§1 Divisors

Divisors come in 3 forms : Cartier divisors

Weil divisors

line bundles

Sometimes the scheme has to satisfy additional assumptions in order to make things work. We shall review the most frequent ones first

A. X = noetherian, integral (= reduced + irreducible), regular in codimension one

(If $\eta \in X$ with $\dim \mathcal{O}_\eta = 1$, then $\mathcal{O}_\eta$ is a regular ring.)

- Geometrically this means that X_{sing} has codim ≥ 2.
- All schemes will be assumed separated.

B. X = noetherian, integral, normal

(For any $x \in X$: $\mathcal{O}_x$ is normal, i.e. $\mathcal{O}_x$ is integrally closed)

Recollections from comm. algebra:

A = noetherian ring is normal

$\underset{\text{Def}}{\iff}$ • $A_\mathfrak{p}$ is normal for all $\mathfrak{p} \in \operatorname{Spec}(A)$,

i.e. $A_\mathfrak{p}$ is integrally closed domain

$\rightsquigarrow$ (B) $\Rightarrow$ (A) $\iff$ • $A_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht} \mathfrak{p} = 1$

• If $\operatorname{ht} \mathfrak{p} \geq 2$, then depth $A_\mathfrak{p} \geq 2$

(By definition $\operatorname{ht}(\mathfrak{p}) = \dim A_\mathfrak{p}$.

Thus, if $\dim A_\mathfrak{p} = 1$ $\Rightarrow$ $A_\mathfrak{p}$ DVR)

A normal domain $\Longrightarrow$ $A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p}$ (*)

(The following will not be used:

- A is <u>Cohen-Macaulay</u> (CM) if
$$\operatorname{depth}(A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) \qquad \forall \mathfrak{p} \in \operatorname{Spec}(A)$$
(usually "$\leq$")

- A normal, $\dim A \leq 2 \Rightarrow$ CM

- Complete intersections are CM
- Complete intersections are normal iff regular in codim 1

($\nearrow$ [Ha, II.8]))

C. X = noetherian, integral, <u>factorial</u>

($A = \mathcal{O}_{X,x}$, $x \in X \Rightarrow A$ UFD ("factorial"))

Recall: A noetherian domain:

- A UFD $\Leftrightarrow$ $\mathfrak{p} \in \operatorname{Spec}(A)$ is principal if ht $\mathfrak{p} = 1$

- A UFD $\Rightarrow$ A normal $\rightsquigarrow$ C $\Rightarrow$ B

Comparison A, C:

$$\text{ht}\,\mathfrak{p} = 1 \quad \begin{array}{l} \xRightarrow{A} \ \mathfrak{p} \subset A_{\mathfrak{p}} \text{ is principal } (\Leftrightarrow \text{regular}) \\ \qquad\qquad \Uparrow \\ \xRightarrow{C} \ \mathfrak{p} \subset A \text{ is principal} \end{array}$$

$\rightsquigarrow$ C $\Rightarrow$ A

- A UFD $\Leftrightarrow$ normal + $\operatorname{Cl}(\operatorname{Spec} A) = 0$ ($\rightarrow$ later)

[Ha, II. Prop. 6.2.]

D. X = noetherian, integral, regular

($\Leftrightarrow$ $\forall x \in X$, $\mathcal{O}_x$ is regular

$\Leftrightarrow$ $m_x \subset \mathcal{O}_x$ generated by $d = \dim \mathcal{O}_x$ elements

$\Leftrightarrow$ $\dim_k m_x/m_x^2 = d$ $\quad k = \mathcal{O}_x/m_x$)

Typical: X = non-singular variety $/k$

D $\Rightarrow$ C $\Rightarrow$ B $\Rightarrow$ A

We will most frequently assume B

(*) geometrically $A = \bigcap_{\mathrm{ht}\, p = 1} A_p$ for normal rings will be used

as $$\Gamma(X, \mathcal{O}_X) \simeq \Gamma(U, \mathcal{O}_X) \text{ w/}$$

$U \subset X$, $\operatorname{codim} X \setminus U \geq 2$

<u>Cartier divisors</u>: X = arbitrary scheme

$\mathcal{K}_X$ sheaf of total quotient rings

$\mathcal{O}_X^* \subset \mathcal{K}_X^* \quad \rightsquigarrow \quad \mathcal{K}_X^*/\mathcal{O}_X^*$ quotient sheaf

$$\underline{Ca(X)} := H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$$

the group of Cartier divisors

$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$ exact sequence

$$\rightsquigarrow \quad H^0(X, \mathcal{K}_X^*) \to \underbrace{H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)}_{= Ca(X)} \to H^1(X, \mathcal{O}_X^*) \qquad (*)$$

