Two coordinate invariant computations:

$$[\tilde{X}, \tilde{Y}] = T_{\tilde{X}}\tilde{Y} - T_{\tilde{Y}}\tilde{X} = T\Psi \cdot (T_X Y - T_Y X) = T\Psi \cdot [X, Y],$$
$$d\omega(X, Y) = (T_X\omega)(Y) - (T_Y\omega)(X) = d\tilde{\omega}(\tilde{X}, \tilde{Y}).$$

Axioms for the covariant derivative:

$$\begin{aligned} &[X, Y] = D_X Y - D_Y X && \text{(Symmetry)} \\ &T_X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z) && \text{(Product Rule)} \end{aligned}$$

Koszul formula:

$$\begin{aligned} &T_Z(g(X, Y)) + T_Y(g(Z, X)) - T_X(g(Y, Z)) + \\ &g([X, Y], Z) - g([Z, X], Y) + g([Y, Z], X) \quad = \quad 2g(D_Y Z, X). \end{aligned}$$

Local formula:
Subtract from the above product rule the local differentiation $T_X(g(Y, Z)) = (T_X g)(Y, Z) + g(T_X Y, Z) + g(Y, T_X Z)$ then insert:

$$D_Y Z := T_Y Z + \Gamma(Y, Z) \text{ and get}$$
$$\begin{aligned} g(\Gamma(Y, Z), X) &:= (T_Z g)(X, Y) + (T_Y g)(Z, X) - (T_X g)(Y, Z) \\ &= g(\Gamma(Z, Y), X) \text{ (Symmetry of } \Gamma). \end{aligned}$$

Parallel vector fields along a curve: $\frac{D}{dt} X \circ c(t) = 0$. (They are a big help.)

Second derivatives of vector fields are not symmetric:

$$\begin{aligned} &D^2_{X,Y} Z - D^2_{Y,X} Z = \\ &(T_X\Gamma)(Y, Z) - (T_Y\Gamma)(X, Z) + \Gamma(X, \Gamma(Y, Z)) - \Gamma(Y, \Gamma(X, Z)). \end{aligned}$$

The **Riemann Curvature Tensor:**

$$R(X, Y)Z := D^2_{X,Y} Z - D^2_{Y,X} Z.$$

Product Rule for $(D^2_{X,Y} - D^2_{Y,X})$:

$$(D^2_{X,Y} - D^2_{Y,X})(A \cdot B) = ((D^2_{X,Y} - D^2_{Y,X})A) \cdot B + A \cdot (D^2_{X,Y} - D^2_{Y,X})B.$$

Symmetries of the curvature tensor:

Skew Symmetry in the 1st pair	$R(X, Y)Z = -R(Y, X)Z$
1. Bianchi Identity	$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
Skew Symmetry in the 2nd pair	$g(R(X, Y)V, W) = -g(R(X, Y)W, V)$
Symmetry in both pairs	$g(R(X, Y)V, W) = g(R(V, W)X, Y)$

Hypersurface Theory is the same as in the Riemannian case, except for signs related to the normal N. Let $F : M \to \mathbb{R}^n, \langle\langle ., . \rangle\rangle$ be a hypersurface immersion with $\langle\langle N, N \rangle\rangle = \pm 1$. Since we assume that the metric is induced from $\mathbb{R}^n$ we differentiate $\langle\langle TF(Y), TF(Z) \rangle\rangle - g(Y, Z) = 0$ and since $F = F^1, \dots, F^n$ is a collection of n functions the above definitions apply: $D^2F(X, Y) = T_X(T_Y F) - T_{D_X Y} F$. Hence

$$\langle\langle D^2F(X, Y), T_Z F \rangle\rangle + \langle\langle T_Y F, D^2F(X, Z) \rangle\rangle = 0.$$

Next we do the same $+, +, -$ cyclic computation as for the Koszul formula and, noting the symmetry $D^2F(X, Y) = D^2F(Y, X)$ we get a first result: $D^2F(Y, Z)$ is normal:

$$\begin{aligned} &2\langle\langle T_X F, D^2F(Y, Z) \rangle\rangle = 0, \text{ hence} \\ &D^2F(Y, Z) = \langle\langle D^2F(Y, Z), N \rangle\rangle / \langle\langle N, N \rangle\rangle \cdot N. \end{aligned}$$

Recall the definition of the **shape operator** (or Weingarten map, or second fundamental tensor) and differentiate $0 = \langle\langle N, T_Y F \rangle\rangle$ to relate the shape operator and $D^2F(X, Y)$:

$$\begin{aligned} &T_Y N =: TF \cdot S \cdot Y \\ &0 = \langle\langle T_X N, T_Y F \rangle\rangle + \langle\langle N, D^2F(X, Y) \rangle\rangle, \text{ or} \\ &g(SX, Y) = -\langle\langle N, D^2F(X, Y) \rangle\rangle. \end{aligned}$$

In particular, the shape operator is g-symmetric. Next, differentiate the definition of S, note the normal and the tangential component of the result and get the Codazzi equation:

$$D^2_{X,Y} N = D^2_{Y,X} N = D^2F(X, SY) + TF((D_X S)Y)$$

Codazzi Equation: $\qquad (D_X S)Y = (D_Y S)X.$

Finally differentiate $g(SY, Z)/\langle\langle N, N \rangle\rangle \cdot N = -D^2F(Y, Z)$ in direction X, then interchange X, Y, subtract and simplify with Codazzi. The difference of the 3rd derivatives is simplified with this product rule:

$$0 = (D^2_{X,Y} - D^2_{Y,X})(T_Z F) = (D^3_{X,Y,Z} - D^3_{Y,X,Z} F) + TF \cdot R(X, Y)Z$$

to get the

Gauss Equation: $g(SY, Z)SX - g(SX, Z)SY - \langle\langle N, N \rangle\rangle R(X, Y)Z = \vec{0}.$

From the full curvature tensor one defines, by taking a trace, a simpler symmetric tensor that will be important for formulating the Einstein equations.

Definition of the Ricci tensor:

$$g(Ric\,Y, Z) = ric(Y, Z) := \sum_i g(R(Y, e_i)e_i, Z)/g(e_i, e_i),$$
$$\mathrm{div}\,(Ric) = \sum_{i,j} (D_{e_j} R)(e_j, e_i)e_i/g(e_j, e_j)/g(e_i, e_i)$$

where $\{e_1, \ldots, e_n\}$ is an orthogonal basis.

Constancy Theorems: Too simple situations become trivial!

