

# Differential forms and characteristic classes

Matthias Lesch & Koen van den Dungen & Matteo Costantini

S2B3 – Hauptseminar Globale Analysis  
S4B3 – Graduate Seminar on Global Analysis  
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## Seminar outline

This seminar is a continuation of the lecture course Global Analysis I (F4B1-V3B3). It is aimed at both Bachelor and Master students with an interest in global analysis and/or differential geometry.

During this seminar, we will study differential forms and de Rham cohomology, with a particular focus on characteristic classes of vector bundles. The following topics will be discussed:

- The de Rham complex (Mayer-Vietoris sequence, Poincaré duality, etc).
- The de Rham theorem, yielding an isomorphism between de Rham cohomology and singular cohomology.
- The Euler class of a vector bundle.
- The Thom isomorphism, relating the de Rham cohomology of a vector bundle with the de Rham cohomology of the base space.
- Characteristic classes of vector bundles, in particular Chern classes (for complex vector bundles) and Pontryagin classes (for real vector bundles).
- Chern-Weil theory.

**Prerequisites:** Basic knowledge of smooth manifolds (incl. differential forms and exterior derivatives) and vector bundles will be assumed. Having completed the lecture course Global Analysis I is not strictly necessary but is strongly recommended.

## Schedule

The seminar is divided into two groups, and is scheduled as follows:

- Group 1: Wednesdays, 10:15–12:00, in Seminarraum N0.007 (Neubau);
- Group 2: Wednesdays, 12:15–14:00, in Seminarraum N0.007 (Neubau).

### Talk 1 – 22.4.2020: de Rham cohomology

Review of de Rham cohomology, the Poincaré Lemma, and the Mayer-Vietoris sequence. [Lee13, Ch.17], [BT82, §1-3], [Bre93, §V.2].

### Talk 2 – 29.4.2020: Good covers

Proofs of the finite-dimensionality of de Rham cohomology groups, Poincaré duality, and the Künneth Formula. [BT82, §5].

### Talk 3 – 6.5.2020: Thom isomorphism

Thom isomorphism, Poincaré Duality and the Thom class, the Global Angular Form and the Euler class for rank 2 vector bundles. [BT82, §6].

### Talk 4 – 13.5.2020: The general Mayer-Vietoris principle

Extending the Mayer-Vietoris sequence to countably many open subsets. [BT82, §8].

**Talk 5 – 20.5.2020: Sphere bundles**

Sphere bundles, Euler class, Global Angular Form, Euler characteristic. [BT82, §11].

**Talk 6 – [25-29].5.2020: Singular (co)homology**

Singular homology and singular cohomology with integer coefficients. [Bre93, §V.5-6] (see also [BT82, §15]).

**Talk 7 – 10.6.2020: de Rham Theorem**

[Lee13, Ch. 18], [BT82, §15], [Bre93, §V.9].

**Talk 8 – 17.6.2020: Chern classes**

[BT82, §20].

**Talk 9 – 24.6.2020: Splitting principle**

Splitting Principle, Whitney Product. [BT82, §21].

**Talk 10 – 1.7.2020: Pontryagin classes**

[BT82, §22].

**Talk 11 – 8.7.2020: Chern-Weil theory I**

Connections and curvature on vector bundles, construction of characteristic classes from invariant polynomials, Chern-Weil Theorem. [MS74, Appendix C, pp.289-299], [Zha01, §§1.2–1.5].

**Talk 12 – 15.7.2020: Chern-Weil theory II**

Recovering the Chern and Pontryagin classes from Chern-Weil theory. Generalised Gauss-Bonnet Theorem. [MS74, Appendix C, pp.299-312], [Zha01, §§1.6.1–1.6.2].

**Recommended literature**

- [Bre93] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer, 1993.
- [BT82] R. Bott and L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer New York, 1982.
- [Lee13] J. M. Lee, *Introduction to smooth manifolds*, 2nd ed., Springer, 2013.
- [MS74] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, No. 76, 1974.
- [Zha01] W. Zhang, *Lectures on Chern-Weil theory and Witten deformations*, Nankai Tracts in Mathematics, vol. 4, World Scientific, 2001.