C^*-ALGEBRAS AND K-THEORY GRADUATE STUDENT SEMINAR SUMMER TERM 2013

MARKUS LAND

This semester we want to study the basic theory of C^*-algebras and their topological K-theory. C^*-algebras are the main objects of study in what is called *non-commutative topology.* A C^*-algebra is a (complex) Banach algebra A with an involution $x \mapsto x^*$ such that the so called *C^*-identity*

$$\|x^*x\| = \|x\|^2$$

is true for all $x \in A$. This harmless looking identity actually has a tremendous effect on the interaction of the topology and the algebraic structure of A, for example the norm on a C^*-algebra A is uniquely determined by its algebraic structure.

The study of K-theory of such algebras contains the usual topological K-theory of (locally) compact spaces as a special case (this is the *commutative* part) but extends to other non-commutative algebras. It is interesting to note that while the topological K-theory of a space X does not tell us too much about the homotopy type of X, the situation for non-commutative algebras is quite different: here, the K-theory often turns out to be a powerful invariant. Classical examples of non-commutative algebras that arise in topology are group algebras as e.g. in the statement of the Baum-Connes Conjecture.

The seminar will roughly be devided into three parts.

In the first part we will talk about general C^*-theory. The goal of this part is to prove the basic theorems about classification and representation theory in order to get a better feeling for the theory. Precisely we will show that every commutative C^*-algebra is of the form $C_0(X)$ for some locally compact space X and that every C^*-algebra admits a faithful representation on some Hilbert space. Hence we will see that every C^*-algebra is a norm closed $*$-subalgebra of the algebra of bounded operators on some Hilbert space $\mathcal{H}$, denoted $\mathcal{B}(\mathcal{H})$. Using this we will be able to define new examples of C^*-algebras, as e.g. the (reduced) Toeplitz algebra, which we will use in the proof of complex Bott-periodicity.

The second part will be devoted to the topological K-theory of C^*-algebras. We will introduce the functors K_0 and K_1 by hand, define suspensions and hence higher K-theories. We will show that K-theory is stable, half-exact, and homotopy invariant. Having introduced the topological K-groups we will continue by proving one of the characteristic properties of topological K-theory, namely Bott-periodicty. The proof we will present is essentially due to J. Cuntz. A variant of it could actually be deduced for

Date: SS 2013.

any half-exact, stable, homotopy invariant functor $E : C^*\mathrm{Alg} \to \mathrm{Ab}$. If time permits we will sketch this more general form of Bott-periodicity.

The third part of the seminar (the last four talks) will be devoted to the study of algebraic K-theory of socalled *stable* C^*-algebras. Those are algebras A where a certain map $A \to A \otimes \mathcal{K}$ is an isomorphism of C^*-algebras ($\mathcal{K}$ denotes the algebra of compact operators). The goal of this last part is to give a sketch of the proof of the Karoubi Conjecture, which is due to combined results of Higson and Suslin-Wodzicki:

Conjecture 1. *For stable C^*-algebras A the natural comparison map*

$$K_*^{\mathrm{alg}}(A) \xrightarrow{c} K_*^{\mathrm{top}}(A)$$

is an isomorphism.

We will only be concerned with the case $* \geq 0$ and hence need not worry about non-connective algebraic K-theory. The case $* < 0$ was proven earlier by Karoubi and uses methods very different from what we are going to see. As one can think, the proof is quite complicated (as it implies that algebraic K-theory is periodic) and involves methods from both algebraic and topological K-theory. The goal is that we will at least see and understand the essential arguments why this conjecture is true. Luckily the input from topological and algebraic K-theory are mostly independent of each other, so that people more interested in homotopy theory might prefer to give one of these talks. The main input from algebraic K-theory will be dealt with in talks 9,10 and 11. The input from topological K-theory is a result of Higson, and is the content of the last talk.

If you want to give a talk, please send me an email (land@math.uni-bonn.de) indicating a choice of 2-3 talks of which you would be willing to give at least one. Please do not hesitate to ask questions via email or also in person concerning your talk or related issues. I will be pleased to help and discuss problems with you. Please also contact me if you feel that you will not be able to provide all theorems that are listed for your talk, so we can see what can be left out without causing too much of harm for the future talks.

Concerning the first four talks, it would be good if you followed my notes as a guideline and use [10] only as additional source. It would definitely be good, if you choose one of the first four talks, for us to meet so that I can give you copies of [6].

