Solutions for exercises, Algebra I (Commutative Algebra) – Week 12

Exercise 61. (Graded rings and modules, 3 points)

1. If the $(a_i)_{i\in I}$ generate A as a A_0-algebra, then they generate A_+ as ideal since any $a \in A_+ \subset A = A_0[(a_i)_{i\in I}]$ is a polynomial in some (finitely many) $a_{i_1}, \ldots, a_{i_n}$ with coefficient in A_0 and with 0 constant term (as $a \in A_+$ is a sum of homogeneous elements in graded pieces > 0).
   Conversely if $(a_i)_{i\in I}$ generate A_+ as an ideal, then elements in A_0 are certainly in $A_0[(a_i)_{i\in I}]$ (constant polynomials). If $a \in A_1 \subset A_+ = (a_i)_{i\in I}$ then $a = \sum_{k=1}^n \alpha_k a_{i_k}$ for some $a_{i_k} \in A_1$ and necessarily $\alpha_k \in A_0$ for grading reason. So $a \in A_0[(a_i)_{i\in I}]$ i.e. $A_1 \subset A_0[(a_i)_{i\in I}]$. So we can proceed by induction: let $n \geq 1$ be such that $A_k \subset A_0[(a_i)_{i\in I}]$ for any $k \leq n$; then for any $a \in A_{n+1} \subset A_+ = ((a_i)_{i\in I})$, we can write $a = \sum_{k=1}^n \alpha_k a_{i_k}$ for some $a_{i_k} \in A_+ = \oplus_{j\geq 1} A_j$ and necessarily $\alpha_k \in \oplus_{0\leq j\leq n-1} A_j$ for grading reason. By induction hypothesis the α_k's are in $A_0[(a_i)_{i\in I}]$. So $a \in A_0[(a_i)_{i\in I}]$ i.e. $A_{n+1} \subset A_0[(a_i)_{i\in I}]$. Thus by induction $A_n \subset A_0[(a_i)_{i\in I}]$ for any n and taking sums $A = A_0[(a_i)_{i\in I}]$.

2. Let $m_1, \ldots, m_n \in M$ be a set of generators of M as A-module, with $m_i \in M_{d_i}$. Let $a_1, \ldots, a_m \in A_+$ be a set of generators of A as A_0-algebra, with $a_i \in A_{r_i}$. By the first question we have $A = A_0[a_1, \ldots, a_m]$. Any element in M, a fortiori in M_k can be written $\sum_{\ell=1}^n b_\ell m_\ell$, with $b_\ell \in A$. For any $1 \leq i \leq n$, $bm_i \in M_k$ if and only if $b \in A_{k-d_i}$. But there are only finitely many monomials $a_1^{\alpha_1} \cdots a_m^{\alpha_m}$ of total degree $\sum r_j \alpha_j = k - d_i$. So M_k is generated over A_0 by the $a_1^{\alpha_1} \cdots a_m^{\alpha_m} m_i$, $i = 1, \ldots, n$ and $\sum r_j \alpha_j = k - d_i$; which then form a finite set of generators.

Exercise 62. (Homogeneous ideals, 2 points)

