Solutions for Exercise sheet 2, Algebra I (Commutative Algebra) – Week 2

The first two exercise sheets will only use material you should be familiar with already. Some of it is covered and recalled by the first three lectures. These two sheets are not compulsory but the points can be counted towards your final score of the necessary 50% to get admitted to the exams.

Exercise 5. (Factor rings of polynomial rings)

1. For $a \in k$, let us consider the evaluation map $ev_a : k[x] \to k$, $P \mapsto P(a)$. We have seen in the previous set of exercises that ev_a is a ring homomorphism (as composition of $k[x] \to \text{Maps}(k,k) \to k$) but we can recall in few words how to show that: we have $ev_a(1) = 1$ and since $((x-a)^i)_{i\in\mathbb{N}}$ is a basis of $k[x]$, for $P, Q, R \in k[x]$, writing them as $P = \sum_i p_i(x-a)^i$, $Q = \sum_i q_i(x-a)^i$ and $R = \sum_i r_i(x-a)^i$, we get

$$\begin{array}{rcl} ev_a(P(Q+R)) & = & ev_a(\sum_{ij\in\mathbb{N}} p_j(q_i+r_i)(x-a)^{i+j}) \\ & = & p_0(q_0+r_0) \\ & = & ev_a(P)(ev_a(Q)+ev_a(R)) \end{array}$$

   So ev_a is a ring homomorphism. It is surjective: for $b \in k$ we have $ev_a((x-a)+b) = b$. Let us analyse its kernel: for $P \in ker(ev_a)$, we have $P(a) = 0$ hence (Euclidean division) $x-a|P$ i.e. $P \in (x-a)$ (where $(x-a)$ designates the principal ideal generated by $x-a$) thus $ker(ev_a) \subset (x-a)$. Moreover, it is easy (ev_a is a ring homomorphism) to see that $(x-a) \subset ker(ev_a)$. Therefore $ker(ev_a) = (x-a)$. In particular there exists an induced isomorphism of rings $\overline{ev_a} : k[x]/ker(ev_a) \xrightarrow{\sim} k$.

2. Let us show that there is a isomorphism of k-vector spaces $k[x]/(f) \simeq k[x]_{\leq d-1}$, where $k[x]_{\leq d-1}$ designates the k-vector space of polynomials of degree at most $d-1$. We define a map $R_f : k[x] \to k[x]_{\leq d-1}$ by Euclidean division by f which ensures that for any $P \in k[x]$ there are a unique $q \in k[x]$ and a unique $r \in k[x]$ with $\deg(r) < \deg(f) = d$ such that $P = qf + r$. The uniqueness of r allows us to define a map R_f as claimed by $P \mapsto r$.
   It is a group homomorphism: for $P_1, P_2 \in k[x]$, consider the data given by Euclidean division by f namely $q_1, q_2 \in k[x]$ and $r_1, r_2 \in k[x]$ with $\deg(r_1) < d$ and $\deg(r_2) < d$, such that $P_i = q_i f + r_i$, $i = 1, 2$; we have $P_1 + P_2 = (q_1+q_2)f + (r_1+r_2)$. But since $\deg(r_1+r_2) \leq \max(\deg(r_1), \deg(r_2)) < d$, by uniqueness of the outputs of Euclidean division, $r_1 + r_2 = R_f(P_1) + R_f(P_2)$ is the remainder of the division of $P_1 + P_2$ by f i.e. $R_f(P_1+P_2) = r_1 + r_2 = R_f(P_1) + R_f(P_2)$. It is clear that R_f commute with multiplication by scalars by uniqueness of the outputs of Euclidean division. So R_f is a linear map.
   The linear map R_f is surjective: for any $r \in k[x]_{\leq d-1}$, we have $r = 0 \cdot f + r$ and $\deg(r) < d = \deg(f)$ so that by uniqueness of the outputs of in Euclidean division $R_f(r) = r$.
   Let us analyse its kernel: for $P \in k[x]$ such that $R_f(P) = 0$, Euclidean division writes $P = qf$ i.e. $P \in (f)$. Conversely, if $P \in (f)$ i.e. $P = Qf$ for some $Q \in k[x]$, by uniqueness of the outputs of in Euclidean division, $R_f(P) = 0$. So $ker(R_f) = (f)$. Therefore there is an induced isomorphism of k-vector spaces $\overline{R_f} : k[x]/ker(R_f) \xrightarrow{\sim} k[x]_{\leq d-1}$. To conclude it is sufficient to notice that $\dim_k(k[x]_{\leq d-1}) = d$ (a basis being $(1, x, \cdots, x^{d-1})$).