TALKS

Talk 1. C^*-algebras I [18.04. – Markus Land]
Define Banach and C^*-algebras. Introduce the unitialization of a non-unital algebra via its left-regular representation. Mention the van Neumann series and deduce openness of invertible elements in a unital Banach algebra. Define the spectrum of an element, show it is non-empty and compact [10, Prop. 2.3 and Thm 2.5]. Define the spectral radius and relate it to the norm. Hence show that in a C^*-algebra the norm is uniquely determined by the algebraic structure. As a corollary, prove that Banach algebras that

are division rings are canonically isomorphic to $\mathbb{C}$. Introduce ideals and quotients of Banach-algebras. State the holomorphic functional calculus [10, Prop. 2.7 and Prop. 2.8]. General references are [10, Chapter 1 and Chapter 2] and also [5, Chapter 6 and 7] for general functional analysis. [10] usually proves things more generally than we need (for (involutive) Banach algebras) but you should state everything in the world of C^*-algebras. Some proofs become simpler then. The notes [6] are more streamlined and might fit your needs better.

Talk 2. Commutative C^*-algebras [18.04. – Christian Wimmer]
The goal of this talk is to prove the Gelfand-Naimark Theorem which states that every commutative C^*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X. Start by introducing the Gelfand-transform [10, Thm 3.11 and Def. 3.12], i.e. introduce the maximal ideal space of Banach algebras (you will need [10, Prop 3.8 and Prop 3.10]. Recall Banach-Alaoglu. If you want, you can prove a result in Fourier-analysis (Theorem of Wiener) as an application, see [6], but you don't have to. Introduce normal, unitary, self-adjoint, idempotent elements. Conclude properties about the spectrum of such elements. Proceed to prove that for C^*-algebras the Gelfand-transform is an isometric isomorphism [10, Thm 4.4]. Finally, prove the spectral theorem for normal elements in arbitrary unital C^*-algebras [10, Prop. 4.6]. Again the standard references include [5, Chapter 8] and [10]. As for the first talk, my notes [6] are a bit more streamlined than [10], so this might also be interesting for you.

Talk 3. C^*-algebras II [02.05 – Ruth Joachimi]
Show that the spectrum of an element is independent of the ambient algebra [10, Prop. 4.8]. In particular show that elements in non-unital algebras have a well-defined spectrum. Introduce *-homomorphisms. Prove that any such homomorphism is continuous (in fact norm-decreasing) [10, Prop. 5.2]. Show that injective homomorphisms between C^*-algebras are norm-increasing [10, Prop. 5.3]. Hence deduce that injective morphisms are isometric. In particular show that every *-homomorphism has closed image (which allows to form cokernels). As a preparation for the next talk, introduce positive elements and the classification of them, precisely prove the following [10, Thm 6.1]: let x be a self-adjoint element in a unital C^*-algebra. Then the following are equivalent: (i) x is positive, (ii) $x = y^*y$, for some $y \in A$. The references [10] and my notes [6] should be sufficient for this talk.

Talk 4. The GNS-construction [02.05 – Benjamin Kuester]
The goal of this talk is to prove that every C^*-algebra may be faithfully represented on some Hilbert space. This is proven by the Gelfand-Naimark-Segal construction. In order to state it, introduce representations, positive linear functionals and states on C^*-algebras [10, Chapter 9]. If you want you can motivate why positive linear functionals classically give rise to representations (if X is a compact space, then a Radon measure μ on X is a positive linear functional on $C(X)$. But Radon measures always give rise to representations of $C(X)$ by multiplication operators on $L^2(X;\mu)$). You should now be able to state the GNS-construction about cyclic representations [10, Thm 9.14]. Proceed to prove that any C^*-algebra admits a faithful representation on Hilbert space ([10, Thm

9.18]). Hence show that we could have defined C^*-algebras to be norm-closed subalgebras of bounded operators of a Hilbert space. Give examples of classical algebras that occur in this fashion: the bounded operators, the compact operators, the Qualkin-algebra, the (abstract) Toeplitz algebra $\mathcal{T}$. Mention that we will need the Toeplitz algebra later on, when we want to prove Bott periodicity. References other than [10] are again my notes [6].

Talk 5. K_0 and Projections [06.06. – Gisa Schaefer]
Introduce matrix algebras over C^*-algebras, use the representation theory to deduce that they are again C^*-algebras. Introduce the different relations on projections: Murray van Neumann equivalence, unitary equivalence and homotopy, show how they are related, as e.g in [4, Def. 4.1.9 ff]. Define the group $K_0(A)$ for unital C^*-algebras and extend this definition to non-unital C^*-algebras briefly. Remark that $K_0(A) \cong K_0^{\mathrm{alg}}(A)$. In particular recall the Theorem of Swan to relate to usual topological K-theory. Prove stability of K_0 as e.g in [4, Lemma 4.2.4]. Prove half-exactness [4, Prop. 4.3.15] (if time permits, you can talk about relative K-theory and excision, but you don't have to). State and prove homotopy invariance [4, Prop 4.4.3]. In general a good reference for K_0 is [4, Chapter 4], but you can also look at [11, Chapter 6] for K_0 and [11, Chapter 5] for the different relations on projections. [1, Chapter 4 and 5] is shorter but more technical. You should try not to get lost in the technicalities concerning all these different relations. Yet another reference for K-theory is [7].