Umbilicity theorem $\quad S = f(p)id \Rightarrow f = const.$

with Codazzi equation: $\quad (D_X S)Y = df(X) \cdot Y = df(Y) \cdot X$

Schur's theorem (dim > 2) $R(X, Y)Z = f(p)(g(Y, Z)X - g(X, Z)Y)$

$$\Rightarrow f = const.$$
$$(D_U R)(X, Y)Z = df(U) \cdot (g(Y, Z)X - g(X, Z)Y)$$

Einstein metric (dim> 2) $\quad ric = f(p)g \Rightarrow f = const.$

2. Bianchi Identity:

$$(D_U R)(X, Y)Z + (D_X R)(Y, U)Z + (D_Y R)(U, X)Z = 0$$

The Jacobi Equation, a fundamental tool

Let $s \mapsto c(s,t)$ be a family of geodesics, i.e. $\frac{D}{ds}\frac{d}{ds}c(s,t) = 0$. Jacobi fields along these geodesics are defined as $s \mapsto \frac{d}{dt}c(s,t) =: J_t(s)$. Their 2nd order linear ODE is obtained by interchanging derivatives, thus bringing in the curvature tensor:

$$\begin{aligned} 0 =& \frac{D}{dt}\frac{D}{ds}\frac{d}{ds}c(s,t) = \frac{D}{ds}\frac{D}{dt}\frac{d}{ds}c(s,t) + R(\frac{d}{dt}c, \frac{d}{ds}c)\frac{d}{ds}c \\ =& \frac{D}{ds}\frac{D}{ds}J_t(s) + R(J_t(s), c')c'. \end{aligned}$$

Estimate the symmetric operator $J \mapsto R(J, c')c'$ via its eigenvalues:

$$\delta \cdot g(J,J) \le g(R(J,c')c', J) \le \Delta \cdot g(J,J),$$

which implies a second order differential inequality:

$$\begin{aligned} \frac{d}{ds}|J| &= g(J, \frac{D}{ds}J)/|J|, \\ \frac{d}{ds}\frac{d}{ds}|J| &= g(J, \frac{D}{ds}\frac{D}{ds}J)/|J| + g(\frac{D}{ds}J, \frac{D}{ds}J)/|J| - g(J, \frac{D}{ds}J)^2/|J|^3 \\ &\ge -g(J, R(J,c')c')/|J| \qquad \text{by Schwarz inequality} \\ &\ge -\Delta \cdot |J|. \end{aligned}$$

The Jacobi equation controls also how the shape operator changes in a geodesically parallel family of hypersurfaces. Let $N(p(t))$ be the normals along a curve in the hypersurface and $s \mapsto c(s,t)$ be the geodesics in these normal directions and $J_t(s) = \frac{d}{dt}c(s,t)$ the Jacobi fields of this family. The Jacobi fields compute S and the curvature tensor controls how S changes:

$$\begin{aligned} \frac{D}{ds}J_t(s) =& \frac{D}{ds}\frac{d}{dt}c(s,t) = \frac{D}{dt}\frac{d}{ds}c(s,t) = \frac{D}{dt}N(c(s,t)) = \\ =& S \cdot \frac{d}{dt}c(s,t) = S \cdot J_t(s), \\ -R(J_t(s), c')c' =& \frac{D}{ds}\frac{D}{ds}J_t(s) = (\frac{D}{ds}S) \cdot J_t(s) + S \cdot \frac{D}{ds}J_t(s) = \\ =& (\frac{D}{ds}S) \cdot J_t(s) + S \cdot S \cdot J_t(s). \end{aligned}$$

Normals of totally geodesic hypersurfaces are Eigenvectors of Ric

If an isometry has a hypersurface M as fixed point set, then M is totally geodesic.

Variations of geodesics in M are tangential, therefore 2nd derivatives of Jacobi fields are tangential, therefore $R(J,c')c'$ is tangential, hence $R(N,c')c'$ is proportional to N. Then, finally: N is eigenvector of Ric.

Jacobi fields have surprisingly many immediate **physical interpretations.**

Examples are: "Tidal forces of gravity", "Perihelion advance of Mercury", various forms of "distance measurements".

Light cone geometry

While the light cones can be pretty complicated, much is similar to the vector space.

1. A geodesic starting in a null direction remains a null geodesic: $\frac{d}{ds}g(\gamma'(s), \gamma'(s)) = 0$.
2. Variations of null geodesics have: $g(\frac{D}{d\epsilon}\gamma'_\epsilon(s), \gamma'_\epsilon(s)) = 0 = g(\frac{D}{ds}J_\epsilon(s), \gamma'_\epsilon(s))$ and $g(\frac{D}{d\epsilon}\gamma_\epsilon(s), \gamma'_\epsilon(s)) = g(J_\epsilon(s), \gamma_\epsilon(s)) = 0$ if this is true initially: Jacobi fields remain tangential.
3. Tangent vectors to the light cone stay tangential when parallel translated along null rays: $\frac{d}{ds}g(u(s), \gamma'(s)) = 0$.
4. Parallel translation of null vectors along any curve c remain null: $\frac{d}{ds}g(u(s), u(s)) = 0$.
5. Arbitrary Jacobi fields J along rays γ can be split as *parallel + tangential*, $J = u + T$.
   Split $J(0) = u(0) + T(0)$, $g(T(0), \gamma'(0)) = 0$, extend u parallel and put $T(s) = J(s) - u(s)$.
   Then $\frac{d}{ds}g(T(s), \gamma'(s)) = g(\frac{D}{ds}J(s), \gamma'(s)) = 0$.

Under conformal changes of the metric, $\tilde{g}(X, Y) = \lambda^{-2}(P)g(X, Y)$, null geodesics remain null geodesics, possibly with a non-affine parametrization.

Assume $\frac{D}{ds}\gamma'(s) = 0$ and reparametrize: $s = \varphi(\sigma), \tilde{\gamma}(\sigma) = \gamma(\varphi(\sigma))$. Then $\frac{d}{d\sigma}\tilde{\gamma}(\sigma) = \gamma'(\varphi(\sigma)) \cdot \varphi'(\sigma)$ and

$$\frac{D}{d\sigma}\frac{d}{d\sigma}\tilde{\gamma}(\sigma) = \frac{D}{ds}\gamma'(\varphi(\sigma))\cdot\varphi'(\sigma)^2 + \gamma'(\varphi(\sigma))\cdot\varphi''(\sigma) = \varphi''(\sigma)/\varphi'(\sigma)\cdot\frac{d}{d\sigma}\tilde{\gamma}(\sigma).$$

Vice versa, if $\frac{D}{d\sigma}\frac{d}{d\sigma}\tilde{\gamma}(\sigma) = m(\sigma) \cdot \frac{d}{d\sigma}\tilde{\gamma}(\sigma)$ is given, then try to find $\sigma = \psi(s)$ such that $\gamma(s) := \tilde{\gamma}(\psi(s))$ satisfies $\frac{D}{ds}\gamma'(s) = 0$.
For this $\psi(s)$ needs to solve the ODE $\quad \psi''(s) + m(\psi(s)) \cdot \psi'(s)^2 = 0$.

Now let $\gamma(s)$ be a null geodesic for the metric g and change the metric conformally to $\tilde{g}(\ ,\ ) = \lambda^{-2}(p) \cdot g(\ ,\ )$. Then we have

$$\tilde{D}_X Y = D_X Y + \Gamma(X, Y) = D_X Y - \frac{T_X\lambda}{\lambda}Y - \frac{T_Y\lambda}{\lambda}X + g(X, Y) \cdot \operatorname{grad}\lambda$$

Therefore $\frac{D}{ds}\gamma'(s) = 0$ implies $\frac{\tilde{D}}{ds}\gamma'(s) = -2\frac{d}{ds}\lambda(\gamma(s))/\lambda(\gamma(s)) \cdot \gamma'(s)$, as claimed.
The second order ODE for the reparametrization can in this case be integrated once and we are left with the 1st order ODE to reparametrize γ:

$$\psi'(\sigma) = \lambda^2(\gamma(\psi(\sigma))), \ \ \tilde{\gamma}(\sigma) := \gamma(\psi(\sigma)).$$

Next I have to explain **redshift as a geometric notion.**
(See Fraunhofer lines - the frequency ratios are the **same** for all objects in the sky.)
Summary:
Let $\gamma_\epsilon(s)$ be a family of null geodesics which join the worldline of the sender S to the worldline of the observer O and let $J_\epsilon(s) = \frac{D}{d\epsilon}\gamma_\epsilon(s)$ be the corresponding Jacobi fields. Let u, v be the timelike unit tangent vectors of the world lines of S, O and assume $J(0) = u$ (by scaling) and $J(1) = \mu \cdot v$. This means:

> Time signals which are sent with sender's time difference 1 are observed with observers time difference μ
> i.e. $\Delta T(S)/\Delta T(O) = 1/\mu$ or for frequencies: $\omega_S/\omega_O = \mu$.

