The local Langlands conjectures for non-quasi-split groups

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Abstract

We present different statements of the local Langlands conjectures for non-quasi-split groups that currently exist in the literature and provide an overview of their historic development. Afterwards, we formulate the conjectural multiplicity formula for discrete automorphic representations of non-quasi-split groups.

Contents

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1 Motivation and review of the quasi-split case

1.1 The basic form of the local Langlands conjecture

Let F be a local field of characteristic zero (see Subsection 1.6 for a brief discussion of this assumption) and let G be a connected reductive algebraic group defined over F. A basic problem in representation theory is to classify the irreducible admissible representations of the topological group $G(F)$. The Langlands classification reduces this problem to that of classifying the tempered irreducible admissible representations of $G(F)$, whose set of equivalence classes will be denoted by $\Pi_{\text{temp}}(G)$. In this paper, we will focus exclusively on tempered representations.

The local Langlands conjecture, as outlined for example in [Bor79], proposes a partition of this set indexed by arithmetic objects that are closely related to representations of the absolute Galois group Γ of F. More precisely, let W_F be the Weil group of F. Then

$$L_F = \begin{cases} W_F, & F \text{ archimedean} \\ W_F \times SU_2(\mathbb{R}), & F \text{ non-archimedean} \end{cases}$$

is the local Langlands group of F, a variant of the Weil-Deligne group suggested in [Lan79a, p.209] and [Kot84, p.647]. Let $\widehat{G}$ be the connected complex Langlands dual group of G, as defined for example in [Bor79, §2] or [Kot84, §1], and let ${}^LG = \widehat{G} \rtimes W_F$ be the Weil-form of the L-group of G. Let $\Phi_{\text{temp}}(G)$ be the set of $\widehat{G}$-conjugacy classes of tempered admissible L-homomorphisms $L_F \to {}^LG$. We recall from [Lan83, IV.2], see also [Bor79, §8], that an L-homomorphism is a homomorphism $\phi : L_F \to {}^LG$ that commutes with the projections to W_F of its source and target. It is called admissible if it is continuous and sends elements of W_F to semi-simple elements of ${}^LG$. It is called tempered if its image projects to a bounded subset of $\widehat{G}$.

The basic form of the local Langlands conjecture is the following.

Conjecture A. *1. There exists a map*

$$LL : \Pi_{\text{temp}}(G) \to \Phi_{\text{temp}}(G), \tag{1}$$

with finite fibers $\Pi_\phi(G) = LL^{-1}(\phi)$.

2. *The fiber $\Pi_\phi(G)$ is empty if and only if ϕ is not relevant, i.e. its image is contained in a parabolic subgroup of ${}^LG$ that is not relevant for G.*

3. *If $\phi \in \Phi_{\rm temp}(G)$ is unramified, then each $\pi \in \Pi_\phi(G)$ is K_π-spherical for some hyperspecial maximal compact subgroup K_π and for every such K there is exactly one K-spherical $\pi \in \Pi_\phi(G)$. The correspondence $\Pi_\phi(G) \leftrightarrow \phi$ is given by the Satake isomorphism.*

4. *If one element of $\Pi_\phi(G)$ belongs to the essential discrete series, then all elements of $\Pi_\phi(G)$ do, and this is the case if and only if the image of ϕ is not contained in a proper parabolic subgroup of ${}^LG$ (or equivalently in a proper Levi subgroup of ${}^LG$).*

5. *If $\phi \in \Phi_{\rm temp}(G)$ is the image of $\phi_M \in \Phi_{\rm temp}(M)$ for a proper Levi subgroup $M \subset G$, then $\Pi_\phi(G)$ consists of the irreducible constituents of the representations that are parabolically induced from elements of $\Pi_{\phi_M}(M)$.*

There are further expected properties, some of which are listed in [Bor79, §10] and are a bit technical to describe here. This basic form of the local Langlands conjecture has the advantage of being relatively easy to state. It is however insufficient for most applications. What is needed is the ability to address individual representations of $G(F)$, rather than finite sets of representations. Ideally this would lead to a bijection between the set $\Pi_{\rm temp}(G)$ and a refinement of the set $\Phi_{\rm temp}(G)$. Moreover, one needs a link between the classification of representations of reductive groups over local fields and the classification of automorphic representations of reductive groups over number fields. Both of these are provided by the refined local Langlands conjecture.

1.2 The refined local Langlands conjecture for quasi-split groups 1

Formulating the necessary refinement of the local Langlands conjecture is a non-trivial task. We will begin with the case when G is quasi-split, in which a statement has been known for some time.

Given $\phi \in \Phi_{\rm temp}(G)$, we consider the complex algebraic group $S_\phi = \mathrm{Cent}(\phi(L_F), \widehat{G})$. The arguments of [Kot84, §10] show that S_ϕ° is a reductive group. Let $\bar{S}_\phi = S_\phi/Z(\widehat{G})^\Gamma$. The first refinement of the basic local Langlands conjecture can now be stated as follows.

Conjecture B. *There exists an injective map*

$$\iota : \Pi_\phi \to \mathit{Irr}(\pi_0(\bar{S}_\phi)), \tag{2}$$

which is bijective if F is p-adic.

We have denoted here by Irr the set of equivalence classes of irreducible representations of the finite group $\pi_0(\bar{S}_\phi)$. Various forms of this refinement appear in the works of Langlands and Shelstad, see for example [She79], as well as Lusztig [Lus83].

A further refinement rests on a conjecture of Shahidi stated in [Sha90, §9]. To describe it, recall that a Whittaker datum for G is a $G(F)$-conjugacy class of pairs (B, ψ), where B is a Borel subgroup of G defined over F with unipotent radical U, and ψ is a non-degenerate character $U(F) \to \mathbb{C}^\times$, i.e. a character whose restriction to each simple relative root subgroup of U is non-trivial. When G is adjoint, it has a unique Whittaker datum. In general, there can be more than one Whittaker datum, but there are always only finitely many. Given a Whittaker datum $\mathfrak{w} = (B, \psi)$, an admissible representation π is called $\mathfrak{w}$-generic if $\mathrm{Hom}_{U(F)}(\pi, \psi) \neq 0$. A strong form of Shahidi's conjecture is the following.

Conjecture C. *Each set $\Pi_\phi(G)$ contains a unique $\mathfrak{w}$-generic constituent.*

This allows us to assume, as we shall do from now on, that ι maps the unique $\mathfrak{w}$-generic constituent of $\Pi_\phi(G)$ to the trivial representation of $\pi_0(\bar{S}_\phi)$. It is then more apt to write $\iota_\mathfrak{w}$ instead of just ι. We shall soon introduce another refinement, which will specify $\iota_\mathfrak{w}$ uniquely. One can then ask the question: How does $\iota_\mathfrak{w}$ depend on $\mathfrak{w}$. This dependence can be quantified precisely [Kal13, §4], but we will not go into this here. We will next state a further refinement that ties the sets $\Pi_\phi(G)$ into the stabilization of the Arthur-Selberg trace formula. It also has the effect of ensuring that the map $\iota_\mathfrak{w}$ is unique (provided it exists).

1.3 Endoscopic transfer of functions

Before we can state the next refinement of the local Langlands conjecture we must review the notion of endoscopic transfer of functions, and for this we must review the notion of endoscopic data and transfer factors. The notion of endoscopic data was initially introduced in [LS87] and later generalized to the twisted case in [KS99]. We will present the point of view of [KS99], but specialized to the ordinary, i.e. non-twisted, case.

Definition 1. 1. *An endoscopic datum is a tuple $\mathfrak{e} = (G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, s^\mathfrak{e}, \eta^\mathfrak{e})$, where $G^\mathfrak{e}$ is a quasi-split connected reductive group defined over F, $\mathcal{G}^\mathfrak{e}$ is a split extension of $\widehat{G}^\mathfrak{e}$ by W_F (but without a chosen splitting), $s^\mathfrak{e} \in \widehat{G}$ is a semi-simple element, and $\eta^\mathfrak{e} : \mathcal{G}^\mathfrak{e} \to {}^LG$ is an L-homomorphism that restricts to an isomorphism of complex reductive groups $\widehat{G}^\mathfrak{e} \to Cent(s^\mathfrak{e}, \widehat{G})^\circ$ and satisfies the following: There exists $s' \in Z(\widehat{G})s^\mathfrak{e}$ such that for all $h \in \mathcal{G}^\mathfrak{e}$, $s'\eta^\mathfrak{e}(h) = \eta^\mathfrak{e}(h)s'$.*

2. *An isomorphism between endoscopic data $\mathfrak{e}_1$ and $\mathfrak{e}_2$ is an element $g \in \widehat{G}$ satisfying $g\eta^{\mathfrak{e}_1}(\mathcal{G}^{\mathfrak{e}_1})g^{-1} = \eta^{\mathfrak{e}_2}(\mathcal{G}^{\mathfrak{e}_2})$ and $gs^{\mathfrak{e}_1}g^{-1} \in Z(\widehat{G}) \cdot s^{\mathfrak{e}_2}$.*

3. *A z-pair for $\mathfrak{e}$ is a pair $\mathfrak{z} = (G_1^\mathfrak{e}, \eta_1^\mathfrak{e})$, where $G_1^\mathfrak{e}$ is an extension of $G^\mathfrak{e}$ by an induced[1] torus with the property that $G_{1,der}^\mathfrak{e}$ is simply connected, and $\eta_1^\mathfrak{e} : \mathcal{G}^\mathfrak{e} \to {}^LG_1^\mathfrak{e}$ is an injective L-homomorphism that restricts to the homomorphism $\widehat{G}^\mathfrak{e} \to \widehat{G}_1^\mathfrak{e}$ dual to the given projection $G_1^\mathfrak{e} \to G^\mathfrak{e}$.*

We emphasize here that the crucial properties of a z-pair are that the representation theory of $G^\mathfrak{e}(F)$ and $G_1^\mathfrak{e}(F)$ are very closely related, and that the map $\eta_1^\mathfrak{e}$ exists. The latter is a consequence of the simply-connectedness of the derived subgroup of $G_1^\mathfrak{e}$.

There are two processes that produce endoscopic data [She83, §4.2], one appearing in the stabilization of the geometric side of the trace formula, and one in the stabilization of the spectral side (or, said differently, in the

[1] We remind the reader that an induced torus is a product of tori of the form $\mathrm{Res}_{E/F}\mathbb{G}_m$ for finite extensions E/F.

spectral interpretation of the stable trace formula). These processes naturally produce the extension $\mathcal{G}^{\mathfrak{e}}$. This extension is however not always isomorphic to the L-group of $G^{\mathfrak{e}}$. The purpose of the z-pair is to circumvent this technical difficulty. It is shown in [KS99, §2.2] that z-pairs always exist.

In some cases the extension $\mathcal{G}^{\mathfrak{e}}$ is isomorphic to ${}^LG^{\mathfrak{e}}$ and the z-pair becomes superfluous. For example, this is the case when G_{der} is simply-connected [Lan79b, Proposition 1]. Further examples are the symplecic and special orthogonal groups. It is then more convenient to work with a hybrid notion that combines an endoscopic datum and a z-pair. Moreover, we can replace in the above definition $s^{\mathfrak{e}}$ by s' without changing the isomorphism class of the endoscopic datum. This leads to the following definition.

Definition 2. *An extended endoscopic triple is a triple $\mathfrak{e} = (G^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}})$, where $G^{\mathfrak{e}}$ is a quasi-split connected reductive group defined over F, $s^{\mathfrak{e}} \in \widehat{G}$ is a semi-simple element, and ${}^L\eta^{\mathfrak{e}} : {}^LG^{\mathfrak{e}} \to {}^LG$ is an L-homomorphism that restricts to an isomorphism of complex reductive groups $\widehat{G}^{\mathfrak{e}} \to Cent(s^{\mathfrak{e}}, \widehat{G})^\circ$ and satisfies $s^{\mathfrak{e}}\,{}^L\eta^{\mathfrak{e}}(h) = {}^L\eta^{\mathfrak{e}}(h)s^{\mathfrak{e}}$.*

The relationship between Definitions 1 and 2 is the following: If $(G^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}})$ is an extended endoscopic triple, then $(G^{\mathfrak{e}}, {}^LG^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}})$ is an endoscopic datum. Moreover, even though $G^{\mathfrak{e}}$ will generally not have a simply-connected derived group, one can take $(G^{\mathfrak{e}}, \text{id})$ as a z-pair for itself.

In this paper we will work with the notion of an extended endoscopic triple. This will allow us to avoid some routine technical discussions. The more general case of an endoscopic datum and a z-pair doesn't bring any substantial changes but comes at the cost of burdening the exposition. Thus we now assume given an extended endoscopic triple $\mathfrak{e}$ for G, as well as a Whittaker datum $\mathfrak{w}$ for G. Associated to these data there is a transfer factor, i.e. a function

$$\Delta[\mathfrak{w}, \mathfrak{e}] : G^{\mathfrak{e}}_{\text{sr}}(F) \times G_{\text{sr}}(F) \to \mathbb{C},$$

where the subscript "sr" means semi-simple and strongly regular (those are the elements whose centralizer is a maximal torus). A variant of the factor Δ was defined in [LS87], then renormalized in [KS99], and slightly modified in [KS] to make it compatible with a corrected version of the twisted transfer factor. We will review the construction, taking these developments into account. For the readers familiar with these references we note that the factor we are about to describe is the factor denoted by Δ'_λ in [KS, (5.5.2)], in the case of ordinary endoscopy. In particular, it does not correspond to the relative factor Δ defined in [LS87]. The difference between the two lies in the inversion of the endoscopic element s. We work with this modified factor in order to avoid having to use inverses later when dealing with inner forms.

We first recall the notion of an admissible isomorphism between a maximal torus $S^{\mathfrak{e}}$ of $G^{\mathfrak{e}}$ and a maximal torus S of G. Let (T, B) be a Borel pair of G defined over F and let $(\widehat{T}, \widehat{B})$ be a Γ-stable Borel pair of $\widehat{G}$. Part of the datum of the dual group is an identification $X_*(T) = X^*(\widehat{T})$. The same is true for $G^{\mathfrak{e}}$ and we fix a Borel pair $(T^{\mathfrak{e}}, B^{\mathfrak{e}})$ of $G^{\mathfrak{e}}$ defined over F and a Γ-stable Borel pair $(\widehat{T}^{\mathfrak{e}}, \widehat{B}^{\mathfrak{e}})$ of $\widehat{G}^{\mathfrak{e}}$. The notion of isomorphism of endoscopic data allows us to assume that $\eta^{-1}(\widehat{T}, \widehat{B}) = (\widehat{T}^{\mathfrak{e}}, \widehat{B}^{\mathfrak{e}})$. Then η induces an isomorphism $X^*(\widehat{T}^{\mathfrak{e}}) \to X^*(\widehat{T})$, and this leads to an isomorphism $T^{\mathfrak{e}} \to T$. An isomorphism $S^{\mathfrak{e}} \to S$ is called admissible, if it is the composition of the following kinds of isomorphisms:

- $\mathrm{Ad}(h) : S^{\mathfrak{e}} \to T^{\mathfrak{e}}$ for $h \in G^{\mathfrak{e}}$.
- $\mathrm{Ad}(g) : S \to T$ for $g \in G$.
- The isomorphism $T^{\mathfrak{e}} \to T$.

Let $\gamma \in G^{\mathfrak{e}}_{\mathrm{sr}}(F)$. Let $S^{\mathfrak{e}} \subset G^{\mathfrak{e}}$ be the centralizer of γ, which is a maximal torus of $G^{\mathfrak{e}}$. Let $\delta \in G_{\mathrm{sr}}(F)$ and let $S \subset G$ be its centralizer. The elements γ and δ are called related if there exists an admissible isomorphism $S^{\mathfrak{e}} \to S$ mapping γ to δ. If such an isomorphism exists, it is unique, and will be called $\varphi_{\gamma,\delta}$.

