Solutions for exercises, Algebra I (Commutative Algebra) – Week 7

Exercise 33. (Extension under flat ring homomorphisms)
(one direction is obvious) Assume $\mathrm{MaxSpec}(A) \subset \mathrm{im}(\varphi)$ and consider a A-module such that $M \otimes B = 0$. If $M \neq 0$, take $0 \neq m \in M$. The cyclic submodule $\langle m \rangle \subset M$ generated by m is isomorphic to $A/\mathfrak{a}$ for $\mathfrak{a} \subsetneq A$ (since $0 \neq m$) the annihilator of m (look at $A \to M$, $a \mapsto am$; its kernel is the annihilator of m and it is surjective onto $\langle m \rangle$ by definition). Since B is a flat A-algebra, we have an induced inclusion $A/\mathfrak{a} \otimes B \hookrightarrow M \otimes B$; thus $A/\mathfrak{a} \otimes B = 0$.
Since B is a flat A-algebra, tensoring the exact sequence

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

with B we get an exact sequence:

$$0 \to \mathfrak{a} \otimes B \to B \to A/\mathfrak{a} \otimes B \to 0.$$

With the previous vanishing we get $B \simeq \mathfrak{a} \otimes B$ as B-modules . Looking at the exact sequence, we see that the isomorphism is givien by $a \otimes b \mapsto a \cdot b = f(a)b$; thus $B \simeq \mathfrak{a} \otimes B$ means $B \simeq \mathfrak{a}B = \mathfrak{a}^e$ as B-modules.
But since $\mathfrak{a} \subsetneq A$, it is contained in a maximal ideal $\mathfrak{m} \in \mathrm{MaxSpec}(A)$. We get $(1) = \mathfrak{a}^e \subset \mathfrak{m}^e$. But by assumption, there is a $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $f^{-1}(\mathfrak{p}) = \varphi(\mathfrak{p}) = \mathfrak{m}$; which yields $\mathfrak{m}^e \subset \mathfrak{p} \subsetneq B$ (as $f(\mathfrak{m}) \subset \mathfrak{p}$ and $\mathfrak{m}^e$ is the smallest ideal containing $f(\mathfrak{m})$). Contradiction. So there is no such $M \ni m \neq 0$ i.e. $M = 0$.

For a counterexample, take $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ the natural inclusion. We know that $\mathbb{Q} \simeq \mathbb{Z}_{(0)}$ is a flat $\mathbb{Z}$-algebra but $\varphi : \mathrm{Spec}(\mathbb{Q}) = (0) \to \mathrm{Spec}(\mathbb{Z})$ is not surjective (as a map from a finite set to an infinite). Then the cyclic $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}$ is non-zero but $\mathbb{Z}/24\mathbb{Z} \otimes \mathbb{Q} = 0$ since $\overline{n} \otimes 1 = \overline{n} \otimes \frac{24}{24} = \overline{24n} \otimes \frac{1}{24} = 0$.

Exercise 34. (Surjectivity of maps induced by flat ring homomorphisms)

1. Let us define $p : N_B \to N$ by $b \otimes n \mapsto bn$ (the later multiplication uses the B-module structure on N). It is a well-defined homomorphism of A-modules (and B-modules) and $p \circ g(n) = p(1 \otimes n) = n$ for any $n \in N$ i.e. $p \circ g = \mathrm{id}_N$. Thus g is injective and presents N as a direct summand of N_B.