1. Let us denote $\mathfrak{a}_i = \mathfrak{a} \cap A_i$ for any $i \geq 0$; by assumption $\mathfrak{a} = \oplus_i \mathfrak{a}_i$ and assume that $\mathfrak{a}$ is proper i.e. $1 \notin \mathfrak{a}$ i.e. $\mathfrak{a}_0 \subsetneq A_0$. The group $\oplus_{i\geq 0} A_i/\mathfrak{a}_i$ is a A_0-algebra: $\overline{1} \in A_0/\mathfrak{a}_0$ is its unit since for any $a \in A_i$, $\overline{1}\overline{a} = (1 + \mathfrak{a}_0)(a + \mathfrak{a}_i) = a + \underbrace{a\mathfrak{a}_0}_{\in \mathfrak{a}\cap A_i = \mathfrak{a}_i} + \mathfrak{a}_i = \overline{a}$. For any $a \in A_i$, $b \in A_j$, $(a + \mathfrak{a}_i)(b + \mathfrak{a}_j) = ab + \underbrace{a\mathfrak{a}_j}_{\in \mathfrak{a}\cap A_{i+j} = \mathfrak{a}_{i+j}} + \underbrace{b\mathfrak{a}_i}_{\in \mathfrak{a}\cap A_{i+j} = \mathfrak{a}_{i+j}}$ so $\overline{a} \cdot \overline{b}$ is well-defined and in $A_{i+j}/\mathfrak{a}_{i+j}$. Associativity and distributivity follows from the rules of A. So $\oplus A_i/\mathfrak{a}_i$ is a ring; the A_0-algebra structure is given by $A_0 \twoheadrightarrow A_0/\mathfrak{a}_0$.
   Let us define $f : A \to \oplus_{i\geq 0} A_i/\mathfrak{a}_i$, by $\sum_{i=0}^n a_i \mapsto \sum_{i=0}^n \overline{a}_i$ where a_i are homogeneous. We have $f(1) = \overline{1}$. It is a ring homomorphism: it is sufficient to check it with homogeneous elements $a \in A_i$, $b \in A_j$, $c \in A_k$; $(a + \mathfrak{a}_i)(b + \mathfrak{a}_j + c + \mathfrak{a}_k) = a(b + c) + (\underbrace{a\mathfrak{a}_j + a\mathfrak{a}_k}_{\in \mathfrak{a}\cap A_{i+j} + \mathfrak{a}\cap A_{i+k}}) + \underbrace{\mathfrak{a}_i(b + c)}_{\in \mathfrak{a}\cap A_{i+j} + \mathfrak{a}\cap A_{i+k}} + \underbrace{\mathfrak{a}_i(\mathfrak{a}_j + \mathfrak{a}_k)}_{\in \mathfrak{a}\cap A_{i+j} + \mathfrak{a}\cap A_{i+k}}$ thus $\overline{a(b+c)} = \overline{a}(\overline{b} + \overline{c})$ i.e. f is a ring homomorphism. It is readily seen to be surjective.
   If $a = \sum_i a_i$, with $a_i \in A_i$ and $A_i \neq A_j$ for any $i \neq j$, is in $\ker(f)$ then $a_i \in \mathfrak{a}_i$, for any i i.e. $a \in \mathfrak{a}$. Conversely, if $a \in \mathfrak{a}$, write $a = \sum_i a_i$, with $a_i \in A_i$ and $A_i \neq A_j$ for any $i \neq j$; as $\mathfrak{a}$ is homogeneous, $a_i \in \mathfrak{a}_i$ for any i so that $f(a) = 0$ i.e. $\ker(f) = \mathfrak{a}$. So $A/\mathfrak{a} \simeq \oplus_i A_i/\mathfrak{a}_i$.

Solutions to be handed in before Monday July 6, 4pm.

2. Let $x \in \sqrt{\mathfrak{a}}$ and write $x = \sum_{i=1}^n x_i$ with $x_i \in A_{k_i}$ homogeneous and $k_1 < \cdots < k_n$. We want to show that $x_i \in \sqrt{\mathfrak{a}}$, $\forall i$. We have $x^N \in \mathfrak{a}$ for some $N > 0$; we can write $x = x_n^N + y$ where $x_n^N \in A_{Nk_n}$ is the term of highest degree and $y \in \oplus_{i<Nk_n} A_i$. Since $\mathfrak{a}$ is homogeneous, $x_n^N \in \mathfrak{a}$ i.e. $x_n \in \sqrt{\mathfrak{a}}$. So $x - x_n = \sum_{i=1}^{N-1} x_i \in \sqrt{\mathfrak{a}}$ (as $\sqrt{\mathfrak{a}}$ is an ideal, in particular a group). So by induction, $x_i \in \mathfrak{a}$, $\forall i$.

Exercise 63. (Proj, 5 points)

1. If every element of A_+ is nilpotent, then $A_+ \subset \cap_{\mathfrak{p}\in \mathrm{Spec}(A)} \mathfrak{p}$; in particular for any homogeneous prime $\mathfrak{p}$, we have $\mathfrak{p} \supset A_+$ i.e. $\mathrm{Proj}(A) = \emptyset$.
Conversely if $\mathrm{Proj}(A) = \emptyset$, then any homogeneous prime ideal contains A_+. If $A_+ \not\subset \mathfrak{N}$, take $a \in A_+ \backslash \mathfrak{N}$; then one of the homogeneous components of a, say a_{i_0}, is not in $\mathfrak{N}$. We have $D_+(a_{i_0}) \subset \mathrm{Proj}(A) = \emptyset$ and since $D_+(a_{i_0}) \simeq \mathrm{Spec}(A_{(a_{i_0})})$ we get $A_{(a_{i_0})} = 0$. So in $A_{(a_{i_0})} \subset A_{a_{i_0}}$, $1 = 0$ i.e. $a_{i_0}^k = 0$ in A for some $k \geq 0$; i.e. $a_{i_0} \in \mathfrak{N}$; contradiction. Thus $A_+ \subset \mathfrak{N}$.