Next, we recall the relationship between pinnings and Whittaker data from [KS99, §5.3]. Extend the Borel pair (T, B) to an F-pinning $(T, B, \{X_\alpha\})$. Here α runs over the set Δ of absolute roots of T in G that are simple relative to B and X_α is a non-zero root vector for α. Each X_α determines a homomorphism $\xi_\alpha : \mathbb{G}_a \to U$ by the rule $d\xi_\alpha(1) = X_\alpha$. Combining all homomorphisms x_α we obtain an isomorphism $\prod_\alpha \mathbb{G}_a \to U/[U, U]$. Composing the inverse of this isomorphism with the summation map $\prod_\alpha \mathbb{G}_a \to \mathbb{G}_a$ we obtain a homomorphism $U \to \mathbb{G}_a$ that is defined over F and hence leads to a homomorphism $U(F) \to F$. Composing the latter with an additive character $\psi_F : F \to \mathbb{C}^\times$ we obtain a character $\psi : U(F) \to \mathbb{C}^\times$ which is generic by construction. Thus (B, ψ) is a Whittaker datum. Since all Whittaker data arise from this construction, we may assume that our choices of pinning and ψ_F were made in such a way that (B, ψ) represents $\mathfrak{w}$.

We can now review the construction of the transfer factor $\Delta[\mathfrak{w}, \mathfrak{e}]$. If γ and δ are not related, we set $\Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta) = 0$. Otherwise, it is the product of terms

$$\epsilon_L(V, \psi_F)\Delta_I^{-1}\Delta_{II}\Delta_{III_2}\Delta_{IV},$$

which we will explain now. Note that the term Δ_{III_1} of [LS87] is missing, as it is being subsumed by Δ_I in the quasi-split case. The letter V stands for the degree 0 virtual Galois representation $X^*(T) \otimes \mathbb{C} - X^*(T^{\mathfrak{e}}) \otimes \mathbb{C}$. The term $\epsilon_L(V, \psi)$ is the local L-factor normalized according to [Tat79, §3.6]. The term Δ_{IV} is the quotient

$$\frac{|\det(\mathrm{Ad}(\delta) - 1|\mathrm{Lie}(G)/\mathrm{Lie}(S))|^{\frac{1}{2}}}{|\det(\mathrm{Ad}(\gamma) - 1|\mathrm{Lie}(G^{\mathfrak{e}})/\mathrm{Lie}(S^{\mathfrak{e}}))|^{\frac{1}{2}}}.$$

To describe the other terms, we need additional auxiliary data. We fix a set of a-data [LS87, §2.2] for the set $R(S, G)$ of absolute roots of S in G, which is a function

$$R(S, G) \to \overline{F}^\times, \alpha \mapsto a_\alpha$$

satisfying $a_{\sigma\lambda} = \sigma(a_\lambda)$ for $\sigma \in \Gamma$ and $a_{-\lambda} = -a_\lambda$. We also fix a set of χ-data [LS87, §2.5] for $R(S, G)$. To recall what this means, let $\Gamma_\alpha = \mathrm{Stab}(\alpha, \Gamma)$ and $\Gamma_{\pm\alpha} = \mathrm{Stab}(\{\alpha, -\alpha\}, \Gamma)$ for $\alpha \in R(S, G)$. Let F_α and $F_{\pm\alpha}$ be the fixed fields of Γ_α and $\Gamma_{\pm\alpha}$ respectively. Then $F_\alpha/F_{\pm\alpha}$ is an extension of degree 1 or 2. A set of χ-data is a set of characters $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ for each $\alpha \in R(S, G)$, satisfying the conditions $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$, $\chi_{-\alpha} = \chi_\alpha^{-1}$, and if $[F_\alpha : F_{\pm\alpha}] = 2$, then $\chi_\alpha|_{F_{\pm\alpha}^\times}$ is non-trivial but trivial on the subgroup of norms from $F_\alpha^\times$.

With these choices, we have

$$\Delta_{II} = \prod_\alpha \chi_\alpha\left(\frac{\alpha(\delta)-1}{a_\alpha}\right),$$

where the product is taken over the set of orbits for the action of Γ on $R(S,G) \smallsetminus \varphi_{\gamma,\delta}^{*,-1}(R(S^\mathfrak{e},G^\mathfrak{e}))$.

The term Δ_I involves the so-called splitting invariant [LS87, §2.3] of S. Let $g \in G$ be such that $gTg^{-1} = S$. Write $\Omega(T,G)$ for the absolute Weyl group. For each $\sigma \in \Gamma$ there exists $\omega(\sigma) \in \Omega(T,G)$ such that for all $t \in T$

$$\omega(\sigma)\sigma(t) = g^{-1}\sigma(gtg^{-1})g.$$

Let $\omega(\sigma) = s_{\alpha_1}\dots s_{\alpha_k}$ be a reduced expression and let n_i be the image of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under the homomorphism $\mathrm{SL}_2 \to G$ attached to the simple root vector X_{α_i}. Then $n(\sigma) = n_1\dots n_k$ is independent of the choice of reduced expression. The splitting invariant of S is the class $\lambda \in H^1(\Gamma, S_{\mathrm{sc}})$ of the 1-cocycle

$$\sigma \mapsto \prod \alpha^\vee(a_\alpha)g(n(\sigma)[g^{-1}\sigma(g)]^{-1})g^{-1}.$$

The product runs over the subset $\{\alpha > 0, \sigma^{-1}\alpha < 0\}$ of $R(S,G)$, with positivity being taken with respect to the Borel subgroup gBg^{-1}. The term Δ_I is defined as

$$\langle \lambda, s^\mathfrak{e}\rangle$$

where the pairing $\langle -,-\rangle$ is the canonical pairing between $H^1(\Gamma, S_{\mathrm{sc}})$ and $\pi_0([\widehat{S}/Z(\widehat{G})]^\Gamma)$ induced by Tate-Nakayama duality. Here we interpret $s^\mathfrak{e}$ as an element of $[Z(\widehat{G}^\mathfrak{e})/Z(\widehat{G})]^\Gamma$, embed the latter into $\widehat{S}^\mathfrak{e}/Z(\widehat{G})$, and use the admissible isomorphism $\varphi_{\gamma,\delta}$ to transport it to $\widehat{S}/Z(\widehat{G})$.

We turn to the term Δ_{III_2}. The construction in [LS87, §2.6] associates to the fixed χ-data a $\widehat{G}$-conjugacy class of L-embeddings $\xi_G : {}^LS \to {}^LG$. This construction is rather technical and we will not review it here. Via the admissible isomorphism $\varphi_{\gamma,\delta}$, the χ-data can be transferred to $S^\mathfrak{e}$ and provides a $\widehat{G}^\mathfrak{e}$-conjugacy class of L-embeddings $\xi_\mathfrak{e} : {}^LS^\mathfrak{e} \to {}^LG^\mathfrak{e}$. The admissible isomorphism $\varphi_{\gamma,\delta}$ provides dually an L-isomorphism ${}^L\varphi_{\gamma,\delta} : {}^LS \to {}^LS^\mathfrak{e}$. The composition $\xi' = {}^L\eta \circ \xi_\mathfrak{e} \circ {}^L\varphi_{\gamma,\delta}$ is then another $\widehat{G}$-conjugacy class of L-embeddings ${}^LS \to {}^LG$. Via conjugation by $\widehat{G}$ we can arrange that ξ_G and ξ' coincide on $\widehat{S}$. Then we have $\xi' = a \cdot \xi_G$ for some $a \in Z^1(W_F, \widehat{S})$. The term Δ_{III_2} is then given by

$$\langle a, \delta\rangle$$

where $\langle -,-\rangle$ is the pairing given by Langlands duality for tori.

We have completed the review of the construction of the transfer factor $\Delta[\mathfrak{w},\mathfrak{e}]$. We now recall the notion of matching functions from [KS99, §5.5].

Definition 3. *Two functions $f^{\mathfrak{w},\mathfrak{e}} \in \mathcal{C}_c^\infty(G^\mathfrak{e}(F))$ and $f \in \mathcal{C}_c^\infty(G(F))$ are called matching (or $\Delta[\mathfrak{w},\mathfrak{e}]$-matching, if we want to emphasize the transfer factor) if for* *all $\gamma \in G^{\mathfrak{e}}_{sr}(F)$ we have*

$$SO_\gamma(f^{\mathfrak{w},\mathfrak{e}}) = \sum_\delta \Delta[\mathfrak{w},\mathfrak{e}](\gamma,\delta)O_\delta(f),$$

where δ runs over the set of conjugacy classes in $G_{sr}(F)$.

We remark that the stable orbital integrals at regular (but possibly not strongly regular) semi-simple elements can be expressed in terms of the stable orbital integrals at strongly regular semi-simple elements by continuity, but one has to be careful with the summation index, see [LS87, §4.3]. The stable orbital integrals at singular elements can be related to the stable orbital integrals at regular elements, see [Kot88, §3].

One of the central pillars of the theory of endoscopy is the following theorem.

Theorem 4. *For each function $f \in \mathcal{C}^\infty_c(G(F))$ there exists a matching function $f^{\mathfrak{w},\mathfrak{e}} \in \mathcal{C}^\infty_c(G^{\mathfrak{e}}(F))$.*

In the case of archimedean F this theorem was proved by Shelstad in [She81] and [She82] in the setting of Schwartz-functions and extends to the setting of smooth compactly supported functions by the results of Bouaziz [Bou94]. In the case of non-archimedean F the proof of this theorem involves the work of many authors, in particular Waldspurger [Wal97], [Wal06], and Ngo [Ngô10].

1.4 The refined local Langlands conjecture for quasi-split groups 2

With the endoscopic transfer of functions at hand we can state the final refinement of the local Langlands conjecture in the setting of quasi-split groups.

Recall that Conjecture B asserted the existence of a map $\iota_{\mathfrak{w}} : \Pi_\phi(G) \to \mathrm{Irr}(\pi_0(\bar{S}_\phi))$. We can write this map as a pairing

$$\langle -,- \rangle : \Pi_\phi \times \pi_0(\bar{S}_\phi) \to \mathbb{C}, \qquad (\pi, s) \mapsto \mathrm{tr}(\iota_{\mathfrak{w}}(\pi)(s)).$$

When F is p-adic, so that the map $\iota_{\mathfrak{w}}$ is expected to be bijective, we may allow ourselves to call this pairing "perfect". Since $\pi_0(\bar{S}_\phi)$ may be non-abelian the word "perfect" is to be interpreted with care, but its definition is simply the one that is equivalent to saying that the map $\iota_{\mathfrak{w}}$, which can be recovered from $\langle -,- \rangle$, is bijective. Using this pairing we can form, for any $\phi \in \Phi_{\mathrm{temp}}(G)$ and $s \in S_\phi$ the virtual character

$$\Theta^s_\phi = \sum_{\pi \in \Pi_\phi(G)} \langle \pi, s \rangle \Theta_\pi, \tag{3}$$

where Θ_π is the Harish-Chandra character of the admissible representation π. Let now $\mathfrak{e} = (G^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}})$ be an extended endoscopic triple and $\phi^{\mathfrak{e}} \in \Phi_{\mathrm{temp}}(G^{\mathfrak{e}})$. Put $\phi = {}^L\eta^{\mathfrak{e}} \circ \phi^{\mathfrak{e}}$. It is then automatic that $s^{\mathfrak{e}} \in S_\phi$.

Conjecture D. *For any pair of matching functions $f^{\mathfrak{w},\mathfrak{e}}$ and f we have the equality*

$$\Theta^1_{\phi^{\mathfrak{e}}}(f^{\mathfrak{w},\mathfrak{e}}) = \Theta^{s^{\mathfrak{e}}}_\phi(f).$$

Note that this statement implies that the distribution Θ_ϕ^1 is stable.

This is the last refinement of the local Langlands conjecture for quasi-split groups. Notice that the linear independence of the distributions Θ_π, together with the disjointness of the packets $\Pi_\phi(G)$, imply that the map $\iota_{\mathfrak{w}}$ of (2) is unique, provided it exists and satisfies Conjectures B and D. On the other hand, these conjectures do not characterize the assignment $\phi \mapsto \Pi_\phi$. The most obvious case is that of those ϕ for which Π_ϕ is a singleton set. For them the content of the refined conjecture is that the constituent of Π_ϕ is generic with respect to each Whittaker datum and its character is a stable distribution. In the case of quasi-split symplectic and special orthogonal groups, Arthur [Art13] shows that the addition of a supplementary conjecture – twisted endoscopic transfer to GL_n – is sufficient to uniquely characterize the correspondence $\phi \mapsto \Pi_\phi$. For general groups such a unique characterization is sill not known.

From now on we will group these four conjectures under the name "refined local Langlands conjecture". In the archimedean case, this conjecture is known by the work of Shelstad. Many statements were derived in [She81], [She82], but with an implicit set of transfer factors instead of the explicitly constructed ones that we have reviewed in the previous section, as those were only developed in [LS87]. The papers [She08a], [She10], and [She08b] recast the theory using the canonical factors of [LS87] and provide many additional and stronger statements. In particular, the refined local Langlands conjecture is completely known for quasi-split real groups. We note here that Shelstad's work is not limited to the case of quasi-split groups. This will be discussed soon.

In the non-archimedean case, much less is known. On the one hand, there are general results for special kinds of groups. The case of GL_n (in which most of the refinements discussed here do not come to bear) is known by the work of Harris-Taylor [HT01] and Henniart [Hen00]. The book [Art13] proves the refined local Langlands conjecture for quasi-split symplectic and odd special orthogonal groups, and a slightly weaker version of it for even special orthogonal groups. Arthur's strategy has been reiterated in [Mok15] to cover the case of quasi-split unitary groups. In these cases, the uniqueness of the generic constituent in Conjecture C is not proved. This uniqueness follows from the works of Moeglin-Waldspurger, Waldspurger, and Beuzart-Plessis, on the Gan-Gross-Prasad conjecture. A short proof can be found in [Ato15]. On the other hand, there are results about special kinds of representations for general classes of groups. The papers [DR09] and [Kal11] cover the case of regular depth-zero supercuspidal representations of unramified p-adic groups, while the papers [RY14] and [Kal15] cover the case of epipelagic representations of tamely ramified groups. Earlier work of Kazhdan-Lusztig [KL87] and Lusztig [Lus95] proves a variant of this conjecture for unipotent representations of split simple adjoint groups, where the representations are not assumed to be tempered and the character identities are not studied.

1.5 Global motivation for the refinement

We now take F to be a number field, and G to be a connected reductive group, defined and quasi-split over F. We fix a Borel subgroup $TU = B \subset$

G and generic character $\psi : U(F) \setminus U(\mathbb{A}_F) \to \mathbb{C}^\times$.

We have the stabilization [Art02, (0.4)] of the geometric side of the trace formula

$$I^G_{\text{geom}}(f) = \sum \iota(G, G^{\mathfrak{e}}) S^{G^{\mathfrak{e}}}(f^{\mathfrak{e}}).$$

Here the sum runs over isomorphism classes of global elliptic extended endoscopic triples $\mathfrak{e} = (G^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}})$, $S^{G^{\mathfrak{e}}}$ is the so called stable trace formula for $G^{\mathfrak{e}}$, and $f^{\mathfrak{e}} = (f^{\mathfrak{e}}_v)$ is a function on $G^{\mathfrak{e}}(\mathbb{A})$ such that $f^{\mathfrak{e}}_v$ matches f_v. We note that global extended endoscopic triples are defined in the same way as in the local case in Definition 2, with only one difference: The condition on $\eta^{\mathfrak{e}}$ is that there exists $a \in Z^1(W_F, Z(\widehat{G}))$ whose class is everywhere locally trivial, so that $s^{\mathfrak{e}}\eta^{\mathfrak{e}}(h) = z(\bar{h})\eta^{\mathfrak{e}}(h)s^{\mathfrak{e}}$ for all $h \in \mathcal{G}^{\mathfrak{e}}$, where $\bar{h} \in W_F$ is the projection of h. One checks that $\eta^{\mathfrak{e}}$ provides a Γ-equivariant injection $Z(\widehat{G}) \to Z(\widehat{G}^{\mathfrak{e}})$. The triple $\mathfrak{e}$ is called *elliptic* if this injection restricts to a bijection $Z(\widehat{G})^{\Gamma,\circ} \to Z(\widehat{G}^{\mathfrak{e}})^{\Gamma,\circ}$.

The trace formula is an identity of the form

$$I^G_{\text{spec}}(f) = I^G_{\text{geom}}(f),$$

where the right hand side is [Art02, (0.1)] and the left hand side is [Art01, (0.2)]. The stabilization of the geometric side has as a formal consequence a stabilization of the spectral side. This allows us to write

$$I^G_{\text{disc}}(f) = \sum \iota(G, G^{\mathfrak{e}}) S^{G^{\mathfrak{e}}}_{\text{disc}}(f^{\mathfrak{e}}).$$

Here I^G_{disc} is the essential part of I^G_{spec}, see [Art01, (3.5)] or [Art88, (4.3)]. It contains the trace of discrete automorphic representations of $G(\mathbb{A})$, but also some contributions coming from Eisenstein series. This is the part of the trace formula one would like to understand in order to study automorphic representations, and the stabilization identity is meant to shed some light on it.

