c(\Phi_f, \xi_0) \Psi_\varepsilon^+(\xi_0) \\
&\cdot \int_{G_\gamma(\mathbb{R})\setminus G(\mathbb{R})} \int_0^\infty \lambda^{3s+3} \varphi(\lambda^2 Q_E(\xi_0)) \\
&\qquad\qquad \cdot \sum_{a \in \mathbb{Z}} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_{\xi_0} x)^t (x^{-1}\gamma_{\xi_0} x) - \frac{\pi}{3}\lambda^2 a^2} d^\times \lambda dx
\end{aligned}$$

Split the integral over λ in one integral over $[0,1]$ and one over $[1,\infty)$. The second one defines an entire function on all of $\mathbb{C}$. For the first one apply Poisson summation to the sum over $a \in \mathbb{Z}$, to obtain

$$\sum_{a \in \mathbb{Z}} e^{-\pi\lambda^2 a^2} = \sum_{b \in \mathbb{Z}} \sqrt{3}\pi^{-\frac{1}{2}} \lambda^{-1} e^{-3\pi^{-1}\lambda^{-2} b^2}.$$

Changing variables $\lambda^{-1} \in [0,1] \leftrightarrow \lambda \in [1,\infty)$, the sum over $b \neq 1$ yields again an entire function. Hence we are left with the term belonging to $b = 0$ to which we may add the integral over $\lambda \in [1,\infty)$ without changing its analytic behaviour. Thus up to an entire function, the above equals

$$\begin{aligned}
&\sqrt{3}\pi^{-\frac{1}{2}} \sum_E \frac{\nu(E)}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi_0 \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} [\mathcal{O}_E : \mathbb{Z}[\xi]] c(\Phi_f, \xi) \Psi_\varepsilon^+(\xi) \\
&\cdot \int_{G_\gamma(\mathbb{R})\setminus G(\mathbb{R})} \int_0^\infty \lambda^{3s+2} \varphi(\lambda^2 Q_E(\xi_0)) e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_{\xi_0} x)^t (x^{-1}\gamma_{\xi_0} x)} d^\times \lambda dx.
\end{aligned}$$

By assumption on ξ_0, γ_{ξ_0} is conjugate to a diagonal matrix over $\mathrm{GL}_3(\mathbb{R})$ so that

$$\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_{\xi_0}x)^t(x^{-1}\gamma_{\xi_0}x)}dx$$
$$= d(\xi_0)^{-\frac{1}{2}} e^{-\pi\lambda^2 Q_E(\xi_0)} \int_{U_0(\mathbb{R})} e^{-\pi\lambda^2(u_1^2+u_2^2+u_3^2)}du$$
$$= d(\xi_0)^{-\frac{1}{2}} e^{-\pi\lambda^2 Q_E(\xi_0)}\lambda^{-3}.$$

Notice that $d(\xi_0)^{-\frac{1}{2}} = [\mathcal{O}_E : \mathbb{Z}[\xi_0]]^{-1} D_E^{-\frac{1}{2}}$ and $\nu(E) D_E^{-\frac{1}{2}} = \operatorname{res}_{s=1} \zeta_E(s) = \rho_E$. Therefore changing λ to $Q_E(\xi_0)^{\frac{1}{2}}\lambda$, the assertion follows upon defining $I(s) = \int_0^\infty \lambda^{3s-1}\varphi(\lambda^2)e^{-\pi\lambda^2}d^\times\lambda$. The last assertion about $I(s)$ follows from the fact that by assumption $\varphi(\lambda^2)e^{-\pi\lambda^2} \in \mathcal{S}(\mathbb{R})$. □

Lemma 71. *Suppose that l is sufficiently large and Φ_f is as in the last lemma. $\mathcal{E}_\varepsilon^{-,l}(s)$ converges absolutely for $\Re s > 2$, and continues to a meromorphic function at least in $\Re s > \frac{3}{2}$ with only pole at $s = 2$. Up to an entire function, $\mathcal{E}_\varepsilon^{-,l}(s)$ can be written for $\Re s > 2$ as the sum of*

$$\sqrt{\frac{3}{\pi}} \frac{\Gamma(\frac{3s+2l-1}{2})}{2\pi^{\frac{3s+2l-1}{2}}} \sum_{E/\mathbb{Q} \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi\in(\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}} c(\Phi_f,\xi)\Psi_\varepsilon^-(\xi)Q_E(\xi)^{\frac{3s-1}{2}}$$

and

$$4\sqrt{3\pi}\frac{\Gamma(\frac{3s+2l}{2})}{\pi^{\frac{3s+2l}{2}}} \sum_{E/\mathbb{Q} \text{ not tot. real}} \rho_E \sum_{\xi\in(\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}} c(\Phi_f,\xi)\Psi_\varepsilon^-(\xi)J(\xi,s)Q_E(\xi)^l$$

with

$$J(\xi,s) = \int_1^\infty (Q_E(\xi) + 4(\Im\tilde{\xi})^2\rho^2)^{-\frac{3s+2l-1}{2}}d\rho$$

for $\tilde{\xi}$ one of complex roots the characteristic polynomial of ξ.

Proof. $\Phi^{\varepsilon,-,l}$ defines a test function $f_s^{\varepsilon,-,l}$ which is differentiable up to a finite order, and we can choose this order arbitrarily high by choosing l sufficiently large. Hence as remarked before, we get on the spectral side holomorphic functions for $\Re s > \frac{n+1}{2}$, which have meromorphic continuations in some larger half plane $\Re s > 2-\delta$ with only pole at $s = 2$ whose residue is given as in Proposition 62. Similarly, as in the proof of Lemma 70, $\mathcal{E}_\varepsilon^{-,l}(s)$ can be written as the sum over all cubic fields $E/\mathbb{Q}$ (of any signature) of

$$\frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|}\sum_{\xi_0\in(\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}} [\mathcal{O}_E:\mathbb{Z}[\xi_0]]c(\Phi_f,\xi_0)\Psi_\varepsilon^-(\xi_0)$$
$$\cdot\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})}\int_0^\infty \lambda^{3s+3}(\lambda^2 Q_E(\xi_0))^l$$
$$\cdot\sum_{a\in\mathbb{Z}} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_{\xi_0}x)^t(x^{-1}\gamma_{\xi_0}x)-\frac{\pi}{3}\lambda^2a^2}d^\times\lambda dx.$$

For totally real extensions, the proof of the last lemma tells us that the respective summand essentially equals

$$\sqrt{\frac{3}{\pi}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \frac{\Gamma(\frac{3s+2l-1}{2})}{2\pi^{\frac{3s+2l-1}{2}}} \sum_{E \text{ tot. real}} \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^-(\xi) Q_E(\xi)^{\frac{3s-1}{2}},$$

since for $\varphi(x) = x^l$, $I(s) = \frac{\Gamma(\frac{3s+2l-1}{2})}{2\pi^{\frac{3s+2l-1}{2}}}$.
For not totally real extensions, we can follows along the same lines, but now the integral $\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma_{\xi_0} x)^t (x^{-1}\gamma_{\xi_0} x)} dx$ equals by Lemma 72

$$8\pi\lambda^{-2} |d(\xi)|^{-\frac{1}{2}} \int_{4(\Im\tilde{\xi})^2}^\infty e^{-\pi\lambda^2 (Q_E(\xi)+\rho^2)} d\rho.$$