Talk 6. Higher topological K-theory [06.06. – Antonio Sartori]
Introduce the functor $A \mapsto K_1(A)$ via homotopy classes of unitaries in A (see [4, Section 4.8]). Define for $n \geq 2$ the group $K_n(A) = K_0(S^n(A))$. Show that K_1 is homotopy invariant. Define the connecting homomorphism classically (for K_1 in the unitary picture). Hence deduce a longer exact sequence in topological K-theory starting at K_1 and ending with K_0. Then show that $K_1(A) \cong K_0(SA)$ and deduce stability and half exactness of K_1. To relate to the higher K-groups define the mapping cone sequence associated to a surjective *-homomorphism $A \to A/J$ as in [4, 4.5.6] and define the boundary map in this context. Hence we get a really long exact sequence involving all K_n. Explain the example for the case of compact operators as ideal in all bounded operators and prove the Toeplitz index theorem [11, Prop. 9.4.2].

Talk 7. Bott periodicity I [20.06 – Alexander Koerschgen]
Introduce spatial tensor products of C^*-algebras ([4, Exercise 3.9.8]) and briefly mention the concept of nuclearity as in [1, Thm 15.8.1 parts 1 and 2] or [4, Chapter 3.3]. Give examples of nuclear algebras (without proof) such as finite dimensional and commutative ones [1, Thm 15.8.2]. Explain the exactness properties of spatial tensor products using the notion of nuclearity ([4, Exercise 3.9.12 and Thm 3.6.6]). Define the exterior products on K-theory in low dimensions, i.e. construct the maps

$$K_0(A) \otimes K_0(B) \to K_0(A \otimes B) \quad \text{and} \quad K_1(A) \otimes K_0(B) \to K_1(B)$$

explicitly as in [6]. Show that they are natural with respect to *-homomorphisms and with respect to the boundary map in the long exact K-theory sequence. Define the Bott element $\beta \in K_1(S\mathbb{C})$ and the Bott map

$$\beta_A : K_0(A) \to K_1(SA) \cong K_2(A)$$

induced by exterior product with the Bott element. Recall the Toeplitz algebra via the unilateral shift and identify the quotient $\mathcal{T}/\mathcal{K}$ with $C(S^1)$ via the spectral theorem for normal operators. If time permits and it has not been done in talk 6 prove the Toeplitz index theorem (else, only state it - you need this to do the computation $\mathcal{T}/\mathcal{K} \cong C(S^1)$). Now you are ready to define the candidate for an inverse of the Bott map which we call s_A.

Talk 8. Bott periodicity II [20.06. – Irakli Patchkoria]
The goal of this talk is to finish the proof of Bott periodicity. For this you have to show separately that the map s_A is a left and a right inverse of β_A. First show that it is a left inverse. Do this firstly for the "universal case" $A = \mathbb{C}$. Then consider the diagram

$$\begin{tikzcd}
K_1(S\mathbb{C}) \times K_0(A) \arrow[r] \arrow[d, "s_{\mathbb{C}} \times \mathrm{id}"'] & K_1(SA) \arrow[d, "s_A"] \\
K_0(\mathbb{C}) \times K_0(A) \arrow[r] & K_0(A)
\end{tikzcd}$$

and show commutativity. It is a formal consequence that s_A is also a left inverse of β_A. Showing that s_A is also a right inverse of β_A is more involved. You can follow [4, Chapter 4.9] which is a bit sketchy, though. Details are available in [6] but since this is my Bachelor's thesis we should browse through this together to capture the essential arguments. If time permits explain the direct proof that $K_*(\mathcal{T}_0 \otimes A) = 0$ which also proves a periodicity phenomenon, but does not relate directly to the Bott map β_A.