However, it is a-priori unclear what the spectral content of $S^{G^{\mathfrak{e}}}_{\text{disc}}(f^{\mathfrak{e}})$ is. The key to understanding this content lies in the refined local Langlands correspondence. Namely, just like the central ingredients of $I^G_{\text{disc}}(f)$ are the characters of discrete automorphic representations, the central ingredients of $S^{G^{\mathfrak{e}}}_{\text{disc}}(f^{\mathfrak{e}})$ are the *stable* characters of discrete automorphic L-packets. This is the content of Arthur's "stable multiplicity formula", as stated for example in [Art13, Theorem 4.1.2]. However, unlike the case of stable orbital integrals, which are defined unconditionally and in an elementary way, stable characters can only be defined once the refined local Langlands correspondence, or at least Conjectures A and B have been established. Granting these, they are the global analogs of the characters Θ^1_ϕ of Equation (3) and can be constructed out of these once a suitable notion of global parameters has been introduced, as was done for example in [Art13]. A global discrete parameter ϕ provides local parameters $\phi_v : L_{F_v} \to {}^LG$ and the associated stable character is then the product of the characters $\Theta^1_{\phi_v}$ over all v. Moreover, to have a chance at proving the stable multiplicity formula, Conjecture D must also be established.

Another crucial ingredient in the interpretation of the spectral side of the stable trace formula is the multiplicity formula for discrete automorphic representations. Given a global discrete parameter ϕ one obtains from the local parameters $\phi_v : L_{F_v} \to {}^LG$ the packets Π_{ϕ_v}. One also obtains a group $\bar{S}_\phi$ with maps $\bar{S}_\phi \to \bar{S}_{\phi_v}$. For each $\pi = \otimes'_v \pi_v$ with $\pi_v \in \Pi_{\phi_v}$ one considers the formula

$$m(\pi, \phi) = |\pi_0(\bar{S}_\phi)|^{-1} \sum_{x \in \pi_0(\bar{S}_\phi)} \prod_v \langle \pi_v, x \rangle.$$

It is then conjectured that the integer $m(\pi, \phi)$ is the ϕ-contribution of π to the discrete spectrum of G, and that the multiplicity of π in the discrete spectrum is equal to the sum of $m(\pi, \phi)$ over all (equivalence classes of) global parameters ϕ. We will discuss this formula in more detail in Section 5, where we will extend it to the case of non-quasi-split groups.

In all of these formulas, the existence of the map $\iota_{\mathfrak{w}} : \Pi_{\phi_v} \to \mathrm{Irr}(\pi_0(\bar{S}_{\phi_v}))$, and hence of the pairing $\langle -, - \rangle$, is crucial. There are further formulas which one can obtain, for example the inversion of endoscopic transfer, which allows one to obtain the characters of tempered representations from the stable characters of tempered L-packets. We refer the reader to [She08b] for a statement of this in the archimedean case, and to [Kal13] for a sample application.

1.6 Remarks on the characteristic of F

We have assumed throughout this section that F has characteristic zero. While it is believed that most of this material carries over in some form for fields (local or global as appropriate) of positive characteristic, most of the literature assumes that F has characteristic zero. For example, the work [LS87], [LS90], and [KS99] is written with this assumption. The later work [KS] is written for arbitrary local fields, which suggests that the definition of transfer factors should work in positive characteristic. However, the descent theory of [LS90] is not worked out in this setting. The fundamental lemma is proved in [Ngô10] in positive characteristic and then transfered to characteristic zero in [Wal09]. But the proof of the transfer theorem (Theorem 4) is only done in characterstic zero [Wal97]. Turning to the global situation, the theory of the trace formula, even before stabilization, for general reductive groups over global fields of positive characteristic is not developed. Thus, while most definitions, results, and conjectures, presented here are expected to hold (either in the same form or with some modifications) in positive characteristic, little factual information is actually present.

2 Non-quasi-split groups: Problems and approaches

We return now to the case of a local field F of characteristic zero and let G be a connected reductive group defined over F, but not necessarily quasi-split. We would like to formulate a refined local Langlands correspondence for G and to have global applications for it similar to the ones outlined in the last section. We are then met with the following problems

- There is no Whittaker datum, hence no canonical normalization of the transfer factor $\Delta(-,-)$.

The transfer factor $\Delta(-,-)$ is still defined in [LS87] and [KS99], but only up to a complex scalar. This has the effect that the notion of matching functions is also only defined up to a scalar. The trouble with this is that Conjecture D can no longer be stated in the precise form given above, and this makes the spectral interpretation of the stable trace formula problematic. Even worse, Arthur notices in [Art06, (3.1)] the following.

- Even the non-canonical normalizations of $\Delta(\gamma,\delta)$ are not invariant under automorphisms of endoscopic data.

This is a problem, because in the stabilization identity we are summing over isomorphism classes of endoscopic groups. The problem can be overcome, but it does indicate that something is not quite right.

- There is no good map $\iota : \Pi_\phi \to \mathrm{Irr}(\pi_0(\bar{S}_\phi))$.

The standard example for this comes from the work of Labesse and Langlands [LL79]. We follow here Shelstad's report [She79]. Let F be p-adic and G the unique inner form of SL_2, so that $G(\mathbb{Q}_p)$ is the group of elements of reduced norm 1 in the unique quaternion algebra over F. We construct a parameter by taking a quadratic extension E/F and a character $\theta : E^\times \to \mathbb{C}^\times$ for which $\theta^{-1}\cdot(\theta\circ\sigma)$ is non-trivial and of order 2, where $\sigma\in\Gamma_{E/F}$ is the non-trivial element. Let $\sigma^\circ\in W_{E/F}$ be a lift of σ. Then

$$\phi(e) = \begin{bmatrix}\theta(e) & 0\\ 0 & \theta(\sigma(e))\end{bmatrix}, e\in E^\times, \qquad \phi(\sigma^\circ) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

is a homomorphism $W_{E/F}\to\mathrm{PGL}_2(\mathbb{C})$. One checks that

$$\bar{S}_\phi = S_\phi = \left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right\} \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}.$$

The packet $\Pi_\phi(\mathrm{SL}_2(F))$ has exactly four elements. However, the packet $\Pi_\phi(G)$ has only one element π. Moreover, no character of χ of S_ϕ can be paired with this π so that the endoscopic character identities hold. In fact, in order to have the desired character identities, one must attach to π the function on $\bar{S}_\phi$ given by

$$f(s) = \begin{cases}2, & s=1\\ 0, & s\neq 1\end{cases}, \tag{4}$$

which is obviously not a character.

This last problem is most severe. Without the pairing between $\pi_0(\bar{S}_\phi)$ and Π_ϕ, we cannot state the global multiplicity formula and we cannot hope for a spectral interpretation of the stable trace formula.

2.1 Shelstad's work on real groups

Despite these problems, we have a very good understanding of the case of real groups thanks to the work of Langlands and Shelstad. Langlands has constructed in [Lan97] the map (1) and has shown that Conjecture

A holds. Shelstad has shown [She82], [She10], [She08b] that once an arbitrary choice of the transfer factor $\Delta(-,-)$ has been fixed, and further choices specific to real groups have been made, there exists an embedding $\iota : \Pi_\phi(G) \to \mathrm{Irr}(\pi_0(\bar{S}_\phi))$, thus verifying Conjecture B, and has moreover shown that the corresponding pairing makes the endoscopic character identities of Conjecture D true. Even more, Shelstad has shown that if one combines the maps ι for multiple groups G, namely those that comprise a so called K-group, then one obtains a bijection between the disjoint union of the corresponding L-packets and the set $\mathrm{Irr}(\pi_0(\bar{S}_\phi))$. For the notion of K-group we refer the reader to [Art99, §1] and [She08b, §4], and we note here only that it is unrelated to the Adams-Barbasch-Vogan notion of strong real forms that we will encounter below.

It may be worth pointing out here that the group $\pi_0(\bar{S}_\phi)$ is always an elementary 2-group in the archimedean case, so that $\mathrm{Irr}(\pi_0(\bar{S}_\phi))$ is in fact the Pontryagin dual group of that elementary 2-group. This work uses the results of Harish-Chandra and Knapp-Zuckerman on the classification of discrete series, and more generally of tempered representations, of real semi-simple groups.

2.2 Arthur's mediating functions

Turning now to p-adic fields, the example of the inner form of SL_2 shows that we cannot expect to have a result in the p-adic case that is similar to that of Shelstad in the real case, because the virtual characters needed in the formulation of Conjecture D for general groups cannot be obtained from characters of $\pi_0(\bar{S}_\phi)$. In his monograph [Art89], Arthur proposes to replace the pairing $\langle -,- \rangle$ by a combination of two objects. The first object is called the "spectral transfer factor", and denoted by $\Delta(\phi^{\mathfrak{e}}, \pi)$. Here again we assume to be given an extended endoscopic triple $\mathfrak{e}$ for G. We moreover assume fixed some arbitrary normalization of the transfer factor Δ, which we now qualify as "geometric", in order to distinguish it from the new "spectral" transfer factor. The spectral transfer factor takes as variables tempered parameters $\phi^{\mathfrak{e}}$ for $G^{\mathfrak{e}}$, as well as tempered representations π of $G(F)$. The role of the spectral transfer factor is to make the identity

$$\Theta^1_{\phi^{\mathfrak{e}}}(f^{\mathfrak{e}}) = \sum_\pi \Delta(\phi^{\mathfrak{e}}, \pi)\Theta_\pi(f)$$

true, whenever $f^{\mathfrak{e}}$ and f are matching with respect to the fixed normalization of the geometric transfer factor. Thus in particular the spectral factor depends on the geometric factor. Moreover, the isomorphisms of endoscopic data have to disturb the spectral factor in the same way that they disturb the geometric factor.

The second object is called the "mediating function", and denoted by $\rho(\Delta, s)$. The role of the mediating function is to make the product $\langle \pi, s \rangle = \rho(\Delta, s) \cdot \Delta(\phi^{\mathfrak{e}}, \pi)$ independent of the choice of geometric factor Δ, invariant under isomorphisms of endoscopic data, and a class function on the group $\pi_0(\bar{S}_\phi)$.

In the later paper [Art06], Arthur modifies this proposition to involve not the group $\bar{S}_\phi$, but rather its preimage S_ϕ^{sc} in the simply-connected cover of $\widehat{G}$, and demands that $\langle \pi, s\rangle$ is not just a class function, but in fact a character of an irreducible representation of $\pi_0(S_\phi^{\text{sc}})$. This is supported by the observation that the function (4) is indeed the character of the unique 2-dimensional irreducible representation of the quaternion group, which is the group S_ϕ^{sc} in the case of the inner forms of SL_2. Besides this observation, the introduction of the group S_ϕ^{sc} has its roots in Kottwitz's theorem [Kot86, Theorem 1.2] that relates the Galois cohomology set $H^1(\Gamma, G)$ to the Pontryagin dual of the finite abelian group $\pi_0(Z(\widehat{G})^\Gamma)$.

Let us be more precise. It is known that there exists a connected reductive group G^*, defined and quasi-split over F, together with an isomorphism $\xi : G^* \to G$ defined over $\overline{F}$ and having the property that for all $\sigma \in \Gamma_F$ the automorphism $\xi^{-1}\sigma(\xi)$ of G^* is inner. It is furthermore known that G^* is uniquely determined by G. Then G is called an inner form of G^* and $\xi : G^* \to G$ is called an inner twist. The inner twist provides an identification of the dual groups of G^* and G. The function $\sigma \mapsto \xi^{-1}\sigma(\xi)$ is an element of $Z^1(\Gamma, G^*_{\text{ad}})$. Kottwitz's theorem interprets this element as a character $[\xi] : Z(\widehat{G}^*_{\text{sc}})^\Gamma \to \mathbb{C}^\times$. Arthur suggests that one choose an arbitrary extension $\Xi : Z(\widehat{G}^*_{\text{sc}}) \to \mathbb{C}^\times$ of this character. Then, for every $\phi \in \Phi_{\text{temp}}(G^*)$, the L-packet $\Pi_\phi(G)$ should be in (non-canonical) bijection with the set $\mathrm{Irr}(\pi_0(S_\phi^{\text{sc}}), \Xi)$ of irreducible representations of the finite group $\pi_0(S_\phi^{\text{sc}})$ that transform under the image of $Z(\widehat{G}^*_{\text{sc}})$ by the character Ξ. For $\pi \in \Pi_\phi(G)$, the character of the representation of $\pi_0(S_\phi^{\text{sc}})$ corresponding to π via this bijection should be the class function $\langle \pi, -\rangle$. For each choice of geometric transfer factor Δ, we should have the expression $\langle \pi, -\rangle = \rho(\Delta, -) \cdot \Delta(\phi^{\mathfrak{e}}, \pi)$ as above.

This conjecture is stated uniformly for archimedean and non-archimedean local fields. In the archimedean case, this conjecture has been settled by Shelstad in [She10] and [She08b], using deep information about the representation theory and harmonic analysis of real reductive groups. In the non-archimedean case, the conjecture is open. The main challenges that impede its resolution are that the conjectural objects $\Delta(\phi^{\mathfrak{e}}, \pi)$ and $\rho(\Delta, s)$ make the extension of the refined local Langlands conjecture to non-quasi-split groups less precise and harder to state, and this leads to a weaker grip on them by the trace formula.

2.3 Vogan's pure inner forms

The work of Adams, Barbasch, and Vogan, [ABV92], introduces the following fundamental idea: When trying to describe L-packets, one should treat all reductive groups in a given inner class together. That is, instead of trying to describe the L-packets of G alone, one should fix the quasi-split inner form G^* of G and then describe the L-packets of all inner forms of G^* (of which G is one) at the same time. Here is a nice numerical example that underscores this idea: For a fixed positive integer n, the real groups $U(p, q)$ with $p+q = n$ constitute an inner class. For any discrete Langlands parameter ϕ one has $|S_\phi| = 2^n$ and $|\bar{S}_\phi| = 2^{n-1}$. On the other hand, one has $|\Pi_\phi(U(p, q))| = \binom{p+q}{q}$. Thus

$$| \sqcup_{p+q=n} \Pi_\phi(U(p, q))| = |S_\phi|.$$

Notice however that $U(p,q)$ and $U(q,p)$ are the same inner form of the quasi-split unitary group G^* (and are isomorphic as groups), but in order for the above equation to work out, we must treat them separately. This is not just a numerical quirk. It hints at a fundamental technical difficulty that will be of crucial importance.

In order to describe this difficulty more precisely, we need to recall a bit of Galois cohomology. The set of isomorphism classes of groups G which are inner forms of G^* is in bijection with the image of $H^1(\Gamma_F, G^*_{\text{ad}})$ in $H^1(\Gamma_F, \text{Aut}(G^*))$. However, this is a badly behaved set. Indeed, we can treat GL_n as an inner form of itself either via the identity map or via the isomorphism $g \mapsto g^{-t}$. Those two identifications clearly have different effects on representations. Thus, if we want to parameterize representations, we should treat these cases separately. This leads to considering not just the groups G which are inner forms of G^*, up to isomorphism, but rather inner twists $\xi : G^* \to G$, up to isomorphism. Here, an isomorphism from $\xi_1 : G^* \to G_1$ to $\xi_2 : G^* \to G_2$ is an isomorphism $f : G_1 \to G_2$ defined over F for which $\xi_2^{-1} \circ f \circ \xi_1$ is an inner automorphism of G^*. According to this definition, $f = \text{id}$ is not an isomorphism between the two inner twists $id : \text{GL}_n \to \text{GL}_n$ and $(-)^{-t} : \text{GL}_n \to \text{GL}_n$. In fact, we have achieved a rigidification of the problem, which means that we have cut down the automorphism group from $\text{Aut}(G)$ to $\text{Aut}(\xi)$, where $\text{Aut}(\xi)$ works out to be the subgroup of $\text{Aut}(G)$ given by $G_{\text{ad}}(F)$. However, as Vogan points out in [Vog93, §2], this rigidification is not enough. Indeed, we run into problems already with a group as simple as $G^* = \text{SL}_2/\mathbb{R}$. Let $\theta = \text{Ad}\begin{bmatrix}1 & \\ & -1\end{bmatrix}$. Then the map $f = \text{id} : G^* \to G^*$ is an isomorphism between the inner twists $id : G^* \to G^*$ and $\theta : G^* \to G^*$. However, θ swaps the constituents of discrete series L-packets (this can be computed explicitly in this example using K-types; it is however a general feature that the action of $G_{\text{ad}}(F)$ on $G(F)$ preserves each tempered L-packet Π_ϕ, as one can see from the stability of Θ^1_ϕ and the linear independence of characters). This is a problem because we would like an isomorphism between inner twists to be compatible with the parameterization of L-packets.