Changing $(Q_E(\xi) + \rho^2)^{\frac{1}{2}}\lambda$ to λ, we obtain for the whole integral

$$8\pi |d(\xi)|^{-\frac{1}{2}} Q_E(\xi_0)^l \int_0^\infty \lambda^{3s+2l} e^{-\pi\lambda^2} d^\times\lambda \int_{4(\Im\tilde{\xi})^2}^\infty (Q_E(\xi) + \rho^2)^{-\frac{3s+2l}{2}+\frac{1}{2}} d\rho$$

from which the assertion follows. □

Lemma 72. *Suppose $\gamma \in \mathrm{GL}_3(\mathbb{Q})_{ell,\ reg}$ is in $\mathrm{GL}_3(\mathbb{R})$ conjugate to $\begin{pmatrix} a & b & \\ -b & a & \\ & & c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$, $b \neq 0$. Then*

$$\int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} e^{-\pi\lambda^2 \operatorname{tr}(x^{-1}\gamma x)^t (x^{-1}\gamma x)} dx = 8\pi\lambda^{-2} |d(\gamma)|^{-\frac{1}{2}} \int_{4b^2}^\infty e^{-\pi\lambda^2(\operatorname{tr}\gamma^2 + \rho^2)} d\rho$$

Proof. The integrand is invariant under $\mathrm{O}(3)$, and we have $G_\gamma(\mathbb{R})\backslash G(\mathbb{R}) \simeq (\mathrm{GL}_{2,\tilde{\gamma}}(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R}) \times \{1\}) U_1(\mathbb{R}) \mathbf{K}_\infty$. Hence the left hand side equals the product of a certain discriminant factor with

$$\int_{\mathrm{GL}_{2,\tilde{\gamma}}(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} \int_{U_1(\mathbb{R})} \exp(-\pi\lambda^2 (\operatorname{tr}(g^{-1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} g)^t (g^{-1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} g) + c^2 + u_1^2 + u_2^2)) du_1 du_2 dg.$$

Since $\mathbb{R}\backslash \mathrm{GL}_{2,\tilde{\gamma}}(\mathbb{R}) \simeq \{\pm 1\}\backslash \mathrm{O}(2)$, using the KAK decomposition, the integrals equals

$$8\pi e^{-\pi\lambda^2 c^2} \int_{\mathbb{R}^2} e^{-\pi\lambda^2(u_1^2+u_2^2)} du_1 du_2 \int_1^\infty e^{-\pi\lambda^2(2a^2 + b^2(\tau^{-4}+\tau^4))} (\tau^2 - \tau^{-2}) d^\times\tau.$$

Substituting $\rho = \frac{\tau^2+\tau^{-2}}{2}$ this is

$$\begin{aligned} 8\pi e^{-\pi\lambda^2 c^2} \int_{\mathbb{R}^2} e^{-\pi\lambda^2(u_1^2+u_2^2)} du_1 du_2 \int_1^\infty e^{-\pi\lambda^2(2(a^2-b^2)+4b^2\rho^2)} d\rho \\ = 8\pi\lambda^{-2} \int_1^\infty e^{-\pi\lambda^2(\operatorname{tr}\gamma^2 + 4b^2\rho^2)} d\rho. \end{aligned}$$

Note that the last integral is well-defined as for any γ, $\operatorname{tr}\gamma^2 + 4b^2\rho^2 \geq \operatorname{tr}\gamma^2 + 4b^2 = \operatorname{tr}\gamma^t\gamma$. Noticing that $4b^2 = d(\begin{pmatrix} a & b \\ -b & a \end{pmatrix})$ and collecting all missing discriminant factors, the assertion follows. □

In the following let

$$\alpha_\varepsilon^\pm(s) = \sum_{E/\mathbb{Q} \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^\pm(\xi) Q_E(\xi)^{\frac{3s-1}{2}}$$

and

$$A_\varepsilon^\pm(X) = \sum_{E/\mathbb{Q} \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z},\ Q_E(\xi) \le X} c(\Phi_f, \xi) \Psi_\varepsilon^\pm(\xi).$$

These functions are related by Mellin transformation and its inverse: We have

$$A_\varepsilon^\pm(X) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha_\varepsilon^\pm(s) \frac{3X^{\frac{3s-1}{2}}}{3s-1} ds$$

for $\sigma_0 \gg 0$, and

$$\alpha_\varepsilon^\pm(s) = \int_1^\infty X^{-\frac{3s-1}{2}} dA_\varepsilon^\pm(X)$$

(cf. [MoVa07]). Moreover, put

$$\gamma_\varepsilon(s) = \sum_{E/\mathbb{Q} \text{ not tot. real}} \rho_E \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^-(\xi) J(\xi, s) Q_E(\xi)^l$$

and

$$\begin{aligned} C_\varepsilon(X) &= \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \gamma_\varepsilon(s) \frac{3X^{\frac{3s-1}{2}}}{3s-1} ds \\ &= \sum_{E/\mathbb{Q} \text{ not tot. real}} \rho_E \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}} c(\Phi_f, \xi) \Psi_\varepsilon^-(\xi) Q_E(\xi)^l \\ &\qquad\qquad \cdot \int_1^{b(\xi, X)} (Q_E(\xi) + 4(\Im\tilde{\xi})^2 \rho^2)^{-l} d\rho \end{aligned}$$

with

$$b(\xi, X) = \begin{cases} \max\{1, \frac{\sqrt{X - Q_E(\xi)}}{2|\Im\tilde{\xi}|}\} & \text{if } Q_E(\xi) \le X \\ 1 & \text{if } Q_E(\xi) > X, \end{cases}$$

where an integral with start and end point equal to 1 is understood to be 0. This together with the definition of $\Psi_\varepsilon^-(\xi)$ ensures that for any X, the sum over E and ξ is in fact finite. From the last form of $C_\varepsilon(X)$, it is clear that if l is even, $C_\varepsilon(X)$ is a non-negative, monotonically increasing function.

Proof of Proposition 68. Throughout we assume that l is sufficiently large and even. By definition of Ψ_ε^+ and Ψ_ε^-, we have $\Psi_\varepsilon^+(\xi) \le 1 \le \Psi_\varepsilon^+(\xi) + \Psi_\varepsilon^-$ for all $\xi \in E$ with E totally real. Hence for any $X > 0$,

$$\begin{aligned} A_\varepsilon^+(X) \le \sum_{E \text{ tot.real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\xi \in (\mathcal{O}_E \setminus \mathbb{Z})/\mathbb{Z}, Q_E(\xi) \le X} c(\Phi_f, \xi) =: \Sigma(X) \\ \le A_\varepsilon^+(X) + A_\varepsilon^-(X). \quad (59) \end{aligned}$$

