Exercise Sheet 5

Discussed on 12.05.2021

Problem 1. Let $f\colon X \to S$ be a map of schemes, let $\mathcal{L}$ be a line bundle on X and let $t_0, \dots, t_n \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$, i.e. the induced map $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective.

(a) Show that the section $t_0, \dots, t_n$ determine a canonical S-morphism $t\colon X \to \mathbb{P}^n_S$.

Hint: Reduce to the case that $S = \operatorname{Spec} A$ is affine, then argue as in the case where S is a field.

(b) Suppose that f is proper. Show that t is a closed immersion if and only if for all $s \in S$, the fiber $t(s)\colon X_s \to \mathbb{P}^n_{\kappa(s)}$ is a closed immersion.

Hint: Note first that t is proper and has finite fibers. You may then use the (non-trivial) fact that these two properties imply finiteness of t (cf. `https://stacks.math.columbia.edu/tag/02LS`). You may now check the surjectivity of $\mathcal{O}_{\mathbb{P}^n_S} \to t_*\mathcal{O}_X$ with Nakayama, which you can reduce to fibers.

Problem 2. Let S be a noetherian scheme and let $f\colon X \to S$ be a proper smooth map all of whose fibers are geometrically connected curves of genus 1. Let furthermore $e\colon S \to X$ be a section of f. Show that locally on S there exist sections $a_1, a_2, a_3, a_4, a_6 \in \Gamma(S, \mathcal{O}_S)$ such that

$$X \cong V_+(y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6) \subset \mathbb{P}^2_S.$$

Hint: Consider the line bundles $\mathcal{O}_X(i \cdot e)$ on X for $i \in \mathbb{Z}$ defined in the lecture. Then use Problem 1.

Problem 3. (a) Let k be a field and let C be a proper smooth connected curve of genus 0 over k. If there is a k-rational point on C then C is isomorphic (over k) to $\mathbb{P}^1_k$.

Hint: Use Riemann-Roch.

(b) Let S be a noetherian scheme and let $f\colon X \to S$ be a proper smooth map all of whose fibers are geometrically connected curves of genus 0. Let furthermore $e\colon S \to X$ be a section of f. Show that locally on S we have $X \cong \mathbb{P}^1_S$.