This leads Vogan to introduce in [Vog93] the notion of a pure inner twist (in fact, Vogan calls it "pure rational form"), which is a pair (ξ, z) with $\xi : G^* \to G$ inner twist and $z \in Z^1(\Gamma, G^*)$ having the property $\xi^{-1}\sigma(\xi) = \text{Ad}(z(\sigma))$. An isomorphism from (ξ_1, z_1) to (ξ_2, z_2) is now a pair (f, δ) with $f : G_1 \to G_2$ an isomorphism over F, $\delta \in G^*$ and satisfying the identities $\xi_2^{-1} f \xi_1 = \text{Ad}(\delta)$ and $z_1(\sigma) = \delta^{-1} z_2(\sigma)\sigma(\delta)$. One can now check that $\text{Aut}((\xi, z)) = G(F)$, thus an automorphism of (ξ, z) fixes each isomorphism class of representations and each rational conjugacy class of elements. We now finally have a shot of trying to parameterize the disjoint union of L-packets $\Pi_\phi((\xi, z))$, where (ξ, z) runs over the set of isomorphism classes of pure inner twists of a given quasi-split group G^*, and where $\Pi_\phi((\xi, z))$ is the L-packet on the group G that is the target of the pure inner twist $(\xi, z) : G^* \to G$. According to Vogan's formulation of the local Langlands correspondence [Vog93, Conjecture 4.3 and Conjecture 4.15], there should exist a bijection

$$\iota_{\mathfrak{w}} : \sqcup_{(\xi,z)} \Pi_\phi((\xi, z)) \to \text{Irr}(\pi_0(S_\phi)).$$

Note that we are not using $\bar{S}_\phi = S_\phi/Z(\widehat{G})^\Gamma$ here. In terms of the example with unitary groups, one checks that $U(p,q)$ and $U(q,p)$, despite being the same group, are not isomorphic pure inner twists of the quasi-split unitary group G^*. In fact, the set of isomorphism classes of pure inner twists of G^* is in bijection with $H^1(\Gamma_F, G^*)$. In the case of unitary groups, this set is precisely the set of pairs (p,q) of non-negative integers such that $p+q=n$.

Note furthermore that now, both in the real and in the p-adic case, the map $\iota_\mathfrak{w}$ is expected to be a bijection. Thus this generalization of Conjecture B makes it more uniform than its version for quasi-split groups. Moreover, it is still normalized to send the unique $\mathfrak{w}$-generic representation in $\Pi_\phi((id,1))$ to the trivial representation of $\pi_0(S_\phi)$, i.e. it is compatible with Conjecture C.

The bijection $\iota_\mathfrak{w}$ is expected to fit in the following commutative diagram

$$\begin{tikzcd} \bigsqcup\limits_{(\xi,z)} \Pi_\phi((\xi,z)) \arrow[r, "\iota_\mathfrak{w}"] \arrow[d] & \mathrm{Irr}(\pi_0(S_\phi)) \arrow[d] \\ H^1(\Gamma, G) \arrow[r] & \pi_0(Z(\widehat{G})^\Gamma)^* \end{tikzcd} \tag{5}$$

The bottom map is Kottwitz's map [Kot86, Theorem 1.2]. The left map sends any constituent of $\Pi_\phi((\xi,z))$ to the class of z. The right map assigns to an irreducible representation of $\pi_0(S_\phi)$ the character by which the group $\pi_0(Z(\widehat{G})^\Gamma)$ acts. When F is p-adic, the bottom map is a bijection. This means that the set $\Pi_\phi((\xi,z))$, which is an L-packet on the pure inner form G of G^* that is the target of the pure inner twist $(\xi,z): G^* \to G$, is in bijection with the corresponding fiber of the right map. When F is real, one can obtain a similar statement by considering K-groups.

We have thus seen that Conjectures B and C generalize beautifully to pure inner twists. It was an observation of Kottwitz that Conjecture D also does. The first step is to construct a natural normalization of the geometric transfer factor for a pure inner twist $(\xi,z): G^* \to G$ and an extended endoscopic triple $\mathfrak{e}$, which we shall call $\Delta[\mathfrak{w},\mathfrak{e},z]$. This was carried out in [Kal11, §2] and we will review it here. Let $\gamma \in G^\mathfrak{e}_{\mathrm{sr}}(F)$ and $\delta \in G_{\mathrm{sr}}(F)$ be related. Using a theorem of Steinberg one can show that there exists $g \in G^*$ such that $\delta = \xi(g\delta^*g^{-1})$ with $\delta^* \in G^*(F)$. By definition, γ and δ^* are also related, so the value $\Delta[\mathfrak{w},\mathfrak{e}](\gamma,\delta^*)$ is non-zero. Moreover, $\sigma \mapsto g^{-1}z(\sigma)\sigma(g)$ is a 1-cocycle of Γ in $S = \mathrm{Cent}(\delta, G)$ whose class we call $\mathrm{inv}[z](\delta^*,\delta)$. We then set

$$\Delta[\mathfrak{w},\mathfrak{e},z](\gamma,\delta) = \Delta[\mathfrak{w},\mathfrak{e}](\gamma,\delta^*) \cdot \langle \mathrm{inv}[z](\delta^*,\delta), s^\mathfrak{e} \rangle, \tag{6}$$

where $s^\mathfrak{e}$ is transported to $\widehat{S}$ via the maps $Z(\widehat{G}^\mathfrak{e})^\Gamma \to \widehat{S}^\mathfrak{e} \to \widehat{S}$, with $S^\mathfrak{e} = \mathrm{Cent}(\gamma, G^\mathfrak{e})$ and the second map coming form the admissible isomorphism $\phi_{\gamma,\delta}$. One then has to check that the function $\Delta[\mathfrak{w},\mathfrak{e},z]$ is indeed a geometric transfer factor and this is done in [Kal11, Proposition 2.2.2]. With the transfer factor and the bijection $\iota_\mathfrak{w}$ in place, we can now state Conjecture D exactly as it was stated in the case of quasi-split groups. We will give the statement of the new versions of Conjectures B, C, and D, together as a new conjecture.

Conjecture E. *Let G^* be a quasi-split connected reductive group defined over F and let $\mathfrak{w}$ be a Whittaker datum for G^*. Let $\phi \in \Phi_{\mathrm{temp}}(G^*)$. For each pure inner twist $(\xi, z) : G^* \to G$ let $\Pi_\phi((\xi, z))$ denote the L-packet $\Pi_\phi(G)$ of conjecture A. Then there exists a bijection $\iota_{\mathfrak{w}}$ making Diagram* (5) *commutative and sending the unique $\mathfrak{w}$-generic constituent of $\Pi_\phi((id, 1))$ to the trivial representation of $\pi_0(S_\phi)$. Moreover, if $\mathfrak{e}$ is an extended endoscopic triple for G^* and if $f^{\mathfrak{e}} \in \mathcal{C}_c^\infty(G^{\mathfrak{e}}(F))$ and $f \in \mathcal{C}_c^\infty(G(F))$ are $\Delta[\mathfrak{w}, \mathfrak{e}, z]$-matching functions, then*

$$\Theta^1_{\phi^{\mathfrak{e}}}(f^{\mathfrak{e}}) = e(G) \sum_{\pi \in \Pi_\phi((\xi,z))} \langle \pi, s^{\mathfrak{e}} \rangle \Theta_\pi(f)$$

provided $\phi^{\mathfrak{e}} \in \Phi_{\mathrm{temp}}(G^{\mathfrak{e}})$ is such that $\phi = {}^L\eta^{\mathfrak{e}} \circ \phi^{\mathfrak{e}}$.

Here $e(G) \in \{\pm 1\}$ is the so-called Kottwitz sign of G, defined in [Kot83]. Note that the set $\Pi_\phi((\xi, z))$ does not depend on z (but we still need to include z in the notation for counting purposes, because the same ξ can be equipped with multiple z). The bijection $\iota_{\mathfrak{w}}$ however does depend on z. We shall specify how later.

This conjecture is very close to the formulation of the local Langlands conjecture given by Vogan in [Vog93], apart from the fact that Vogan does not discuss endoscopic transfer. In the real case, it can be shown using Shelstad's work that this conjecture is true. We refer the reader to [Kal16, §5.6] for details. In the p-adic case, its validity has been checked in [DR09] and [Kal11] for regular depth-zero supercuspidal L-packets. It has also been checked in [Kal15] for the L-packets consisting of epipelagic representations [RY14]. In fact, the latter work is valid in the broader framework of isocrystals with additional structure, which will be discussed next.

The relationship between the statements of Conjectures B and D given here and those suggested by Arthur in [Art89] is straightforward. One has to replace S_ϕ^{sc} with S_ϕ and demand $\rho(\Delta[\mathfrak{w}, \mathfrak{e}, z], s^{\mathfrak{e}}) = 1$. This specifies the function $\rho(\Delta, s^{\mathfrak{e}})$ uniquely and Arthur's formulation of the conjectures follows from the one given here.

It thus appears that pure inner twists provide a resolution to all problems obstructing a formulation of the refined local Langlands conjecture for general reductive groups. Unfortunately, this is not quite true. The theory is perfect for inner twists whose isomorphism class, which is an element of $H^1(\Gamma_F, G^*_{\mathrm{ad}})$, is in the image of the natural map $H^1(\Gamma_F, G^*) \to H^1(\Gamma_F, G^*_{\mathrm{ad}})$. However, since this map is in general not surjective, not every group G can be described as the target of a pure inner twist $(\xi, z) : G^* \to G$ of a quasi-split group G^*. Basic examples are provided by the groups of units of central simple algebras. These are inner forms of the quasi-split group $G^* = \mathrm{GL}_n$. However, the generalized Hilbert 90 theorem states that $H^1(\Gamma, G^*) = \{1\}$. Thus no non-trivial inner form of G^* can be made pure. There are also other examples, involving inner forms of symplectic and special orthogonal groups.

2.4 Work of Adams, Barbasch, and Vogan

The fact that pure inner forms are not sufficient to describe the refined local Langlands conjecture for all connected reductive groups begs the question of whether there exists a notion that is more general than pure inner forms yet still has the necessary structure as to allow a version of Conjecture E to be stated. In the archimedean case, such a notion is presented by Adams, Barbasch, and Vogan, in [ABV92]. It is the notion of a "strong rational form". The set of equivalence classes of strong rational forms contains the set of equivalence classes of pure inner forms. At the same time it is large enough to encompass all inner forms. Moreover, in [ABV92] a bijection

$$\iota : \sqcup_x \Pi_\phi(x)) \to \mathrm{Irr}_{\mathrm{alg}}(\pi_0(\tilde{S}_\phi))$$

is constructed, where $\tilde{S}_\phi$ is the preimage of S_ϕ in the universal covering of $\widehat{G}$. When $\widehat{G}$ is semi-simple, this covering is just $\widehat{G}_{\mathrm{sc}}$, but when $\widehat{G}$ is a torus, this covering is affine space. In general, it is a mix of these two cases.

Thus, the book [ABV92] contains a proof of suitable generalizations of conjectures B and C. It does not discuss the character identities stated as Conjecture D. The main focus of [ABV92] is in fact the study of how non-tempered representations interface with the conjectures of Langlands and Arthur. This is a fascinating topic that is well beyond the scope of our review.

2.5 Kottwitz's work on isocrystals with additional structure

The notion of "strong rational forms" introduced by Adams, Barbasch, and Vogan, resolved in the archimedean case the problem that pure inner forms are not sufficient to allow a statement of Conjecture E that encompasses all connected reductive groups. It thus became desirable to find an analogous notion in the non-archimedean case. This was formally formulated as a problem in [Vog93, §9], where Vogan lists the desired properties that this conjectural notion should have. The solution in the archimedean case did not suggest in any way whether a solution in the non-archimedean case exists and where it might be found, as the construction of strong rational forms in [ABV92] made crucial use of the fact that Gal($\mathbb{C}/\mathbb{R}$) has only one non-trivial element.

Led by his and Langlands' work on Shimura varieties, Kottwitz introduced in [Kot85] and [Kot97] the set $B(G)$ of equivalence classes of isocrystals with G-structure, for any connected reductive group G defined over a non-archimedean local field. The notion of an isocrystal plays a central role in the classification of p-divisible groups. Let F be a p-adic field and F^u its maximal unramified extension, and L its completion. An isocrystal is a finite-dimensional L-vector space V equipped with a Frobenius-semi-linear bijection. According to Kottwitz, an isocrystal with G-structure is a $\otimes$-functor from the category of finite-dimensional representations of the algebraic group G to the category of isocrystals. This can be given a cohomological description. Indeed, the set of isomorphism classes of n-dimensional isocrystals can be identified with $H^1(W_F, \mathrm{GL}_n(\bar{L}))$, and the set of isomorphism classes of isocrystals with G-structure can be identified with $H^1(W_F, G(\bar{L}))$.

Manin has shown that the category of isocrystals is semi-simple and the simple objects are classified by the set $\mathbb{Q}$ of rational numbers. The rational number corresponding to a given simple object is called its slope. A general isocrystal is thus given by a string of rational numbers, called its slope decomposition. The objects of constant slope, i.e. the isotypic objects, are called *basic* isocrystals. Kottwitz generalizes this notion to the case of isocrystals with G-structure. The set $B(F)_{\mathrm{bas}}$ of equivalence classes of basic isocrystals with additional structure is a subset of $B(G)$.

Kottwitz shows that there exists a functorial injection $H^1(\Gamma, G) \to B(G)_{\mathrm{bas}}$. He furthermore shows that each element $b \in B(G)_{\mathrm{bas}}$ leads to an inner form G^b of G. More precisely, one needs to take b to be a representative of the equivalence class given by an element of $B(G)_{\mathrm{bas}}$, and then one obtains an inner twist $\xi : G \to G^b$. We will call the pair (ξ, b) an extended pure inner twist, for a lack of a better name.

The bijection $H^1(\Gamma, G) \to \pi_0(Z(\widehat{G})^\Gamma)^*$ used in Diagram (5) extends to a bijection $B(G)_{\mathrm{bas}} \to X^*(Z(\widehat{G})^\Gamma)$. This allows one to conjecture the existence of a diagram similar to (5), but with $B(G)_{\mathrm{bas}}$ in place of $H^1(\Gamma, G)$. In order to be able to state an analog of Conjecture E, the last missing ingredient is the normalization of the transfer factor. This has been established in [Kal14, §2]. We will not review the construction here, as it is quite analogous to the one reviewed in the section on pure inner forms. The analog of Conjecture E in the context of isocrystals is then the following conjecture made by Kottwitz.

Conjecture F. *Let G^* be a quasi-split connected reductive group defined over F, $\mathfrak{w}$ a fixed Whittaker datum for G^*, and $\phi \in \Phi_{\mathrm{temp}}(G^*)$. Let $S_\phi^\natural = S_\phi/[S_\phi \cap [\widehat{G}]_{der}]^\circ$. For each extended pure inner twist $(\xi, b) : G^* \to G$ let $\Pi_\phi((\xi, b))$ denote the L-packet $\Pi_\phi(G)$ provided by conjecture A. Then there exists a commutative diagram*

$$\begin{tikzcd} \bigsqcup_{(\xi,b)} \Pi_\phi((\xi,b)) \arrow[r, "\iota_{\mathfrak{w}}"] \arrow[d] & Irr(S_\phi^\natural) \arrow[d] \\ B(G^*)_{bas} \arrow[r] & X^*(Z(\widehat{G})^\Gamma) \end{tikzcd} \tag{7}$$

in which the top arrow is bijective. We have used Irr to denote the set of irreducible algebraic representation of the disconnected reductive group $S_\phi^\natural$. The image of the unique $\mathfrak{w}$-generic constituent of $\Pi_\phi((id, 1))$ is the trivial representation of $S_\phi^\natural$.