The coefficients $\frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} c(\Phi_f, \xi)\Psi_\varepsilon^+(\xi)$ in the Dirichlet series $\alpha_\varepsilon^+(s)$ are non-negative and the series (58) converges absolutely for $\Re s > 2$ with meromorphic continuation at least in $\Re s > \frac{3}{2}$. Moreover, the only pole in this region is at $s = 2$, and we can choose φ such that $I(s)$ is non-vanishing on $\Re s = 2$ and at any given point in $\Re s > 2$. Hence an application of the Wiener-Ikehara Tauberian Theorem [MoVa07, Corollary 8.7] yields that the first sum in 59 satisfies an asymptotic $\sim \beta_\varepsilon X^{\frac{5}{2}} + o(X^{\frac{5}{2}})$ for $X \to \infty$. Therefore, $\liminf_{X\to\infty} X^{-\frac{5}{2}}\Sigma(X) \geq \beta_\varepsilon$ for any $\varepsilon > 0$. The constant β_ε can be given explicitly as an integral over the Schwartz-Bruhat function $\Phi^{\varepsilon,+}$ as in Proposition 62. In particular, by definition, $\beta_\varepsilon \to \beta_0$ as $\varepsilon \searrow 0$, with

$$\beta_0 = \frac{3\sqrt{\pi}\zeta(3)}{5\cdot 12\sqrt{3}I(2)} \int_{x\in \operatorname{Mat}_{3\times 3}(\mathbb{A}), d(x_\infty)>0} e^{-\pi \operatorname{tr} x_\infty^t x_\infty} \varphi(\operatorname{tr} x_\infty^2 - \frac{1}{3}(\operatorname{tr} x_\infty)^2)\Phi_f(x_f)dx.$$

Hence

$$\liminf_{X\to\infty} X^{-\frac{5}{2}}\Sigma(X) \geq \beta_0.$$

To show the reverse inequality, we have to work harder. Consider the function $\mathcal{E}_\varepsilon^{-,l}(s)$. It has a simple pole at $s = 2$, and is holomorphic elsewhere in some half plane $\Re s > 2 - \delta$. As $4\sqrt{3}\pi \frac{\Gamma(\frac{3s+2l}{2})}{\pi^{\frac{3s+2l}{2}}}$ is holomorphic and non-zero in that half plane, the function

$$\begin{aligned}\frac{\pi^{\frac{3s+2l}{2}}}{4\sqrt{3}\pi\Gamma(\frac{3s+2l}{2})}\mathcal{E}_\varepsilon^{-,l}(s) &= \frac{1}{8\pi}\frac{\sqrt{\pi}\Gamma(\frac{3s+2l-1}{2})}{\Gamma(\frac{3s+2l}{2})}\alpha_\varepsilon^-(s) + \gamma_\varepsilon(s) \\ &= \frac{1}{8\pi}\beta_l(s)\alpha_\varepsilon^-(s) + \gamma_\varepsilon(s)\end{aligned}$$

has the same properties as $\mathcal{E}_\varepsilon^{-,l}$ with

$$\beta_l(s) = \int_{\mathbb{R}} (1+x^2)^{-\frac{3s+2l}{2}} dx = 2\int_1^\infty y^{-\frac{3s+2l}{2}} d\sqrt{y-1},$$

and the residue β_ε^- at $s = 2$ is given by a constant multiple of

$$\int_{\operatorname{Mat}_{3\times 3}(\mathbb{A})} \Phi_\infty^{\varepsilon,-,l}(x_\infty)\Phi_f(x_f)dx,$$

which tends to 0 as $\varepsilon \searrow 0$.

For $X > 0$ and $\sigma_0 \gg 0$ sufficiently large, let

$$B_l(X) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \beta_l(s)\frac{3X^{\frac{3s-1}{2}}}{3s-1}ds$$

and

$$AB_{l,\varepsilon}(X) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \beta_l(s)\alpha_\varepsilon^-(s)\frac{3X^{\frac{3s-1}{2}}}{3s-1}ds.$$

In particular, $B_l(X) = 2\int_1^X y^{-(l-1)} d\sqrt{y-1}$. From the definitions it is clear that $C_\varepsilon(X) \geq 0$, $B_l(X) \geq 0$ and $AB_{l,\varepsilon}(X) \geq 0$ and the functions are monotonically increasing. Hence an application of the Wiener-Ikehara Theorem gives $\lim_{X\to\infty} X^{-\frac{5}{2}}(AB_{l,\varepsilon}(X) + C_\varepsilon(X)) = \beta_\varepsilon^-$, and, as everything is positive, $AB_{l,\varepsilon}(X) \leq \beta_\varepsilon^- X^{\frac{5}{2}} + R_\varepsilon(X)$ for $R_\varepsilon(X)$ a suitable error function with $R_\varepsilon(X) \to 0$ as $X \to \infty$. Therefore, by definition

$$X^{\frac{5}{2}}\beta_\varepsilon^- + R_\varepsilon(X) \geq \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma+i\infty} \beta_l(s)\alpha_\varepsilon(s)\frac{3X^{\frac{3s-1}{2}}}{3s-1}ds$$

and the right hand side can be written as

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma+i\infty} \alpha_\varepsilon^-(s)\left(\int_1^\infty v^{-\frac{3s-1}{2}} dB_l(v)\right)\frac{3X^{\frac{3s-1}{2}}}{3s-1}ds \\
&= \int_1^\infty \left(\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma+i\infty} \alpha_\varepsilon^-(s)\left(\frac{X}{v}\right)^{\frac{3s-1}{2}}\frac{3ds}{3s-1}\right) dB_l(X) \\
&\qquad = \int_1^\infty A_\varepsilon^-(\frac{X}{v})dB_l(v).
\end{aligned}$$

As A_ε^- is monotonically increasing, the last integral can be bounded from below as

$$\geq \int_2^3 A_\varepsilon^-(\frac{X}{v})dB_l(v) \geq A_\varepsilon^-(\frac{X}{3})\int_2^3 dB_l(v) > 0$$

for X sufficiently large. Hence there exists a constant $c > 0$ such that for any $\varepsilon > 0$, $\limsup_{X\to\infty} X^{-\frac{5}{2}}A_\varepsilon^-(X) \leq c\beta_\varepsilon^-$, and thus

$$\limsup_{X\to\infty} X^{-\frac{5}{2}}A_\varepsilon^-(X) \longrightarrow 0$$

for $\varepsilon \searrow 0$. Therefore,

$$\begin{aligned}
\limsup_{X\to\infty} X^{-\frac{5}{2}}\Sigma(X) &= \limsup_{X\to\infty} X^{-\frac{5}{2}}A_\varepsilon^+(X) + \limsup_{X\to\infty} X^{-\frac{5}{2}}A_\varepsilon^-(X) \\
&\leq \beta_\varepsilon^+ + c\beta_\varepsilon^- \longrightarrow \beta_0
\end{aligned}$$

for $\varepsilon \searrow 0$, which finishes the proof of the asymptotic. As $\Sigma(X)$ is independent of φ, β_0 likewise is independent of φ, and hence can be computed to be of the form asserted in the proposition. □

V.ii.iv Bounds for the residues of Dedekind zeta functions and a sequence of test functions

As $c_0(\gamma) \geq 1$ for all $\gamma \in \mathrm{GL}_3(\mathbb{Q})_{\text{ell, reg}}$ by Corollary 67, an immediate consequence of Proposition 68 is the following.

Corollary 73. *There exists $\alpha > 0$ such that*

$$\limsup_{X\to\infty} X^{-\frac{5}{2}} \sum_{\substack{E \text{ tot.real, } [E:\mathbb{Q}]=3 \\ m_1(E)\le X}} \operatorname*{res}_{s=1} \zeta_E(s) \le \alpha. \tag{60}$$

To complement this upper bound we shall later show the following lower bound.

Proposition 74. *For any $\varepsilon > 0$ we have*

$$\liminf_{X\to\infty} X^{-\frac{5}{2}+\varepsilon} \sum_{E,\ m_1(E)\le X} \operatorname*{res}_{s=1} \zeta_E(s) = \infty$$

where E runs over all totally real cubic fields.