We proved in 5. above that

$$g(u, \gamma'(0)) = g(J(0), \gamma'(0)) \overset{(5.)}{=} g(J(1), \gamma'(1)) = g(\mu \cdot v, \gamma'(1))$$

hence

$$1 + z := \frac{\omega_S}{\omega_O} = \mu = \frac{g(u, \gamma'(0))}{g(v, \gamma'(1))}.$$

Special Relativity Application:

$$\gamma' = (1,0,0,1),\ u = (0,0,0,1),\ v = (a,0,0,1)/\sqrt{1-a^2},\ \mu = \sqrt{\frac{1+a}{1-a}}$$

Next, if we change the metric conformally, $\tilde{g}(X,Y) = \lambda^{-2} g(X,Y)$, then we get a very simple formula

$$\frac{\tilde{\omega}_S}{\tilde{\omega}_O} = \frac{\omega_S}{\omega_O} \cdot \frac{\lambda(S)}{\lambda(O)}$$

Note that unit vectors change as $\tilde{u}(P) = \lambda(P) \cdot u(P)$. The g-null geodesic $\gamma(s)$, $\frac{D}{ds}\gamma'(s) = 0$ is still a null geodesic for $\tilde{g}$, but not with an affine parametrization. The reparametrization $\tilde{\gamma}(\sigma) = \gamma(\psi(\sigma))$ satisfies $\psi'(\sigma) = \lambda^2(\tilde{\gamma}(\sigma))$, so that for example

$$\tilde{g}(\tilde{u}, \tilde{\gamma}'(\sigma_0)) = \lambda^{-2}(S) \cdot g(\lambda(S)u, \lambda^2(S)\gamma'(0)) = \lambda(S) \cdot g(u, \gamma'(0)).$$

Example 1: Schwarzschild Metric

Ansatz: $M := (a,b) \times \mathbb{S}^2 \times \mathbb{R}$

$$ds^2 := d\rho^2 + G^2(\rho)d\sigma^2 - F^2(\rho)dt^2 \quad \text{(later: } F'(\rho) > 0\text{)}$$

where $d\sigma^2$ is the standard metric on $\mathbb{S}^2$.

This Ansatz is intended to give a relativistic model of the empty space outside a rotationally symmetric and static star. The star itsself is not part of the model.
The worldline of a person standing, say, on the surface of the earth is given by ρ and σ being constant. The coordinate time t is **not** what atomic clocks there would measure, but $\tau = F(\rho) \cdot t$. Since F is an increasing function, more **measured** time passes for clocks standing at larger ρ.

• This means that distances measured via the travelling time of reflected time signals give larger radial distances for the astronomers further out. – We have used reflected radar signals to measure the distance to Venus.

The totally geodesic Riemannian submanifold $\{t = const\}$ could be called "our space". But typical measurement procedures do not measure the Riemannian distance. We perceive directions in our 3-dim space, but light signals arrive via null-geodesics on our backwards light cone. We use the unit tangent vector of our world line to relate directions in our space with light rays.

• The best known distance measurement: Given two stars A,B; the distance to A is known and B is one quarter as bright as A, then B is twice as far away as A. This is easy to fit into the 4-dim picture: B is twice as far back on our backwards light cone. This method needs a start.

• The most precise start comes from the 1987 supernova: That star was surrounded by a huge disk of dust and the light of the explosion made the disk shine bright. The disk was tilted towards our line of sight, therefore we saw the closest part of the disk shine bright first and then could watch until the whole disk was bright. By using the known speed of light one could precisely compute the size of this disk. Measuring the angular size as seen from earth gave the distance. – In the case of the Scharzschild geometry the curvatures along the light cone have different sign, therefore known spherical balls are seen as ellipses.

• The older method for starting the brightness comparison used parallaxe measurments with a diameter of the earth orbit as base. Again, the 4-dim computation depends on Jacobi fields along the light cone. In the presence of curvature the measured distance depends on the orientation of the base.

Determination of the functions G and F: for quantitative predictions

The hypersurfaces $\{\rho = const\}$ have the product metric $G^2(\rho)d\sigma^2 - F^2(\rho)dt^2$ with hypersurface curvature $1/G^2$ tangential to $\mathbb{S}^2$ and hypersurface curvature 0 for the $e_\sigma \wedge e_4$-planes. The eigenvalues of the shape operator are obtained from Jacobi fields which are restrictions of Killing fields. These Jacobi fields are obtained from variations in 2-dimensional totally geodesic subspaces, which implies $J(s)/|J(s)|$ is a *parallel* field. Therefore we do not need Christoffel symbols to compute ($s = \rho$)

$$\frac{D}{ds}(\frac{D}{ds}J(s)) = |J(s)|'' \cdot \frac{J(s)}{|J(s)|} = -R(J(s), c'(s))c'(s), \quad c'(s) = e_1.$$

$$|J_4(\rho)| = F(\rho), \quad |J_\sigma(\rho)| = G(\rho).$$

$$R(e_4, e_1)e_1 = -\frac{F''(\rho)}{F(\rho)}e_4, \quad R(e_\sigma, e_1)e_1 = -\frac{G''(\rho)}{G(\rho)}e_\sigma.$$

$$S \cdot e_4 = \frac{F'}{F}e_4, \quad S \cdot e_\sigma = \frac{G'}{G}e_\sigma.$$

$$R(e_2, e_3)e_3 = (\frac{1}{G^2} - (\frac{G'}{G})^2)e_2, \quad R(e_3, e_2)e_2 = (\frac{1}{G^2} - (\frac{G'}{G})^2)e_3,$$

$$R(e_4, e_\sigma)e_\sigma = 0 - (\frac{G'}{G})(\frac{F'}{F})e_4, \quad R(e_\sigma, e_4)e_4 = 0 - (-\frac{F'}{F})(\frac{G'}{G})e_\sigma.$$

Note that in the Gauss equations, because of $g(Sy, y)Sx = \lambda_y g(y, y)\lambda_x x$, it matters which vectors are timelike and which are spacelike.
From the above curvature tensor data we obtain the eigenvalues of Ricci:

$$Ric(e_1) = \lambda_1 e_1 = (-\frac{F''}{F} - 2\frac{G''}{G}) \cdot e_1,$$

$$Ric(e_\sigma) = \lambda_\sigma e_\sigma = (\frac{1}{G^2} - (\frac{G'}{G})^2 - \frac{G''}{G} - \frac{F'G'}{FG}) \cdot e_\sigma,$$

$$Ric(e_4) = \lambda_4 e_4 = (-\frac{F''}{F} - 2\frac{F'G'}{FG}) \cdot e_4.$$

Einstein vacuum equations: $\Lambda = \lambda_1 = \lambda_\sigma = \lambda_4$. ($|\Lambda| << 1$ given constant)

The Schwarzschild geometry is obtained by solving this ODE-system.