Given an extended pure inner twist $(\xi, b) : G^* \to G$ and an extended endoscopic triple $\mathfrak{e}$ for G^*, for any $\Delta[\mathfrak{w}, \mathfrak{e}, b]$-matching functions $f^\mathfrak{e} \in \mathcal{C}_c^\infty(G^\mathfrak{e}(F))$ and $f \in \mathcal{C}_c^\infty(G(F))$ the equality

$$\Theta^1_{\phi^\mathfrak{e}}(f^\mathfrak{e}) = e(G) \sum_{\pi \in \Pi_\phi((\xi,b))} \langle \pi, s^\mathfrak{e} \rangle \Theta_\pi(f)$$

holds, where $\phi^\mathfrak{e} \in \Phi_{\mathrm{temp}}(G^\mathfrak{e})$ is such that $\phi = {}^L\eta^\mathfrak{e} \circ \phi^\mathfrak{e}$.

A version of this conjecture was stated in [Rap95, §5], and later in [Kal14, §2.4]. A verification of this conjecture was given in [Kal14] for regular depth-zero supercuspidal parameters, and in [Kal15] for epipelagic parameters. Moreover, while we have only considered non-archimedean fields so far, the conjecture also makes sense for archimedean fields thanks to Kottwitz's recent construction [Kot14] of $B(G)$ for all local and global fields.

Given this conjecture, there is the following obvious question: How much bigger is $B(G^*)_{\text{bas}}$ than $H^1(\Gamma, G^*)$? Is it enough to treat all reductive groups?

The answer is the following: When $Z(G^*)$ is connected, Kottwitz has shown that the natural map $B(G^*)_{\text{bas}} \to H^1(\Gamma, G^*_{\text{ad}})$ is surjective. In other words, every inner form can be enriched with the datum of an extended pure inner twist. For such groups G^*, conjecture F provides a framework to treat all their inner forms. Important examples of such groups G^* are the group $\mathrm{GL}(N)$, whose inner forms are the multiplicative groups of central simple algebras of degree N; the unitary groups $U_{E/F}(N)$ associated to quadratic extensions E/F; as well as the similitude groups $GU_{E/F}(N)$, GSp_N, and GO_N.

At the other end of the spectrum are the semi-simple groups. For them, the natural injection $H^1(\Gamma, G^*) \to B(G^*)_{\text{bas}}$ is surjective. Thus the set $B(G^*)_{\text{bas}}$ does not provide any additional inner forms beyond the pure ones, and Conjecture F is the same as Conjecture E. In particular, no inner forms of $\mathrm{SL}(N)$ and $\mathrm{Sp}(N)$ can be reached by either conjecture.

3 The canonical Galois gerbe and its cohomology

In [LR87], Langlands and Rapoport introduced the notion of a "Galois gerbe". Their motivation is the study of the points on the special fiber of a Shimura variety. In [Kot97], Kottwitz observed that the set $B(G)$ can be described using the cohomology of certain Galois gerbes. This lead to the idea that it might be possible to overcome the limitations of the set $B(G)$ discussed in the previous section by using different Galois gerbes.

In this section, we are going to describe the construction of a canonical Galois gerbe over a local field of characteristic zero and discuss its properties. We will see in the next section how this gerbe leads to a generalization of Conjecture E that encompasses all connected reductive groups.

3.1 The canonical Galois gerbe

Langlands and Rapoport define [LR87, §2] a Galois gerbe to be an extension of groups

$$1 \to u \to W \to \Gamma \to 1$$

where u is the set of $\overline{F}$-points of an affine algebraic group and Γ is the absolute Galois group of F. Given such a gerbe, one can let it act on $G^*(\overline{F})$ through its map to Γ and consider the cohomology group $H^1(W, G^*)$.

From now on, let F be a local field of characteristic zero. A simple example of a Galois gerbe can be obtained as follows. The relative Weil group of a finite Galois extension E/F is an extension of topological groups

$$1 \to E^\times \to W_{E/F} \to \Gamma_{E/F} \to 1.$$

Pulling back along the natural surjection $\Gamma_F \to \Gamma_{E/F}$ and then pushing out along the natural injection $E^\times \to \bar{F}^\times$ provides a Galois gerbe

$$1 \to \mathbb{G}_m \to \mathcal{E}_{E/F} \to \Gamma_F \to 1.$$

These are called Dieudonne gerbes in [LR87, §2] and are the ones that Kottwitz uses in [Kot97, §8] to provide an alternative description of the set $B(G^*)$. More precisely, Kottwitz shows that if T is an algebraic torus defined over F and split over E, then there is a natural isomorphism

$$H^1_{\text{alg}}(\mathcal{E}_{E/F}, T) \to B(T),$$

where H^1_{alg} is the subgroup of H^1 consisting of the classes of those 1-cocycles whose restriction to $\mathbb{G}_m$ is a homomorphism $\mathbb{G}_m \to T$ of algebraic groups.

One could hope that using more sophisticated Galois gerbes might lead to a cohomology theory that allows an analog of Conjecture E to be stated that applies to all reductive algebraic groups. For this to work, the gerbe needs to have the following properties.

1. It should be naturally associated to any local field F of characteristic zero, so as to provide a uniform statement of the conjecture.
2. In order to have a well-defined cohomology group $H^1(W, G^*)$, the gerbe W needs to be rigid, i.e. have no unnecessary automorphisms. This amounts to the requirement $H^1(\Gamma, u) = 1$.
3. In order to be able to capture all reductive groups, the gerbe W has to have the property that $H^1(W, G^*)$ comes equipped with a natural map $H^1(W, G^*) \to H^1(\Gamma, G^*_{\text{ad}})$ which is *surjective*.
4. In order to be useful for endoscopy, there needs to exist a Tate-Nakayama type isomorphism identifying $H^1(W, G^*)$ with an object definable in terms of $\widehat{G}^*$.

There is of course no a priori reason or even a hint that a Galois gerbe satisfying these conditions should exist. In fact, some experimentation reveals that conditions 2 and 3 seem to pull in opposite directions.

However, it turns out that if one slightly enlarges the scope of consideration, a suitable gerbe does exist. Namely, one has to give up the requirement that u is an affine algebraic group and rather allow it to be a profinite algebraic group, whose $\bar{F}$-points will then carry the natural profinite topology. The pro-finite group u that we are going to consider is the following.

$$u = \varprojlim_{n, E/F} (\mathrm{Res}_{E/F}\, \mu_n)/\mu_n.$$

This is a profinite algebraic group that encodes in a certain way the arithmetic of F. One can show the following [Kal16, Theorem 3.1].

Proposition 5. *We have the canonical identification*

$$H^2(\Gamma, u) = \begin{cases} \widehat{\mathbb{Z}}, & F \textit{ is non-arch.} \\ \mathbb{Z}/2\mathbb{Z}, & F = \mathbb{R} \end{cases}, \qquad H^1(\Gamma, u) = 1.$$

Here the continuous cohomology groups are taken with respect to the natural topology on $u(\overline{F})$ coming from the inverse limit.

Thus there exists a canonical isomorphism class of extensions of Γ by u, and each extension in this isomorphism class has as its group of automorphisms only the inner automorphisms coming from u. This means that if we take W to be any extension in the canonical isomorphism class and consider the set $H^1(W, G)$, this set will be independent of the choice of W.

However, it turns out that this is not quite the right object to consider. For example, it does not come equipped with a map to $H^1(\Gamma, G_{\mathrm{ad}})$ when G is a connected reductive group. The following slight modification is better suited for our purposes: Define $\mathcal{A}$ to be the category of injections $Z \to G$, where G is an affine algebraic group and Z is a finite central subgroup. For an object $[Z \to G] \in \mathcal{A}$, let $H^1(u \to W, Z \to G)$ be the subset of $H^1(W, G)$ consisting of those classes whose restriction to u takes image in Z. This provides a functor $\mathcal{A} \to \mathsf{Sets}$ and there is an obvious natural transformation $H^1(u \to W, Z \to G) \to H^1(\Gamma, G/Z)$ between functors $\mathcal{A} \to$ Sets. Furthermore, when G is reductive, we have the obvious map $H^1(\Gamma, G/Z) \to H^1(\Gamma, G_{\mathrm{ad}})$.

3.2 Properties of $H^1(u \to W, Z \to G)$

The basic properties of the functor $H^1(u \to W, Z \to G)$ are summarized in the following commutative diagram [Kal16, (3.6)]

$$
\begin{tikzcd}
1 \arrow[r] & H^1(\Gamma, Z) \arrow[r, "\mathrm{Inf}"] \arrow[d] & H^1(u \to W, Z \to Z) \arrow[r, "\mathrm{Res}"] \arrow[d] & \mathrm{Hom}(u, Z)^\Gamma \arrow[d, equal] & \\
1 \arrow[r] & H^1(\Gamma, G) \arrow[r, "\mathrm{Inf}"] \arrow[d, equal] & H^1(u \to W, Z \to G) \arrow[r, "\mathrm{Res}"] \arrow[d, "a"] & \mathrm{Hom}(u, Z)^\Gamma \arrow[r] \arrow[d, "b"] & * \arrow[d, equal] \\
 & H^1(\Gamma, G) \arrow[r] \arrow[d] & H^1(\Gamma, G/Z) \arrow[r] \arrow[d] \arrow[dl] & H^2(\Gamma, Z) \arrow[r] \arrow[d] & * \\
 & H^1(\Gamma, G/Z(G)) & 1 & 1 &
\end{tikzcd}
$$

where $*$ is to be taken as $H^2(\Gamma, G)$ if G is abelian and disregarded otherwise. The three rows are exact, and so is the outer arc (after identifying the two copies of $\mathrm{Hom}(u, Z)^\Gamma$). The middle column is exact, and the map b is surjective. The middle exact sequence is an inflation-restriction-type sequence. By itself it already gives some information about the set $H^1(u \to W, Z \to G)$. First, it shows that $H^1(u \to W, Z \to G)$ contains as a subset $H^1(\Gamma, G)$, thus it faithfully captures the set of equivalence classes of pure inner forms. Second, it tells us that $H^1(u \to W, Z \to G)$ fibers over $\mathrm{Hom}(u, Z)^\Gamma$. One easily sees that the latter is finite, which implies

- $H^1(u \to W, Z \to G)$ is finite.

Using the basic twisting argument in group cohomology, one sees that the fibers of this fibration are of the form $H^1(\Gamma, G^\dagger)$, where $G^\dagger$ runs over suitable inner forms of G. In particular, we obtain the disjoint union decomposition

- $H^1(u \to W, Z \to G) = \bigsqcup H^1(\Gamma, G^\dagger)$.

This allows one to effectively compute $H^1(u \to W, Z \to G)$ using the standard tools of Galois cohomology. One can moreover ask, what is the meaning of $\mathrm{Hom}(u, Z)^\Gamma$. This question is answered by the map b. When Z

is split (that is, when $X^*(Z)$ has trivial Γ-action), the map b in the above diagram is bijective. Thus, in a slightly vague sense, the group u represents the functor $Z \mapsto H^2(\Gamma, Z)$ restricted to the category of split finite multiplicative algebraic groups (the group u is itself of course not finite). Note that any continuous homomorphism $u \to Z$ factors through a finite quotient of u and is automatically algebraic, so we can write $\mathrm{Hom}(u, Z)^\Gamma = \mathrm{Hom}_F(u, Z)$. On the larger category of general finite multiplicative algebraic groups, one sees easily that the functor $Z \mapsto H^2(\Gamma, Z)$ is not representable, even in the above more vague sense, as it is not left exact. Nonetheless, the map b is surjective, so we can think of u as coming close to representing that functor. In other words, $H^1(u \to W, Z \to G)$ interpolates between $H^1(\Gamma, G)$ and $H^2(\Gamma, Z)$. Moreover, the surjectivity of b leads to the surjectivity of a. When G is reductive and Z is large enough, the map $H^1(\Gamma, G/Z) \to H^1(\Gamma, G_{\mathrm{ad}})$ is also surjective. For example, this is true as soon as $Z = Z(G_{\mathrm{der}})$. For some purposes it is thus sufficient to fix $Z = Z(G_{\mathrm{der}})$. In general, the flexibility afforded by allowing Z to vary is quite useful. For example, fixing Z would not provide a functorial assignment, and this would make basic operations like parabolic descent unnecessarily complicated.

3.3 Tate-Nakayama-type isomorphism

We have thus seen that the Galois gerbe W satisfies the first three of the four required properties listed in Section 3.1. The fourth property – the Tate-Nakayama-type isomorphism, is the most crucial. Luckily, the gerbe W satisfies that property too.

To give the precise statement, we let $\mathcal{R} \subset \mathcal{A}$ be the subcategory consisting of those $[Z \to G]$ for which G is connected and reductive. We have the functor

$$\mathcal{R} \to \mathsf{Sets}, \qquad [Z \to G] \mapsto H^1(u \to W, Z \to G).$$

We now define a second functor. Given $[Z \to G] \in \mathcal{R}$, let $\bar{G} = G/Z$. The isogeny $G \to \bar{G}$ provides an isogeny of Langlands dual groups $\widehat{\bar{G}} \to \widehat{G}$. Let $Z(\widehat{\bar{G}})^+$ denote the preimage in $\widehat{\bar{G}}$ of $Z(\widehat{G})^\Gamma$. Then $\pi_0(Z(\widehat{\bar{G}})^+)$ is a finite abelian group and one checks easily that

$$\mathcal{R} \to \mathsf{Sets}, \qquad [Z \to G] \mapsto \mathrm{Hom}(\pi_0(Z(\widehat{\bar{G}})^+), \mathbb{C}^\times)$$

is a functor.

The following theorem, proved in [Kal16, §4], contains the precise statement how the gerbe W satisfies the expected property 4 of Section 3.1.

Theorem 6.

- *There is a unique morphism between the two above functors that extends the Tate-Nakayama isomorphism between the restrictions of these functors to the subcategory consisting of objects $[1 \to T]$, where T is an algebraic torus, and that lifts a certain natural morphism Hom$(\pi_0(Z(\widehat{\bar{G}})^+), \mathbb{C}^\times) \to$ Hom$_F(u, Z)$.*
- *The morphism is an isomorphism between the restrictions of the above functors to the subcategory consisting of objects $[Z \to T]$, where T is an algebraic torus.*

- *The morphism is an isomorphism between the above functors when F is non-archimedean.*
- *The kernel and cokernel of the morphism can be explicitly described when F is archimedean.*
- *The morphism restricts to Kottwitz's map on the subcategory of objects $[1 \to G]$.*

The fact that the morphism is not an isomorphism when F is archimedean is not surprising. If it were, it would endow each set $H^1(u \to W, Z \to G)$, and in particular each set $H^1(\Gamma, G)$, with the structure of a finite abelian group in a functorial way. However, it is generally not possible to endow $H^1(\Gamma, G)$ with a group structure in such a way that natural maps, like $H^1(\Gamma, G) \to H^1(\Gamma, G_{\text{ad}})$, are group homomorphisms.

When F is p-adic, this theorem does endow the set $H^1(u \to W, Z \to G)$ with the structure of a finite abelian group in a functorial way. It furthermore gives a simple way to effectively compute the set $H^1(u \to W, Z \to G)$. The most important consequences of the theorem for us will however be to the theory of endoscopy. More precisely, the theorem will allow us to construct a normalization of the geometric transfer factor and to state a conjecture analogous to Conjecture E that encompasses all connected reductive groups.

4 Local rigid inner forms and endoscopy

In this section we are going to see how the Galois gerbe W constructed in the previous section leads to a generalization of Conjecture E that encompasses all connected reductive groups. Just like Conjecture E, its statement will be uniform for all local fields of characteristic zero.

We begin with a few simple definitions, essentially modeling those for pure inner forms. Let F be a local field of characteristic zero and let G^* be a quasi-split connected reductive group defined over F.

Definition 7. 1. *A rigid inner twist $(\xi, z) : G^* \to G$ is a pair consisting of an inner twist $\xi : G^* \to G$ and an element $z \in Z^1(u \to W, Z \to G^*)$, for some finite central $Z \subset G^*$, such that $\xi^{-1}\sigma(\xi) = Ad(\bar{z}(\sigma))$, where $\bar{z} \in Z^1(\Gamma, G^*_{ad})$ is the image of z.*

2. *Given two rigid inner twists $(\xi_i, z_i) : G^* \to G_i$, $i = 1, 2$, an isomorphism $(f, \delta) : (\xi_1, z_1) \to (\xi_2, z_2)$ of rigid inner twists is a pair consisting of an isomorphism $f : G_1 \to G_2$ defined over F and an element $\delta \in G^*$, satisfying the identities $\xi_2^{-1} f \xi_1 = Ad(\delta)$ and $z_1(w) = \delta^{-1} z_2(w) \sigma_w(\delta)$.*

Here σ_w is the image of $w \in W$ in Γ, and $\bar{z}$ is the image of $z \in Z^1(u \to W, Z \to G)$ in $Z^1(\Gamma, G/Z)$. It is again straightforward to check that $\text{Aut}(\xi, z) = G(F)$.