Solutions to be handed in before Monday April 20, 4pm.

3. Let us begin by proving that φ_a is a ring homomorphism (in an inelegant way). For a n-uple $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, we denote $\underline{x}^{\underline{d}}$ the monomial $x_1^{d_1} \cdots x_n^{d_n}$ and we denote $a^{\underline{d}}$ the element $a_1^{d_1} \cdots a_n^{d_n} \in k$. We have $\varphi_a(1) = 1$ and for $P = \sum_{\underline{d} \in \mathbb{N}^n} p_{\underline{d}} \underline{x}^{\underline{d}} \in k[x_1, \ldots, x_n]$ and $Q = \sum_{\underline{d} \in \mathbb{N}^n} q_{\underline{d}} \underline{x}^{\underline{d}}$, we have

$$\begin{array}{rcl} \varphi_a(P+Q) & = & \sum_{\underline{d}} (p_{\underline{d}} + q_{\underline{d}}) a^{\underline{d}} \\ & = & \sum_{\underline{d}} p_{\underline{d}} a^{\underline{d}} + q_{\underline{d}} a^{\underline{d}} \\ & = & \varphi_a(P) + \varphi_a(Q) \end{array}$$

and

$$\begin{array}{rcl} \varphi_a(PQ) & = & \sum_{\underline{d},\ \underline{d}'} p_{\underline{d}} q_{\underline{d}'} a^{\underline{d}+\underline{d}'} \\ & = & \sum_{\underline{d},\ \underline{d}'} p_{\underline{d}} a^{\underline{d}} q_{\underline{d}'} a^{\underline{d}'} \\ & = & (\sum_{\underline{d}} p_{\underline{d}} a^{\underline{d}}) \cdot (\sum_{\underline{d}'} q_{\underline{d}'} a^{\underline{d}'}) \\ & = & \varphi_a(P) \cdot \varphi_a(Q) \end{array}$$

So φ_a is a ring homomorphism. It is easily seen to be surjective: for $b \in k$, we can consider the constant polynomial $b \in k[x_1, \ldots, x_n]$, for which we have $\varphi_a(b) = b$. Observe that for any $i \in \{1, \ldots, n\}$ and any $P \in k[x_1, \ldots, x_n]$, we have $\varphi_a((x_i - a_i)P) = \varphi_a(x_i - a_i)\varphi_a(P) = 0\varphi_a(P) = 0$. So that the ideal

$$(x_1 - a_1, \ldots, x_n - a_n) = \{P \in k[x_1, \ldots, x_n],\ \exists (q_1, \ldots, q_n) \in k[x_1, \ldots, x_n]^n,\ P = \sum_i (x_i - a_i) q_i\}$$

generated by the $x_i - a_i$'s is contained in the kernel of φ_a.
To conclude, we can either prove that $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal - which will yield, since $\varphi_a \neq 0$, $ker(\varphi_a) = (x_1 - a_1, \ldots, x_n - a_n)$ - or directly prove the reverse inclusion (or any other solution that works).
For the first solution, notice that for $a = (0, \ldots, 0)$, it is obvious that $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \simeq k$ so that $(x_1, \ldots, x_n)$ is a maximal ideal of $k[x_1, \ldots, x_n]$. Now given a $b = (b_1, \ldots, b_n) \in k^n$ define the linear map $t_b : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ defined by (extend linearly) $\underline{x}^{\underline{d}} \mapsto (x_1 + b_1)^{d_1} \cdots (x_n + b_n)^{d_n}$. Then by the same kind of calculation as above, t_b is a ring homomorphism. Moreover $t_b \circ t_{-b} = id_{k[x_1, \ldots, x_n]}$ so that t_b is an isomorphism of rings for any $b \in k^n$. We have $(x_1 - a_1, \ldots, x_n - a_n) = t_a^{-1}((x_1, \ldots, x_n))$ so $(x_1 - a_1, \ldots, x_n - a_n)$ is a prime ideal. Moreover, as t_a is an isomorphism, it is immediate to deduce maximality of $(x_1 - a_1, \ldots, x_n - a_n)$ from maximality of $(x_1, \ldots, x_n)$.