2. If φ is surjective then given a $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, there is a $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}$. Thus $f(\mathfrak{m}) \subset \mathfrak{p}$ and $\mathfrak{m}^e \subset \mathfrak{p} \subsetneq B$ ($\mathfrak{m}^e$ is the smallest ideal containing $f(\mathfrak{m})$).
Conversely assume that for any $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, $\mathfrak{m}^e \subsetneq (1)$ and take a $\mathfrak{m} \in \mathrm{MaxSpec}(A)$. Since $f(\mathfrak{m}) \subset \mathfrak{m}^e$, we have $\mathfrak{m} \subset f^{-1}(\mathfrak{m}^e)$. Now if there is a $x \in f^{-1}(\mathfrak{m}^e) \backslash \mathfrak{m}$, then $\overline{x} \in A/\mathfrak{m}$ is non-zero thus invertible (since $A/\mathfrak{m}$ is a field) i.e. there is a $y \in A$ and a $m \in \mathfrak{m}$, such that $xy = 1 + m$. Applying f, we get $f(x)f(y) = 1 + f(m)$; but $f(m) \in f(\mathfrak{m}) \subset \mathfrak{m}^e$ and $f(x) \in \mathfrak{m}^e$ by assumption, hence $1 = f(x)f(y) - f(m) \in \mathfrak{m}^e$. Contradiction. So $f^{-1}(\mathfrak{m}^e) = \mathfrak{m}$. Then by Corollary 9.15, we have $\mathfrak{m} \in \mathrm{im}(\varphi)$. As a consequence $\mathrm{MaxSpec}(A) \subset \mathrm{im}(\varphi)$.
Now let $\mathfrak{p} \in \mathrm{Spec}(A)$. By Corollary 9.15, it is sufficient to prove that $f^{-1}(\mathfrak{p}^e) = \mathfrak{p}$ to have $\mathfrak{p} \in \mathrm{im}(\varphi)$.
By definition $\mathfrak{p} \subset f^{-1}(\mathfrak{p}^e)$ so let us consider the A-module $M = f^{-1}(\mathfrak{p}^e)/\mathfrak{p}$. Since B is a flat A-algebra, tensoring

$$0 \to f^{-1}(\mathfrak{p}^e) \to A \to A/f^{-1}(\mathfrak{p}^e)A \to 0$$

Solutions to be handed in before Monday May 25, 4pm.

with B, we get an exact sequence of B-modules:

$$0 \to f^{-1}(\mathfrak{p}^e) \otimes B \to B \to A/f^{-1}(\mathfrak{p}^e)A \otimes B \to 0.$$

But $A/f^{-1}(\mathfrak{p}^e)A \otimes B \simeq B/f^{-1}(\mathfrak{p}^e)^e B$ and (check it) $f^{-1}(\mathfrak{p}^e)^e = \mathfrak{p}^e$ so $A/f^{-1}(\mathfrak{p}^e)A \otimes B \simeq B/\mathfrak{p}^e B$. Thus the exactness of the above sequence means that $f^{-1}(\mathfrak{p}^e) \otimes B \simeq \mathfrak{p}^e$ (by $a \otimes b \mapsto ab$) as B-modules.
Likewise, using flatness of B, we have an exact sequence of B-modules:

$$0 \to \mathfrak{p} \otimes B \to B \to A/\mathfrak{p} \otimes B \to 0.$$

Again $A/\mathfrak{p} \otimes B \simeq B/\mathfrak{p}^e B$ (by $a \otimes b \mapsto ab$) so that the exactness of the above sequence means $\mathfrak{p} \otimes B \simeq \mathfrak{p}^e$.
Now by definition, we have an exact sequence

$$0 \to \mathfrak{p} \to f^{-1}(\mathfrak{p}^e) \to M \to 0$$

and since B is flat, we get an exact sequence of B-modules

$$0 \to \mathfrak{p} \otimes B \to f^{-1}(\mathfrak{p}^e) \otimes B \to M \otimes B \to 0.$$

By what we have seen the two first terms are both isomorphic to $\mathfrak{p}^e$ and the isomorphisms are compatible with the natural inclusion. Thus the first map of the exact sequence is an isomorphism; which means $M \otimes B = 0$. By the previous exercise, we get $M = 0$ i.e. $\mathfrak{p} = f^{-1}(\mathfrak{p}^e)$. Now, Corollary 9.15 tells us that $\mathfrak{p} \in \mathrm{im}(\varphi)$. Hence φ is surjective.