2. For $k[x] = \oplus_{i\geq 0} k \cdot x^i$, we have $k[x]_+ = (x)$. We know that $\mathrm{Spec}(k[x]) = \{(0)\} \cup \{(f),\ f \in k[x] \text{ irreducible}\}$. Let $f = \sum_i^d a_i x^i \in k[x]$ be an irreducible polynomial ($d = \deg(f)$). If (f) is an homogeneous ideal, then since $f \in (f)$, for any i, $a_i x^i \in (f)$, in particular $a_d x^d \in (f)$. Since $a_d \neq 0$ is a unit, $x^d \in (f)$ and since (f) is a prime ideal $x \in (f)$; but then $a_0 = f - x(\sum_{i\geq 1} a_i x^{i-1}) \in (f)$ which, as (f) is a proper ideal, means $a_0 = 0$ i.e. $x|f$. Since f is irreducible, we must have $f = x$ (up to scaling); thus the only homogeneous prime ideals in $k[x]$ are (0) and $(x) = k[x]_+$. So $\mathrm{Proj}(k[x]) = \{(0)\}$.

3. Let $\mathfrak{p} \in \mathbb{P}_k^n = \mathrm{Proj}(k[x_0, \ldots, x_n])$ be a closed point; then $k[x_0, \ldots, x_n]_+ \subsetneq \mathfrak{p}$ i.e. there is a $f \in k[x_0, \ldots, x_n]_+$ such that $f \notin \mathfrak{p}$. Since $k[x_0, \ldots, x_n]_+$ is generated by $(x_0, \ldots, x_n)$ there is a i such that $x_i \notin \mathfrak{p}$ i.e. $\mathfrak{p} \in D_+(x_i) \simeq \mathrm{Spec}(k[\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}])$ and it is a closed point; thus a maximal ideal of $k[\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}]$ and since k is algebraically closed there is a n-uple $(a_0, \ldots, \widehat{a_i}, \ldots, a_n) \in k^n$ such that $\mathfrak{p} = (\frac{x_0}{x_i} - a_0, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i} - a_n) \subset D_+(x_i)$. But the contraction of $(\frac{x_0}{x_i} - a_0, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i} - a_n)$ by $k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n]_{(x_i)}$ is $(x_0 - a_0 x_i, \ldots, x_{i-1} - a_{i-1} x_i, x_{i+1} - a_{i+1} x_i, \ldots, x_n - a_n x_i)$. So associated to $(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) \neq (0, \ldots, 0)$ under the map of the exercise; showing that it is surjective.

   Assume that for $(a_0, \ldots, a_n) \in k^{n+1} \backslash \{(0, \ldots, 0)\}$ and $(b_0, \ldots, b_n) \in k^{n+1} \backslash \{(0, \ldots, 0)\}$, we have $(a_i x_j - a_j x_i)_{i,j} = (b_i x_j - b_j x_i)_{i,j}$. Then for $i_0 \neq j_0$, $a_{i_0} x_{j_0} - a_{j_0} x_{i_0} \in (b_i x_j - b_j x_i)_{i,j}$. For degree reasons, $a_{i_0} x_{j_0} - a_{j_0} x_{i_0} = \sum_i \lambda_{i,j}(b_i x_j - b_j x_i)$ for some $\lambda_{i,j} \in k$. Evaluating at $x_i = 0$ $i \neq i_0, j_0$, we get $a_{i_0} x_{j_0} - a_{j_0} x_{i_0} = \lambda_{i_0,j_0}(b_{i_0} x_{j_0} - b_{j_0} x_{i_0})$ thus $a_{i_0} = \lambda_{i_0,j_0} b_{i_0}$ and $a_{j_0} = \lambda_{i_0,j_0} b_{j_0}$. It is so for any pair (i_0, j_0).
   Since $(b_0, \ldots, b_n) \in k^{n+1} \backslash \{(0, \ldots, 0)\}$ there is a $b_i \neq 0$. For simplicity, we can assume $b_0 \neq 0$. Then for any $i, j > 0$, looking at $a_0 x_i - a_i x_0$ and $a_0 x_j - a_j x_0$ we have $\lambda_{i,0} = \frac{a_0}{b_0} = \lambda_{j,0}$ thus $a_i = \frac{a_0}{b_0} b_i$ (and $a_j = \frac{a_0}{b_0} b_j$) for any i. If $a_0 = 0$ then we get $(a_0, \ldots, a_n) = 0$ so $\frac{a_0}{b_0} \neq 0$ and $(a_0, \ldots, a_n) = \frac{a_0}{b_0}(b_0, \ldots, b_n)$.

