HOMOTOPY & COHOMOLOGY

GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

Winter School

“From Field Theories to Elliptic Objects”

February, 28th till March, 4th 2006
Schloss Mickeln, Düsseldorf

GRK-Sprecher:
Prof. Dr. C.-F. Bödigheimer
Arbeitsgruppe Topologie
Beringstrasse 1
D- 53115 Bonn
Email: grk1150@math.uni-bonn.de

GRK-Sekretariat:
Frau S. George
Tel: 0228-73 77 88
Fax: 0228-73 64 90

Talk No. 12

Speaker: Moritz Wiethaupt

universitätbonn
Rheinische Friedrich-Wilhelms-Universität Bonn

Outline of Talk:

I proof of theorem

- spaces of EFTs
- original proof
  (using C^*-algebra theory)
- new proof
  (using Fredholm operators
  & spaces of configurations)
  - relating Fredholm to configurations
  - Dold-Thom theory of quasifibrations

II the partition function of susy EFTs

(III Thom class, family index)

Theorem: The space of real $(1|1)$-dim. susy EFTs of degree n represents the real K-theory functor KO^{-n}.
Analogous statement for the complex case.

What is the space of EFTs?

Have many reasonable choices:

$$\begin{array}{ccccc}
\{\text{susy EFTs of degree } n\} & & \mathrm{Conf}^{\text{f, odd}}_{C_n}(\overline{\mathbb{R}},\infty) & \cong & \begin{array}{c}\text{space of}\\ \text{generators}\end{array} \\
\updownarrow \scriptstyle{1-1} & & \cap\wr & & \\
\mathrm{ssg}(\mathbb{R}^{1|1}_{\geq 0}, HS^{sa}_{C_n}) & \xrightarrow{\simeq} & \mathrm{Conf}^{EFT}_{C_n}(\overline{\mathbb{R}},\infty) & \cong & \begin{array}{c}\text{space of}\\ \text{generators}\end{array} \\
\cap\wr & & \cap\wr & & \\
\mathrm{ssg}(\mathbb{R}^{1|1}_{\geq 0}, K^{sa}_{C_n}) & \cong & \mathrm{Conf}^{\text{odd}}_{C_n}(\overline{\mathbb{R}},\infty) & \cong & \begin{array}{c}\text{space of}\\ \text{generators}\end{array}
\end{array}$$

Which K-theory spectrum to use?

a) C^*-homomorphisms

<u>**Thm**</u> (Higson-Guentner):
Let H_n be a real separable Hilbert space, graded module over C_n, containing each irred. module infinitely often, let $C_0(\mathbb{R})$ be the C^*-algebra of functions vanishing at ∞ (grading induced by $t \mapsto -t$, $t \in \mathbb{R}$).
Then the space of grading preserving C^*-morphisms

$$C^*(C_0(\mathbb{R}), K_{C_n}(H_n))$$

represents KO^{-n}.

Relation to EFTs:

$$\text{EFTs} \longrightarrow C^*(C_0(\mathbb{R}), K_{C_n}(H_n))$$

$$\text{generator } D \longmapsto (f \mapsto f(D))$$

(functional calculus)

$$C^*(C_0(\mathbb{R}), K_{C_n}(H_n)) \longrightarrow \text{EFTs}$$

Given $\varphi: C_0(\mathbb{R}) \to K_{C_n}(H_n)$,
apply spectral theorem to the pairwise commuting family $\varphi(f)$, $f \in C_0(\mathbb{R})$, to decompose H_n into simultaneous eigenspaces of the $\varphi(f)$.

If E is one such eigenspace, we have

$$\varphi(f)|_E = \lambda(f).$$

$f \mapsto \lambda(f)$ is given by evaluation at some $t \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
Get configuration by labelling E with $\cancel{\pm}\, t$.

b) Fredholm operators

$\widehat{H}_{n-1}$ ungraded Hilbert space,
w. C_{n-1}-module structure,
$e_i \in C_{n-1}$ acting through skew adjoint operators: $e_i^* = -e_i$.

$$\widehat{\mathcal{F}}_n = \{ T \in \mathrm{Fred}(\widehat{H}_{n-1}) \mid T^* = -T, \; Te_i = -e_i T \},$$

if $n \not\equiv 3 \ (4)$.