In fact, one expects that the limit of the left hand side in (60) exists and equals and appropriate constant $\alpha > 0$ and by Proposition 74 this can not be too far from the truth. To proof this, one has to know more about the finite coefficients $c(\Phi_f, \xi)$ in Proposition 68. Let $\Phi_f^0 \in \mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{A}_f), \mathbf{K}_f)$ be the characteristic function of $\mathrm{Mat}_{3\times3}(\hat{\mathbb{Z}})$. Using this as the finite part of the Schwartz-Bruhat function, one could try to compute the resulting coefficients $c_0(\gamma)$ explicitly, as is done for the case GL(2). However, this is considerably more difficult for GL(3) and we will not attempt to do so. Instead try to apply a limiting process by choosing an appropriate sequence of Schwartz-Bruhat functions Φ_f. Even though the sequence we shall construct, yields coefficients tending to 1, we have not yet succeeded in proving the existence of the limit above: For that one would have to show that the sequence converges to 1 uniformly in ξ.

We shall nevertheless give the construction of this sequence of functions, as there is a good chance to eventually prove the uniform convergence of it.

Proposition 75. *For any totally real, cubic $E/\mathbb{Q}$ and $\xi \in (\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z}$, there is a monotonically decreasing sequence $(c_{\mathfrak{m}}(\xi))_{\mathfrak{m}\in\mathrm{Div}^+(\mathbb{Q})}$ such that $c_{\mathfrak{m}}(\xi) \to 1$ as a net on $\mathrm{Div}^+(\mathbb{Q})$ for any ξ and such that*

$$\sum_{E \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\substack{\xi\in(\mathcal{O}_E\backslash\mathbb{Z})/\mathbb{Z} \\ Q_E(\xi)\le X}} c_{\mathfrak{m}}(\xi) = \alpha_{\mathfrak{m}} X^{\frac{5}{2}} + o(X^{\frac{5}{2}})$$

as $X \to \infty$ for suitable $\alpha_{\mathfrak{m}} > 0$. Moreover, $(\alpha_{\mathfrak{m}})_{\mathfrak{m}\in\mathrm{Div}^+(\mathbb{Q})}$ converges as a net on $\mathrm{Div}^+(\mathbb{Q})$ with limit $\alpha > 0$, which can be given as an Euler product (see (62)).

For the construction of the sequence we first define functions, which are not actual Schwartz-Bruhat functions, but will be used to build up test functions. Let $p \in \mathbb{Z}$ be a prime, and $\Phi_p^0 : \mathrm{Mat}_{3\times3}(\mathbb{Q}_p) \longrightarrow \mathbb{C}$ the characteristic function of $\mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$. $x \in \mathrm{Mat}_{3\times3}(\mathbb{Q}_p)$ is called regular if x has pairwise distinct eigenvalues in $\overline{\mathbb{Q}_p}$, i.e. $d(X) \neq 0$. Hence the set of non-regular elements in $\mathrm{Mat}_{3\times3}(\mathbb{Q}_p)$ has measure 0. Define the function $\tilde{\Phi}_p : \mathrm{Mat}_{3\times3}(\mathbb{Q}_p) \longrightarrow \mathbb{C}$ by

$$\tilde{\Phi}_p(x) = \begin{cases} \frac{[\mathcal{O}_{\mathbb{Q}_p[x]}:\mathbb{Z}_p[x]]}{\int_{\mathrm{GL}_{3,x}(\mathbb{Q}_p)\backslash\mathrm{GL}_3(\mathbb{Q}_p)} \Phi_p^0(g^{-1}xg)dg} & \text{if } x \text{ is regular, and } x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p), \\ 0 & \text{else.} \end{cases}$$

This function is locally constant in $\mathrm{Mat}_{3\times3}(\mathbb{Q}_p)\backslash\{x \mid d(x) = 0\}$ by Krasner's Lemma. Note that for $x \in \mathrm{Mat}_{3\times3}(\mathbb{Q}_p)$ with $\tilde{\Phi}_p(x) \neq 0$,

$$\frac{1}{[\mathcal{O}_{\mathbb{Q}_p[x]} : \mathbb{Z}_p[x]]} \int_{\mathrm{GL}_{3,x}(\mathbb{Q}_p)\backslash\mathrm{GL}_3(\mathbb{Q}_p)} \tilde{\Phi}_p(g^{-1}xg)dg = 1$$

so that in fact, one would actually like to use $\tilde{\Phi}_f$ for building up test functions. However, as $\tilde{\Phi}_f \notin \mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{A}_f))$, we construct a sequence of Schwartz-Bruhat functions, approximating $\tilde{\Phi}_f$.

Lemma 76. *For $x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$ regular we have*

$$0 \leq \tilde{\Phi}_p(x) \begin{cases} = 1 & \textit{if } |d(x)|_p = 1 \textit{ or } p^{-1} \\ \leq 1 & \textit{else.} \end{cases}$$

Proof. We first reduce to the case of an irreducible characteristic polynomial with the help of Lemma 65, and then use the expansion of orbital integrals from Lemma 66. Let $x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$ be regular. Hence $\mathbb{Q}_p[x] \simeq E_1 \oplus \ldots \oplus E_r$ with $E_i = \mathbb{Q}_p(\theta_i)$ fields of degree n_i, $\sum_{i=1,\ldots,r} n_i = 3$, $\theta_i \in \mathcal{O}_{E_i}$. By Lemmas 65 and 66 the inverse $\tilde{\Phi}_p(x)^{-1}$ therefore equals the product over $i = 1, \ldots, r$ of

$$\frac{[\mathcal{O}_{E_i}^\times : \mathbb{Z}_p[\theta_i]^\times]}{[\mathcal{O}_{E_i} : \mathbb{Z}_p[\theta_i]]} |\operatorname{Frac}^0(\mathbb{Z}_p[\theta_i])/P(\mathbb{Z}_p[\theta_i])| \\ + \frac{1}{[\mathcal{O}_{E_i} : \mathbb{Z}_p[\theta_i]]} \sum_{\mathbb{Z}_p[\theta_i]\subsetneq\mathfrak{o}\subseteq\mathcal{O}_{E_i}} |\operatorname{Frac}^0(\mathfrak{o})/P(\mathfrak{o})| [\mathcal{O}_{E_i}^\times : \mathfrak{o}^\times].$$

This sum is always ≥ 1 by Corollary 67. The condition $|d(x)|_p = 1$ implies that $|d(\theta_i)|_p = 1$ for all i, and hence $\mathbb{Z}_p[\theta_i] = \mathcal{O}_{E_i}$ so that $\tilde{\Phi}_p(x) = 1$. If $|d(x)|_p = p^{-1}$, then either $|d(\theta_i)|_p = 1$ or $= p^{-1}$. In the first case the factor above again is 1. But $|d(\theta_i)|_p = p^{-1}$ can only occur if θ_i is divisible by the prime element $\varpi_i \in \mathcal{O}_{E_i}$, i.e. $\varpi_i^{-1}\theta_i \in \mathcal{O}_{E_i}$ or if the extension $E_i/\mathbb{Q}_p$ is ramified with $|D_{E_i}|_p = p^{-1}$ because of the equation $\mathbb{Z} \ni [\mathcal{O}_{E_i} : \mathbb{Z}_p[\theta_i]] = |D_{E_i} d(\theta_i)^{-1}|_p^{\frac{1}{2}}$ for $\theta_i \notin \varpi_i\mathcal{O}_{E_i}$. If θ_i is divisible by ϖ_i, then $\mathbb{Z} \ni |d(\varpi_i^{-1}\theta_i)|_p^{-1} = q_i^{\delta_i}|d(\theta_i)|_p^{-1} = q_i^{\delta_i}p$ with δ_i the degree of the discriminant as a homogeneous polynomial in the roots and q_i the cardinality of $\mathcal{O}_{E_i}/\varpi_i\mathcal{O}_{E_i}$. Thus for this to be an integer, we must have $n_i = 1$ and therefore $\mathbb{Z}_p[\theta_i] = \mathbb{Z}_p = \mathcal{O}_{E_i}$. Hence in both cases θ_i again generates the maximal order so that we again get 1 for the corresponding factor.