$(*1): \lambda_1 = \lambda_4 \Leftrightarrow \dfrac{F'}{F} = \dfrac{G''}{G'} \Leftrightarrow (\dfrac{F}{G'})' = 0 \Leftrightarrow \dfrac{F}{G'} = const.$

By scaling the t-coordinate we can have $const = 1$, hence

$$F = G'.$$

This leaves only one function to be determined. Setting all three eigenvalues equal and inserting $F = G'$ gives a third order ODE for G which we can integrate twice for a first order ODE for G that contains two parameters, one is the cosmological constant, the other will be called m for mass of the central star.

$$\begin{aligned} (*2): \qquad & \lambda_\sigma = (\lambda_4 + \lambda_1)/2 \\ \Longleftrightarrow\ & 0 = \frac{F''}{F} + \frac{1}{G^2} - (\frac{G'}{G})^2 = \frac{G'''}{G'} + \frac{1}{G^2} - (\frac{G'}{G})^2 \\ \Longleftrightarrow\ & \Big(-\frac{1}{G^2} + (\frac{G'}{G})^2 + 2\frac{G''}{G}\Big)' = 2\frac{G'}{G}\left(\frac{G'''}{G'} + \frac{1}{G^2} - (\frac{G'}{G})^2\right) = 0. \end{aligned}$$

If we compare the obtained first integral with λ_σ, we find that the value of this constant function is $-\Lambda$:

$$\Big(-\frac{1}{G^2} + (\frac{G'}{G})^2 + 2\frac{G''}{G}\Big) = -\Lambda.$$

We add the third order ODE and multiply by G^2G' to get

$$(\frac{G'''}{G'} + 2\frac{G''}{G})G^2G' = -\Lambda G^2 G',$$

hence another constant function:

$$\big(G^2G'' + \frac{\Lambda}{3}G^3\big)' = 0.$$

Define $\qquad m := \big(G^2G'' + \frac{\Lambda}{3}G^3\big),$

and observe $\qquad m \ = \ G^2G'' + \frac{\Lambda}{3}G^3 = \frac{G}{2}(1 - G'^2) - \frac{\Lambda}{6}G^3.$

So we arrived at the desired first order ODE for G:

$$\boxed{G'^2 = 1 - \frac{2m}{G} - \frac{\Lambda}{3}G^2}$$

Note that this ODE implies the third order ODE and hence all other used identities.

We make the change to the historic coordinates $r := G(\rho)$, $dr = G'(\rho)d\rho$ and recall that the historic Schwarzschild solution has $\Lambda = 0$. The metric is

$$\boxed{ds^2 = (1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\sigma^2 - (1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)dt^2.}$$

Curvatures in terms of m, G

We have computed above the Jacobi part of the curvature tensor in terms of F, G, now we use the ODE to compute these curvatures in terms of m, Λ. Note $(G'/G)(F'/F) = G''/G = m/G^3 - \Lambda/3$, $(1 - G'^2)/G^2 = 2m/G^3 + \Lambda/3$ and $-F''/F = -G'''/G' = (1 - G'^2)/G^2 = 2m/G^3 + \Lambda/3$. To obtain in addition to the six values already listed also $R(e_1, N)N$ with N one of the totally geodesic hypersurface normals e_4, e_σ used above, note that we proved already $R(e_1, N)\tilde{N} = 0$ if $N \perp \tilde{N}$ are any two of those normals. This says that $R(e_1, N)N$ is a multiple of e_1 and $g(R(e_1, N)N,\ e_1)$ is already known. This gives the following list

$$\begin{aligned}
&R(e_4, e_1)e_1 = (2m/G^3 + \Lambda/3)e_4, \quad R(e_\sigma, e_1)e_1 = (-m/G^3 + \Lambda/3)e_\sigma,\\
&R(e_2, e_3)e_3 = (2m/G^3 + \Lambda/3)e_2, \quad R(e_3, e_2)e_2 = (2m/G^3 + \Lambda/3)e_3,\\
&R(e_4, e_\sigma)e_\sigma = (-m/G^3 + \Lambda/3)e_4, \quad R(e_\sigma, e_4)e_4 = (+m/G^3 - \Lambda/3)e_\sigma,\\
&R(e_1, e_\sigma)e_\sigma = (-m/G^3 + \Lambda/3)e_1, \quad R(e_1, e_4)e_4 = (-2m/G^3 - \Lambda/3)e_1.
\end{aligned}$$

Quotient Geometry

In particular, the eigenvalues of $[R]$ agree up to sign.
The eigenvalues of $[R]$ will depend on $\vec{n}_1$. We put

$$\vec{n}_1 = (x^\rho, x^\sigma, 0, x^t) \quad \text{with} \quad (x^\rho)^2 + G(\rho)^2(x^\sigma)^2 - F(\rho)^2(x^t)^2 = 0.$$

Next we choose two tangent vectors to the light cone through $\vec{n}_1$

$$\vec{u} := (0, 0, x^3, 0) \perp \vec{n}_1,\ \vec{v} \perp \vec{u}, \vec{n}_1$$

Recall that $R(\vec{X}, \vec{Y})\vec{Z} = 0$ if we insert three orthogonal vectors tangent to the factors of $M = (a, b) \times \mathbb{S}^2 \times \mathbb{R}$. So we get

$$\begin{aligned}
R(\vec{u}, \vec{n}_1)\vec{n}_1 &= (x^\rho)^2 R(\vec{u}, \vec{e}_1)\vec{e}_1 + G(\rho)^2(x^\sigma)^2 R(\vec{u}, \vec{e}_2)\vec{e}_2\\
&\quad + F(\rho)^2(x^t)^2(\vec{u}, \vec{e}_4)\vec{e}_4\\
&= \frac{3m}{G}(x^\sigma)^2 \cdot \vec{u}
\end{aligned}$$

Accelerations, Christoffel Symbol

On the underlying product manifold $M = (a,b) \times \mathbb{S}^2 \times \mathbb{R}$ we have the product metric $d\rho^2 + d\sigma^2 - dt^2$. We can work with its covariant derivative $D^\times$ without introducing local coordinates on $\mathbb{S}^2$. We denote by $\Gamma(.,.)$ the difference tensor between the Schwarzschild covariant derivative D and $D^\times$:

$$D_X Y \;=\; D^\times_X Y + \Gamma(X,Y).$$

Again, Γ is computed with the $(+,+,-)$ cyclic permutation trick:

$$\begin{aligned} T_Z(g(X,Y)) &= g(D_Z X, Y) + g(X, D_Z Y) \\ &= g(D^\times_Z X, Y) + g(X, D^\times_Z Y) + (D^\times_Z g)(X,Y). \end{aligned}$$

With the notation $X = (X^\rho, X^\sigma, X^t)$, $g(X,Y) = X^\rho Y^\rho + G(\rho)^2\langle X^\sigma, Y^\sigma\rangle - F(\rho)^2 X^t Y^t$
we have

$$\begin{aligned} &(D^\times_Z g)(X,Y) = 2G(T_Z G)\langle X^\sigma, Y^\sigma\rangle - 2F(T_Z F)X^t \cdot Y^t, \\ &T_Z G = G'(\rho)\cdot Z^\rho, \quad T_Z F = F'(\rho)\cdot Z^\rho \\ &g(\Gamma(X,Y),Z) = \frac{1}{2}\big(-(D^\times_Z g)(X,Y) + (D^\times_X g)(Y,Z) + (D^\times_Y g)(Z,X)\big) \end{aligned}$$