3. We can use the previous question to solve this one. Remember that the ring $A_\mathfrak{p}$ is local i.e. only one maximal ideal which is $\mathfrak{p}_\mathfrak{p}$. Suppose $\mathfrak{p}_\mathfrak{p}^e = (1)$. Then we can find $p \in \mathfrak{p}$, $s \in A\backslash\mathfrak{p}$, $t \in B\backslash\mathfrak{q}$ and $b \in B$ such that $\frac{1}{1} = \frac{bf(p)}{f(s)t} \in B_\mathfrak{q}$.; which means that $t'tf(s) = t'bf(p)$ in B for some $t' \in B\backslash\mathfrak{q}$. But on one hand $f(p) \in \mathfrak{p}^e \subset \mathfrak{q}$ which yields $t'fb(p) \in \mathfrak{q}$ and on the other, $t't \in B\backslash\mathfrak{q}$ and $s \in A\backslash\mathfrak{p} = A\backslash f^{-1}(\mathfrak{q})$ i.e. $f(s) \in B\backslash\mathfrak{q}$, contradicting the fact that $\mathfrak{q}$ is prime. So $\mathfrak{p}_\mathfrak{p}^e \subsetneq (1)$. It remains to prove that $f_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{q}$ is flat. By Corollary 8.28, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Set $S = f(A\backslash\mathfrak{p}) = f(A\backslash f^{-1}(\mathfrak{q})) \subset B$ and $S' = B\backslash\mathfrak{q}$. We have $\ker(f) = f^{-1}(0) \subset f^{-1}(\mathfrak{q}) = \mathfrak{p}$ so S is a multiplicative subset of B and $S \subset S'$. Moreover by definition of the A-module structure on B, $S^{-1}B \simeq B_\mathfrak{p}$; let us denote $h : B \to S^{-1}B$ the localization. Let us prove that $B_\mathfrak{q}$ is the localization of $B_\mathfrak{p}$ with respect to $h(S')$.
Let us define $g : S^{-1}B \to S'^{-1}B$ by $\frac{b}{f(s)} \mapsto \frac{b}{f(s)}$. It is well-defined since $S \subset S'$: if $\frac{b}{f(s)} = \frac{b'}{f(s')} \in S^{-1}B$ then $f(t)f(s')b = f(t)b'f(s)$ in B; but since $f(t)f(s'),\ f(t)f(s) \in S \subset S'$ this tells us that $\frac{b}{f(s)} = \frac{b'}{f(s')} \in S'^{-1}B$. It is easy to see that it is a ring homomorphism. Moreover for $\frac{t}{f(s)} \in h(S')$, we have $g(\frac{t}{f(s)}) = \frac{t}{f(s)} \in S'^{-1}B$ is invertible.
Now given a ring homomorphism $q : S^{-1}B \to C$ such that $q(h(S')) \subset C^*$, define $\overline{q} : S'^{-1}B \to C$ by $\frac{b}{s} \mapsto q(\frac{b}{1})q(\frac{s}{1})^{-1}$. It is a well defined map: for $\frac{b}{s} = \frac{b'}{s'} \in S'^{-1}B$ we have $ts'b = tsb$ in B for a $t \in S'$; which yields $q(\frac{t}{1})(q(\frac{s'}{1})q(\frac{b}{1}) - q(\frac{s}{1})q(\frac{b'}{1})) = 0$ in C. But $q(\frac{t}{1}) \in C^*$ by assumption; so $q(\frac{s'}{1})q(\frac{b}{1}) = q(\frac{s}{1})q(\frac{b'}{1}$ in C. Again $q(\frac{s'}{1}),\ q(\frac{s}{1}) \in C^*$ by assumption; thus $q(\frac{b'}{1})q(\frac{s'}{1})^{-1} = q(\frac{b}{1})q(\frac{s}{1})^{-1}$.
It is a ring homomorphism (left to check) and for any $b \in B$, $\overline{q}(g(\frac{b}{1})) = \overline{q}(\frac{b}{1}) = q(\frac{b}{1})$. Since for $f(s) \in S \subset S'$, $\frac{f(s)}{1} \in S^{-1}B$ is invertible, we get $q(\frac{1}{f(s)}) = q(\frac{f(s)}{1})^{-1}$ in C; likewise $\overline{q}(\frac{1}{f(s)}) = q(\frac{f(s)}{1})^{-1}$. So for $\frac{b}{f(s)} \in S^{-1}B$,

$$\overline{q}(g(\frac{b}{f(s)})) = \overline{q}(g(\frac{b}{1})g(\frac{1}{f(s)})) = \overline{q}(g(\frac{b}{1}))\overline{q}(g(\frac{1}{f(s)})) = q(g(\frac{b}{1}))q(\frac{f(s)}{1})^{-1} = q(\frac{b}{f(s)}).$$