To prove the reverse inclusion directly, instead: take $P \in ker(\varphi_a)$ and write the Euclidean division of P by $(x_n - a_n)$ in the polynomial ring $k(x_1, \ldots, x_{n-1})[x_n]$ (where $k(x_1, \ldots, x_{n-1})$ is the field of fractions of the integral ring $k[x_1, \ldots, x_{n-1}]$): $P = (x_n - a_n)\frac{A_1}{B_1} + \frac{P_1}{R_1}$ where $A_1 \in k[x_1, \ldots, x_{n-1}][x_n] \simeq k[x_1, \ldots, x_n]$ and $B_1, P_1, R_1 \in k[x_1, \ldots, x_{n-1}]$ with B_1, R_1 monic and the fractions are irreducible. We have

$$P(x_1, \ldots, x_{n-1}, a_n) = \frac{P_1}{R_1};$$

the left hand side being a polynomial, we get $R_1 = 1$ (the fraction is irreducible). So rewriting the result of the Euclidean division as:

$$P - P_1 = (x_n - a_n)\frac{A_1}{B_1}$$

we see that $B_1 = 1$ (left hand side polynomial). So the Euclidean division gives in fact an equality of polynomials $P = (x_n - a_n)A_1 + P_1$ with $P_1 \in k[x_1, \ldots, x_{n-1}]$. Moreover we have

$$0 = P(a_1, \ldots, a_n) = (a_n - a_n)A_1(a_1, \ldots, a_{n-1}) + P_1(a_1, \ldots, a_{n-1}) = P_1(a_1, \ldots, a_{n-1}).$$

So let $i \geq 1$ such that P can be written $P = \sum_{j=0}^{i-1}(x_{n-j} - a_{n-j})A_{j+1} + P_i$ with $A_j \in k[x_1, \ldots, x_n]$ and $P_i \in k[x_1, \ldots, x_{n-i}]$ such that $P_i(a_1, \ldots, a_{n-i}) = 0$. Write the Euclidean division of P_i by $(x_{n-i} - a_{n-i})$ in $k(x_1, \ldots, x_{n-i-1})[x_{n-1}]$:

$$P_i = (x_{n-i} - a_{n-i})\frac{A_{i+1}}{B_i} + \frac{P_{i+1}}{R_i}$$

with $A_{i+1} \in k[x_1, \ldots, x_{n-i-1}][x_{n-i}]$ and $B_i, P_{i+1}, R_i \in k[x_1, \ldots, x_{n-i-1}]$ with B_i, R_i monic and the fractions irreducible. We have $P_i(x_1, \ldots, x_{n-i-1}, a_{n-1}) = \frac{P_{i+1}}{R_i}$ so that $R_i = 1$ and again rewriting the Euclidean division, we get $B_i = 1$ i.e. $P_i = (x_{n-i} - a_{n-i})A_{i+1} + P_{i+1}$ in $k[x_1, \ldots, x_{n-i}]$, with $P_{i+1} \in k[x_1, \ldots, x_{n-i-1}]$ and $P_{i+1}(a_1, \ldots, a_{n-i-1}) = 0$ (as we can see evaluating the equality at $(a_1, \ldots, a_{n-i})$). Thus, by induction we have proved that we can write $P \in ker(\varphi_a)$ as $\sum_i (x_i - a_i)A_i$ with $A_i \in k[x_1, \ldots, x_n]$ i.e. that $ker(\varphi_a) = (x_1 - a_1, \ldots, x_n - a_n)$.
In particular, φ_a induces a isomorphism of rings $\overline{\varphi_a} : k[x_1, \ldots, x_n]/ker(\varphi_a) \xrightarrow{\sim} k$. As k is a field, we get that $ker(\varphi_a) = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal (indeed if $\alpha \in k[x_1, \ldots, x_n] \backslash ker(\varphi_a)$ then $\overline{\varphi_a}(\alpha) \neq 0$ in k so there is a $\beta \in k[x_1, \ldots, x_n]$ such that $\alpha\beta = 1 \bmod (x_1 - a_1, \ldots, x_n - a_n)$ i.e. $1 \in (x_1 - a_1, \ldots, x_n - a_n, \alpha)$).

