Prof. Dr. Daniel Huybrechts
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Exam: Commutative Algebra (V3A1, Algebra I)

The exam will be marked by Sunday August 2 and the grades entered into basis. To review your exam and the correction you have to write an email to this address einsicht@math.uni-bonn.de (with your student number). You will then be assigned a time slot during the week August 2-7 (or possibly the week after).

Exercise A. (Points: 3+2)
Assume A is a commutative ring such that for every element $a \in A$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$.

(i) Show that $\dim(A) = 0$.

(ii) Describe an explicit example of such a ring that is not a field.

Exercise B. (Points: 5)
Consider the ring $A := k[x,y]/(x(y+1), x(y+x^2))$, with $\mathrm{char}(k) \neq 2$. Describe all connected components of $\mathrm{Spec}(A)$, decide which ones consist of just one closed point and which ones have a non-empty intersection with $\mathrm{Spec}(A_{x+y})$.

Exercise C. (Points: 2+4)
Consider the ring $A = k[x,y,z]/(xyz, y^2)$.

(i) Show that the ideals $(\bar{x}) \subset A$ and $(\bar{z}) \subset A$ are both primary ideals and determine their radicals.

(ii) Determine a primary decomposition of the ideal $(0) \subset A$ and decide which associated prime ideals are isolated and which are embedded.

Exercise D. (Points: 4+4)
Consider $A = k[x,y,z]/(xy, xz)$ as a graded ring with $\deg(\bar{x}) = \deg(\bar{y}) = \deg(\bar{z}) = 1$.

(i) Compute the Poincaré series $P(A,t)$ and determine the dimension of A.[1]

(ii) Is $A_{(x,y,z)}$ regular or Cohen–Macaulay?

Exercise E. (Points: 4)
Consider the ring $A := k[x]$ and the A-module $M := \mathrm{coker}(\psi)$, where $\psi\colon A^{\oplus 2} \to A^{\oplus 2}$ is given by the matrix $\psi = \begin{pmatrix} x-1 & 1-x \\ 1-x & x-1 \end{pmatrix}$. Determine $\mathrm{Ass}(M)$ and $\mathrm{Supp}(M)$.

Exercise F. (Points: 2+2)
Describe explicitly Noether normalization for the k-algebras $k[x,y,z]/(xy)$ and $k[x,x^{-1}]$.

Exercise G. (Points: 3)
Let $\mathfrak{a} \subset A$ be an ideal and $f\colon M \to N$ an A-module homomorphism such that the induced $A/\mathfrak{a}$-module homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. Assume N is a finite A-module and show that there exists an $a \in \mathfrak{a}$ for which $M_b \to N_b$ is surjective, where $b = 1 + a$.

All rings are commutative with a unit and $1 \neq 0$.

[1] You will have to use that there are $\binom{2+n}{2}$ monomials of degree n in three variables.