Thus $q = \overline{q} \circ g$. Uniqueness of the factorization through g is checked likewise (looking first at $\frac{b}{1}$ and then taking the inverses). So $g : S^{-1}B \to S'^{-1}B$ is the localization of

$S^{-1}B$ with respect to $h(s')$. But $S'^{-1}B \simeq B_\mathfrak{q}$ by definition. Thus $B_\mathfrak{q}$ is a flat $B_\mathfrak{p}$-algebra and the later is a flat $A_\mathfrak{p}$-algebra, as a result $B_\mathfrak{q}$ is a flat $A_\mathfrak{p}$-algebra. And we can apply the previous question.

Exercise 35. (Algebras of invariants)
Notice that B^G is indeed an A-algebra: denoting $f : A \to B$ the ring homomorphism giving the structure of A-algebra, we have, for $a \in A$ and $g_i \in G$, we have $g(f(a)) = f(a)g(1) = f(a)\cdot 1 = f(a)$ since g_i is a homomorphism of A-algebras (i.e. an A-linear ring homomorphism) i.e. $f(A) \subset B^G$. Moreover for $b, b' \in B^G$, $g(b+b') = g(b)+g(b') = b+b'$ and $g(bb') = g(b)g(b') = bb'$.
We have $f(A) \subset B^G \subset B$ with B of finite type over A, the later being Noetherian. So if we knew that B was a finite B^G-module, Proposition 11.24 would tell us that B^G is Noetherian. So Let us prove that B is a finite B^G-module.
Since $f(A) \subset B^G$ and B is of finte type over A, we get that B is a finite type over B^G. Thus by Corollary 11.11, it is sufficient to prove that B is integral over B^G to get that B is a finite B^G-module.
Now let $b \in B$. It is annihilated by $(x-b) \in B[x]$ thus it is also annihilated by the monic polynomial $P = \Pi_{g\in G}(x - g(b)) \in B[x]$. Let us prove that $P \in B^G[x]$ actually: the usual expansion (B commutative) of P gives $P = \sum_{i=0}^{|G|} \sigma_{|G|-i}((g(b))_{g\in G})x^i$ where σ_k (set $\sigma_0 = 1$) designates the k^{th} elementary symmetric function on $|G|$-variables $\sigma_k : (X_1, \ldots, X_{|G|}) \mapsto \sum\limits_{1\le i_1<i_2\cdots<i_k\le|G|} \Pi_{j=1}^k X_{i_j}$. But since the $g_i \in G$ are A-algebras homomorphisms (respect sums and products) and for any $g \in G$, $G \to G$, $g' \mapsto gg'$ is a bijection (G is a finite group; injectivity is clear and conclude by cardinal), for any $g \in G$ (set $g_0 = \mathrm{id}_B$) and k,

$$\begin{aligned} g(\sigma_k(b, g_1(b), \ldots, g_{|G|-1}(b))) = \sum_{1\le i_1<i_2\cdots<i_k\le|G|} \Pi_{j=1}^k g(g_{i_j}(b)) &= \sum_{1\le i'_1<i'_2\cdots<i'_k\le|G|} \Pi_{j=1}^k g_{i'_j}(b) \\ &= \sigma_k(b, g_1(b), \ldots, g_{|G|-1}(b)) \end{aligned}$$

proving that $\sigma_k(b, g_1(b), \ldots, g_{|G|-1}(b)) \in B^G$ for any k i.e. $P \in B^G[x]$ and is monic. So b is integral over B^G and since b was arbitrary B is integral over B^G which allows us to use Corollary 11.11 and Proposition 11.24 to conclude.