For $n \equiv 3 \ (4)$: In addition require that the self adjoint, C_{n-1}-linear operator

$$e_1 \cdots e_{n-1} T$$

is <u>neither</u> essentially positive

<u>nor</u> essentially negative.

Thm (Atiyah-Singer): Jänich

$\hat{\mathfrak{F}}_n$ represents KO^{-n}.

Analogous statement for KU.

We need graded version of $\hat{\mathfrak{F}}_n$:

H_n graded module over C_n.

$$\hat{\mathfrak{F}}_n \cong \mathfrak{F}_n = \{ S \in \mathrm{Fred}(H_n) : S^* = S,\ S \text{ odd},\ S\ C_n\text{-linear},\ e_n S|_{H_n^0} \in \hat{\mathfrak{F}}_n \}$$

$$e_n S|_{H_n^0} \longleftarrow\!\shortmid\ S$$

$$T \longmapsto \begin{pmatrix} 0 & -Te_n \\ -e_n T & 0 \end{pmatrix}$$

Let $T \in \mathcal{F}_a$. Then the essential spectrum of T has a gap around 0.

More precisely:

$$\sigma_{ess}(T) \cap (-\varepsilon(T), \varepsilon(T)) = \emptyset,$$

where $\varepsilon(T) = \| q(T)^{-1} \|^{-1}$,

$$q: \mathcal{B} \to \mathcal{B}/\mathcal{K}.$$

$\Rightarrow$ get configuration

$$c_T \in \mathrm{Conf}^{odd}_{C_a}([-1,1], \pm 1) \cong \mathrm{Conf}^{odd}_{C_a}(\bar{\mathbb{R}}, \pm\infty),$$

$$c_T(\lambda) = \begin{cases} E_{\varepsilon(T)\lambda}(T) & \lambda \in (-1,1) \\ E_{[\varepsilon(T),\infty)}(T) & \lambda = 1 \\ E_{(-\infty,-\varepsilon(T)]}(T) & \lambda = -1 \end{cases}$$

Have deformation retraction

$$(T, t) \longmapsto \frac{1}{1 - t + t\,\varepsilon(T)}\, T$$

of $\mathcal{F}_n$ onto $\mathcal{F}_n^{\varepsilon = 1}$, subspace of operators with essential spectral gap 1.

Have map

$$\mathrm{Conf}^{odd}_{Cl_n}([-1,1], \pm\infty) \longrightarrow \mathcal{F}_n^{\varepsilon=1}$$

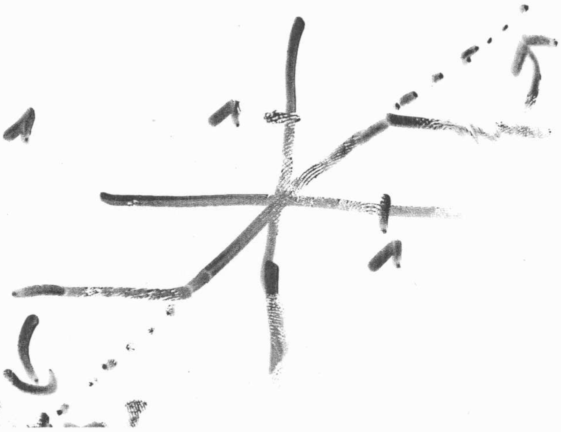
$$c \longmapsto T_c = \sum_{\lambda \in [-1,1]} \lambda \, pr_{c(\lambda)}.$$

$n \not\equiv 3\ (4)$

Now $c_{T_c} = c$,

$$T_{c_T} = f(T), \quad T \in \mathcal{F}_n^{\varepsilon=1}, \text{ where}$$

$$f(\lambda) = \begin{cases} -1 & \lambda \leq -1 \\ \lambda & -1 \leq \lambda \leq 1 \\ 1 & \lambda \geq 1 \end{cases}$$



$\Rightarrow F_n \simeq F_n^{\varepsilon=1}$

$\simeq \mathrm{Conf}^{odd} ([-1, 1] \ldots$

So for $n \not\equiv 3\ (4)$:

$$\mathcal{F}_n \simeq \mathrm{Conf}_{C_n}^{f,odd}([-1,1], \pm\infty) \simeq \underset{\substack{\| \\ \mathrm{Conf}_n}}{\mathrm{Conf}_{C_n}^{f,odd}(\widehat{\mathbb{R}}, \pm\infty)}$$

$n \equiv 3\ (4)$:

$$\mathcal{F}_n \simeq \mathrm{Conf}_n \subseteq \mathrm{Conf}_{C_n}^{f,odd}(\widehat{\mathbb{R}}, \pm\infty)$$

Remains to show that the obvious map

$$p: \mathrm{Conf}_{C_n}^{f,odd}(\widehat{\mathbb{R}}, \pm\infty) \longrightarrow \mathrm{Conf}_{C_n}^{f,odd}(\overline{\mathbb{R}}, \infty)$$

is a homotopy equivalence when restricted to Conf_n.