□

Lemma 77. *There exists $\rho > 0$ such that*

$$1 - \rho\frac{1}{p^2} \leq \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Q}_p)} \tilde{\Phi}_p(x)dx \leq 1$$

for all primes p. Hence the Euler product

$$\int_{\mathrm{Mat}_{3\times 3}(\mathbb{A}_f)} \tilde{\Phi}_f(x)dx = \prod_{p<\infty} \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Q}_p)} \tilde{\Phi}_p(x)dx$$

converges to some number $\beta \in \mathbb{R}$ with $0 < \beta \leq 1$.

Proof. The second inequality is clear by the last lemma. For the first write

$$\begin{aligned}\int_{\mathrm{Mat}_{3\times 3}(\mathbb{Q}_p)} \tilde{\Phi}_p(x)dx &= \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)} \tilde{\Phi}_p(x)dx \\ &= \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)} dx - \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)} (1 - \tilde{\Phi}_p(x))dx.\end{aligned}$$

By the last lemma $0 \leq 1 - \tilde{\Phi}_p(x) \leq 1$ for all x, and $1 - \tilde{\Phi}_p(x)$ vanishes if $|d(x)|_p \in \{1, p^{-1}\}$. As $0 \leq 1 - \tilde{\Phi}_p(x) \leq 1$ for all x, it therefore suffices to bound the volume of the set of all x with $d(x) \equiv 0 \mod p^2$ from above. Consider the commutative diagram

$$\begin{array}{ccc}\mathrm{Mat}_{3\times 3}(\mathbb{Z}_p) & \longrightarrow & \mathrm{Mat}_{3\times 3}(\mathbb{F}_p) \\ \downarrow & & \downarrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{F}_p\end{array}$$

with vertical maps given by the discriminant. If $\bar{a} \in \mathrm{Mat}_{3\times 3}(\mathbb{F}_p)$, the fibre above $\bar{a}$ has volume p^{-9} in $\mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)$. For $x \in \mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)$ to satisfy $d(x) \equiv 0 \mod p^2$, its reduction $\bar{x} \in \mathrm{Mat}_{3\times 3}(\mathbb{F}_p)$ must satisfy $d(\bar{x}) = 0$, which is equivalent to the condition that $\bar{x}$ has at least two identical eigenvalues in $\mathbb{F}_p$. The union of the preimages of conjugacy classes of such elements all have volume $\leq p^{-3} + O(p^{-4})$ except for the conjugacy classes generated by $\bar{x} = \left(\begin{smallmatrix} t_1 & 1 & \\ & t_1 & \\ & & t_2 \end{smallmatrix}\right)$ with $t_1, t_2 \in \mathbb{F}_p$, $t_1 \neq t_2$. The union over preimages of such classes has volume $p^{-1}(1 - p^{-2} - p^{-3} + p^{-5})$. Let $x \in \mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)$ be in the fibre above $\bar{x}$. Then $d(x + py) = d(x) + p\Delta d_x y + O(p^2)$ for Δd_x the gradient of d at the point x. As $t_1 \neq t_2$, $\Delta d_x \not\equiv 0 \mod p$, and thus for x fixed, there are $O(p^8)$ possibilities for y to satisfy $d(x + py) \equiv 0 \mod p^2$. Hence $\mathrm{vol}(\{x \in \mathrm{Mat}_{3\times 3}(\mathbb{Z}_p) \mid d(x) \equiv 0 \mod p^2\}) = O(p^{-2})$ with implied constant not depending on p. □

We now define a sequence of Schwartz-Bruhat functions at the finite places, which yield our desired test functions. Let $\Sigma \subseteq \mathrm{Mat}_{3\times 3}(\mathbb{Z}_p)$ be the set of all $x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$ such that $d(x) = 0$. For $m \in \mathbb{N}_0$ define a function $\Phi_p^m : \mathrm{Mat}_{3\times3}(\mathbb{Q}_p) \longrightarrow \mathbb{C}$ by

$$\Phi_p^m(x) = \begin{cases} 1 & \text{if } x \in \Sigma + p^m \mathrm{Mat}_{3\times3}(\mathbb{Z}_p) \\ \tilde{\Phi}_p(x) & \text{if } x \notin \Sigma + p^m \mathrm{Mat}_{3\times3}(\mathbb{Z}_p). \end{cases}$$

In particular, Φ_p^0 coincides with the characteristic function of $\mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$. For $x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p)\backslash(\Sigma + p^m \mathrm{Mat}_{3\times3}(\mathbb{Z}_p))$, the stabiliser of $M_3(\mathbb{Z}_p)$ in $\mathrm{GL}_3(\mathbb{Q}_p)$ is canonically isomorphic to $\mathbf{K}_p$. Moreover, $\mathbb{Z}_p[x]$ is a finite index subgroup of $\mathrm{Mat}_{3\times3}(\mathbb{Z}_p)$ whose index is bounded in terms of m. Hence, if we identify $\mathrm{Stab}_{\mathrm{GL}_3(\mathbb{Q}_p)}(\mathbb{Z}_p[x])$ with a subgroup of $\mathbf{K}_p$, and set

$$K^m = \bigcap_{x \in \mathrm{Mat}_{3\times3}(\mathbb{Z}_p)\backslash p^m \mathrm{Mat}_{3\times3}(\mathbb{Z}_p) \text{ regular}} \mathrm{Stab}_{\mathrm{GL}_3(\mathbb{Q}_p)}(\mathbb{Z}_p[x]),$$

then $K^m \subseteq \mathbf{K}_p$ is a subgroup of finite index, and by construction, $\Phi_p^m \in \mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{Q}_p), K^m)$. Let $\mathfrak{m} = (m_p)_{p<\infty} \in \mathrm{Div}^+(\mathbb{Q})$. Define the function $\Phi_f^{\mathfrak{m}} : \mathrm{Mat}_{3\times3}(\mathbb{A}_f) \longrightarrow \mathbb{C}$ by

$$\Phi_f^{\mathfrak{m}} = \prod_{p<\infty} \Phi_p^{m_p}$$

which is contained in $\mathcal{S}(\mathrm{Mat}_{3\times3}(\mathbb{A}_f), K_{\mathfrak{m}})$ with $K_{\mathfrak{m}} = \prod_{p<\infty} K^{m_p}$. Let $\Phi_f^0 = \prod_{p<\infty} \Phi_p^0$ be the characteristic function of $\mathrm{Mat}_{3\times3}(\hat{\mathbb{Z}}) \subseteq \mathrm{Mat}_{3\times3}(\mathbb{A}_f)$.