Finally, the difference tensor is

$$\boxed{\Gamma(X,Y) = \begin{pmatrix} FF'X^tY^t - GG'\langle X^\sigma, Y^\sigma\rangle \\ (G'/G)(X^\rho Y^\sigma + Y^\rho X^\sigma) \\ (F'/F)(X^\rho Y^t + Y^\rho X^t) \end{pmatrix}}$$

The first application is the radial acceleration of the Killing observers. Their world lines are

$$\gamma(s) := (\rho_0, \sigma_0, s/F(\rho_0)), \gamma'(s) = (0,0,1/F(\rho_0)),$$

$$\frac{D}{ds}(\gamma'(s)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \Gamma\begin{pmatrix} 0 & 0 \\ 0 & , \; 0 \\ 1/F(\rho_0) & 1/F(\rho_0) \end{pmatrix} = \begin{pmatrix} (F'/F)(\rho_0) \\ 0 \\ 0 \end{pmatrix},$$

and more explicitly in the Schwarzschild geometry

$$\frac{F'}{F}(\rho) = \frac{G''}{G'}(\rho) = \frac{m/G^2 - (\Lambda/3)G}{\sqrt{1 - 2m/G - (\Lambda/3)G^2}}(\rho) = \Big|_{\Lambda=0} \frac{m}{G^2}\frac{1}{\sqrt{1-2m/G}}(\rho).$$

Circular Planetary Observers, Kepler's 3rd law

For the world lines of circling observers we have
(with $\sigma_\rho(.)$ a great circle in $\mathbb{S}^2$)

$$\gamma(s) = (\rho, \sigma_\rho(s), 0, \tau(\rho) \cdot s), \quad \gamma'(s) = (0, \omega(\rho), 0, \tau(\rho)) \quad \text{with}$$
$$-1 = g(\gamma'(s), \gamma'(s)) \;=\; G^2(\rho)\omega(\rho)^2 - F^2(\rho)\tau(\rho)^2.$$

The world line of an infinitesimal planet is in addition geodesic, i.e.

$$\begin{aligned}\frac{D}{ds}\gamma'(s) &= \begin{pmatrix}0\\0\\0\end{pmatrix} + \Gamma(\gamma'(s), \gamma'(s)) \\ &= \begin{pmatrix} FF'(\rho)\tau(\rho)^2 - GG'(\rho)\omega(\rho)^2 \\ 0 \\ 0 \end{pmatrix} \overset{(!)}{=} \begin{pmatrix}0\\0\\0\end{pmatrix}\end{aligned}$$

The **geodesic condition**, with proper time parameter s and using $G' = F$, therefore is:

$$GG''(\rho)\tau(\rho)^2 - G(\rho)^2\omega(\rho)^2 = 0, \quad G'(\rho)^2\tau(\rho)^2 - G(\rho)^2\omega(\rho)^2 = +1,$$

which gives for the constants in γ' :

$$\tau(\rho)^2 = (G'(\rho)^2 - GG''(\rho))^{-1} = (1 - \frac{3m}{G})^{-1},$$
$$\omega(\rho)^2 = \tau(\rho)^2\frac{G''}{G}(\rho) = \frac{(m/G^3 - \Lambda/3)}{(1 - 3m/G)}\Bigg|_{\Lambda=0} = \frac{m}{G^3}(1 - \frac{3m}{G})^{-1}.$$

We compare these results with **Kepler's third law** – presently under the agreement that a *Killing observer signals* when orbits are completed (see next lecture):

$$(\text{Proper Period Time, Planetary Clock})^2 = \Big(\frac{2\pi}{\omega(\rho)}\Big)^2 = \frac{4\pi^2}{m}G^3\Big(1 - \frac{3m}{G}\Big),$$

2m for the sun can be computed from this equation:
$Tunit = 3.3 \cdot 10^{-6} sec, 1year = 9.46 \cdot 10^{12} Tunits,\; G_{earth} = 1.5 \cdot 10^8 km \Longrightarrow 2m = 3\,km.$

Example 2: The standard cosmological model

The Einstein Equation

$$8\pi T = G + \Lambda \mathrm{id}\,.$$

For the Ricci Tensor I use different names for its bilinear version: $ric(v,w)$ and its 1-1-Tensor version: $Ric(v)$, and of course: $ric(v,w) = g(Ric(v),\ w)$. The divergence free part of the Ricci Tensor is the Einstein Tensor G:

$$G := Ric - \frac{1}{2}(\mathrm{trace}\, Ric)\cdot \mathrm{id}\,, \qquad \mathrm{trace}\, G = -\mathrm{trace}\, Ric, \qquad \dim = 4.$$

Stress-Energy Tensor of Perfect Fluid :

$$T \cdot W = (p \cdot W + (\rho + p) g(U, W) \cdot U.$$

The time unit vector field U determines the rest frame of the fluid.

Use the Einstein equation to get Ric from T:

$$\mathrm{trace}\,(G) = 8\pi \cdot \mathrm{trace}\,(T) - 4\Lambda$$

$$Ric = G - \frac{1}{2}(\mathrm{trace}\, G)\cdot \mathrm{id} \; = 8\pi T - 4\pi \cdot \mathrm{trace}\,(T)\mathrm{id} \, + (2-1)\Lambda \mathrm{id}$$

$$\boxed{Ric(U) = (\Lambda - 4\pi(\rho + 3p))\cdot U, \quad Ric\big|_{U^\perp} = (\Lambda + 4\pi(\rho - p))\cdot \mathrm{id}\,\big|_{U^\perp}.}$$

By looking at the Ricci tensor we can now recognize whether some Lorentz manifold has as its matter content a perfect fluid. The quadratic examples of lecture 2 do not model such type of matter.

A Lorentz manifold with such a Ricci tensor is still more general than the standard model. We need more input from observations.

Immediate consequences of $\mathrm{div}\,(T) = 0$

Recall that, when Einstein wrote down the above field equation, physicists had already met stress energy tensors of materials and they were convinced that T would be divergence free for all materials. Therefore Einstein constructed the right side of the equation to be divergence free. We learn some facts about perfect fluids by computing the divergence of T:

$$\mathrm{div}\,(T) := \sum_i \frac{(D_{e_i}T)\cdot e_i}{g(e_i,\ e_i)} \Longrightarrow g(\mathrm{div}\,(T), W) = \sum_i \frac{g((D_{e_i}T)\cdot W, e_i)}{g(e_i,\ e_i)}$$

$$g(\mathrm{div}\,(T), W) = T_W p + (p+\rho) g(W, D_U U) + g(W, U)\mathrm{div}\,((p+\rho)U).$$

If we *use* $\mathrm{div}\,(T) = 0$ and apply this computation for $W \perp U$, then we get

$$D_U U = -(\mathrm{grad}\, p)/(p+\rho), \quad \mathrm{grad} \ = \mathrm{grad}^{\,Restspace}$$

in particular, in the case of dust, we get **geodesic world lines** for the dust particles. In general the acceleration is caused by the pressure gradient (in the rest space).
If we use the computation for $W = U$ in the dust case, we get $\mathrm{div}\,(\rho \cdot U) = 0$, a **conservation of mass** result.