<u>Lemma:</u> $p^{-1}(c)$ contractible $\forall c$, $n \not\equiv 3\ (4)$

$p^{-1}(c) \cap \mathrm{Conf}_n$ contractible $\forall c$, $n \equiv 3\ (4)$

Proof: $p^{-1}(c) =$ decompositions of $V_\infty := c(\infty)$ as $V_\infty = V \perp \alpha V$ (α grading involution)

$\tilde{c} \in p^{-1}(c)$

$\downarrow$

$V = \tilde{c}(-\infty)$

$\downarrow$

$\beta|_V = 1, \beta|_{\alpha V} = -1$

$\downarrow$

$\beta_0 = \beta|_{V_\infty^{ev}}$

$$\cong \{\beta : V_\infty \to V_\infty : \beta \ C_n\text{-linear}, \beta^2 = 1, \beta = \beta^*, \alpha\beta = -\beta\alpha\}$$

$$\cong \{\beta_0 : V_\infty^{ev} \to V_\infty^{odd} \mid \beta_0 \ C_n^{ev}\text{-linear}, \beta_0 \text{ orthogonal}\}$$

$$\simeq * \quad (C_n^{ev}\text{-linear Kuiper})$$

$\beta_0 \Longleftrightarrow \beta_1 = e_1 \ldots e_n \beta_0 : V_\infty^{ev} \circlearrowleft$ s.a., Cl_n-linear, $\beta_1^2 = 1$

$$p^{-1}(c) \cap Conf_n \cong \{\beta_1 : V_\infty^{ev} \circlearrowleft : \text{s.a.}, Cl_n\text{-linear}, \beta_1^2 = 1, \dim E_{\pm 1}(\beta_1) = \infty\}$$

$$\cong U_{Cl_n^{ev}}(V_\infty^{ev}) / U_{Cl_n^{ev}}(V_{\infty,1}^{ev}) \times U_{Cl_n^{ev}}(V_{\infty,2}^{ev})$$

Final step:

Lemma: $p|_{\mathrm{Conf}_n}$ is a quasi fibration

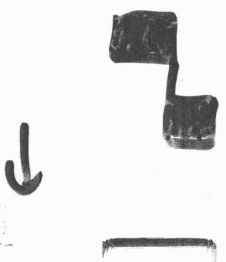
Defi: $p : E \to B$ quasifibration:

$p_* : \pi_k(E, p^{-1}(b)) \xrightarrow{\cong} \pi_k(B, b)$

$\forall b \in B, k \geq 1$

($\Rightarrow$ exact homotopy sequence)

Example:



E

B

quasifibration, not a fibration.

Thm (Dold-Thom): $p : E \to B$ is a quasifibration if there is a filtration

$F_0 \subseteq F_1 \subseteq \dots$ of B s.t.

i) $p|_{F_i \setminus F_{i-1}}$ is a fibration, $i \geq 1$

ii) There is a neighborhood N_i of F_i inside F_{i+1} and a deformation h of N_i to F_i

iii) h is covered by $H : p^{-1}(N_i) \times I \to p^{-1}(N_i)$

s.t. $H_0 = id$,

$H_1 : p^{-1}(x) \xrightarrow{\simeq} p^{-1}(h_1(x))$,

$x \in N_i$.

<u>Proof (Lemma):</u>

$$F_i := \{ c : \dim \bigoplus_{\lambda \in \mathbb{R}} c(\lambda) \leq 2i \}$$

$$N_i := \{ c \in F_{i+1} \mid \dim c(0) \neq 2i+1, 2i+2 \}$$

i) as long as dimensions don't jump, $p|_{\mathrm{Conf}_n}$ is a fibre bundle

ii) deformation pushes one pair of labels off to infinity

iii) this is covered by the analogous deformation in Conf_n, H_1 is a map between contractible spaces.