By definition we have for all $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Div}^+(\mathbb{Q})$ with $\mathfrak{m} \geq \mathfrak{m}'$, and all $x \in \mathrm{Mat}_{3\times3}(\mathbb{A}_f)$

$$0 \leq \tilde{\Phi}_f(x) \leq \Phi_f^{\mathfrak{m}}(x) \leq \Phi_f^{\mathfrak{m}'}(x) \leq \Phi_f^0(x) \leq 1. \tag{61}$$

Moreover, $\lim_{\mathfrak{m}} \Phi_f^{\mathfrak{m}}(x) = \tilde{\Phi}_f(x)$ for any x. Similarly, the finite functions $\Phi_p^{m_p}$ are monotonically decreasing with limit function $\tilde{\Phi}_p$ so that the local integrals

$$\int_{\mathrm{Mat}_{3\times3}(\mathbb{Q}_p)} \Phi_p^{m_p}(x)dx \quad \text{and} \quad \int_{\mathrm{GL}_{3,\gamma}(\mathbb{Q}_p)\backslash \mathrm{GL}_3(\mathbb{Q}_p)} \Phi_p^{m_p}(g^{-1}\gamma g)dx,$$

$\gamma \in \mathrm{GL}_3(\mathbb{Q})_{\text{ell, reg}}$, converge to

$$\int_{\mathrm{Mat}_{3\times3}(\mathbb{Q}_p)} \tilde{\Phi}_p(x)dx \quad \text{and} \quad \int_{\mathrm{GL}_{3,\gamma}(\mathbb{Q}_p)\backslash \mathrm{GL}_3(\mathbb{Q}_p)} \tilde{\Phi}_p(g^{-1}\gamma g)dx = 1,$$

respectively. The infinite products over all those integrals exist by Lemma 77. Hence

$$\lim_{\mathfrak{m}} \int_{\mathrm{Mat}_{3\times3}(\mathbb{A}_f)} \Phi_f^{\mathfrak{m}}(x)dx = \int_{\mathrm{Mat}_{3\times3}(\mathbb{A}_f)} \tilde{\Phi}_f(x)dx$$

and

$$\lim_{\mathfrak{m}} \int_{\mathrm{GL}_{3,\gamma}(\mathbb{A}_f)\backslash \mathrm{GL}_3(\mathbb{A}_f)} \Phi_f^{\mathfrak{m}}(g^{-1}\gamma g) = 1.$$

In particular,

$$\alpha = \frac{2\pi^{\frac{5}{2}}\zeta(3)}{15\sqrt{3}} \int_{y\in \mathrm{Mat}_{3\times 3}(\mathbb{R}), d(y)>0} e^{-\pi \operatorname{tr} y^t y} dy \prod_{p<\infty} \int_{\mathrm{Mat}_{3\times 3}(\mathbb{Q}_p)} \tilde{\Phi}_p(x)dx. \quad (62)$$

This finishes the construction for Proposition 75.

Remark 78. *If one has established an asymptotic of the form*

$$\Sigma(X) := \sum_{E \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\substack{\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z} \\ Q_E(\xi)\le X}} 1 \sim \alpha X^{\frac{5}{2}},$$

one easily gets that

$$\Sigma_{prim}(X) := \sum_{E \text{ tot. real}} \frac{\rho_E}{|\operatorname{Aut}(E/\mathbb{Q})|} \sum_{\substack{\xi\in((\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z})_{prim} \\ Q_E(\xi)\le X}} 1 \sim \zeta(5)^{-1}\alpha X^{\frac{5}{2}}.$$

This is because by definition of primitivity, $\Sigma(X) = \sum_{n\in\mathbb{N}} \Sigma_{prim}(\frac{X}{n^2})$ so that by Möbius inversion,

$$\Sigma_{prim}(X) = \sum_{m\in\mathbb{N}} \mu(m)\Sigma(\frac{X}{m^2}) \sim \alpha X^{\frac{5}{2}} \sum_{m\in\mathbb{N}} \mu(m)m^{-5} = \zeta(5)^{-1}\alpha X^{\frac{5}{2}}.$$

To prove the lower bound from Proposition 74, we first need to bound a part of the sum we are not interested in by the following.

Lemma 79. *(i) Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice with positive definite quadratic form $Q : \Lambda \longrightarrow \mathbb{R}$. Then*

$$\sum_{\gamma\in\Lambda,\ Q(\gamma)\le X} 1 = \frac{2\pi X}{\sqrt{d(Q)}} + R(\frac{X^{\frac{1}{2}}}{d(Q)^{\frac{1}{4}}})$$

for $X \to \infty$ where $R(\frac{X^{\frac{1}{2}}}{d(Q)^{\frac{1}{4}}})$ is some error term of order $O(\frac{X^{\frac{1}{2}}}{d(Q)^{\frac{1}{4}}})$.

(ii) For all $\varepsilon > 0$

$$\sum_{E:\ m_2(E)\le X} \rho_E \sum_{\substack{\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z} \\ Q_E(\xi)\le X}} 1 = O(X^{2+\varepsilon})$$

where the first sum extends over all cubic, totally real fields $E/\mathbb{Q}$.

Proof. (i) Consider the set of all points $\gamma \in \Lambda$ with $Q(\gamma) \leq X$ or, equivalently, all Λ-points within the ellipse $\{x \in \mathbb{R}^2 \mid Q(x) \leq X\}$. By a well-known theorem of Gauss [Co62, p.161], the number of such points is equal to the volume of the ellipse $\frac{2\pi X}{\sqrt{d(Q)}}$ plus some small error term of order $\frac{X^{\frac{1}{2}}}{d(Q)^{\frac{1}{4}}}$. As for the number of lattice points only the ratio $\frac{X}{\sqrt{d(Q)}}$ is relevant, the asserted upper bound follows.

(ii) By Minkowski's second theorem (see, e.g. [Ca97, VIII.4.3]), we have $D_E \asymp m_1(E)m_2(E)$ (we consider the successive minima of a quadratic form) so that $m_1(E) \leq m_2(E) \leq X$ implies $c_0 D_E \leq m_1(E)m_2(E) \leq 16X^2$ for some $c_0 > 0$ Hence there is by (i) some constant $C > 0$ such that

$$\sum_{\xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z},\ Q_E(\xi) \leq X} 1 \leq C \frac{X}{\sqrt{d(Q_E)}}$$

for all E with $m_1(E) \leq m_2(E) \leq X$. By the Brauer-Siegel Theorem [La86, XVI, §4 Theorem 4], there exists for all $\varepsilon > 0$ some number $C_\varepsilon > 0$ such that $\rho_E = \mathrm{res}_{s=1}\, \zeta_E(s) = 4D_E^{-\frac{1}{2}} h_E R_E \leq C_\varepsilon D_E^{\varepsilon}$ for all totally real cubic fields E. Hence the left hand side of (ii) equals

$$\sum_{E:\ m_2(E) \leq X} \rho_E \sum_{\xi \in \mathcal{O}_E \backslash \mathbb{Z},\ Q_E(\xi) \leq X} 1 \leq C C_\varepsilon \sqrt{3} \sum_{E:\ m_2(E) \leq X} X D_E^{\varepsilon - \frac{1}{2}}.$$

This can bounded by

$$C C_\varepsilon \sqrt{3} X \sum_{E:\ D_E \leq 16X^2} D_E^{\varepsilon - \frac{1}{2}} \leq C C_\varepsilon \sqrt{3} X^{1+\varepsilon} \sum_{E:\ D_E \leq 16X^2} D_E^{-\frac{1}{2}}.$$

By [DaHe71, Theorem 1] or [DaWr88, Theorem I.1], $\sum_{E:\ D_E \leq X} 1 = c_0 X + o(X)$ for some $c_0 > 0$ so that

$$C C_\varepsilon \sqrt{3} X^{1+\varepsilon} \sum_{E:\ D_E \leq 16X^2} D_E^{-\frac{1}{2}} \leq 16 c_0 C C_\varepsilon \sqrt{3} X^{2+\varepsilon} + o(X^{2+\varepsilon})$$

which is the assertion.