Exercise 36. (Localization of integral ring homomorphisms)
Notice first that $k[x]$ is indeed integral over over $k[x^2-1]$: x is annihilated by the monic polynomial $X^2 - (x^2-1) + 1 \in k[x^2-1][X]$ so it is integral over $k[x^2-1]$. Hence $k[x^2-1][x] = k[x]$ is a finite $k[x^2-1]$-module by Proposition 11.6 and the same proposition gives us integrality of any element in $k[x]$.
Since $x-1$ is irreducible $(x-1)$ is a prime ideal and $(x-1)^c = (x-1) \cap k[x^2-1]$. If $f \in k[x^2-1]$ it can be written $f = a_0 + \sum_{i\ge1} a_i(x^2-1)^i$ with $a_i \in k$. Iff is in $(x-1)^c$, it vanishes at 1 thus $a_0 = 0$. Conversely since $x^2-1 = (x-1)(x+1)$ any $f \in k[x^2-1]$ which has no constant term is in $(x-1)$. Thus $(x-1)^c = (x-1) \cap k[x^2-1] = (x^2-1)$.
Since $char(k) \ne 2$, we have $1 \ne -1$; as a consequence $x+1 \notin (x-1)$ (because any polynomial in the principal ideal vanishes at 1 and $x+1$ does not). Thus $\frac{1}{x+1} \in k[x]_{(x-1)}$. Assume $\frac{1}{x+1}$ is integral over $k[x^2-1]_{(x^2-1)}$. Then we have $\frac{1}{(x+1)^n} + \sum_{i\le n-1} \frac{f_i}{g_i}\frac{1}{(x+1)^i} = 0 \in k[x]_{(x-1)}$ for some $\frac{f_i}{g_i} \in k[x^2-1]_{(x^2-1)}$. We have

$$0 = \frac{1}{(x+1)^n} + \sum_{i\le n-1} \frac{f_i}{g_i}\frac{1}{(x+1)^i} = \frac{(\Pi_k g_k) + \sum_{i\le n} \Pi_{k\ne i} g_k f_i (x+1)^{n-i}}{\Pi_k g_k (x+1)^n}$$

which means $g((\Pi_k g_k) + \sum_{i\le n-1} \Pi_{k\ne i} g_k f_i (x+1)^{n-i}) = 0$ in $k[x]$ for some $g \notin (x-1)$. In particular $g \ne 0$, thus ($k[x]$ integral domain) $(\Pi_k g_k) + \sum_{i\le n-1} \Pi_{k\ne i} g_k f_i (x+1)^{n-i} = 0$ in $k[x]$. Now $(x+1)|\Pi_{k\ne i} g_k f_i (x+1)^{n-i}$ for $i \le n-1$, thus $(x+1)|\Pi_k g_k$. But $g_k \notin (x^2-1)$ for any k which contradicts the fact that $(x+1)$ is a prime ideal. So $\frac{1}{x+1}$ is not integral over $k[x^2-1]_{(x^2-1)}$.

Exercise 37. (Noetherian topological spaces)

1. Assume A is Noetherian. Let $V_1 \supset V_2 \supset V_3 \cdots \supset V_n \supset \cdots$ be a descending chain of closed subsets of $\mathrm{Spec}(A)$. By definition of Zariski topology, we can find ideals $(\mathfrak{a}_i)_{i\in\mathbb{N}}$ of A such that $V_i = V(\mathfrak{a}_i)$ for any $i \in \mathbb{N}$. Now the inclusion $V(\mathfrak{a}_{i+1}) \subset V(\mathfrak{a}_i)$ is equivalent to $\sqrt{\mathfrak{a}_i} \subset \sqrt{\mathfrak{a}_{i+1}}$ for any i. Thus the descending chain of closed subsets gives rise to an ascending chain of ideals of A:

$$\sqrt{\mathfrak{a}_1} \subset \sqrt{\mathfrak{a}_2} \subset \cdots \subset \sqrt{\mathfrak{a}_n} \subset \cdots$$