A Schur Theorem for conformally flat perfect fluids

LEMMA: *A conformally flat perfect fluid is curvature isotropic.*
We write more explicitly what we mean by **"curvature isotropic with respect to U",** i.e., by the property that *the curvature tensor distinguishes no directions in the rest spaces $U^\perp$ of the matter* – in agreement with observations. Clearly, such a curvature tensor has to have the following properties:

$$X,Y,Z \perp U \Longrightarrow R(X,Y)Z = k(p)(g(Y,Z)X - g(X,Z)Y),$$
$$R(X,U)U = \mu(p)\cdot X,$$

with the immediate consequences: $R(X,Y)U = 0,$

$$R(U,X)Y = -\mu(p)\cdot g(X,Y)\cdot U.$$

(Note that $g(R(U,X)Y,Z) = 0$ for all $Z \perp U$ and $g(R(U,X)Y,U) = g(R(X,U)U,\ Y)$.)

This is enough information to check that any curvature isotropic curvature tensor has its Weyl conformal curvature tensor vanish. Moreover, we find for the Ricci tensor (of such a curvature tensor):

$$ric(U,U) = 3\mu(p) \qquad = -\lambda_U = (-\Lambda + 4\pi(\rho+3p))$$
$$ric(U,Y) = 0$$
$$ric(X,Y) = (2k-\mu)g(X,Y) = \lambda_{U^\perp} = (\Lambda + 4\pi(\rho - p)).$$

This shows that the eigenspace decomposition is the correct one for a perfect fluid (we also need to satisfy $0 \le 3p \le \rho$). Note:

$$6k - 2\Lambda = 16\pi\rho, \quad 4\mu - 2k + 2\Lambda = 16\pi p, \quad \mu + k = 4\pi(p+\rho).$$

Theorem of Schur type. Let M^4 be curvature isotropic for a time like unit vector field U so that M^4 models a perfect fluid. We also assume $\rho > 0$, since otherwise one cannot everywhere define the local rest frame of the matter, namely $U, U^\perp$. Then:

a) $U^\perp$ is an **integrable** distribution.
b) The 3-dim integral manifolds have intrinsically constant curvature.
c) A matter equation $F(p,\rho) = 0$, $\frac{\partial}{\partial p}F \neq 0$ implies $D_U U = 0$ so that extrinsically these integral manifolds are parallel hypersurfaces with the matter world lines as the orthogonal geodesics.

This shows: if ρ, hence k, are not konstant then the levels of ρ are the integral manifolds of the distribution $U^\perp$.

One may look for a cosmological model assuming these conclusions without obtaining them first as a Pseudo-Riemannian theorem.

Model assumptions, for **Friedman-Robertson-Walker universes:**

Matter content.

The matter of the model is a perfect fluid. Mostly we assume the matter equation for **dust**, $p = 0$. To illustrate how the type of matter changes the model we will also deal with the matter equation for a **photon gas**, $3p = \rho$.

Symmetry.

Other observers on matter world lines should see the universe as we do, and, roughly speaking, the observations do not distinguish special rest space directions (i.e. orthogonal to matter world lines). We turn this into the assumption: *the curvature tensor distinguishes no directions in the rest spaces.*

Ansatz.

From these assumptions we concluded that the matter world lines are geodesics and that the orthogonal distribution is integrable, giving space slices of constant intrinsic and extrinsic curvature. This foliation also defines a global time function τ and the curvatures as well as ρ and p depend on τ.

The underlying manifold therefore is

$$M^4 = M^3_\kappa \times (a, b),$$

with (a, b) to be determined and M^3_κ a space of constant curvature κ.

M^4 has a warped product metric (prefered Ansatz in Physics)

$$\bar{g} = a^2(\tau) g_\kappa(.,.) - d\tau^2.$$

We introduce a new time function t and define what will turn out to be the conformal factor:

$$dt := \frac{d\tau}{a(\tau)} \quad \text{and} \quad \lambda(t) := a(\tau(t))^{-1}. \quad d\tau = \frac{dt}{\lambda(t)}. \qquad \text{Note} \;\; \lambda(today) = 1.$$

This transforms the above Ansatz metric in a conformally flat form:

$$\bar{g} = a(\tau)^2 g_\kappa - d\tau^2 = \lambda(t)^{-2}(g_\kappa - dt^2).$$

From the definition of $t, \lambda(t)$ follows (with $\frac{d}{d\tau} h(\tau) = h'(\tau), \;\; \frac{d}{dt} h(t) = \dot{h}(t)$):

$$\frac{a'}{a}(\tau) = -\dot{\lambda}(t), \qquad \frac{a''}{a} = -\lambda\ddot{\lambda} + \dot{\lambda}^2.$$

These relations suffice to translate the (Einstein) differential equations for $a(\tau)$ into differential equations for $\lambda(t)$. But we will derive the equations for $\lambda(t)$ from scratch.

Einstein equations for the conformally flat cosmological model

Curvature tensor, Ricci tensor and Einstein tensor for the product metric $g = g_\kappa - dt^2$ are easily obtained (observe that U is globally parallel for g):

$$\begin{aligned}
&R(*,*)U = 0, \quad R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y),\\
&Ric(U) = 0, \quad Ric(X) = 2\kappa X, \quad \frac{1}{2}\text{trace}\,(Ric) = 3\kappa,\\
&G(U) = -3\kappa U, \quad G(X) = -\kappa X, \text{ with } \ g(U,X) = 0.
\end{aligned}$$

For the conformally changed metric $\bar{g} = \lambda^{-2} g$ we compute the Einstein tensor with the conformal-change-formula at the end of last lecture.
Note $\text{grad}_g\lambda = -\dot{\lambda}U$, $\; D\text{grad}_g\lambda = -\ddot{\lambda}g(U,.)U$

$$\begin{aligned}
(\overline{G} + \Lambda)(X) &= \big(\lambda^2(-\kappa + 0 - \frac{3\dot{\lambda}^2}{\lambda^2} + \frac{2\ddot{\lambda}}{\lambda}) + \Lambda\big)X \overset{(!)}{=} pX\\
(\overline{G} + \Lambda)(U) &= \big(\lambda^2(-3\kappa - \frac{2\ddot{\lambda}}{\lambda} - \frac{3\dot{\lambda}^2}{\lambda^2} + \frac{2\ddot{\lambda}}{\lambda}) + \Lambda\big)U \overset{(!)}{=} -\rho U
\end{aligned}$$

This gives the differential equations:

(Einstein-p) $\quad 2\lambda\ddot{\lambda} - 3\dot{\lambda}^2 - \kappa\lambda^2 + \Lambda = 0,$

(Einstein-ρ) $\quad \rho(t) = 3\dot{\lambda}^2 + 3\kappa\lambda^2 - \Lambda = 2\kappa\lambda^2 + 2\lambda\ddot{\lambda}.$

Hence: $\quad \dot{\rho} = 6\dot{\lambda}(\kappa\lambda + \ddot{\lambda}) = 3\frac{\dot{\lambda}}{\lambda}(2\kappa\lambda^2 + 2\lambda\ddot{\lambda}) = 3\frac{\dot{\lambda}}{\lambda}\rho,$

and: $\quad \rho(t) = \rho(T)\cdot\lambda(t)^3,$ $\quad$ Abbreviate $T :=$ *today* henceforth.