□

Hence we can restrict our attention to the cubic fields with $m_1(E) \leq X < m_2(E)$. If E is such a field, there are exactly two primitive vectors $\pm\xi_0$ in $(\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z}$ with $Q_E(\xi_0) \leq X$ both satisfying $Q_E(\pm\xi_0) = m_1(E)$, and for any $\xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z}$ with $Q_E(\xi) \leq X$, there is a a unique $n \in \mathbb{Z}\backslash\{0\}$ with $\xi = n\xi_0$.

Proof of Proposition 74. We first show that

$$\liminf_{X \to \infty} X^{-\frac{5}{2}+\varepsilon} \sum_{E, \xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z},\ Q_E(\xi) \leq X} \rho_E = \infty \tag{63}$$

for any $\varepsilon > 0$. Let $\varepsilon > 0$. By the Brauer-Siegel Theorem there exists $A_\varepsilon > 0$ such that $\rho_E \geq A_\varepsilon D_E^{-\frac{\varepsilon}{2}}$ for all E. Thus this sum is bounded from below by $A_\varepsilon X^{-\frac{\varepsilon}{2}} \sum_{E,\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z},\ Q_E(\xi)\leq X} 1$. Hence it will certainly suffice to show that there exists $C > 0$ such that

$$\sum_{E,\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z},\ Q_E(\xi)\leq X} 1 \sim CX^{\frac{5}{2}}$$

as $X \to \infty$. (We do not attempt to compute the constant, but there is a master's thesis by Gero Brockschneider in preparation in which the asymptotic with exact coefficients is determined.) Using Lemma 63, the map associating with the pair $E, \xi \in (\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z}$ the characteristic polynomial $T^3 + a_2T^2 + a_1T + a_0$ of ξ is $3-1$ or $1-1$, and the coefficient $a_2 = 0$. As E is totally real, we have $d(\xi) = -4a_1^3 - 27a_0^2 > 0$, or $a_0^2 \leq -\frac{27}{4}a_1^3$. Since $X \geq Q_E(\xi) = -2a_1 > 0$, this implies

$$-\frac{X}{2} \leq a_1 < 0 \quad \text{and} \quad 0 < a_0 \leq \sqrt{-\frac{27}{4}a_1^3} \leq \frac{3\sqrt{3}}{4\sqrt{2}}X^{\frac{3}{2}}.$$

Hence, ignoring constants, there are $a_1^{\frac{3}{2}}$ many a_0 and

$$\int_1^{X/2} a_1^{\frac{3}{2}}\,da_1 = \frac{1}{10\sqrt{2}}X^{\frac{5}{2}} - \frac{2}{5}$$

many a_1 satisfying all the conditions. On the other hand, any irreducible polynomial with integer coefficients satisfying such inequalities defines (an equivalence class of) a cubic field E and ξ as before. Thus we only need to show that the reducible polynomials with coefficients satisfying above inequalities do not contribute to $CX^{\frac{5}{2}}$. If $T^3 + a_1T + a_0$ is reducible, write it as a product $(T^2 + b_1T + b_0)(T + c)$ with $b_1, b_0, c \in \mathbb{Z}$. Hence $c = -b_1$, $cb_0 = a_0$ and $b_0 - c^2 = a_1$. Hence if we fixed a_0 (for which there are at most $O(X^{\frac{3}{2}})$ possibilities), there are at most $O(a_0^\delta) \leq O(X^\delta)$ possibilities for c and b_0 for any $\delta > 0$. Thus there are only $O(X^{\frac{3}{2}+\delta})$ reducible polynomials satisfying above constraints.

Now split the sum over E in the following parts: One belonging to E such that $m_1(E) > X$, one with $m_1(E) \leq X < m_2(E)$, and the last one with $m_1(E) < m_2(E) \leq X$. For E with $m_1(E) > X$, there are no ξ contributing to the sum in (63) so that the sum on the left hand side of (63) equals

$$X^{-\frac{5}{2}+\varepsilon} \sum_{E,m_1(E)\leq X<m_2(E)} \rho_E \sum_{\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z},\ Q_E(\xi)\leq X} 1 + X^{-\frac{5}{2}+\varepsilon} \sum_{E,m_1(E)\leq m_2(E)\leq X} \rho_E \sum_{\xi\in(\mathcal{O}_E\setminus\mathbb{Z})/\mathbb{Z},\ Q_E(\xi)\leq X} 1.$$

By Lemma 79(ii), the second sum tends to 0 for $X \to \infty$ provided $\varepsilon < \frac{1}{2}$. Hence the first sum is not bounded from below as $X \to \infty$ for any $\varepsilon > 0$.

As $m_1(E) \leq X < m_2(E)$, any $\xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z}$ with $Q_E(x) \leq X$ is of the form $\xi = n\xi_0$ for some $n \in \mathbb{N}$, and ξ_0 one of the two primitive vectors in $(\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z}$. Thus

$$\begin{aligned}
&\sum_{E, m_1(E) \leq X < m_2(E)} \rho_E \sum_{\xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z},\ Q_E(\xi) \leq X} 1 \\
&= \sum_{n \in \mathbb{N}} \sum_{E, m_1(E) \leq X < m_2(E)} \rho_E \sum_{\xi_0 \in ((\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z})_{\mathrm{prim}},\ Q_E(\xi_0) \leq \frac{X}{n^2}} 1 \\
&\qquad = 2 \sum_{n \in \mathbb{N}} \sum_{E, m_1(E) \leq \frac{X}{n^2} < m_2(E)} \rho_E
\end{aligned}$$

Suppose there are $\kappa > 0$ and $c_0 > 0$ such that

$$\liminf_{X \to \infty} X^{-\frac{5}{2}+\kappa} \sum_{E, m_1(E) \leq X < m_2(E)} \rho_E = c_0.$$

Then

$$\begin{aligned}
X^{-\frac{5}{2}+\kappa} &\sum_{E, m_1(E) \leq X < m_2(E)} \rho_E \sum_{\xi \in (\mathcal{O}_E \backslash \mathbb{Z})/\mathbb{Z},\ Q_E(\xi) \leq X} 1 \\
&= 2 \sum_{n \in \mathbb{N}} n^{-5+2\kappa} (\frac{X}{n^2})^{-\frac{5}{2}+\kappa} \sum_{E, m_1(E) \leq \frac{X}{n^2} < m_2(E)} \rho_E,
\end{aligned}$$

and for any n, $\liminf_{X \to \infty} (\frac{X}{n^2})^{-\frac{5}{2}+\kappa} \sum_{E, m_1(E) \leq \frac{X}{n^2} < m_2(E)} \rho_E = c_0$ so that the limit inferior of the above is $2c_0\zeta(5 - 2\kappa)$ in contradiction to the unboundedness of the limit inferior of the first sum as $X \to \infty$. □