4. The map $\varphi : \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$ is given by $\mathfrak{p} \mapsto \mathfrak{p} \cap A_0$ (induced by the ring homomorphism $A_0 \to A$). Let $a \in A_0$, it is homogeneous and $\varphi^{-1}(D(a)) = \{\mathfrak{p} \in \mathrm{Proj}(A),\ a \notin \mathfrak{p} \cap A_0\} = D_+(a)$ so φ is continuous.

Exercise 64. (Numerical polynomials, 4 points)

1. Since $\deg(\binom{T}{r}) = r$ (with $\binom{T}{0} = 1$) the family $(\binom{T}{r})_{r\geq 0}$ is a basis of $\mathbb{Q}[T]$. So any

$P \in \mathbb{Q}[T]$ can be written $\sum_i c_i \binom{T}{i}$ with $c_i \in \mathbb{Q}$. We have the identity

$$\begin{aligned}\binom{T+1}{r} - \binom{T}{r} &= \frac{\prod_{k=0}^{r_1}(T+1-k)}{r!} - \frac{\prod_{k=0}^{r-1}(T-k)}{r!} \\ &= \frac{\prod_{k=-1}^{r_2}(T-k)}{r!} - \frac{\prod_{k=0}^{r-1}(T-k)}{r!} \\ &= \frac{\prod_{k=0}^{r_2}(T-k)}{r!}(T+1-(T-(r-1))) \\ &= \binom{T}{r-1}.\end{aligned}$$

If the numerical polynomial P has degree 0, since $P(n) \in \mathbb{Z}$ for $n >> 0$, this constant term is an integer. So let $d \geq 0$ be an integer such that all numerical polynomials of degree $\leq d$ are of the desired form. Now, let $P \in \mathbb{Q}[T]$ be a numerical polynomial of degree $d+1$. Since $(\binom{T}{r})_{r \geq 0}$ is a basis of $\mathbb{Q}[T]$, we can write $P = \sum_{i=0}^{d+1} c_{d+1-i}\binom{T}{i}$ with $c_i \in \mathbb{Q}$. Now look at $Q(T) = P(t+1) - P(T) \in \mathbb{Q}[T]$. It is a numerical polynomial (for $n >> 0$, $P(n+1), P(n) \in \mathbb{Z}$) and $Q(T) = \sum_{i=1}^{d+1} c_{d+1-i}(\binom{T+1}{i} - \binom{T}{i}) = \sum_{i=1}^{d+1} c_{d+1-i}\binom{T}{i-1}$ so Q has degree d. So by induction hypothesis $c_i \in \mathbb{Z}$ for any $i \leq d$. Now take $n >> 0$ of the form $n = (d+1)!k$ (i.e. $k >> 0$) then $P(n) = c_{d+1} + \sum_i i = 1^{d+1} c_{d+1-i}\frac{(d+1)!k((d+1)!k-1)\cdots((d+1)!k-i+1)}{i!}$ where we see that $\frac{(d+1)!k((d+1)!k-1)\cdots((d+1)!k-i+1)}{i!} \in ZZ$ since $i \leq d+1$. So $c_{d+1} = P(n) - \sum_{i=1}^{d+1} c_{d+1-i}\frac{(d+1)!k((d+1)!k-1)\cdots((d+1)!k-i+1)}{i!} \in \mathbb{Z}$ for $k >> 0$ i.e. $c_{d+1} \in \mathbb{Z}$; concluding the induction step.

2. Let us write $Q(T) = \sum_{i=0}^{d} c_{d-i}\binom{T}{i}$ with $c_i \in \mathbb{Z}$ by the previous question. Set $P = \sum_{i=0}^{d} c_{d-i}\binom{T}{i+1} \in \mathbb{Q}(T)$. It is a numerical polynomial. A direct calculation shows that $P(T+1) - P(T) = Q(T)$ and $\deg(P) = \deg(Q)$. So $\Delta f(n) = Q(n) = \Delta P(n)$ for $n >> 0$. Let $n_0 \in \mathbb{N}$ such that $forall n \geq n_0$, $\Delta(f)(n) = \Delta(P)(n)$ i.e. $(f-P)(n+1) = (f-P)(n)$ so $\forall n \geq n_0$, $\mathbb{Z} \ni (f-P)(n) = (f-P)(n_0)$. Since P is a numerical polynomial $P' = P + (f - P(n_0)) \in Q[T]$ is also a numerical polynomial and $f(n) = P'(n)$ for $n >> 0$.