$\operatorname{Im}(H^0(X, \mathcal{K}_X^*) \to Ca(X)) = \{$<u>principal</u> Cartier divisors$\}$

$\rightsquigarrow$ equivalence relation "$\sim$"

$$\underline{CaCl(X)} := Ca(X)/\sim$$

(In [KMM] one uses $Div(X)$ instead of $Ca(X)$.)

<u>Rem</u>: Cartier divisors can be pulled-back under dominant morphisms. If: $f: Y \to X$ dominant, then

$$\begin{array}{ccc} Ca(X) & \xrightarrow{f^*} & Ca(Y) \\ \downarrow & & \downarrow \\ CaCl(X) & \xrightarrow[f^*]{} & CaCl(Y) \end{array} \qquad Y \text{ irreducible!}$$

$D \triangleq \{(f_i \in \Gamma(U_i, \mathcal{K}_X^*)) \mid f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)\}$

$\longmapsto f^*D \triangleq \{(f_i \circ f \in \Gamma(f^{-1}U_i, \mathcal{K}_Y^*))\}$

<u>Line bundles</u> $\quad X$ arbitrary scheme

$$\underline{\mathrm{Pic}(X)} = \{\text{invertible sheaves}\} \qquad \text{Picard group}$$
$$\cong H^1(X, \mathcal{O}_X^*)$$

<u>Rem</u>: Line bundles can be pulled-back under arbitrary morphisms:

$$f: Y \to X \rightsquigarrow f^*: \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$
group homom.

The exact sequence ⊛ yields

$$\begin{tikzcd}
\mathrm{Ca}(X) \arrow[rr] \arrow[dr, two heads] & & \mathrm{Pic}(X) \\
& \mathrm{CaCl}(X) \arrow[ur, hook] &
\end{tikzcd}$$

X projective$/k \Rightarrow \mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$

X integral $\Rightarrow \mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$
([Ha, II. 6.15])
($\mathcal{K}(X)^*$ constant sheaf $\Rightarrow$ flasque
$\Rightarrow H^1(X, \mathcal{K}(X)^*) = 0$)

<u>Weil divisors</u>: Suppose X satisfies Ⓐ.

<u>prime divisor</u>: $Y \subset X$ closed integral subscheme, codim $= 1$

$$Z(X) = \{ \sum n_i [Y_i] \text{ finite sums } / Y_i \subset X \text{ prime} \}$$

Ⓐ is needed in order to introduce "principal" divisors

If $Y \subset X$ prime divisor, $\eta \in Y$ generic

$\Rightarrow \dim \mathcal{O}_\eta = \operatorname{codim} Y = 1$

Ⓐ $\Rightarrow$ $\mathcal{O}_\eta$ regular $\Rightarrow$ DVR

(Recollections on valuations:

K field, $\quad v : K \setminus \{0\} \longrightarrow \mathbb{Z}$ <u>discrete valuation</u>

if i) $v(xy) = v(x) + v(y)$

ii) $v(x+y) \geq \min \{v(x), v(y)\}$

$v \rightsquigarrow R := \{x \mid v(x) \geq 0\} \cup \{0\} \subset K$

"<u>valuation ring</u>"

with

$m := \{x \mid v(x) > 0\} \cup \{0\} \subset R$ maximal ideal

Let R = noeth., local, $\dim R = 1$. Then

R DVR $\iff$ integrally closed (normal)

$\iff$ regular

$\iff$ maximal ideal is principal.)

Consider the induced valuation

$$v_Y : Q(\mathcal{O}_\eta^*) = K(X)^* \longrightarrow \mathbb{Z}$$

Suppose $f \in K(X)^*$

$\leadsto \quad (f) = \sum v_Y(f)\,[Y]$,

where the sum runs over all prime divisors.

Divisors of the form (f) are called

<u>principal Weil divisors</u>

Note: · $(f \cdot g) = (f) + (g)$

· The sum is indeed finite ([Ha, II. 6.1])

$\leadsto$ "$\sim$" equivalence relation on $Z(X)$

$$\leadsto \quad Cl(X) := Z(X)/\sim$$

<u>divisor class group</u>

(Rem: Not any discrete valuation is induced by some prime divisor $Y \subset X$.)