Exercise 6. (Quotient modules)

1. Let us define $\varphi : M/M_1 \to M/M_2$ by $m \bmod M_1 \mapsto m \bmod M_2$. It is a well-defined map since for $m \in M$ and $m_1 \in M_1$, $\varphi(m + m_1 \bmod M_1) = m + m_1 \text{mod } M_2 = m \bmod M_2 = \varphi(m)$ (as $M_1 \subset M_2$). It is a homomorphism of A-modules since for $m, n \in M$, and $a \in A$, we have

$$\begin{array}{rcl} \varphi(a(m-n) \bmod M_1) & = & a(m-n) \bmod M_2 \\ & = & am - an \bmod M_2 \\ & = & a\varphi(m \bmod M_1) - a\varphi(n \bmod M_1) \bmod M_2 \end{array}.$$

   Moreover, φ is surjective: for $m \bmod M_2 \in M/M_2$ with $m \in M$ a representative, we have $\varphi(m \bmod M_1) = m \bmod M_2$. Let us analyse its kernel: if $m \in M$ is such that $m \bmod M_1 \in ker(\varphi)$, then $m = 0 \bmod M_2$ i.e. $m \in M_2$. So $ker(\varphi) \subset M_2/M_1$. Conversely for $\overline{m} \in M_2/M_1$, we can take a representative $m \in M_2$ then we have $\varphi(\overline{m}) = \varphi(m \bmod M_1) = m \bmod M_2 = 0 \bmod M_2$. So $ker(\varphi) = M_2/M_1$. So φ induces an isomorphism of A-modules $\overline{\varphi} : (M/M_1)/(M_2/M_1) \xrightarrow{\sim} M/M_2$.

2. Let us define $\varphi : M_2 \to (M_1 + M_2)/M_1$ to be the composition of the inclusion $M_2 \hookrightarrow M_1 + M_2$ and the quotient $M_1 + M_2 \to (M_1 + M_2)/M_1$. Then φ is a homomorphism of A-modules as composition of homomorphisms of A-modules. Moreover, φ is surjective: indeed, for a $\alpha \in (M_1 + M_2)/M_1$ take a representative $\sum_i m_i^1 + m_i^2 \in M_1 + M_2$ of α with $m_i^1 \in M_1$ and $m_i^2 \in M_2$, $\forall i$; then we have

$$\varphi(\sum_i m_i^2) = \sum_i m_i^2 \bmod M_1 = \sum_i m_i^1 + m_i^2 \bmod M_1 = \alpha.$$

   Let us analyse the kernel of φ: if $m \in ker(\varphi)$, then $m \bmod M_1 = 0$ i.e. $m \in M_1$, thus $m \in M_1 \cap M_2$. Conversely, for $m \in M_1 \cap M_2$, we have $\varphi(m) = m \bmod M_1 = 0$ so that $ker(\varphi) = M_1 \cap M_2$. As a consequence, φ induces a isomorphism of A-modules $\overline{\varphi} : M_2/(M_1 \cap M_1) \xrightarrow{\sim} (M_1 + M_2)/M_1$.

Exercise 7. (Module homomorphisms)

1. Let us first prove that $\mathrm{Hom}_A(M, N)$ has a structure of commutative group. We define $0_{\mathrm{Hom}} \in \mathrm{Hom}_A(M, N)$ by $m \mapsto 0$ and for $f \in \mathrm{Hom}_A(M, N)$, $-f \in \mathrm{Hom}_A(M, N)$ by $m \mapsto -f(m)$. For $f, g \in \mathrm{Hom}_A(M, N)$, we define $f +_{\mathrm{Hom}} g \in \mathrm{Hom}_A(M, N)$ by $m \mapsto f(m) + g(m)$. Then associativity of $+_{\mathrm{Hom}}$ follows from associativity of $+$:

$$\begin{aligned}(f +_{\mathrm{Hom}} g) +_{\mathrm{Hom}} h &= [m \mapsto (f+g)(m) + h(m)] = [m \mapsto (f(m) + g(m)) + h(m)] \\ &= [m \mapsto f(m) + (g(m) + h(m))] = f +_{\mathrm{Hom}} (g +_{\mathrm{Hom}} h)\end{aligned}$$

for $f, g, h \in \mathrm{Hom}_A(M, N)$. We also have

$$f +_{\mathrm{Hom}} 0_{\mathrm{Hom}} = [m \mapsto f(m) + 0] = [m \mapsto f(m)] = f$$

and

$$f +_{\mathrm{Hom}} (-f) = [m \mapsto f(m) + (-f(m))] = [m \mapsto f(m) - f(m)] = [m \mapsto 0] = 0_{\mathrm{Hom}}.$$