Exercise 65. (Grothendieck group, 5 points)

1. Notice first that for any additive function $\lambda : \mathcal{C} \to \mathbb{Z}$, $\lambda(0) = \lambda(0) + \lambda(0)$ since the sequence $0 \to 0 \to 0 \to 0 \to 0$ is exact so $\lambda(0) = 0$.
   Notice also that if $M \simeq N$, $\lambda(M) = \lambda(N)$ and $[M] = [N] \in K(\mathcal{C})$ since then the isomorphism sits in the exact sequence $0 \to M \to N \to 0 \to 0$ (and we have seen $\lambda(0) = 0$).

   If $\overline{\lambda} : K(\mathcal{C}) \to \mathbb{Z}$ is a group homomorphism. Define $\lambda : \mathcal{C} \to \mathbb{Z}$, $C \mapsto \overline{\lambda}([C])$. Since for any short exact sequence $0 \to M' \to M \to M'' \to 0$, $[M] - [M'] - [M''] = 0 \in K(\mathcal{C})$, we get additivity of λ.

   Conversely, given an additive function $\lambda : \mathcal{C} \to \mathbb{Z}$. We can naturally extend by additivity λ to a group homomorphism from the free abelian group $\lambda' : \oplus_{M \in \mathrm{Obj}(\mathcal{C})}\mathbb{Z} \cdot M \to \mathbb{Z}$, $nM \mapsto n\lambda(M)$. Then as λ is additive, $M - M' - M'' \in \ker(\lambda')$ for any M, M', M'' appearing in an exact sequence $0 \to M' \to M \to M'' \to 0$. So the subgroup K generated by such sums is contained in $\ker(\lambda')$. So there is an induced group homomorphism $\overline{\lambda} : K(\mathcal{C}) \simeq \oplus_{M \in \mathrm{Obj}(\mathcal{C})}\mathbb{Z} \cdot M/K \to \mathbb{Z}$. Moreover it is easy to see that the additive function associated to $\overline{\lambda}$ is λ.

2. Define the group homomorphism $\varphi : \mathbb{Z} \to K(\mathrm{Vec}_{fd}(k))$, $1 \mapsto [k]$. Notice that for $n > 0$, by induction and decomposing $M^{\oplus n}$ into short exact sequence, $[M^{\oplus n}] = n[M]$ in $K(\mathcal{C})$. Like wise $[M \oplus N] = [M] + [N]$ in $K(\mathcal{C})$.
   Notice that for any $M \in Vec_{fd}(k)$, $M \simeq k^{\oplus d}$ where $d = \dim_k(M)$; thus $[M] \simeq [k^d] = d[k]$ in $K(\mathcal{C})$. So φ is surjective.

We can define a group homomorphism $\phi : \oplus_{M\in \mathrm{Obj}(\mathrm{Vec}_{\mathrm{fd}}(k))}\mathbb{Z}\cdot M \to \mathbb{Z}$, by (extend linearly) $M \mapsto dim_k(M)$. Then the subgroup K generated by the $M' - M + M''$ for M, M', M'' appearing in an exact sequence $0 \to M' \to M \to M'' \to 0$ is contained in the kernel of ϕ. So there is an induced group homomorphism $K(\mathcal{C}) \to \mathbb{Z}$. We have $\phi \circ \varphi = \mathrm{id}_{\mathbb{Z}}$ so φ is injective.

3. $\mathcal{C} = \mathrm{mod}(A_0)$. The proof goes exactly as in the lecture notes; the only difference is the use of $\mathcal{C} \to K(\mathcal{C})$ instead of $\mathcal{C} \to \mathbb{Z}$.

   The exact sequence $0 \to K_n \to M_n \overset{a_N\cdot}{\to} M_{n+d} \to C_{n+d} \to 0$ can be broken in two exact sequences : $0 \to K_n \to M_n \overset{a_N\cdot}{\to} \mathrm{im}(a_N\cdot) \to 0$ and $0 \to \mathrm{im}(a_N\cdot) \to M_{n+d} \to C_{n+d} \to 0$. So $[M_n] - [K_n] = [\mathrm{im}(a_N\cdot)] = [M_{n+d}] - [C_{n+d}]$ in $K(\mathcal{C})$ which gives $[M_n] - [M_{n+d}] = [K_n] - [C_{n+d}]$ in $K(\mathcal{C})$ (as with the additive function in the lecture notes).