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Gödel's Completeness Theorem[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article contains the proof of a simplified completeness theorem for a countable relational language without equality.

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The notation and terminology used in this paper are introduced in the following articles: [19], [13], [21], [2], [4], [11], [16], [1], [17], [10], [23], [14], [22], [24], [12], [15], [18], [20], [3], [8], [5], [9], [7], and [6].

1. Henkin's Theorem

For simplicity, we adopt the following convention: X, Y denote subsets of CQC-WFF, n denotes a natural number, p, q denote elements of CQC-WFF, x, y denote bound variables, A denotes a non empty set, J denotes an interpretation of A, v denotes an element of $\boldsymbol{V}(A)$, f_1 denotes a finite sequence of

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elements of CQC-WFF, C_1, C_2, C_3 denote consistent subsets of CQC-WFF, J_1 denotes a Henkin interpretation of C_1, and a denotes an element of A.

Let us consider X. We say that X is negation faithful if and only if:

(Def. 1) $X \vdash p$ or $X \vdash \neg p$.

Let us consider X. We say that X has examples if and only if:

(Def. 2) For all x, p there exists y such that $X \vdash \neg \exists_x p \vee p(x, y)$.

One can prove the following propositions:

(1) If C_1 is negation faithful, then $C_1 \vdash p$ iff $C_1 \nvdash \neg p$.

(2) For every finite sequence f of elements of CQC-WFF such that $\vdash f \frown \langle \neg p \vee q \rangle$ and $\vdash f \frown \langle p \rangle$ holds $\vdash f \frown \langle q \rangle$.

(3) If X has examples, then $X \vdash \exists_x p$ iff there exists y such that $X \vdash p(x, y)$.

(4) Suppose if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$. Suppose C_1 is negation faithful and has examples. Then $J_1, \mathrm{valH} \models \neg p$ if and only if $C_1 \vdash \neg p$.

(5) If $\vdash f_1 \frown \langle p \rangle$ and $\vdash f_1 \frown \langle q \rangle$, then $\vdash f_1 \frown \langle p \wedge q \rangle$.

(6) $X \vdash p$ and $X \vdash q$ iff $X \vdash p \wedge q$.

(7) Suppose that

(i) if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$, and

(ii) if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models q$ iff $C_1 \vdash q$.

Suppose C_1 is negation faithful and has examples. Then $J_1, \mathrm{valH} \models p \wedge q$ if and only if $C_1 \vdash p \wedge q$.

(8) Let given p. Suppose the number of quantifiers in $p \leq 0$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(9) $J, v \models \exists_x p$ iff there exists a such that $J, v(x{\restriction}a) \models p$.

(10) $J_1, \mathrm{valH} \models \exists_x p$ iff there exists y such that $J_1, \mathrm{valH} \models p(x, y)$.

(11) $J, v \models \neg \exists_x \neg p$ iff $J, v \models \forall_x p$.

(12) $X \vdash \neg \exists_x \neg p$ iff $X \vdash \forall_x p$.

(13) The number of quantifiers in $\exists_x p =$ (the number of quantifiers in $p) + 1$.

(14) The number of quantifiers in $p =$ the number of quantifiers in $p(x, y)$.

In the sequel a denotes a set.

The following three propositions are true:

(15) Let given p. Suppose the number of quantifiers in $p = 1$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(16) Let given n. Suppose that for every p such that the number of quantifiers in $p \leq n$ holds if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$. Let given p. Suppose the number of quantifiers in $p \leq n + 1$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(17) For every p such that C_1 is negation faithful and has examples holds $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

2. Satisfiability of Consistent Sets of Formulas with Finitely Many Free Variables

The following proposition is true

(18) WFF is countable.

The subset ExCl of CQC-WFF is defined by:

(Def. 3) $a \in \mathrm{ExCl}$ iff there exist x, p such that $a = \exists_x p$.

The following propositions are true:

(19) CQC-WFF is countable.

(20) ExCl is non empty and ExCl is countable.

Let p be an element of WFF. Let us assume that p is existential. The functor ExBound(p) yielding a bound variable is defined as follows:

(Def. 4) There exists an element q of WFF such that $p = \exists_{\mathrm{ExBound}(p)} q$.

Let p be an element of CQC-WFF. Let us assume that p is existential. The functor ExScope(p) yielding an element of CQC-WFF is defined by:

(Def. 5) There exists x such that $p = \exists_x \mathrm{ExScope}(p)$.

Let F be a function from $\mathbb{N}$ into CQC-WFF and let a be a natural number. The bound in $F(a)$ yields a bound variable and is defined as follows:

(Def. 6) If $p = F(a)$, then the bound in $F(a) = \mathrm{ExBound}(p)$.

Let F be a function from $\mathbb{N}$ into CQC-WFF and let a be a natural number. The scope of $F(a)$ yields an element of CQC-WFF and is defined by:

(Def. 7) If $p = F(a)$, then the scope of $F(a) = \mathrm{ExScope}(p)$.

Let us consider X. The functor snb(X) yields an element of 2^{BoundVar} and is defined by:

(Def. 8) $\mathrm{snb}(X) = \bigcup\{\mathrm{snb}(p) : p \in X\}$.

Next we state a number of propositions:

(21) If $p \in X$, then $X \vdash p$.

(22) $\mathrm{ExBound}(\exists_x p) = x$ and $\mathrm{ExScope}(\exists_x p) = p$.

(23) $X \vdash \mathrm{VERUM}$.

(24) $X \vdash \neg\mathrm{VERUM}$ iff X is inconsistent.

(25) For all finite sequences f, g of elements of CQC-WFF such that $0 < \mathrm{len}\, f$ and $\vdash f \frown \langle p \rangle$ holds $\vdash (\mathrm{Ant}(f)) \frown g \frown \langle \mathrm{Suc}(f) \rangle \frown \langle p \rangle$.

(26) $\mathrm{snb}(\{p\}) = \mathrm{snb}(p)$.

(27) $\mathrm{snb}(X \cup Y) = \mathrm{snb}(X) \cup \mathrm{snb}(Y)$.

(28) For every element A of 2^{BoundVar} such that A is finite there exists x such that $x \notin A$.

(29) If $X \subseteq Y$, then $\mathrm{snb}(X) \subseteq \mathrm{snb}(Y)$.

(30) For every finite sequence f of elements of CQC-WFF holds $\mathrm{snb}(\mathrm{rng}\, f) = \mathrm{snb}(f)$.

(31) If $\mathrm{snb}(C_1)$ is finite, then there exists C_2 such that $C_1 \subseteq C_2$ and C_2 has examples.

(32) If $X \vdash p$ and $X \subseteq Y$, then $Y \vdash p$.

(33) If C_1 has examples, then there exists C_2 such that $C_1 \subseteq C_2$ and C_2 is negation faithful and has examples.

In the sequel J_2 denotes a Henkin interpretation of C_3, J denotes an interpretation of A, and v denotes an element of $\boldsymbol{V}(A)$.

We now state the proposition

(34) If $\mathrm{snb}(C_1)$ is finite, then there exist C_3, J_2 such that $J_2, \mathrm{valH} \models C_1$.

3. Gödel's Completeness Theorem

We now state four propositions:

(35) If $J, v \models X$ and $Y \subseteq X$, then $J, v \models Y$.

(36) If $\mathrm{snb}(X)$ is finite, then $\mathrm{snb}(X \cup \{p\})$ is finite.

(37) If $X \models p$, then $J, v \not\models X \cup \{\neg p\}$.

(38) If $\mathrm{snb}(X)$ is finite and $X \models p$, then $X \vdash p$.

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