So $\mathrm{Hom}_A(M, N)$ is a commutative (checked by the same kind of computations) group.
Let us define a structure of A-module by the following rule: for $a \in A$ and $f \in \mathrm{Hom}_A(M, N)$, define $af := [m \mapsto af(m)]$.
Then for $a, b \in A$, and $f, g \in \mathrm{Hom}_A(M, N)$, we have

$$\begin{aligned} a(f +_{\mathrm{Hom}} g) = [m \mapsto a(f(m) + g(m))] &= [m \mapsto af(m) + ag(m)] \text{ } N \text{ is a } A\text{−module} \\ &= af +_{\mathrm{Hom}} ag \end{aligned}$$

and

$$\begin{aligned} (a + b)f = [m \mapsto (a + b)f(m)] &= [m \mapsto af(m) + bf(m)] \text{ } N \text{ is a } A\text{−module} \\ &= af +_{\mathrm{Hom}} bf \end{aligned}$$

and

$$\begin{aligned} (ab)f = [m \mapsto (ab)f(m)] &= [m \mapsto a(bf(m))] \text{ } N \text{ is a } A\text{−module} \\ &= a(bf) \end{aligned}$$

and finally

$$\begin{aligned} 1 \cdot f = [m \mapsto 1 \cdot f(m)] &= [m \mapsto f(m)] \text{ } N \text{ is a } A\text{−module} \\ &= f \end{aligned}$$

As a conclusion, $\mathrm{Hom}_A(M, N)$ admits a natural structure of A-module.

2. Let us define $\varphi : \mathrm{Hom}_A(A, M) \to M$ by $f \mapsto f(1)$. Then for $a \in A$ and $f, g \in \mathrm{Hom}_A(A, M)$, we have

$$\begin{aligned} \varphi(a(f + g)) = a(f(1) + g(1)) &= af(1) + ag(1) \text{ } M \text{ is a } A\text{−module} \\ &= a\varphi(f) + a\varphi(g) \end{aligned}$$

so φ is a homomorphism of A-modules. This homomorphism is injective: if $\varphi(f) = 0$ then for any a, $f(a) = f(a \cdot 1) = af(1) = a\varphi(f) = 0$ since f is a homomorphism of A-modules, so $f = 0_{\mathrm{Hom}}$.
The homomorphism is also surjective: given a $m \in M$, define $f_m : a \mapsto am$. Then for $a, b, c \in A$, $f(a(b+c)) = a(b+c)m = abm+acm = af(b)+af(c)$ since M is a A-module, as a consequence $f \in \mathrm{Hom}_A(A, M)$ and $\varphi(f) = m$. As a conclusion, φ is a isomorphism of A-modules.

3. For $n > 1$, $\mathbb{Z}/n\mathbb{Z} \neq \{0\}$ is a commutative group and as such it is a $\mathbb{Z}$-module. Now take $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$, we have

$$\begin{aligned} \mathbb{Z} \ni 0 = f(\overline{0}) = f(\underbrace{\overline{1} + \cdots + \overline{1}}_{n\text{−times}}) &= \underbrace{f(\overline{1}) + \cdots + f(\overline{1})}_{n\text{−times}} \text{ } f \text{ group homomorphism} \\ &= nf(1) \in \mathbb{Z} \end{aligned}$$

so ($\mathbb{Z}$ integral domain) $f(1) = 0$. So for a $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$, take a representative $a \in \{0, \ldots, n-1\}$ then

$$f(\overline{a}) = f(\underbrace{\overline{1} + \cdots + \overline{1}}_{a\text{−times}}) = \underbrace{f(\overline{1}) + \cdots + f(\overline{1})}_{a\text{−times}} = af(1) = 0$$

so $f = 0_{\mathrm{Hom}}$. So $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

4. In terms of abelian groups, the first item says that for G_1, G_2 abelian groups, $\mathrm{Hom}_{\mathbb{Z}}(G_1, G_2)$ is again an abelian group. The second item says that for any abelian group G, there is a isomorphism of groups $G \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G)$.