which, since A is Noetherian, becomes stationary i.e. there is a $n \in \mathbb{N}$, such that $\sqrt{\mathfrak{a}_m} = \sqrt{\mathfrak{a}_n}$ for any $m \geq n$, which means $V_m = V(\mathfrak{a}_m) = V(\mathfrak{a}_n) = V_n$ for any $m \geq m$ i.e. the descending chain of closed subsets becomes stationary. Hence $\mathrm{Spec}(A)$ is a Noetherian topological space.
The typical example of a non-Noetherian ring is the polynomial ring in infinitely many variables $A = k[(x_i)_{i\in\mathbb{N}_{>0}}]$ but its spectrum is not easy to describe. But let us consider $B = A/(x_1, x_2^2, x_3^3, \ldots, x_n^n, \ldots)$. We have $\mathrm{Spec}(B) = V((x_1, x_2^2, x_3^3, \ldots, x_n^n, \ldots)) \subset \mathrm{Spec}(A)$ and $(x_1, x_2, x_3, \ldots, x_n, \ldots) \subset \mathfrak{N}_B$. Since for any $\mathfrak{p} \in \mathrm{Spec}(B)$, $\mathfrak{N}_B \subset \mathfrak{p}$, we get $\mathrm{Spec}(B) = \{(x_1, x_2, x_3, \ldots, x_n, \ldots)\}$ (since $(x_1, x_2, x_3, \ldots, x_n, \ldots) \in \mathrm{Spec}(A)$ is maximal) so $\mathrm{Spec}(B)$ is a Noetherian topological space. But

$$(\overline{x_2}) \subset (\overline{x_2}, \overline{x_3}) \subset (\overline{x_2}, \overline{x_3}, \overline{x_4}) \subset \cdots (\overline{x_2}, \overline{x_3}, \ldots \overline{x_n}) \subset \cdots$$

is ascending chain of ideals which is not stationary.

2. Let $\mathfrak{p} \in \mathrm{im}(\varphi) \subset \mathrm{Spec}(A)$ then $\varphi^{-1}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(B),\ f^{-1}(\mathfrak{q}) = \mathfrak{p}\}$. Now $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ is equivalent to $f(\mathfrak{p}) = \mathrm{im}(f) \cap \mathfrak{q}$ and $\ker(f) \subset \mathfrak{p}$: indeed if $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ then $f(\mathfrak{p}) \subset \mathfrak{q} \cap \mathrm{im}(f)$ and if $y \in \mathfrak{q} \cap \mathrm{im}(f)$ then there is a $x \in A$ $\mathfrak{q} \ni y = f(x)$ but this means $x \in f^{-1}(\mathfrak{q}) = \mathfrak{p}$; thus $y = f(\mathfrak{p})$ i.e. $f(\mathfrak{p}) = \mathrm{im}(f) \cap \mathfrak{q}$. Moreover $\ker(f) = f^{-1}(0) \subset f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Conversely, if $f(\mathfrak{p}) = \mathrm{im}(f) \cap \mathfrak{q}$ and $\ker(f) \subset \mathfrak{p}$ then $\mathfrak{p} \subset f^{-1}(\mathfrak{q})$ and if $x \in f^{-1}(\mathfrak{q})$, we have $f(x) \in \mathfrak{q} \cap \mathrm{im}(f) = f(\mathfrak{p})$ i.e. there is a $x' \in \mathfrak{p}$ such that $f(x) = f(x')$; then $x = x' + (x - x') \in \mathfrak{p} + \ker(f) = \mathfrak{p}$.
Next, $f(\mathfrak{p}) = \mathrm{im}(f) \cap \mathfrak{q}$ and $\ker(f) \subset \mathfrak{p}$ if and only if $\mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \emptyset$ and $f(\mathfrak{p}) \subset \mathfrak{q}$: indeed if $f(\mathfrak{p}) = \mathrm{im}(f) \cap \mathfrak{q}$ (then obviously $f(\mathfrak{p}) \subset \mathfrak{q}$) and $\ker(f) \subset \mathfrak{p}$, then if $y \in \mathfrak{q} \cap f(A \backslash \mathfrak{p}) \subset \mathfrak{q} \cap \mathrm{im}(f)$ then we can write $y = f(x)$ with $x \in A \backslash \mathfrak{p}$ and $y = f(x')$ with $x' \in \mathfrak{p}$. So $x - x' \in \ker(f) \subset \mathfrak{p}$; thus $x \in \mathfrak{p}$ contradiction. Conversely, if $\mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \emptyset$ and $f(\mathfrak{p}) \subset \mathfrak{q}$, since $f(\ker(f)) = \{0\} \subset \mathfrak{q}$ we get $\ker(f) \subset \mathfrak{p}$. We have $\mathrm{im}(f) \backslash f(\mathfrak{p}) \subset f(A \backslash \mathfrak{p})$ and if $y \in f(A \backslash \mathfrak{p})$ then we can write $y = f(x)$ for $x \in A \backslash \mathfrak{p}$. If $y \in f(\mathfrak{p})$, we can also write $y = f(x')$ with $x' \in \mathfrak{p}$; then $x = x' + (x - x') \in \mathfrak{p} + \ker(f) \subset \mathfrak{p}$; contradiction. So $\mathrm{im}(f) \backslash f(\mathfrak{p}) = f(A \backslash \mathfrak{p})$. Thus $\mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \emptyset$ means $\mathfrak{q} \cap \mathrm{im}(f) \subset f(\mathfrak{p})$. But as we have $f(\mathfrak{p}) \subset \mathfrak{q}$, we get $\mathfrak{q} \cap \mathrm{im}(f) = f(\mathfrak{p})$.