4.1 Refined endoscopic data and canonical transfer factors

The fact that the Tate-Nakayama-type isomorphism pairs the cohomology set $H^1(u \to W, Z \to G)$ not with elements of $\widehat{G}$, but rather of $\widehat{\bar{G}}$, leads to the necessity to modify the notion of endoscopic data. The notion of an endoscopic datum was reviewed in Section 1.3. Let $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$. A refinement of $\mathfrak{e}$ is a tuple $\dot{\mathfrak{e}} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\dot{\mathfrak{e}}}, \eta^{\mathfrak{e}})$. The only difference is the element $s^{\dot{\mathfrak{e}}}$, which should be an element of $\widehat{\bar{G}}$ that lifts $s^{\mathfrak{e}}$. This refinement also suggests a modification of the notion of an isomorphism. Namely, an isomorphism between $\dot{\mathfrak{e}}_1$ and $\dot{\mathfrak{e}}_2$ is now an element $g \in \widehat{G}$ that satisfies two conditions. The first is $g\eta^{\mathfrak{e}_1}(\mathcal{G}^{\mathfrak{e}_1})g^{-1} = \eta^{\mathfrak{e}_2}(\mathcal{G}^{\mathfrak{e}_2})$, which is the same as before. To describe the second, let $H_i = G^{\mathfrak{e}_i}$. We use the canonical embedding $Z(G) \to Z(H_i)$ to form $\bar{H}_i = H_i/Z$. Then $\mathrm{Ad}(g)$ provides an isomorphism $\widehat{\bar{H}_1} \to \widehat{\bar{H}_2}$, which induces an isomorphism $\pi_0(Z(\widehat{\bar{H}_1})^+) \to \pi_0(Z(\widehat{\bar{H}_2})^+)$. The element $s^{\dot{\mathfrak{e}}_i}$ provides an element $\bar{s}^{\dot{\mathfrak{e}}_i} \in \pi_0(Z(\widehat{\bar{H}_i})^+)$ and we require that $\mathrm{Ad}(g)\bar{s}^{\dot{\mathfrak{e}}_1} = \bar{s}^{\dot{\mathfrak{e}}_2}$.

One checks that every endoscopic datum can be refined, and there are only finitely many isomorphism classes of refined endosocpic data that lead to isomorphic unrefined endoscopic data. This allows one to refine sums over isomorphism classes of endoscopic data by sums over isomorphism classes of refined endoscopic data.

One can analogously define the notion of a refined extended endoscopic triple, but we leave this to the reader.

The notion of a refined endoscopic data can be used, together with Theorem 6, to obtain a canonical normalization of the geometric transfer factor. The construction of the factor is essentially the same as the one for pure inner twists given by Equation (6). Given a rigid inner twist $(\xi, z) : G^* \to G$ and a refined extended endoscopic triple $\dot{\mathfrak{e}}$, let $\gamma \in G^{\mathfrak{e}}(F)$ and $\delta \in G(F)$ be semi-simple strongly regular related elements, and let $\delta^* \in G^*(F)$ and $g \in G^*$ be as in Equation (6). Then $g^{-1} \cdot z(w) \cdot \sigma_w(g)$ is an element of $Z^1(u \to W, Z \to S)$, whose class we call $\mathrm{inv}[z](\delta^*, \delta)$, and we set

$$\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, z](\gamma, \delta) = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta^*) \cdot \langle \mathrm{inv}[z](\delta^*, \delta), s^{\dot{\mathfrak{e}}} \rangle, \tag{8}$$

where now the pairing is between $H^1(u \to W, Z \to S)$ and $\pi_0([\widehat{\bar{S}}]^+)$ and is given by the Tate-Nakayama-type isomorphism of Theorem 6.

One can then prove [Kal16, §5.3] the following.

Theorem 8. *The function $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, z]$ is indeed a transfer factor. Moreover, it is invariant under all automorphisms of $\dot{\mathfrak{e}}$.*

We see that the notion of refined endoscopic data and their isomorphisms resolves the problem of non-invaraince of transfer factors under isomorphism noted by Arthur in [Art06].

4.2 Conjectural structure of tempered L-packets

We are now ready to state the refined local Langlands conjecture for general connected reductive groups. Again we take G^* to be a quasi-split connected reductive group defined over F and we fix a Whittaker datum $\mathfrak{w}$ for it. We fix a finite central subgroup $Z \subset G^*$ and set as before $\bar{G}^* = G^*/Z$. Let $\phi \in \Phi_{\mathrm{temp}}(G^*)$. We are of course interested in the L-packet for ϕ on non-quasi-split groups G that occur as inner forms of G^*. Recall $S_\phi = \mathrm{Cent}(\phi, \widehat{G}^*)$. Set

$$S_\phi^+ = S_\phi \times_{\widehat{G}^*} \widehat{\bar{G}^*},$$

which is simply the preimage of S_ϕ under the isogeny $\widehat{\bar{G^*}} \to \widehat{G}^*$.

Conjecture G. *For each rigid inner twist $(\xi, z) : G^* \to G$ with $z \in Z^1(u \to W, Z \to G^*)$ let $\Pi_\phi((\xi, z))$ denote the L-packet $\Pi_\phi(G)$ whose existence is asserted by conjecture A. Then there exists a commutative diagram*

$$\begin{tikzcd} \displaystyle\coprod_{(\xi,z)} \Pi_\phi((\xi,z)) \arrow[r, "\iota_{\mathfrak{w}}"] \arrow[d] & \mathit{Irr}(\pi_0(S_\phi^+)) \arrow[d] \\ H^1(u \to W, Z \to G^*) \arrow[r] & \pi_0(Z(\widehat{\bar{G}^*})^+)^* \end{tikzcd} \tag{9}$$

in which the top arrow is bijective. The image of the unique $\mathfrak{w}$-generic constituent of $\Pi_\phi((id, 1))$ is the trivial representation of $\pi_0(S_\phi^+)$.

Given a rigid inner twist $(\xi, z) : G^* \to G$ and a refined endoscopic triple $\dot{\mathfrak{e}}$ for G^*, for any $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, z]$-matching functions $f^{\dot{\mathfrak{e}}} \in \mathcal{C}_c^\infty(G^{\mathfrak{e}}(F))$ and $f \in \mathcal{C}_c^\infty(G(F))$ the equality

$$\Theta^1_{\phi^{\mathfrak{e}}}(f^{\dot{\mathfrak{e}}}) = e(G) \sum_{\pi \in \Pi_\phi((\xi,z))} \langle \pi, s^{\dot{\mathfrak{e}}} \rangle \Theta_\pi(f)$$

holds, where $\phi^{\mathfrak{e}} \in \Phi_{\text{temp}}(G^{\mathfrak{e}})$ is such that $\phi = {}^L\eta^{\mathfrak{e}} \circ \phi^{\mathfrak{e}}$, and $\langle \pi, - \rangle = tr(\iota_{\mathfrak{w}}(\pi)(-))$.

If we are interested in a particular fixed non-quasi-split group G, then we endow it with the datum of a rigid inner twist $(\xi, z) : G^* \to G$ and consider the fiber over the class of z of the diagram. On the left, this fiber is the L-packet on G (or rather, the K-group of G when F is archimedean), and on the right, this fiber consists of those irreducible representations which transform under $\pi_0(Z(\widehat{\bar{G}^*})^+)$ by the character determined by z.

Note that when G^* is split and semi-simple, the group S_ϕ^+ coincides with the group S_ϕ^{sc} suggested by Arthur in [Art06]. However, when G is a general connected reductive group, in particular a torus, then S_ϕ^+ is quite different, and in fact more closely related to the group used in [ABV92].

We emphasize also that the group $\pi_0(S_\phi^+)$ is in general more complicated than the groups $\pi_0(\bar{S}_\phi)$ or $\pi_0(S_\phi)$. Indeed, in the archimedean case the latter two groups are elementary 2-groups, while the former need not be a 2-group. It is still abelian, however. In the non-archimedean case it is known that the latter two groups may be non-abelian, but the former is non-abelian much more often. Indeed, already in the case of SL_2 the octonian group occurs as the group $\pi_0(S_\phi^+)$ for the parameter discussed in Section 2.

We have formulated the endocsopic character identities in Conjecture G only for refined extended endoscopic triples. For a formulation in the slightly more general context of refined endoscopic data and z-pairs we refer the reader to [Kal16, §5.4].

4.3 Results for real groups

So far we have not addressed the question of how the rigid inner forms we have defined, when specialized to the case $F = \mathbb{R}$, compare to the strong rational forms defined in [ABV92]. A-priori the two constructions are very different and in fact the construction of rigid inner twists was initially motivated by non-archimedean examples. Nonetheless, we have the following result [Kal16, §5.2].

Theorem 9. *There is an equivalence between the category of rigid inner twists of a real reductive group and the category of strong rational forms of that group.*

We will not discuss here the precise definition of these categories and refer the reader to [Kal16, §5.2] for their straightforward definition.

Another natural question to ask is: What can be said about Conjecture G when $F = \mathbb{R}$? As we discussed in Section 2.1, the structure of tempered L-packets and their endoscopic character identities are very well understood for real groups by the work of Shelstad. A careful study of her arguments leads to the following result [Kal16, §5.6].

Theorem 10. *Conjecture G holds when $F = \mathbb{R}$.*

It is easy and instructive to explicitly compute the extension $1 \to u \to W \to \Gamma \to 1$ in the case of $F = \mathbb{R}$. In that case, $u(\mathbb{C}) = u(\mathbb{R})$ is the trivial Γ-module $\widehat{\mathbb{Z}}$ and the class of this extension can be represented by the 2-cocycle ξ determined by $\xi(\sigma, \sigma) = 1$, where $\sigma \in \Gamma$ is the non-trivial element. Recalling that the Weil group of $\mathbb{R}$ is an extension $1 \to \mathbb{C}^\times \to W_{\mathbb{C}/\mathbb{R}} \to \Gamma \to 1$ whose class can be represented by the 2-cocycle c determined by $c(\sigma, \sigma) = -1$, we see that it can be recovered as the pushout of W along the map $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}/2\widehat{\mathbb{Z}} \cong \{\pm 1\} \subset \mathbb{C}^\times$.

This computation shows that the extension $1 \to u \to W \to \Gamma \to 1$ is very closely related to the Weil group $W_{\mathbb{C}/\mathbb{R}}$. While for any finite Galois extension E/F of p-adic fields the relative Weil group $W_{E/F}$ has a similar structure as $W_{\mathbb{C}/\mathbb{R}}$, the absolute Weil group $W_{\overline{F}/F}$ is not an extension of the absolute Galois group Γ, but rather a dense subgroup of it. One can thus think of the extension $1 \to u \to W \to \Gamma \to 1$ as a closer analog for p-adic fields of the absolute Weil group of $\mathbb{R}$.

4.4 Dependence on the choice of z

In Conjecture G we defined $\Pi_\phi((\xi, z))$ to be the L-packet $\Pi_\phi(G)$, where $(\xi, z) : G^* \to G$ is a rigid inner twist with $z \in Z^1(u \to W, Z \to G^*)$. It is clear from this definition that the set $\Pi_\phi((\xi, z))$ does not depend on z. What does depend on z is the representation of $\pi_0(S_\phi^+)$ that $\iota_{\mathfrak{w}}$ assigns to $\pi \in \Pi_\phi((\xi, z))$, and hence the value $\langle \pi, s^{\dot{\mathfrak{e}}} \rangle$ that enters the endoscopic character identity. This dependence can be quantified precisely.

Let $\xi : G^* \to G$ be an inner twist and let $z_1, z_2 \in Z^1(u \to W, Z \to G^*)$ be two elements such that (ξ, z_1) and (ξ, z_2) are rigid inner twists. According to Definiton 7 and the diagram in Subsection 3.2 we have $z_2 = x z_1$ with $x \in Z^1(u \to W, Z \to Z) = Z^1(W, Z)$.

Let $\widehat{Z}$ denote the kernel of the isogeny $\widehat{\overline{G^*}} \to \widehat{G}^*$. It is shown in [Kal18b, §6] that the finite abelian groups $H^1(W, Z)$ and $Z^1(\Gamma, \widehat{Z})$ are in canonical duality. Moreover, this duality is compatible with the duality between $H^1(u \to W, Z \to T)$ and $\pi_0([\widehat{\overline{T}}]^+)$ of Theorem 6.

Consider the map

$$(-d) : S_\phi^+ \to Z^1(\Gamma, \widehat{Z}), \qquad s \mapsto \phi(w_\sigma) s^{-1} \phi(w_\sigma)^{-1} s,$$

where $w_\sigma \in L_F$ is any lift of $\sigma \in \Gamma$. The result is independent of the lift because the finiteness of $\widehat{Z}$ implies $Z^1(L_F, \widehat{Z}) = Z^1(\Gamma, \widehat{Z})$. One can show that $(-d)$ is a group homomorphism. Moreover, since $[S_\phi^+]^\circ \subset \mathrm{Cent}(\phi, \widehat{\bar{G}^*})$, we see that $(-d)$ factors through $\pi_0(S_\phi^+)$. One can then show [Kal18b, Lemma 6.2] that if $\pi \in \Pi_\phi(G)$ and if $\langle \pi, s^{\dot{\mathfrak{c}}} \rangle_1$ and $\langle \pi, s^{\dot{\mathfrak{c}}} \rangle_2$ are the values of $\mathrm{tr}(\iota_{\mathfrak{w}}(\pi)(s^{\dot{\mathfrak{c}}}))$ obtained by considering π as an element of $\Pi_\phi((\xi, z_1))$ and $\Pi_\phi((\xi, z_2))$ respectively, then the validity of Conjecture G implies

$$\langle \pi, s^{\dot{\mathfrak{c}}} \rangle_2 = \langle [x], (-d)s^{\dot{\mathfrak{c}}} \rangle \langle \pi, s^{\dot{\mathfrak{c}}} \rangle_1.$$

4.5 Comparison with isocrystals

Even though Conjecture F cannot be stated for arbitrary connected reductive groups, as we discussed at the end of Subsection 2.5, it is still a very important part of the theory, due to the geometric significance of Kottwitz's theory of isocrystals with additional structure. For example, Conjecture F is the basis of Kottwitz's conjecture [Rap95, Conjecture 5.1] on the realization of the local Langlands correspondence in the cohomology of Rapoport-Zink spaces. Moreover, Fargues and Fontaine [FF18] have recently proved that G-bundles on the Fargues-Fontaine curve are parameterized by the set $B(G)$. Based on that, Fargues [Far16] has outlined a geometric approach that would hopefully lead to a proof of Conjecture F. It it therefore desirable to understand the relationship between conjectures F and G. This relationship is examined in [Kal18b].

The simplest qualitative statement that can be made is the following: The validity of Conjecture F for all connected reductive groups with connected center is equivalent to the validity of G for all conneced reductive groups.

Let us now be more specific. Let G be a connected reductive group. Define

$$H^1(u \to W, Z(G) \to G) = \varinjlim H^1(u \to W, Z \to G)$$

where Z runs over the finite subgroups of $Z(G)$ defined over F. Then there exists a canonical map [Kal18b, (3.14)]

$$B(G)_{\text{bas}} \to H^1(u \to W, Z(G) \to G). \tag{10}$$

One can give an explicit formula for the dual of this map. For this, we need some preparation. Let $Z_n \subset Z(G)$ be the preimage in $Z(G)$ of the group of n-torsion points of the torus $Z(G)/Z(G_{\text{der}})$. The Z_n form an exhaustive tower of finite subgroups of $Z(G)$ and we can use this tower to form the above limit. Set $G_n = G/Z_n$. Then $G_n = G_{\text{ad}} \times Z(G_n)$ and $Z(G_n) = Z(G_1)/Z(G_1)[n]$, where $Z(G_1) = Z(G)/Z(G_{\text{der}})$. Dually we have $\widehat{G}_n = \widehat{G}_{\text{sc}} \times \widehat{C}_n$, where $\widehat{C}_n$ is the torus dual to $Z(G_n)$. Since $Z(G_1)$ is the maximal torus quotient of G, its dual $\widehat{C}_1$ is the maximal normal torus of $\widehat{G}$, i.e. $Z(\widehat{G})^\circ$. It will be convenient to represent $\widehat{C}_n$ as $\widehat{C}_1 = Z(\widehat{G})^\circ$, and then the natural quotient map $\widehat{C}_m \to \widehat{C}_n$ for $n|m$ becomes the m/n-power map $\widehat{C}_1 \to \widehat{C}_1$. Set $\widehat{C}_\infty = \varprojlim \widehat{C}_n$.

