Exercise Session 3

Ramification

Given $f: X \to Y$ of smooth curves$/k$. For all closed $x \in X$ we defined

$f_x := [\kappa(x) : \kappa(f(x))]$

e_x is s.t. $\pi_Y = u \cdot \pi_X^{e_x}$ for some uniformizers $\pi_Y \in \mathcal{O}_{Y,y}$, $\pi_X \in \mathcal{O}_{X,x}$.

Intuition:

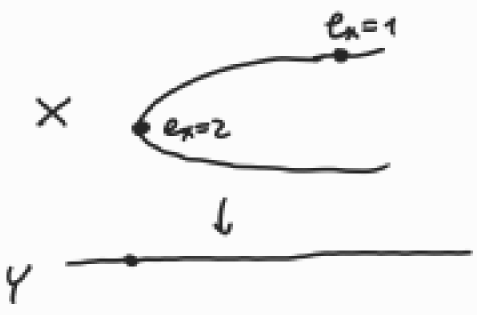
- $[\kappa(x) : k]$ = "# geometric points of X corresponding to x"

  = "# $\bar{x} \in X_{\bar{k}}$ s.t. $\bar{x} \mapsto x$ under $X_{\bar{k}} \to X$"

  Example: Let $X = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[T]$. Let $x = (T^2+1) \in X$. Geometrically, this x corresponds to the points i and $-i$.

- e_x = "number of branches of f near x"

E.g.



Example: Let $k = \bar{k}$ field, char $k \notin \{2,3\}$. Let E be the closure of

$$E^\circ := V(\underbrace{y^2 - x(x-1)(x-\lambda)}_{f_\lambda}) \subseteq \mathbb{A}^2_k$$

in $\mathbb{P}^2_k$, where λ is any element in k.

(0) Compute E: $E = V(y^2z - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2_k$.

(i) Under what condition on λ is E smooth?

- E^0: $\frac{df_\lambda}{dx} = x(x-1) + x(x-\lambda) + (x-1)(x-\lambda)$

$$\frac{df_\lambda}{dy} = 2y$$

Jacobian criterion: E^0 smooth $\Leftrightarrow \left(\frac{df_\lambda}{dx}, \frac{df_\lambda}{dy}\right)$ does not vanish on E^0

$\frac{df_\lambda}{dy} = 0 \Leftrightarrow y = 0 \rightsquigarrow x$ satisfies $0 = x(x-1)(x-\lambda)$
i.e. $x \in \{0, 1, \lambda\}$.

Need to make sure that this is not a root of $\frac{df_\lambda}{dx}$

$\Leftrightarrow \lambda \neq 0, 1$.

- $E \setminus E^0 = \{[0:1:0]\} \rightsquigarrow$ smooth indep. of λ

(ii) Consider the projection $p_x : E^0 \to \mathbb{A}^1_k$ to the x-axis.
Extend p_x to a map $p_x : E \to \mathbb{P}^1_k$. What does p_x look like on a neighbourhood of $E \setminus E^0$?

Nbhd of $E \setminus E^0$: $\underbrace{\operatorname{Spec} k[x,z]/(z - x(x-z)(x-\lambda z))}_{E^\infty}$

$p_x : E^\infty \to \mathbb{P}^1_k, \ [x:\underset{\neq 0}{y}:z] \mapsto$ "$[x:z]$" = "$\frac{x}{z}$" = "$\frac{1}{(x-z)(x-\lambda z)}$"

$= [y^2 : (x-z)(x-\lambda z)]$

(iii) $\deg p_x = 2$

$$\begin{array}{c} \| \\ [k(E) : k(\mathbb{P}^1_k)] \end{array} \qquad k(\mathbb{P}^1) \xrightarrow{p_x^*} k(E)$$

$$\nearrow \qquad \nwarrow k(t) \qquad\qquad t \longmapsto x$$

$\mathrm{Frac}\left(k[x,y]/\left((y^2 - x(x-1)(x-\lambda))\right)\right)$

$$= \underbrace{k(x)[y]/\left((y^2 - x(x-1)(x-\lambda)\right)}_{\text{quadr. ext. of } k(x) = k(t)} \rightsquigarrow \deg p_x = 2$$

(iv) Compute e_ω for all $\omega \in E$ (wrt. $p_x : E \to \mathbb{P}^1_k$)

- Use $2 = \deg p_x = \sum_{\omega \in p_x^{-1}(x)} e_\omega$

$$\rightsquigarrow e_\omega = \begin{cases} 1 & \text{if } |p_x^{-1} p_x \omega| = 2 \\ 2 & \text{else} \end{cases}$$

$$e_\omega = 2 \iff \{(0,0), (1,0), (\lambda, 0), \overset{[0:1:0]}{\overset{\shortparallel}{\infty}}\}$$

<u>Exercise</u>: • Compute e_ω directly using the definition.