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Summary

The Arthur-Selberg trace formula is an identity $J_{\text{geom}}(f) = J_{\text{spec}}(f)$ of two distributions, the so-called geometric and spectral side, on a space of test functions defined on the adelic points of a reductive algebraic group. It is an important tool in the theory of automorphic forms, and it is itself the object of extensive studies. We are going to study different aspects of this trace formula.
For potential applications it is of importance to have large spaces of test functions and meaningful expansions of the distributions available. We are giving a modified version of Arthur's fine geometric expansion for the group GL(3), and we are showing that this modified geometric expansion converges absolutely for a large space of test functions. This space of test functions is in some sense natural and corresponding results for the geometric side of the trace formula for GL(2) resp. for the spectral side of the trace formula for a general reductive group, have been shown by Finis and Lapid (2011) resp. Finis, Lapid und Müller (2011).
This space of test functions contains for GL(n) certain special functions depending on a complex parameter s (with $\Re s \gg 0$), which are of special arithmetic interest. For example, the main part of the spectral side yields a sum of automorphic L-functions by the theory of Godement-Jacquet. It is therefore natural to consider the trace formula for GL(n) for such test functions as a function of s. We give meromorphic continuations for the spectral terms to larger half planes and determine their first poles.
As an application we show the following asymptotic: For certain Schwartz-Bruhat functions $\Phi_f \neq 0$ on the space of 3×3-matrices over the finite adeles, there exists $\alpha > 0$ such that

$$\sum_{\substack{E/\mathbb{Q} \text{ totally real} \\ [E:\mathbb{Q}]=3}} \operatorname*{res}_{s=1} \zeta_E(s) \sum_{\substack{\xi\in\mathcal{O}_E\setminus\mathbb{Z} \\ \operatorname{tr}_{E/\mathbb{Q}}\xi^2\leq X}} \frac{I(\Phi_f,\xi)}{[\mathcal{O}_E:\mathbb{Z}[\xi]]} = \alpha X^{\frac{5}{2}} + o(X^{\frac{5}{2}}) \tag{64}$$

as $X\to\infty$. Here $I(\Phi_f,\xi)\geq 0$ are coefficients associated with Φ_f and ξ by means of orbital integrals. For GL(n) in general, one expects similar asymptotics for sums of n-dimensional field extensions of certain signature, provided that the convergence of the required trace formula can be shown. A consequence of (64) is an upper bound for the limit superior of

$$X^{-\frac{5}{2}} \sum_{\substack{E \text{ tot. real, } [E:\mathbb{Q}]=3 \\ m_1(E)\leq X}} \operatorname*{res}_{s=1} \zeta_E(s)$$

as $X\to\infty$, where $m_1(E)$ is the second successive minimum of the positive quadratic form $\xi\mapsto \operatorname{tr}_{E/\mathbb{Q}}\xi^2$ on $\mathcal{O}_E$. The limit of this sum is actually expected to exists and to be non-zero.
The case of GL(2) serves as a model and we shall study it in detail: Because of the aforementioned results, we are allowed to use our special test functions in this case as well. In particular, it turns out that the main part of the

geometric side is constituted by the Shintani zeta function with the help of which Shintani was able to show asymptotics for class numbers of binary quadratic forms.

Zusammenfassung

Die Arthur-Selberg-Spurformel ist eine Identität $J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f)$ zweier Distributionen, der sogenannten geometrischen und spektralen Seite, auf einem geeigneten Raum von Testfunktionen, welche auf den adelischen Punkten einer reduktiven algebraischen Gruppe definiert sind. Sie stellt ein wichtiges Werkzeug in der Theorie der automorphen Formen dar, und ist selbst Gegenstand weitreichender Untersuchungen. Wir werden verschiedene Aspekte dieser Spurformel untersuchen.

Für mögliche Anwendungen ist es von Bedeutung, große Räume von Testfunktionen und sinnvolle Entwicklungen der Distributionen zur Verfügung zu haben. Wir stellen für die Gruppe GL(3) eine modifizierte Form der von Arthur angegebenen feinen geometrischen Entwicklung auf und zeigen, dass die so modifizierte geometrische Seite für einen großen Raum von Testfunktionen absolut konvergent ist. Dieser Raum von Testfunktionen ist in gewisser Weise natürlich und entsprechende Resultate für die geometrische Seite der Spurformel von GL(2) bzw. für die Spektralseite der Spurformel einer allgemeinen Gruppe wurden von Finis und Lapid (2011) bzw. Finis, Lapid und Müller (2011) bewiesen.

In diesem Raum von Testfunktionen befinden sich für die Gruppen GL(n) insbesondere spezielle, von einem komplexen Parameter s (mit $\Re s \gg 0$) abhängige Funktionen, die von besonderer arithmetischer Bedeutung sind. Für solche Funktionen ergibt sich beispielsweise mit Hilfe der Theorie von Godement-Jacquet für den Hauptteil der Spektralseite eine Summe von automorphen L-Funktionen. Es ist dementsprechend naheliegend, die Spurformel für GL(n) für diese Testfunktionen als Funktion von s zu betrachten. Wir werden meromorphe Fortsetzungen für die spektralen Terme auf größere Halbebenen zeigen und deren erste Pole bestimmen.

Als Anwendung werden wir folgende Asymptotik zeigen: Für bestimmte Schwartz-Bruhat Funktionen $\Phi_f \neq 0$ auf dem Raum der 3×3-Matrizen über den endlichen Adelen existieren Konstanten $\alpha > 0$, so dass

$$\sum_{\substack{E/\mathbb{Q} \text{ total reell} \\ [E:\mathbb{Q}]=3}} \operatorname*{res}_{s=1} \zeta_E(s) \sum_{\substack{\xi \in \mathcal{O}_E \setminus \mathbb{Z} \\ \operatorname{tr}_{E/\mathbb{Q}} \xi^2 \leq X}} \frac{I(\Phi_f, \xi)}{[\mathcal{O}_E : \mathbb{Z}[\xi]]} = \alpha X^{\frac{5}{2}} + o(X^{\frac{5}{2}}) \tag{65}$$

für $X \to \infty$. Hierbei sind $I(\Phi_f, \xi) \geq 0$ durch Bahnintegrale zu Φ_f und ξ assoziierte Koeffizienten. Für GL(n) im Allgemeinen erwartet man ähnliche Asymptotiken solcher Summen für n-dimensionale Körpererweiterungen bestimmter Signatur, sofern die Konvergenz der geometrischen Seite für die jeweils benötigte Spurformel gezeigt werden kann. Eine Konsequenz von (65) ist eine obere Schranke für den Limes Superior von

$$X^{-\frac{5}{2}} \sum_{\substack{E \text{ tot. reell, } [E:\mathbb{Q}]=3 \\ m_1(E) \leq X}} \operatorname*{res}_{s=1} \zeta_E(s)$$

für $X \to \infty$, wobei $m_1(E)$ das zweite sukzessive Minimum der positiv definiten quadratischen Form $\xi \mapsto \operatorname{tr}_{E/\mathbb{Q}} \xi^2$ auf $\mathcal{O}_E$ ist. Es ist zu erwarten, dass

der Grenzwert dieser Summe existiert und nicht verschwindet.
Der Fall GL(2) dient hierbei als Vorbild und wir werden ihn im Detail untersuchen: Aufgrund der oben genannten Resultate, dürfen wir auch hier unsere speziellen Testfunktionen verwenden. Es wird sich unter anderem herausstellen, dass der Hauptteil der geometrischen Seite durch die Shintani-Zetafunktion gegeben ist, mit deren Hilfe Shintani in der Lage war, Asymptotiken für Klassenzahlen von binär-quadratischen Formen anzugeben.