Consider the group $Z(\widehat{G}_{\text{sc}}) \times \widehat{C}_\infty$. Elements of it are of the form $(a, (b_n)_n)$, where $a \in Z(\widehat{G}_{\text{sc}})$ and $b_n \in \widehat{C}_1$ is a sequence satisfying $(b_m)^{m/n} = b_n$ for all $n|m$. We have the obvious map

$$Z(\widehat{G}_{\mathrm{sc}}) \times \widehat{C}_\infty \to Z(\widehat{G}), \qquad (a,(b_n)) \mapsto a_{\mathrm{der}} \cdot b_1,$$

where a_{der} is the image in $Z(\widehat{G}_{\mathrm{der}})$ of a. Let $(Z(\widehat{G}_{\mathrm{sc}}) \times \widehat{C}_\infty)^+$ be the subgroup consising of those elements whose image in $Z(\widehat{G})$ is Γ-fixed. One can show the the duality pairing of Theorem 6 is compatible with the limit and becomes a pairing [Kal18b, (3.12)]

$$\pi_0((Z(\widehat{G}_{\mathrm{sc}}) \times \widehat{C}_\infty)^+) \times H^1(u \to W, Z(G) \to G) \to \mathbb{C}^\times.$$

Now consider the map

$$(Z(\widehat{G}_{\mathrm{sc}}) \times \widehat{C}_\infty)^+ \to Z(\widehat{G}), \qquad (a,(b_n)) \mapsto \frac{a_{\mathrm{der}} \cdot b_1}{N_{E/F}(b_{[E:F]})}, \qquad (11)$$

where E/F is any finite Galois extension so that Γ_E acts trivially on $Z(\widehat{G})$. The choice of E/F doesn't matter and one can show that the above map factors through $\pi_0((Z(\widehat{G}_{\mathrm{sc}}) \times \widehat{C}_\infty)^+)$ and is the map dual to (10), see [Kal18b, Proposition 3.3].

We now turn to the comparison of Conjectures F and G. Assume first that G^* is a quasi-split connected reductive group with connected center. Let $\xi : G^* \to G$ be an inner twist. There exists a representative b of an element of $B(G^*)_{\mathrm{bas}}$ such that (ξ, b) is an extended pure inner twist. Via the map (10) (which also works on the level of cocycles) we obtain from b an element $z \in Z^1(u \to W, Z(G^*) \to G^*)$ so that (ξ, z) is a rigid inner twist. Then one can show [Kal18b, §4] that conjecture F for (ξ, b) is equivalent to Conjecture G for (ξ, z). Not only that, but one can explicitly relate the internal parameterization of the L-packets $\Pi_\phi((\xi, b))$ and $\Pi_\phi((\xi, z))$. This is realized by an explicit bijection

$$\mathrm{Irr}(S_\phi^\natural, b) \to \mathrm{Irr}(\pi_0(S_\phi^+), z),$$

where $\mathrm{Irr}(S_\phi^\natural, b)$ is the subset of those irreducible algebraic representations of $S_\phi^\natural$ which transform under $Z(\widehat{G})$ via the character determined by b, and $\mathrm{Irr}(\pi_0(S_\phi^+), z)$ is defined analogously. This bijection is given as the pull-back of representations under a group homomorphism

$$\pi_0(S_\phi^+) \to S_\phi^\natural$$

that can be defined as follows. We may take as the finite central subgroup $Z \subset G^*$ one of the groups Z_n defined above. Moreover, we can take it so that n is a multiple of the degree $k = [E : F]$ of some finite Galois extension E/F as above. Then $S_\phi^+ \subset \widehat{G}_n = \widehat{G}_{\mathrm{sc}} \times \widehat{C}_n$ and we define the above map to send $(a, b_n) \in S_\phi^+$ to $[a_{\mathrm{der}} \cdot b_n^n] N_{E/F}(b_n^{-\frac{n}{k}})$. In other words, we use the same formula as for (11).

We have thus compared Conjectures F and G for a fixed quasi-split group G^* with connected center. In order to obtain the above qualitative statement, we must now reduce the proof of Conjecture G to the case of groups with connected center. This is possible [Kal18b, §5] and involves a construction, called a z-embedding, which embeds the connected reductive group G^* into another connected reductive group $\tilde{G}^*$ whose center is connected and whose endoscopy is comparable. One can then show that Conjecture G for G^* is equivalent to Conjecture G for $\tilde{G}^*$, see [Kal18b, §5.2].

4.6 Relationship with Arthur's formulation

The formulation of the refined local Langlands conjecture due to Arthur, that we briefly discussed in Subsection 2.2, is quite different from Conjecture G. For example, the group S_ϕ^{sc} that Arthur proposes is in general different from $\pi_0(S_\phi^+)$. Nonetheless, it turns out [Kal18a, §4.6] that Conjecture G implies a strong form of Arthur's formulation. Let G^* be a quasi-split connected reductive group and let $\xi : G^* \to G$ be an inner twist. From ξ one obtains the 1-cocycle $\sigma \to \xi^{-1}\sigma(\xi)$, an element of $Z^1(\Gamma, G^*_{\text{ad}})$. According to Kottwitz's theorem the class of this element provides a character $[\xi] : Z(\widehat{G}^*_{\text{sc}})^\Gamma \to \mathbb{C}^\times$. Arthur suggests that one should choose an arbitrary extension $\Xi : Z(\widehat{G}^*_{\text{sc}}) \to \mathbb{C}^\times$. Then, for any $\phi \in \Phi_{\text{temp}}(G^*)$ there should be a non-canonical bijection between $\text{Irr}(S_\phi^{\text{sc}}, \Xi)$ and the L-packet $\Pi_\phi(G)$.

In order to relate Conjecture G to Arthur's formulation, it is not enough to choose $z \in Z^1(u \to W, Z \to G)$ so that (ξ, z) becomes a rigid inner twist. Rather, we consider the inner twist $\xi : G^*_{\text{sc}} \to G_{\text{sc}}$ on the level of simply connected covers induced by ξ and fix an element $z_{\text{sc}} \in Z^1(u \to W, Z(G^*_{\text{sc}}) \to G^*_{\text{sc}})$ so that $(\xi, z_{\text{sc}}) : G^*_{\text{sc}} \to G_{\text{sc}}$ becomes a rigid inner twist. According to the duality of Theorem 6, the class of $[z_{\text{sc}}]$ provides a character $Z(\widehat{G}^*_{\text{sc}}) \to \mathbb{C}^\times$ that extends the character $[\xi] : Z(\widehat{G}^*_{\text{sc}})^\Gamma \to \mathbb{C}^\times$. Thus, we see that from our current point of view the choice of extension Ξ of the character $[\xi]$ corresponds to the choice of z_{sc} lifting the cocycle $\sigma \mapsto \xi^{-1}\sigma(\xi)$. In fact, when F is p-adic the class of $[z_{\text{sc}}]$ and the extension Ξ determine each other. When F is real, however, the class $[z_{\text{sc}}]$ is the primary object, because it determines Ξ, but is not determined by it.

The real strenght of the new point of view comes from the fact that z_{sc} provides not just the character Ξ, but at the same time a normalization of the Langlands-Shelstad transfer factor Δ, namely $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, z]$, where $z \in Z^1(u \to W, Z(G^*_{\text{der}}) \to G^*)$ is the image of z_{sc}. In this way it specifies the mediating function $\rho(\Delta, -)$ and the spectral transfer factor $\Delta(\phi^{\mathfrak{e}}, \pi)$. Namely, $\rho(\Delta[\mathfrak{w}, \dot{\mathfrak{e}}, z], s^{\dot{\mathfrak{e}}}) = 1$ and $\Delta(\phi^{\mathfrak{e}}, \pi) = \langle \pi, s^{\dot{\mathfrak{e}}} \rangle$.

Let us now show that the internal parameterization of the L-packet $\Pi_\phi(G)$ given by Conjecture G implies the parameterization expected by Arthur. Let $\bar{G}^* = G^*/Z(G^*_{\text{der}}) = G^*_{\text{ad}} \times Z(G^*)/Z(G^*_{\text{der}})$. Then dually $\widehat{\bar{G}^*} = \widehat{G}^*_{\text{sc}} \times Z(\widehat{G}^*)^\circ$. We have $Z(\widehat{\bar{G}^*}) = Z(\widehat{G}^*_{\text{sc}}) \times Z(\widehat{G}^*)^\circ$ and the subgroup $Z(\widehat{\bar{G}^*})^+$ can be described as the set of pairs (a, z) such that $a_{\text{der}} \cdot z \in Z(\widehat{G}^*)$ is Γ-fixed, where $a_{\text{der}} \in Z(\widehat{G}^*)$ is the image of a. Similarly, the subgroup $S_\phi^+ \subset \widehat{\bar{G}^*}$ can be described as the set of pairs $(a, z) \in \widehat{G}^*_{\text{sc}} \times Z(\widehat{G}^*)^\circ$ with the property that $a_{\text{der}} \cdot z \in S_\phi$, where $a_{\text{der}} \in \widehat{G}^*$ is the image of a. One checks that the map

$$S_\phi^+ \oplus_{Z(\widehat{\bar{G}^*})^+} Z(\widehat{G}^*_{\text{sc}}) \to S_\phi^{\text{sc}}, \qquad ((a, z), x) \mapsto ax$$

is an isomorphism of groups. If $\rho \in \text{Irr}(\pi_0(S_\phi^+), [z])$, then the representation $\rho \otimes [z_{\text{sc}}]$ of $S_\phi^+ \times Z(\widehat{G}^*_{\text{sc}})$ descends to the quotient $S_\phi^+ \oplus_{Z(\widehat{\bar{G}^*})^+} Z(\widehat{G}^*_{\text{sc}})$ and via the above isomorphism becomes a representation of S_ϕ^{sc}. This gives a bijection

$$\text{Irr}(\pi_0(S_\phi^+), [z]) \to \text{Irr}(\pi_0(S_\phi^{\text{sc}}), \Xi). \tag{12}$$

5 The automorphic multiplicity formula

In Section 1.5 we discussed that the internal structure of L-packets is a central ingredient in the multiplicity formula for discrete automorphic representations of quasi-split connected reductive groups defined over number fields. In this section we shall formulate the multiplicity formula for general (i.e. not necessarily quasi-split) connected reductive groups, using the conjectural internal structure of tempered L-packets given by Conjecture G. Since we are only considering tempered L-packets locally, the multiplicity formula will be limited to the everywhere tempered automorphic representations. This restriction is just cosmetic – one can incorporate non-tempered automorphic representations by replacing local L-packets with local Arthur packets in the same way as is done in the quasi-split case.

When one attempts to use the local results of the previous sections to study automorphic representations, one realizes that the local cohomological constructions are by themselves not sufficient. They need to be supplemented by a parallel global cohomological construction that ensures that the local cohomological data at the different places of the global field behave coherently. We shall thus begin this section with a short overview of the necessary results. We will then state the multiplicity formula, beginning first with the case of groups that satisfy the Hasse principle, for which the notation simplifies and the key constructions become more transparent, and treating the general case afterwards.

5.1 The global gerbe and its cohomology

Let F be a number field, $\overline{F}$ a fixed algebraic closure, and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. For each place v of F let F_v denote the completion, $\overline{F_v}$ a fixed algebraic closure, and $\Gamma_v = \mathrm{Gal}(\overline{F_v}/F_v)$. Fixing an embedding $\overline{F} \to \overline{F_v}$ over F (which we think of as a place $\dot{v}$ of $\overline{F}$ over v) provides a closed embedding $\Gamma_v \to \Gamma$, whose image we call $\Gamma_{\dot{v}}$.

It is shown in [Kal18a] that there exists a set of places $\dot{V}$ of $\overline{F}$ lifting the places of F, a pro-finite algebraic group P (depending on $\dot{V}$), and an extension

$$1 \to P \to \mathcal{E} \to \Gamma \to 1$$

with the following properties. For an affine algebraic group G and a finite central subgroup $Z \subset G$, both defined over F, let $H^1(P \to \mathcal{E}, Z \to G) \subset H^1(\mathcal{E}, G)$ be defined analogously to the local set $H^1(u \to W, Z \to G)$ of Section 3.1. In fact, let us denote the local set now by $H^1(u_v \to \mathcal{E}_v, Z \to G)$ to emphasize the local field F_v. Then for each $v \in V$ there is a localization map

$$\mathrm{loc}_v : H^1(P \to \mathcal{E}, Z \to G) \to H^1(u_v \to \mathcal{E}_v, Z \to G). \tag{13}$$

This map is functorial in $Z \to G$. Moreover, it is already well-defined on the level of 1-cocycles, up to coboundaries of Γ_v valued in Z, that is there is a well-defined map

$$\mathrm{loc}_v : Z^1(P \to \mathcal{E}, Z \to G) \to Z^1(u_v \to \mathcal{E}_v, Z \to G)/B^1(\Gamma_v, Z), \tag{14}$$

that induces (13).

Let now G be connected and reductive. For a fixed $x \in H^1(P \to \mathcal{E}, Z \to G)$, the class $\mathrm{loc}_v(x)$ is trivial for almost all v. Thus we have the total localization map

$$H^1(P \to \mathcal{E}, Z \to G) \to \coprod_v H^1(u_v \to \mathcal{E}_v, Z \to G), \tag{15}$$

where we have used the coproduct sign to denote the subset of the product consisting of tuples almost all of whose entries are trivial. One can show that the kernel of this map coincides with the kernel of the usual total localization map

$$H^1(\Gamma, G) \to \coprod_v H^1(\Gamma_v, G).$$

One can also characterize the image of the total localization map (15). This is based on the duality between $H^1(u_v \to \mathcal{E}_v, Z \to G)$ and $\pi_0(Z(\widehat{\bar{G}})^{+v})$ from Theorem 6, as well as an analogous global duality [Kal18a, §3.7]. Recall that $\bar{G} = G/Z$ and that $Z(\widehat{\bar{G}})^{+v}$ is the subgroup of $Z(\widehat{\bar{G}})$ consisting of those elements whose image in $Z(\widehat{G})$ is Γ_v-fixed. In the same way we define $Z(\widehat{\bar{G}})^{+}$, where we now demand that the image in $Z(\widehat{G})$ is Γ-fixed. The obvious inclusions $Z(\widehat{\bar{G}})^{+} \to Z(\widehat{\bar{G}})^{+v}$ lead on the level of characters to the summation map

$$\bigoplus_v \pi_0(Z(\widehat{\bar{G}})^{+v})^* \to \pi_0(Z(\widehat{\bar{G}})^{+})^*.$$

Then the image of (15) is the kernel of the composition

$$\coprod_v H^1(u_v \to \mathcal{E}_v, Z \to G) \to \bigoplus_v \pi_0(Z(\widehat{\bar{G}})^{+v})^* \to \pi_0(Z(\widehat{\bar{G}})^{+})^*. \tag{16}$$

Finally, we remark that when Z is sufficiently large (for example when it contains $Z(G_{\mathrm{der}})$) then the natural map $H^1(P \to \mathcal{E}, Z \to G) \to H^1(\Gamma, G_{\mathrm{ad}})$ is surjective.

5.2 Global parameters

It is conjectured [Kot84, §12] that there exists a topological group L_F, called the Langlands group of the global field F, which is an extension of the Weil group W_F by a compact group, such that the irreducible complex n-dimensional representations of L_F parameterize the cuspidal automorphic representations of GL_n/F. For each place v of F there should exist an embedding $L_{F_v} \to L_F$, well-defined up to conjugation in L_F. We shall admit the existence of this group in order to have a clean formulation of global parameters. In the case of classical groups the use of L_F can be avoided using Arthur's formal parameters, see [Art13, §1.4].