Talk 9. Algebraic K-Theory [04.07 – Markus Hausmann]
You should recall the definition of algebraic K-theory of (unital) rings via the plus construction. Then show that for a C^*-algebra A the space $K_0^{\mathrm{top}}(A) \times BGL(A)$ has the right homotopy type for *topological* K-theory. Remark that this model allows us to construct a map from the algebraic K-theory space to $BGL(A)$ and hence define the comparison map $c : K_*^{\mathrm{alg}}(A) \to K_*^{\mathrm{top}}(A)$. Explain how to define algebraic K-theory of non-unital rings, and introduce the *excision* property. State that $K_0^{\mathrm{alg}}(A) \cong K_0^{\mathrm{top}}(A)$ and identify the map $K_1^{\mathrm{alg}}(A) \to K_1^{\mathrm{top}}(A)$. Introduce *relative* algebraic K-theory as the homotopy fiber of the comparison map. If time permits, explain some of the results on lower K-theory as e.g. [8, Thm 12] or [8, Thm 13].

Talk 10. Excision in mod p algebraic K-theory [04.07. – Dominik Ostermayr]
Introduce algebraic K-theory with finite coefficients following [12]. The goal of this talk is to show that $\mathbb{Q}$-algebras (and hence C^*-algebras) satisfy excision in mod p K-theory for any prime p, so you should keep your focus on the results in [12] you need for this. Another reference is [9, Lemma 1.9], it is shorter, but seems to rely on the fact that C^*-algebras satisfy property $AH_{\mathbb{Z}}$ (this is suggested by [9, Lemma 1.8]). If this does not fill 75 minutes, you should talk to the person giving the talk on excision in rational algebraic K-theory and give as much of prerequisites for that talk as possible. So it

would be good if the two speakers of the excision results talked to each other about this.

Talk 11. Excision in rational algebraic K-theory [18.07. – Karol Szumilo]
This is probably a difficult talk. The main line of argument seems to be as follows. You should mention (if time permits sketch) that C^*-algebras A are H-unital as in [13] or [8, 3.3.2]. Use the functional calculus to show that $A^2 = A$ (this should be in [4]). Then introduce the property AH_Λ (we probably only need $AH_\mathbb{Q}$). You will want to use [9, Corollary 1.8] in order to show that C^*-algebras satisfy excision in rational algebraic K-theory. So proceed to show that C^*-algebras have property $AH_\mathbb{Q}$. For this the main results are [9, Prop. 6.1 (b)] and [9, Thm 2.10]. Explain why only $\widetilde{GL}(A)$ is used in [9, Chapter 2] and not also $\widetilde{\widetilde{GL}}(A)$ as in the definition of property AH_Λ.

Talk 12. Higson's result and the conclusion [18.07. – Markus Land]
The first goal of this talk is to sketch Higson's result (a rigidity result, if you will) that certain kinds of functors from C^*-algebras to abelian groups are automatically homotopy invariant in the C^*-algebraic sense [3, Chapter 3, Thm 3.2.2], [8, Thm 3.2]. This talk involves ideas from KK-theory, so whoever chooses to do this talk, should either have heard of Fredholm-modules and analytic K-homology or be willing to work in this direction. Then you should put the pieces together and prove the Karoubi conjecture, following either [8, Thm 3.3] or [9, Chapter 10, Thm 10.9].

References

[1] B. Blackadar, *K-theory for Operator Algebras*, Mathematical Sciences Research Publications, 1998.
[2] T. Goodwillie, *Relative Algebraic K-Theory and Cyclic Homology*, Annals of Mathematics, Vol. 124, 1986, available online.
[3] N. Higson, *Algebraic K-Theory for stable C^*-algebras*, Advances in Mathematics 67, available online.
[4] N. Higson, J. Roe, *Analytic-K-Homology*, Oxford Science Publications, 2000.
[5] F. Hirzebruch, W. Scharlau, *Einführung in die Funktionalanalysis*, Spektrum Akademischer Verlag, 1971.
[6] M. Land, *Personal Notes.*
[7] M. Rørdam, F. Larsen, N. Laustsen, *An Introduction to K-theory of C^*-Algebras*, Cambridge University Press, 2000.
[8] J. Rosenberg, *Comparison Between Algebraic and Topological K-Theory for Banach and C^*-Algebras*, Handbook of K-theory, 2005.
[9] A.A. Suslin, M. Wodzicki, *Excision in algebraic K-Theory*, Annals of Mathematics, Vol. 136, 1992, available online.
[10] M. Takesaki, *Theory of Operator Algebras I*, Springer, 1979.
[11] N.E. Wegge-Olsen, *K-Theory and C^*-algebras - A friendly Approach*, Oxford University Press, 1993.
[12] C.A. Weibel, *Mayer-Viertories Sequences and Mod P K-theory*, available online.
[13] M. Wodzicki, *The long exact sequence in cyclic homology associated with an extension of algebras*, C.R. Acad. Sci. Paris Sér. I Math., 306(9):399-403, 1988.