As in the first description $\rho(t)$ scales expectedly with $\lambda(t)^3$ so that scaling sizes of space slices that intersect matter world lines at $\gamma(t)$ can equivalently be expressed in terms of matter densities, more precisely $\rho(t)^{1/3}$, along γ.
The just established fact that $\rho(t)\lambda(t)^{-3}$ is a constant translates into a first order ODE for λ (namely: $(3\dot{\lambda}^2 + 3\kappa\lambda^2 - \Lambda)\lambda^{-3} = \rho(T)$) that has the two Einstein equations we started with as consequences:

$$\begin{aligned}
0 = \frac{d}{dt}(\frac{1}{3}\rho(t)\lambda(t)^{-3}) &= \frac{d}{dt}\big(\dot{\lambda}^2\lambda^{-3} + \kappa\lambda^{-1} - \frac{\Lambda}{3}\lambda^{-3}\big)\\
&= 2\dot{\lambda}\ddot{\lambda}\lambda^{-3} - 3\dot{\lambda}^3\lambda^{-4} - \kappa\dot{\lambda}\lambda^{-2} + \Lambda\dot{\lambda}\lambda^{-4}\\
&= \dot{\lambda}\lambda^{-4}\big(2\lambda\ddot{\lambda} - 3\dot{\lambda}^2 - \kappa\lambda^2 + \Lambda\big) \quad = \quad 0.
\end{aligned}$$

So finally we have reached

The Equation of the Cosmological Model

$$\dot\lambda^2 = \frac{\rho(T)}{3}\cdot\lambda^3 - \kappa\lambda^2 + \frac{\Lambda}{3}, \quad \rho(t) = \rho(T)\cdot\lambda(t)^3,$$
$$\overline{g} = \frac{1}{\lambda^2}(g_\kappa - dt^2) = \left(\frac{\rho(T)}{\rho(t)}\right)^{2/3}\cdot(g_\kappa - dt^2).$$

For $\Lambda \neq 0$ this ODE for $\lambda(t)$ is the ODE of an elliptic function while for $\Lambda = 0$ an explicit integration in terms of elementary transcendental functions is possible. We therefore assume in the following $\Lambda = 0$ whenever reference to the explicit solution is made. We use abbreviations for $\sin(\sqrt{\kappa}\,t)/\sqrt{\kappa}$ and similar functions as follows:

$$\begin{array}{ll} \mathbf{s}_\kappa'' + \kappa\mathbf{s}_\kappa = 0, \quad \mathbf{s}_\kappa(0) = 0, \quad \mathbf{s}_\kappa'(0) = 1. & \text{Note: } (\mathbf{s}_\kappa')^2 + \kappa\mathbf{s}_\kappa^2 = 1 \\ \mathbf{c}_\kappa'' + \kappa\mathbf{c}_\kappa = 0, \quad \mathbf{c}_\kappa(0) = 1, \quad \mathbf{c}_\kappa'(0) = 0. & (\mathbf{c}_\kappa')^2 + \kappa\mathbf{c}_\kappa^2 = \kappa \\ \mathbf{c}_\kappa = \mathbf{s}_\kappa' & \end{array}$$

Summary. In the case $\Lambda = 0$ we have the following **explicit solution** of the model ODE:

$$\lambda(t) := \mathbf{s}_\kappa(\frac{T-t_0}{2})^2\cdot\mathbf{s}_\kappa(\frac{t-t_0}{2})^{-2} \qquad (\text{Recall } T = \textit{today}),$$
with
$$\rho(T) = 3\mathbf{s}_\kappa(\frac{T-t_0}{2})^{-2}, \quad \rho(t) = 3\mathbf{s}_\kappa(\frac{T-t_0}{2})^4\cdot\mathbf{s}_\kappa(\frac{t-t_0}{2})^{-6},$$
$$\bar g = \lambda^{-2}\cdot(g_\kappa - dt^2).$$

Here t_0 is the time where the mass density becomes infinite. There is no harm in setting $t_0 = 0$. To prove the claim compute $(\dot\lambda)^2/\lambda^2 + \kappa$ with the help of $(\mathbf{s}_\kappa')^2 + \kappa\mathbf{s}_\kappa^2 = 1$ and find it equal to $(\mathbf{s}_\kappa(T/2)^{-2}\cdot\lambda(t)$, hence $\rho(T)/3 = \mathbf{s}_\kappa(T/2)^{-2}$.

As a first observation we have a **Big Bang prediction.**

Red Shift prediction

For the physically unimportant product metric $g = g_\kappa - dt^2$ we have that the vector field U is a time like Killing field of constant length. Therefore we have no red shift between observers represented by the integral curves of U. Under the conformal change to the physically relevant metric $\bar{g} = \frac{1}{\lambda^2} g$ these integral curves become the world lines of the matter particles of that model. We have computed the red shift caused by a conformal change and found:

$$1 + z = \frac{\omega_{Source}}{\omega_{Observer}} = \frac{\lambda(t)}{\lambda(T)} = \frac{a(\tau = today)}{a(\tau(t))} = \frac{\mathbf{s}_\kappa(T/2)^2}{\mathbf{s}_\kappa(t/2)^2} = \left(\frac{\rho(t)}{\rho(T)}\right)^{1/3}.$$

This has an immediate interpretation: The red shift of light received from 'distant' galaxies tells us how much denser the universe was at time t of emission than at time $T = today$ of reception. Since t is unknown this is not a quantitative prediction.

We look at **another matter equation**, at a **photon gas,** $\rho = 3p$. If we insert this into the above eigenvalue computations for the Einstein tensor in the conformally flat description we get

(1)	$\frac{\rho(t)}{3} = p(t) = 2\lambda\ddot{\lambda} - 3\dot{\lambda}^2 - \kappa\lambda^2 + \Lambda,$
(2)	$\rho(t) = 3\dot{\lambda}^2 + 3\kappa\lambda^2 - \Lambda,$
(1) + (2)	$\frac{4}{3}\rho = 2\lambda\ddot{\lambda} + 2\kappa\lambda^2.$
$((1) - (2)/3)\dot{\lambda}/\lambda^5$	$0 = \frac{d}{dt}\left(\frac{\dot{\lambda}^2}{\lambda^4} + \frac{\kappa}{\lambda^2} - \frac{\Lambda}{3\lambda^4}\right) = \frac{d}{dt}\frac{\rho(t)}{3\lambda(t)^4},$
Differentiating (2)	$\dot{\rho} = 6\dot{\lambda}(\kappa\lambda + \ddot{\lambda}) = 4\frac{\dot{\lambda}}{\lambda}\rho.$
Finally:	$\rho(t) = \rho(T)\cdot\lambda(t)^{\mathbf{4}}.$

Again we end up with a first order ODE for the scaling function $\lambda(t)$, but a *different* power dependence, $\rho(t) \sim \lambda(t)^{\mathbf{4}}$, than for dust. Vice versa, this power law for ρ and the first order ODE for $\lambda(t)$ imply the two Einstein equations (1) and (2).