As a consequence,

$$\{\mathfrak{q} \in \mathrm{Spec}(B),\ f^{-1}(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q} \in \mathrm{Spec}(B),\ \mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \emptyset \text{ and } f(\mathfrak{p}) \subset \mathfrak{q}\} = \{\mathfrak{q} \in \mathrm{Spec}(B_\mathfrak{p}),\ f(\mathfrak{p}) \subset \mathfrak{q}\}$$

since by Proposition 9.14 $\mathrm{Spec}(B_\mathfrak{p}) \simeq \{\mathfrak{q} \in \mathrm{Spec}(B),\ \mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \emptyset\}$. Now, since $\mathfrak{q}$ is qn ideal, $f(\mathfrak{p}) \subset \mathfrak{q}$ means $\mathfrak{p}^e \subset \mathfrak{q}$ thus

$$\{\mathfrak{q} \in \mathrm{Spec}(B_\mathfrak{p}),\ f(\mathfrak{p}) \subset \mathfrak{q}\} = \{\mathfrak{q} \in \mathrm{Spec}(B_\mathfrak{p}),\ \mathfrak{p}^e \subset \mathfrak{q}\} = V(\mathfrak{p}^e) = \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}^e B_\mathfrak{p}) = \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p} B_\mathfrak{p}).$$

But $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \simeq B_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p} \simeq B \otimes_A A_\mathfrak{p} \otimes_{A_\mathfrak{p}} Q(A/\mathfrak{p}) \simeq B \otimes_A Q(A/\mathfrak{p})$. Thus

$$\{\mathfrak{q} \in \mathrm{Spec}(B),\ f^{-1}(\mathfrak{q}) = \mathfrak{p}\} = \mathrm{Spec}(B \otimes_A A_\mathfrak{p}/\mathfrak{p}).$$

Now let us choose a surjective homomorphism of A-algebra $g : A[x_1, \ldots, x_n] \twoheadrightarrow B$; tensoring with $Q(A/\mathfrak{p})$ we get a surjective $Q(A/\mathfrak{p})[x_1, \ldots, x_n] \twoheadrightarrow B \otimes_A Q(A/\mathfrak{p})$. The field $Q(A/\mathfrak{p})$ is Noetherian so $B \otimes_A Q(A/\mathfrak{p})$ is Noetherian. Hence according to the first question, $\{\mathfrak{q} \in \mathrm{Spec}(B),\ f^{-1}(\mathfrak{q}) = \mathfrak{p}\} = \mathrm{Spec}(B \otimes_A A_\mathfrak{p}/\mathfrak{p})$ is Noehterian.