Exercise 8. (Spectrum of a ring)

1. Case $A = \mathbb{F}_p[x]$. It is known (see Examples 3.5 in the lecture) that $\mathbb{F}_p[x]$ is a principal ideal domain. According to Lemma 3.6 the maximal ideals of A are of the form (f) for $f \in \mathbb{F}_p[x]$ an irreducible polynomial. So $\mathrm{MaxSpec}(A) = \{(f),\ f \in \mathbb{F}_p[x],\ f \text{ irreducible}\}$. Let $(f) \subset \mathbb{F}_p[x]$ ($\mathbb{F}_p[x]$ is a principal ideal domain) be a prime ideal, then f is a prime element. As $\mathbb{F}_p[x]$ is a domain, f is irreducible. So as A is an integral domain, $\mathrm{Spec}(A) = \mathrm{MaxSpec}(A) \cup \{(0)\}$.

2. Case $A = k[x]/(x^3)$. We use the bijection between $\mathrm{Spec}(A)$ and $\{(x^3) \subset \mathfrak{p},\ \mathfrak{p} \subset k[x] \text{ is prime}\}$ and since $k[x]$ is a principal ideal domain, $\mathrm{Spec}(A)$ is in bijection with $\{(f),\ (x^3) \subset (f),\ f \in k[x] \text{ is prime}\} = \{(f),\ f|x^3,\ f \in k[x] \text{ is prime}\} = \{(x)\}$. Since $k[[x]]/(x) \simeq k$ (the constant term), (x) is a maximal ideal. So $\mathrm{MaxSpec}(A) = \{(x)\} = \mathrm{Spec}(A)$.

3. Case $A = k[[x]]$. Let $f = \sum_{i \geq 0} a_i x^i \in A$ such that $a_0 \neq 0$ then there is a $g \in k[[x]]$ such that $fg = 1$ ($g = \sum_i b_i x^i$ with $b_i \in k$ defined by induction, $b_0 = a_0^{-1}$ and $b_{i+1} = -a_0^{-1} \sum_{j=0}^{i} a_{i+1-j} b_j$). As a consequence, a proper ideal $\mathfrak{a}$ of $k[[x]]$ cannot contain such an element so that we have $\mathfrak{a} \subset (x)$ for any proper ideal $\mathfrak{a} \subset k[[x]]$.
   Now, let $0 \neq \mathfrak{a} \subset k[[x]]$ be a proper ideal and $0 \neq f = \sum_i a_i x^i \in \mathfrak{a}$. Set $d := \min\{i,\ a_i \neq 0\} > 0$ (by the previous observation). We have $f = x^d(\sum_{i \geq 0} a_{d+i} x^i)$; since $\sum_{i \geq 0} a_{d+i} x^i$ has non-zero constant term, we can find a $g \in k[[x]]$ such $g(\sum_{i \geq 0} a_{d+i} x^i) = 1$ so that $\mathfrak{a} \ni fg = x^d(g(\sum_{i \geq 0} a_{d+i} x^i)) = x^d$.
   Set now $0 < d_{\mathfrak{a}} := \min\{\min\{i,\ a_i(f) \neq 0\}, 0 \neq f \in \mathfrak{a}\}$ where $f = \sum_i a_i(f) x^i$. Then by the previous computation, $(x^{d_{\mathfrak{a}}}) \subset \mathfrak{a}$ and by definition of $d_{\mathfrak{a}}$, $\mathfrak{a} \subset (x^{d_{\mathfrak{a}}})$ so that $\mathfrak{a} = (x^{d_{\mathfrak{a}}})$. But among the (x^d)'s ($d > 0$), the only prime ideal is (x). Looking at the terms of least degree in a product, we see that $k[[x]]$ is an integral domain i.e. (0) is a prime ideal, so $\mathrm{Spec}(A) = \{(0), (x)\}$. The only maximal ideal is (x).