• Do the same for $p_y : E \to \mathbb{P}^1_k$.

Riemann-Hurwitz Formula

① k field, $f: X \to Y$ separable map of proper smooth curves/k.

(a) $0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$ is exact.

By 1a on last sheet, there is an exact sequence

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

Left to show: $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ injective. Since both $f^*\Omega^1_{Y/k}$ and $\Omega^1_{X/k}$ are locally $\cong \mathcal{O}_X$, so reduces to showing that map on stalks at $\eta(X)$ is inj.

(Locally on $\operatorname{Spec} A \subseteq X$, the map corresponds to a map $A \to A$. Since A integral, this map is injective $\Leftrightarrow$ non-zero)

$\rightsquigarrow$ Check that $\Omega^1_{k(Y)/k} \underset{k(Y)}{\otimes} k(X) \to \Omega^1_{k(X)/k}$ non-zero.

$$\underset{\text{by above sequence}}{\operatorname{coker}} = \Omega^1_{k(X)/k(Y)} \underset{k(X)/k(Y) \text{ separable.}}{=} 0$$

(b) $\dim_k \Omega^1_{X/Y,x} = (e_x - 1) \cdot [k(x) : k]$.

Need to understand $\Omega^1_{\mathcal{O}_{Y,y}/k} \underset{\mathcal{O}_{Y,y}}{\otimes} \mathcal{O}_{X,x} \to \Omega^1_{\mathcal{O}_{X,x}/k}$.

Claim: $\Omega^1_{\mathcal{O}_{X,x}/k} \cong \mathcal{O}_{X,x} \cdot d\pi_x$ for any uniformizer $\pi_x \in \mathcal{O}_{X,x}$.

(Know it's free, just need to show that $d\pi_x$ is a generator, i.e. $d\pi_x \not\equiv 0 \bmod \pi_x$)

Thus, above map becomes

$$0 \to \mathcal{O}_{X,x}\, d\pi_y \to \mathcal{O}_{X,x}\, d\pi_x \to \Omega^1_{X/Y,x} \to 0$$

But $\pi_y = \pi_x^{e_x} \cdot u$, $u \in \mathcal{O}_{X,x}^\times$ by def. of e_x.

$$\Rightarrow d\pi_y = d(\pi_x^{e_x} \cdot u) = e_x \pi_x^{e_x - 1} \cdot u\, d\pi_x + \pi_x^{e_x} du$$

$$\Rightarrow \Omega^1_{X/Y,x} = \mathcal{O}_{X,x}\, d\pi_x \big/ \left(u e_x \cdot \pi_x^{e_x - 1} d\pi_x + \pi_x^{e_x} du\right)$$

$$\cong \mathcal{O}_{X,x} / \pi_x^{e_x - 1}$$

use $e_x \neq 0$ since $\operatorname{char} k = 0$.

$\Rightarrow \dim \Omega^1_{X/Y,k} = (e_x - 1)\,[k(x):k]$.

<u>Corollary:</u> Let char $k = 0$, $f: X \to Y$ as above s.t. X, Y geom. conn. ($\Leftrightarrow H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = k$). Then

$$2g_X - 2 = (2g_Y - 2)\cdot \deg f + \sum_{x \in X} (e_x - 1)\cdot [k(x):k]$$

<u>Proof:</u> By above exact sequence we have

$$\underbrace{\chi(\Omega^1_{X/k})}_{= \deg \Omega^1_{X/k} + 1 - g} = \underbrace{\chi(f^*\Omega^1_{Y/k})}_{= \deg f^*\Omega^1_{Y/k} + 1 - g} + \underbrace{\chi(\Omega^1_{X/Y})}_{= h^0(\Omega^1_{X/Y}) \overset{16}{=} \sum_x (e_x - 1)[k(x):k]}$$

$$\Rightarrow \underbrace{\deg \Omega^1_{X/k}}_{= 2g_X - 2} = \underbrace{\deg f^*\Omega^1_{Y/k}}_{= \deg f \cdot \underbrace{\deg \Omega^1_{Y/k}}_{= 2g_Y - 2}} + \sum \ldots$$

□

<u>Example:</u> $\underbrace{2g_E - 2}_{=0} = \underbrace{\deg p_x}_{=2} \cdot \underbrace{(2g_{\mathbb{P}^1} - 2)}_{-2} + \underbrace{\sum_{\omega \in E} (e_\omega - 1)}_{=4}$