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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 7

Exercise 33. (Extension under flat ring homomorphisms, 3 points)
Let $f\colon A \to B$ be a flat ring homomorphism for which the induced map $\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Show that an A-module M is zero if and only if $M \otimes_A B = 0$. In fact, it suffices to assume that $\mathrm{MaxSpec}(A) \subset \mathrm{im}(\varphi)$. Give an example that shows that the assumption on the surjectivity of φ cannot be dropped.

Exercise 34. (Surjectivity of maps induced by flat ring homomorphisms, 5 points)
Let $f\colon A \to B$ be a flat ring homomorphism.

(i) Given N a B-module, define the B-module $N_B := B \otimes_A {}_AN$ (where ${}_AN$ denotes the restriction of scalars). Show that the homomorphism of A-modules $g : N \to N_B$ given by $n \mapsto 1 \otimes n$ is injective and that $\mathrm{im}(g) \subset N_B$ is a direct summand of N_B.
Remark: flatness of B is not needed.

(ii) Show that $\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective if and only if for all maximal ideals $\mathfrak{m} \subset A$ one has $\mathfrak{m}^e \neq (1)$.
Hint: For the 'if' direction, show that $\mathrm{MaxSpec}(A) \subset \mathrm{im}(\varphi)$, then use the previous exercise to show that the other prime ideals of A are contained in $\mathrm{im}(\varphi)$.

(iii) Let $\mathfrak{q} \subset B$ be a prime ideal and $\mathfrak{p} := \mathfrak{q}^c \subset A$. Show that the induced ring homomorphism $A_\mathfrak{p} \to B_\mathfrak{q}$ yields surjective map $\mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$.

Exercise 35. (Algebras of invariants, 2 points)
Let A be a Noetherian ring and B a finite type A-algebra. Suppose $G = \{g_i\}$ is a finite group of A-algebra homomorphisms $g_i\colon B \to B$. Show that $B^G := \{b \in B \mid \forall i\colon g_i(b) = b\}$ is a finite type A-algebra.

Exercise 36. (Localization of integral ring homomorphisms, 3 points)
Suppose $A \to B$ is integral. For a maximal ideal $\mathfrak{n} \subset B$ let $\mathfrak{m} := \mathfrak{n}^c \subset A$ (which is again maximal, as will be shown in class). Is then the induced ring homomorphism $A_\mathfrak{m} \to B_\mathfrak{n}$ always integral? *Hint*: Consider $k[x^2 - 1] \subset k[x]$ ($char(k) \neq 2$) and $\mathfrak{n} = (x - 1)$.

Exercise 37. (Noetherian topological spaces, 3 points)
A topological space X is *Noetherian* if every ascending chain of open subsets $U_1 \subset U_2 \subset \dots$ becomes stationary (i.e. $\bigcup U_i = U_n$ for $n \gg 0$) or, equivalently, if every descending chain of closed sets $V_1 \supset V_2 \supset \dots$ becomes stationary (i.e. $\bigcap V_i = V_n$ for $n \gg 0$).

(i) Show that $\mathrm{Spec}(A)$ of a Noetherian ring is a Noetherian topological space and find a counter-example for the converse.

(ii) Show that for a finite type A-algebra B the fibres $\varphi^{-1}(\mathfrak{p})$ of $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are Noetherian topological spaces.

Solutions to be handed in before Monday May 25, 4pm.