For **comparison with the literature** we need to discuss the model parameters. One parameter is the cosmological constant Λ, but I do not know how to discuss its connection with observations. Our Ansatz had todays space slice curvature κ as one model parameter, and the integration gave a second parameter, either the age T of the universe or equivalently the matter density today, $\rho(T)$. None of these parameters is used in the literature. The expanding universe discussion suggests why the *Hubble function*$(\tau) = a'(\tau)/a(\tau) = -\dot{\lambda}(t)$ was defined. Its value today is the *Hubble Constant* H. It is one of the most prominent astronomical constants and it is one of the usual model parameters. We have:

$$H^2 := \dot{\lambda}(T)^2 \overset{(ODE)}{=} -\kappa + \frac{\rho(T)+\Lambda}{3} \Big|_{\Lambda=0}$$
$$= \left(\frac{\dot{\mathbf{s}}_\kappa}{\mathbf{s}_\kappa}(T/2)\right)^2 = -\kappa + \frac{1}{\mathbf{s}_\kappa^2(T/2)}.$$

One can introduce H instead of any of the other parameters to specify the model in the family. The second model parameter in the physics literature also comes from sympathy for Taylor approximations. The parameter is called *acceleration parameter* q and defined via $a''|_{today}$:
(recall $a''/a = -\lambda\ddot{\lambda} + \dot{\lambda}^2$ and $2\lambda\ddot{\lambda} = 3\dot{\lambda}^2 + \kappa\lambda^2 - \Lambda = \rho(t) - 2\kappa\lambda(t)^2$)

$$q := -\frac{a''}{a}\Big|_{today} \cdot \frac{1}{H^2} = \frac{\ddot{\lambda}}{\lambda}\left(\frac{\lambda}{\dot{\lambda}}\right)^2 - 1 = \frac{1}{2\dot{\lambda}^2}(\dot{\lambda}^2 + \kappa\lambda^2 - \Lambda)$$
$$= \frac{1}{6H^2}(\rho(T) - 2\Lambda),$$
$$(2q-1)H^2 = \kappa - \Lambda.$$

With these equations one can choose, in terms of which parameters one wants the model to be specified. To me, H and $\rho(T)$ seem closest to direct observations.

Killing Fields

A map $A : M \to M$ is called a Pseudo-Riemannian isometry (often Lorentz isometry for short) if it satisfies for arbitrary tangent vectors Y, Z

$$g(TA \cdot Y, TA \cdot Z) = g(Y, Z).$$

Let A_t be a family of Lorentz isometries with $A_0 = \mathrm{id}$.

Definition of a **Killing field** $X(p) := \frac{\partial}{\partial t} A_t(p)|_{t=0}$.

The covariant differential DX of a Killing field is a skew-symmetric endomorphism field, i.e. $g(D_Y X, Y) = 0$. And vice versa, the flow of a vector field X with skew-symmetric DX consists of Lorentz isometries.

Choose a curve $p(s),\ p(0) = p,\ p'(0) = Y$. Then

$$\begin{aligned}
0 = \frac{d}{dt} g(TA_t \cdot Y, TA_t \cdot Y)|_{t=0} &= 2g(\frac{D}{dt}\big(\frac{d}{ds} A_t(p(s))|_{s=0}\big), TA_t \cdot Y)|_{t=0} \\
&= 2g(\frac{D}{ds}\big(\frac{d}{dt} A_t(p(s))|_{t=0}\big)|_{s=0},\ Y) \\
&= 2g(\frac{D}{ds}\big(X(p(s))\big)|_{s=0},\ Y) \\
0 = \qquad\qquad &= 2g(D_Y X(p), Y).
\end{aligned}$$

Conserved Quantities. If X is a Killing field and γ is a geodesic (for example a force free world line) then

$$g(X(\gamma(s)), \gamma'(s)) = const$$

Proof: $\frac{d}{ds} g(X(\gamma(s)), \gamma'(s)) = g(\frac{D}{ds} X(\gamma(s)), \gamma'(s)) + g(X(\gamma(s)), \frac{D}{ds}\gamma'(s)) = 0.$

We will have to discuss the question: why do we observe conserved quantities even though our cosmological Lorentz manifold has no Killing fields. The following will be needed.

Second Order PDE for Killing Fields.
The flow of a Killing field moves each geodesic through a family of geodesics, so that the restriction of a Killing field to any geodesic is a Jacobi field. That

means, for every tangent vector of a geodesic, γ', the Killing field X has to satisfy

$$D^2_{\gamma',\gamma'}X + R(X,\gamma')\gamma' = 0$$

Hence we have for all tangent vectors Y, Z

$$\begin{array}{ll}
 & D^2_{Y+Z,Y+Z}X + R(X,Y+Z)Y+Z = 0, \\
 & D^2_{Y,Y}X + R(X,Y)Y = 0, \quad D^2_{Z,Z}X + R(X,Z)Z = 0, \\
\text{by subtraction} & D^2_{Y,Z}X + D^2_{Z,Y}X + R(X,Y)Z + R(X,Z)Y = 0, \\
\text{by definition} & D^2_{Y,Z}X - D^2_{Z,Y}X = R(Y,Z)X, \\
\text{by addition} & 2D^2_{Y,Z}X + R(X,Y)Z = R(Z,X)Y + R(Y,Z)X, \\
\text{with 1.Bianchi} & \qquad D^2_{Y,Z}X + R(X,Y)Z = 0.
\end{array}$$

REMARK. If one tries to *construct* X by solving Jacobi equations along radial geodesics, then one can guarantee the correct second derivative of X only in the direction of those geodesics – while the second order PDE (derived above) requires much more.

Quotient Geometry on the Light Cone.

The induced metric on any light cone LC is degenerate: let $c(s)$ be a null geodesic on LC, then we have for **all** $v \in T_{c(s)}LC$ that $g(c'(s), v) = 0$. It is therefore useful to introduce the quotient geometry by defining

$$\text{equivalence classes:} \qquad [v] := v + \mathbb{R}c'(s).$$

We have a well defined positive definite scalar product on the quotient of $T_{c(s)}LC$:

$$\begin{aligned} g([v],[w]) &:= g(v + \lambda c', w + \mu c') \\ &= g(v,w) + \lambda g(c', w) + \mu g(v, c') + \lambda\mu g(c', c') = g(v,w). \end{aligned}$$

We have a covariant derivative on the quotient bundle along $c(s)$:
Let $[v](s) = [v(s) + \lambda(s)c'(s)]$ then we can define
(with $(\frac{D}{ds}v(s))^{tang}$ denoting the LC-tangential component of $\frac{D}{ds}v(s)$)

$$\frac{D}{ds}[v](s) := [(\frac{D}{ds}v(s))^{tang} + \lambda'(s)c'(s)] = [(\frac{D}{ds}v(s))^{tang}]$$

and hence have

$$\begin{aligned} \frac{d}{ds}g([v](s),[w](s)) = \frac{d}{ds}g(v(s),w(s)) &= g(\frac{D}{ds}v(s), w(s)) + g(v(s), \frac{D}{ds}w(s)) \\ &= g(\frac{D}{ds}[v](s), [w](s)) + g([v](s), \frac{D}{ds}[w](s)). \end{aligned}$$

Since $R(v + \lambda c', c')c' = R(v, c')c' \in T_{c(s)}TC$ we can define

$$[R]([v], c')c' := [R(v + \lambda c', c')c'],$$

so that $[R]$ is a symmetric operator on the 2-dimensional quotient space at $c(s)$. Finally, since a tangential Jacobi field $J(s)$ has a tangential covariant derivative, we have the twodimensional quotient Jacobi equation

$$\frac{D}{ds}(\frac{D}{ds}[J(s)] + [R]([J](s), c')c' = 0.$$

The two eigenvalues of $[R]$ will be important for measurement discussions.

homepage: www.math.uni-bonn.de/people/karcher

http://www.math.uni-bonn.de/people/karcher/TalksOnRelativity.html

http://www.math.uni-bonn.de/people/karcher/ShellPhiladelphia.pdf