$\blacksquare$

The partition function of susy EFTs:

Let E be a susy EFT (of degree n), generated by odd operator D.

The partition function of E is defined by $Z_E(t) = E(S_t^{per} \times \mathbb{R}^{0|1})$

<u>Prop</u>: $Z_E(t)$ is an integer, independent of t.

<u>Proof</u>:
$$\begin{aligned} Z_E(t) &= \mathrm{str}_{C_n} E(I_t) \\ &= \mathrm{str}_{C_n} e^{-tD^2} \\ &= \sum_\lambda e^{-t\lambda^2}\, \mathrm{sdim}_{C_n} E_\lambda(D) \\ &= \mathrm{sdim}_{C_n} \ker D \in \mathbb{Z}. \end{aligned}$$

Here $\mathrm{str}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \mathrm{tr}\, A - \mathrm{tr}\, D$,

$\mathrm{sdim}(E) = \mathrm{str}\, \mathrm{pr}_E \quad (= \dim E^0 - \dim E^1)$

The family index

$\pi : Z \to X$ fiber bundle of fibre dimension n, spin structure on τ, the tangent bundle along the fibers.

$\Rightarrow \quad \pi_! : KO(Z) \to KO^{-n}(X)$.

$\xi \to Z$ real vector bundle.

$S \to Z$ $\quad C\ell(\tau) - C_n$ bimodule bundle representing spin structure on τ.

$\rightsquigarrow$ C_n-bundle on X with fiber

$$H_x = L^2(Z_x, (S \otimes \xi)_{|Z_x}) \text{ over } x \in X.$$

$D_x \otimes \xi_{|Z_x}$ C_n-linear on $H_x \cong H_{\mathbb{R}}$

$$\rightsquigarrow \quad X \to \mathrm{sog}\left(\mathbb{R}_{>0}^{\mathbb{Z}/4}, HS^{sa}_{C_n}(H_{\mathbb{R}})\right)$$

$$x \mapsto \left((t, \Theta) \mapsto f_{t,\Theta}(D_x \otimes \xi) \right)$$

$$\left(f_{t,\Theta}(x) = e^{-tx^2} + \Theta x e^{-tx^2} \right).$$

This represents $\pi_![\xi] \in KO^{-n}(X)$

The Thom class

$\pi : \xi \to X$ spin vector bundle of rank n,
$C(\xi) - C_n$ - bimodule bundle $S \to X$ representing the spin structure.

Embed $S \hookrightarrow X \times H_{\mathbb{R}}$ (trivial C_n-linear Hilbert bundle)

$v \in \xi_x$: $\quad c(v) : S_x \to S_x$ skew adjoint

$\qquad \varepsilon\, c(v) : S_x \to S_x$ self adjoint

$C_{-n} = C(-\mathbb{R}^n)$ acts on the right on S_x if w acts by $\varepsilon\, c(w)$

$\Rightarrow$ $\varepsilon\, c(v)$ is selfadjoint C_{-n}-linear

$\rightsquigarrow$ get map

$$\xi \to EFT_{-n}^{\mathbb{R}}$$

$$v \mapsto \cancel{\varepsilon c(v)} \quad (t,\theta) \mapsto f_{t,\theta}(\varepsilon\, c(v)) \in HS^{sa}_{C_{-n}}(S_x) \subset HS^{sa}_{C_{-n}}(H_{\mathbb{R}}$$

This extends to Thom space

$\rightsquigarrow$ $X^{\xi} \to EFT_{-n}^{\mathbb{R}}$

representing $Th(\xi) \in KO^n(X^{\xi})$

$$Z_E : \pi_0\, \mathcal{E}\mathcal{F}\mathcal{I}_n^{\mathbb{F}} \cong \begin{cases} KU_n(*) & \mathbb{F}=\mathbb{C} \\ KO_n(*) & \mathbb{F}=\mathbb{C} \end{cases}$$

$$E \longmapsto [Z_E]$$

$$KU_*(pt) \cong \hat{\mathfrak{M}}_n / i^* \hat{\mathfrak{M}}_{n+1}$$

$\hat{\mathfrak{M}}_n$ = K-group of graded $\mathbb{C}\ell_n$-modules.