There are natural maps (group homom.)

$$\begin{array}{ccc} Ca(X) & \longrightarrow & Z(X) \\ \downarrow & & \downarrow \\ CaCl(X) & \longrightarrow & Cl(X) \end{array}$$

$$(f_i) \longmapsto \sum \underbrace{v_{Y \cap U_i}(f_i)}\,[Y]$$

if $U_i \cap Y \neq \emptyset$. Independent of i,

for $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$

If Ⓒ, then
$$\begin{array}{ccc} Ca(X) & \cong & Z(X) \\ \downarrow & & \downarrow \\ CaCl(X) & \cong & Cl(X) \end{array}$$

[Ha, II. Prop. 6.11]

Roughly: Im $(Ca(X) \to Z(X))$ are the Weil divisors, that can be defined (locally) by one equation!
For factorial schemes this holds for all divisors.
(Ⓑ $\Rightarrow$ injective)

$Y \subset X$ prime divisor, $U = X \setminus Y$.

$$\Rightarrow \quad Cl(X) / _{[Y]\mathbb{Z}} \cong Cl(U)$$

[Ha, II. Prop. 6.5]

Rem: Weil divisors cannot be pulled-back in general! (The preimage $f^{-1}Y$ of $Y \subset X$ gets components of smaller dimensions.)

Intersection numbers

Let X be a complete scheme, e.g. a projective variety. If $L_1, \dots, L_r \in \mathrm{Pic}(X)$, then

$$(m_1, \dots, m_r) \longmapsto \chi(X, L_1^{m_1} \otimes \dots \otimes L_r^{m_r}) = \sum (-1)^i h^i(\quad)$$

is a *numerical polynomial* $(r = \dim X)$

(i.e. polynomial in $m_1, \dots, m_r$ with integer values for $(m_1 \dots m_r) \in \mathbb{Z}^r$) of degree r

($\nearrow$ [Ha, Ample subvarieties], Kleiman
[Debarre] for projective.

More generally, $\chi(X, L_1^{m_1} \otimes \dots \otimes L_r^{m_r} \otimes \mathcal{F})$ is a numerical polynomial for $\mathcal{F} \in \mathrm{Coh}(X)$ with $\dim \operatorname{supp} \mathcal{F} = r$

Def. $(L_1 . \dots . L_r)$ = coeff of the monomial $m_1 \dots m_r$

Often this is instead written for Cartier divisors $D_1, \dots, D_r$ as

$$(D_1 . \dots . D_r) := (\mathcal{O}(D_1) . \dots . \mathcal{O}(D_r))$$

Rem: A priori, intersection numbers don't make sense for Weil divisors! (Except if X is factorial.)

- $(\underbrace{D \cdot \ldots \cdot D}_{r\text{-times}}) = \frac{1}{r!}$ coefficient of $\chi(\mathcal{O}(D)^m)$
  $=: D^r$!

- One also writes $\chi(mD) := \chi(\mathcal{O}(D)^m)$
  and $\chi(m_1 D_1 + \ldots + m_r D_r)$ . .

- Clearly, $\chi(mD) = \frac{m^r}{r!} D^r$ + lower order terms

__Exercise:__ $f: Z \to X$ birational morphism between normal varieties Z, X. (X normal enough!)

If $Y_1, \dots Y_k \subset Z$ exceptional prime divisors, i.e. $f(Y_1), \dots f(Y_k) \subset X$ have codim ≥ 2, then

$$\bigoplus \mathbb{Z}[Y_i] \hookrightarrow Cl(X).$$

Otherwise $\exists \; h \in K(Z)^* = K(X)^*$

$$(h) = \sum a_i [Y_i]$$

$$\Rightarrow \quad h \in \Gamma(X \setminus \cup f(Y_i), \mathcal{O})$$

$$= \Gamma(X, \mathcal{O}) \qquad \text{normality of } X$$

$$\Rightarrow \quad h = f^* h_0 \qquad h_0 \in \Gamma(X, \mathcal{O})$$

$$\Rightarrow \quad h \in \Gamma(Z, \mathcal{O}_Z)$$

Similar for h^{-1} $\Rightarrow$ no zeroes no poles