Let G^* be a quasi-split connected reductive group defined over F and let $\xi : G^* \to G$ be an inner twist. A discrete generic global parameter is a continuous semi-simple L-homomorphism $\phi : L_F \to {}^LG^*$ with bounded projection to $\widehat{G}^*$, whose image is not contained in a proper parabolic subgroup of ${}^LG^*$. Given such ϕ and a place v of F, let ϕ_v be the restriction of

ϕ to L_{F_v}, a tempered (but usually not discrete) local parameter. Define the adelic L-packet $\Pi_\phi(G,\xi)$ as

$$\Pi_\phi(G,\xi) = \{\pi = \otimes'_v \pi_v | \pi_v \in \Pi_{\phi_v}(G),\ \pi_v \text{ is unramified for a.a. } v\}$$

where the local L-packet $\Pi_{\phi_v}(G)$ is the one from Conjecture A. Note that we are using ξ to identify $\widehat{G}^*$ with $\widehat{G}$.

The question we want to answer in the following sections is this: Which elements $\pi \in \Pi_\phi(G,\xi)$ are discrete automorphic representations and what is their multiplicity in the discrete spectrum? More precisely, let $\chi : Z(G)(\mathbb{A}) \to \mathbb{C}^\times$ denote the central character of π. The locally compact topological group $G(\mathbb{A})$ is unimodular. We endow $G(\mathbb{A})$ with a Haar measure and the discrete group $G(F)$ with the counting measure and obtain a $G(\mathbb{A})$-invariant measure on the quotient space $G(F) \setminus G(\mathbb{A})$. Denote by $L^2_\chi(G(F) \setminus G(\mathbb{A}))$ the space of those square-integrable functions on the quotient $G(F)\setminus G(\mathbb{A})$ that satisfy $f(zg) = \chi(z)f(g)$ for $z \in Z(G)(\mathbb{A})$. The question we want to answer is this: What is the multiplicity of π as a closed subrepresentation of this space?

The answer to this question will be given in terms of objects that depend on G, ξ, and π. However, the construction of these objects will use the global cohomology set $H^1(P \to \mathcal{E}, Z \to G^*)$. In preparation for this, we define a global rigid inner twist $(\xi, z) : G^* \to G$ to consist of an inner twist $\xi : G^* \to G$ and $z \in Z^1(P \to \mathcal{E}, Z \to G^*)$, where $Z \subset G^*$ is a finite central subgroup defined over F, so that the image of z in $Z^1(\Gamma, G^*_{\text{ad}})$ equals $z_{\text{ad}}(\sigma) = \xi^{-1}\sigma(\xi)$.

5.3 Groups that satisfy the Hasse principle

Let G be a connected reductive group defined over F. Recall that G is said to satisfy the Hasse principle if the total localization map

$$H^1(\Gamma, G) \to \coprod_v H^1(\Gamma_v, G)$$

is injective. This is always true if G is semi-simple and either simply connected or adjoint, see [PR94, Theorems 6.6, 6.22]. Other groups that are known to satisfy the Hasse principle are unitary groups and special orthogonal groups. It was shown by Kottwitz [Kot84, §4] that G satisfies the Hasse principle if and only if the restriction map

$$H^1(\Gamma, Z(\widehat{G})) \to \bigoplus_v H^1(\Gamma_v, Z(\widehat{G}))$$

is injective.

We assume now that G satisfies the Hasse principle. Let G^* be the unique quasi-split inner form of G and let $\xi : G^* \to G$ be an inner twist. Let $z_{\text{ad}}(\sigma) \in Z^1(\Gamma, G^*_{\text{ad}})$ be given by $z_{\text{ad}}(\sigma) = \xi^{-1}\sigma(\xi)$. Fix $z \in Z^1(P \to \mathcal{E}, Z(G^*_{\text{der}}) \to G^*)$ lifting z_{ad}. For every place v let $z_v \in Z^1(u_v \to \mathcal{E}_v, Z(G^*_{\text{der}}) \to G^*)$ be the localization of z, well defined up to $B^1(\Gamma_v, Z(G^*_{\text{der}}))$. Then $(\xi, z_v) : G^* \to G$ is a (local) rigid inner twist.

Let $\phi : L_F \to {}^LG^*$ be a discrete generic global parameter. For such ϕ, the centralizer $S_\phi = \text{Cent}(\phi, \widehat{G}^*)$ is finite modulo $Z(\widehat{G}^*)^\Gamma$. For any place

v of F we have the tempered local parameter $\phi_v = \phi|_{L_{F_v}}$ and $S_\phi \subset S_{\phi_v}$. Let $\pi \in \Pi_\phi(G,\xi)$ and let χ be its central character. We interpret π_v as an element of $\Pi_{\phi_v}((\xi_v, z_v))$ and obtain from Conjecture G the class function $\langle \pi_v, - \rangle$ on $\pi_0(S^+_{\phi_v})$. Let S^+_ϕ be the preimage in $\widehat{\bar{G}^*}$ of S_ϕ and let $\langle \pi, - \rangle$ be the product over all places v of the pull-back to $\pi_0(S^+_\phi)$ of $\langle \pi_v, - \rangle$. It is a consequence [Kal18a, Proposition 4.2] of the description (16) of the image of (15) that this class function descends to the quotient $\pi_0(\bar{S}_\phi) := \pi_0(S^+_\phi/Z(\widehat{\bar{G}^*})^+) = \pi_0(S_\phi/Z(\widehat{G}^*)^\Gamma)$ and is moreover independent of the choice of z. It is the character of a finite-dimensional representation of $\pi_0(\bar{S}_\phi)$.

Conjecture H. *The natural number*

$$\sum_\phi |\pi_0(\bar{S}_\phi)|^{-1} \sum_{x \in \pi_0(\bar{S}_\phi)} \langle \pi, x \rangle,$$

where ϕ runs over the $\widehat{G}$-conjugacy classes of discrete generic global parameters satisfying $\pi_v \in \Pi_{\phi_v}(G)$, is the multiplicity of π in $L^2_\chi(G(F) \setminus G(\mathbb{A}))$.

This conjecture is is essentially the one from [Kot84, §12]. The only addition here is that we have explicitly realized the global pairing $\langle \pi, - \rangle$ as a product of normalized local pairings $\langle \pi_v, - \rangle$ with the help of the local and global Galois gerbes, and we have built in the simplifications implied by the Hasse principle.

In order to apply the stable trace formula to the study of this conjecture one needs to have a coherent local normalization of the geometric transfer factors. Let $\mathfrak{e} = (G^\mathfrak{e}, s^\mathfrak{e}, {}^L\eta^\mathfrak{e})$ be a global elliptic extended endoscopic triple. Due to the validity of the Hasse principle for G we may and will assume that $s^\mathfrak{e} \in Z(\widehat{G}^\mathfrak{e})^\Gamma$. To this triple, Kottwitz and Shelstad associate [KS99, §7.3] a canonical adelic transfer factor

$$\Delta_\mathbb{A} : G^\mathfrak{e}_{\mathrm{sr}}(\mathbb{A}) \times G_{\mathrm{sr}}(\mathbb{A}) \to \mathbb{C}.$$

Note however that the original definition needs a correction, as explained in [KS]. We assume henceforth that $\Delta_\mathbb{A}$ is the corrected global factor corresponding to the local factors Δ' of [KS, §5.4].

Choose a lift $s^{\dot{\mathfrak{e}}} \in \widehat{\bar{G}^\mathfrak{e}}$ of $s^\mathfrak{e}$. For each place v of F, $\dot{\mathfrak{e}}_v = (G^\mathfrak{e}, s^{\dot{\mathfrak{e}}}, {}^L\eta^\mathfrak{e})$ is a refined local extended endoscopic triple and we have the normalized transfer factor $\Delta[\mathfrak{w}, \dot{\mathfrak{e}}_v, \xi_v, z_v]$.

Theorem 11 ([Kal18a, Proposition 4.1])**.** *For $\delta \in G_{sr}(\mathbb{A})$ and $\gamma \in G^\mathfrak{e}_{sr}(\mathbb{A})$ one has*

$$\Delta_\mathbb{A}(\gamma, \delta) = \prod_v \Delta[\mathfrak{w}, \dot{\mathfrak{e}}_v, \xi_v, z_v](\gamma_v, \delta_v).$$

5.4 General groups

We shall now explain how to modify Conjecture H and Theorem 11 in the case when G does not satisfy the Hasse principle. In order to handle this case, it is not enough to choose $z \in Z^1(P \to \mathcal{E}, Z(G^*_{\mathrm{der}}) \to G^*)$ lifting z_{ad}. Instead, we consider the inner twist $\xi : G^*_{\mathrm{sc}} \to G_{\mathrm{sc}}$ on the level of the simply connected covers of the derived subgroups. Let $z_{\mathrm{sc}} \in Z^1(P \to \mathcal{E}, Z(G^*_{\mathrm{sc}}) \to$

$G^*_{\mathrm{sc}})$ lift z_{ad}. Let $z_{\mathrm{sc},v} \in Z^1(u_v \to \mathcal{E}_v, Z(G^*_{\mathrm{sc}}) \to G^*_{\mathrm{sc}})$ denote the localization of z_{sc}, well defined up to $B^1(\Gamma_v, Z(G^*_{\mathrm{sc}}))$. Let $z_v \in Z^1(u_v \to \mathcal{E}_v, Z(G^*_{\mathrm{der}}) \to G^*)$ be the image of $z_{\mathrm{sc},v}$.

Let $\phi : L_F \to {}^LG^*$ be a discrete generic global parameter. The group $\bar{S}_\phi = S_\phi/Z(\widehat{G}^*)^\Gamma$ that we used when G satisfied the Hasse principle is now not adequate any more. The reason is that two global parameters ϕ_1 and ϕ_2 are considered equivalent not only when they are $\widehat{G}^*$-conjugate, but when there exists $a \in Z^1(L_F, Z(\widehat{G}^*))$ whose class is everywhere locally trivial, and $g \in \widehat{G}^*$, so that $\phi_2(x) = a(x) \cdot g^{-1}\phi_1(x)g$, see [Kot84, §10]. Then the group of self-equivalences S_ϕ of a global parameter ϕ is defined to consist of those $g \in \widehat{G}^*$ for which $x \mapsto g^{-1}\phi(x)g\phi(x)^{-1}$ takes values in $Z(\widehat{G}^*)$ (then it is a 1-cocycle for formal reasons) and its class is everywhere locally trivial. This group contains not just $Z(\widehat{G}^*)^\Gamma$, but all of $Z(\widehat{G}^*)$, and we set $\bar{S}_\phi = S_\phi/Z(\widehat{G}^*)$.

As before we have for each $\pi \in \Pi_\phi(G,\xi)$ the local representation π_v as an element of $\Pi_{\phi_v}((\xi_v, z_v))$ and hence the class function $\langle \pi_v, - \rangle$ on $\pi_0(S^+_{\phi_v})$. We want to produce from these class functions a class function on $\pi_0(\bar{S}_\phi)$. Let $x \in \bar{S}_\phi$. Choose a lift $x_{\mathrm{sc}} \in \widehat{G}^*_{\mathrm{sc}}$ and let x_{der} be its image in $\widehat{G}^*_{\mathrm{der}}$. For each place v there exists $y_v \in Z(\widehat{G}^*)$ so that $x_{\mathrm{der}}y_v \in S_{\phi_v}$. Write $y_v = y'_v y''_v$ with $y'_v \in Z(\widehat{G}^*_{\mathrm{der}})$ and $y''_v \in Z(\widehat{G}^*)^\circ$ and choose a lift $\dot{y}'_v \in Z(\widehat{G}^*_{\mathrm{sc}})$. Since $\bar{G}^* = G^*/Z(G^*_{\mathrm{der}})$ we have $\widehat{\bar{G}^*} = \widehat{G}^*_{\mathrm{sc}} \times Z(\widehat{G}^*)^\circ$. Then $(x_{\mathrm{sc}}\dot{y}'_v, y''_v) \in S^+_{\phi_v}$. The reason we had to choose z_{sc} is that now the class $[z_{\mathrm{sc},v}] \in H^1(u_v \to \mathcal{E}_v, Z(G^*_{\mathrm{sc}}) \to G^*_{\mathrm{sc}})$ becomes a character of $Z(\widehat{G}^*_{\mathrm{sc}})$, which we can evaluate on $\dot{y}'_v$. It can be shown [Kal18a, Proposition 4.2] that the product $\langle \pi, x \rangle = \prod_v \langle [z_{\mathrm{sc},v}], \dot{y}'_v \rangle^{-1} \langle \pi_v, (x_{\mathrm{sc}}\dot{y}'_v, y''_v) \rangle$ is a class function on $\pi_0(\bar{S}_\phi)$ that is independent of the choices of z_{sc}, x_{sc}, $\dot{y}'_v$, and y''_v, and is the character of a finite-dimensional representation. We note here that each individual factor $\langle [z_{\mathrm{sc},v}], \dot{y}'_v \rangle^{-1} \langle \pi_v, (x_{\mathrm{sc}}\dot{y}'_v, y''_v) \rangle$, as a function of x_{sc}, is the character of an irreducible representation of the finite group $\pi_0(S^{\mathrm{sc}}_{\phi_v})$ discussed in Section 4.6. In fact, it is precisely the character of $\pi_0(S^{\mathrm{sc}}_{\phi_v})$ that is the image of the character $\langle \pi_v, - \rangle$ under the map (12).

Conjecture I. *The natural number*

$$\sum_\phi |\pi_0(\bar{S}_\phi)|^{-1} \sum_{x \in \pi_0(\bar{S}_\phi)} \langle \pi, x \rangle,$$

where ϕ runs over the equivalence classes of discrete generic global parameters satisfying $\pi_v \in \Pi_{\phi_v}(G)$, is the multiplicity of π in $L^2_\chi(G(F) \setminus G(\mathbb{A}))$.

A similar procedure is necessary in order to decompose the canonical adelic transfer factor $\Delta_{\mathbb{A}}$ into a product of normalized local transfer factors. Let $\mathfrak{e} = (G^{\mathfrak{e}}, s^{\mathfrak{e}}, {}^L\eta^{\mathfrak{e}})$ be a global elliptic extended endoscopic triple. Choose a lift $s_{\mathrm{sc}} \in \widehat{G}^*_{\mathrm{sc}}$ of the image of $s^{\mathfrak{e}}$ in $\widehat{G}^*_{\mathrm{ad}}$, and let $s_{\mathrm{der}} \in \widehat{G}^*_{\mathrm{der}}$ be the image of s_{sc}. For each place v there is $y_v \in Z(\widehat{G}^*)$ so that $s_{\mathrm{der}}y_v \in Z(\widehat{G}^{\mathfrak{e}})^{\Gamma_v}$. Here we have identified $\widehat{G}^{\mathfrak{e}}$ as a subgroup of $\widehat{G}^*$ via ${}^L\eta^{\mathfrak{e}}$. Write $y_v = y'_v y''_v$ with $y'_v \in Z(\widehat{G}^*_{\mathrm{der}})$ and $y''_v \in Z(\widehat{G}^*)^\circ$ and choose a lift $\dot{y}'_v \in Z(\widehat{G}^*_{\mathrm{sc}})$. Then $(s_{\mathrm{sc}}\dot{y}'_v, y''_v) \in Z(\widehat{\bar{G}^{\mathfrak{e}}})^{+_v}$, so $\dot{\mathfrak{e}}_v = (\widehat{G}^{\mathfrak{e}}, (s_{\mathrm{sc}}\dot{y}'_v, y''_v), {}^L\eta^{\mathfrak{e}})$ is a refined local extended endoscopic triple.

Theorem 12 ([Kal18a, Proposition 4.1]). *For $\delta \in G_{sr}(\mathbb{A})$ and $\gamma \in G^{\mathfrak{e}}_{sr}(\mathbb{A})$ one has*

$$\Delta_{\mathbb{A}}(\gamma, \delta) = \prod_v \langle [z_{sc,v}], \dot{y}'_v \rangle^{-1} \Delta[\mathfrak{w}, \dot{\mathfrak{e}}_v, \xi_v, z_v](\gamma_v, \delta_v).$$

5.5 Known cases

There are a few cases in which Conjecture H has been established. In [KMSW14] this conjecture is verified for pure inner forms of unitary groups. In [Taï19] this conjecture has been verified in the following setting. One considers non-quasi-split symplectic and orthogonal groups G for which there exists a finite set S of real places such that at $v \in S$ the real group $G(F_v)$ has discrete series, and for $v \notin S$ the local group $G \times F_v$ is quasi-split. For those groups, Taïbi studies the subspace $L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A}))^{S-\text{alg.reg.}}$ of discrete automorphic representations whose infinitesimal character at each place $v \in S$ is regular algebraic and shows that Conjecture H is valid for this subspace.

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