AN ARITHMETIC SIEGEL–WEIL FORMULA FOR CENTRAL SIMPLE ALGEBRAS

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Contents

1. Introduction

The purpose of this paper[1] is to prove an arithmetic Siegel–Weil (ASW) formula for the non-singular Fourier coefficients in a unitary central simple algebra (CSA) setting. More precisely, we adapt the unitary ASW formalism of Kudla [12] and Kudla–Rapoport [16] to unitary Shimura varieties that are defined by a CSA with involution of the second kind over an imaginary-quadratic field. We define arithmetic 0-cycles on the integral models of such varieties and relate their degrees to the Fourier coefficients of the first derivative of an incoherent Eisenstein series.

Our result is purely global in the following sense: The 0-cycles we consider are supported over the supersingular locus at non-split primes, which is completely analogous to the situation in [16]. A theorem of Landherr states that the CSA in question is split at such a place. Using Morita equivalence, this implies that the local theory of our setting is identical with that of Kudla–Rapoport [15]. In particular, the theorem of Li–Zhang [20] (formerly KR Conjecture) as well as its variants (see below) apply and express the degrees in question in terms of derivatives of local Whittaker functions. Our main addition is then a Siegel–Weil formula (for the non-singular coefficients) that allows to relate these quantities to Eisenstein series.

The initial motivation for our article came from [22, §8]. There, Madapusi introduces a general formalism for generating series of special cycles on Shimura varieties. These series are indexed by the symmetric or hermitian elements in central simple algebras with involution. Moreover, he formulates a modularity conjecture for such series [22, Conjecture 8.4]. Our setting is a special case of his formalism, and our main result, which we next describe in more detail, is closely related to his conjecture.

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[1]This is still a draft version. Sections 1–5 are already finished, Sections 6 and 7 will be finalized soon.

1.1. **The Siegel–Weil formula.** Let E be a number field, let D/E be a central simple algebra, and let $*: D \to D$ be an involution of the second kind. Denote by $F = E^{*=\mathrm{id}}$ the fixed field of $*|_E$ and by $n = \dim_E(D)^{1/2}$ the degree of D. Consider a hermitian right D-module V and denote by $G = U_D(V)$ its unitary group. We assume that V is free of rank m as D-module. Also consider the quasi-split skew-hermitian left D-module $W = D^\ell \oplus D^\ell$ and set $H = U_D(W)$. Then (G, H) forms a Howe dual pair in $Sp(V \otimes_D W)$ and there is the $H(\mathbb{A})$-equivariant Rallis map

$$\lambda : S(V(\mathbb{A})^\ell) \longrightarrow I(s_0, \chi), \qquad s_0 = \frac{n(m-\ell)}{2} \tag{1.1}$$

from Schwartz functions on $V(\mathbb{A})^\ell$ to an induced representation of $H(\mathbb{A})$. Assume that V is free of rank m as D-module and that D is of degree n over E. Attached to a standard section $\Phi \in I(s, \chi)$, there is an Eisenstein series $E(h, s, \Phi)$, where $h \in H(\mathbb{A})$ and $s \in \mathbb{C}$. It has a Fourier expansion of the form

$$E(h, s, \Phi) = \sum_{\xi \in \mathrm{Herm}_\ell(D)} E_\xi(h, s, \Phi) \tag{1.2}$$

where $\mathrm{Herm}_\ell(D)$ denotes the hermitian $(\ell \times \ell)$-matrices with values in D. Further assume that $\xi \in \mathrm{Herm}_\ell(D)$ is invertible, i.e. $\xi \in GL_\ell(D)$. Then there is also the theta integral (when convergent)

$$I_\xi(h, \Phi) = \frac{1}{2} \int_{[G]} \sum_{x \in V^\ell,\ (x,x)=\xi} \omega(h)\Phi(g^{-1}x) dg, \qquad h \in H(\mathbb{A}). \tag{1.3}$$

If V^ℓ does not represent ξ, then we simply have $I_\xi(h, \Phi) = 0$. The following is our main result in the current setting.

Proposition 1.1. *Assume that $m \geq \ell$ and that $\xi \in \mathrm{Herm}_\ell(D)$ is invertible. Then $E_\xi(h, s, \Phi)$ is holomorphic at $s = s_0$, and $I_\xi(h, \Phi)$ is absolutely convergent. Moreover,*

$$E_\xi(h, s_0, \Phi) = \kappa I_\xi(h, \Phi) \tag{1.4}$$

with $\kappa = 1$ if $m > \ell$ and $\kappa = 2$ if $m = \ell$.

If $D = M_n(E)$ with standard involution, then Proposition 1.1 is a special case of the unitary Siegel–Weil formula of Ichino [10]. Our proof follows closely his ideas. In general, the case of reductive dual pairs for central simple algebras has already been considered by Weil [29, 30] and he proves an identity of the form (1.4) when $m > 2n$ (see Theorem 4.1 in the text).

1.2. **The Arithmetic Siegel–Weil formula.** We now set $\boldsymbol{k} = E$ and assume it is an imaginary-quadratic field. We also assume that the involution on D is positive, and that V is free of rank 1 and of signature $(n-1, 1)$. Then, the unitary group $G = U_D(V)$ is a (form of a) unitary group in n variables. Let $\widetilde{G} = \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}(\mathbb{G}_m) \times G$ be its RSZ variant as introduced by Kudla–Rapoport [16]. There is a natural way to define a Shimura variety for $\widetilde{G}$, which is an $(n-1)$-dimensional variety M over $\boldsymbol{k}$. Our case of interest is when $D \not\cong M_n(\boldsymbol{k})$. Then $U_D(V)$ is anisotropic so M is proper. Let d be the order of D in the Brauer group. Concretely, if $D \cong M_m(D_0)$ for a division algebra $D_0/\boldsymbol{k}$, then $d = \dim_{\boldsymbol{k}}(D_0)^{1/2}$. Then there are natural special cycles in all codimensions which are a multiple of d.

With suitable choices of integral data, this Shimura variety has an integral model $\mathcal{M} \to \mathrm{Spec}\, O_{\boldsymbol{k}}$ that parametrizes pairs (E, A) consisting of an elliptic curve E with CM by $O_{\boldsymbol{k}}$, and a polarized abelian variety A with action by a fixed maximal order $O_D \subset D$. For every such pair,

$$L(E, A) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_{O_{\boldsymbol{k}}}(E, A)$$

is a hermitian D-module. Thus for every hermitian element $\xi \in D$, there is a cycle $\mathcal{Z}(\xi) \to \mathcal{M}$ that parametrizes triples (E, A, x) where $x \in L(E, A)$ with $(x, x) = \xi$. Note that if $D = M_n(\boldsymbol{k})$ with standard involution, then the definition specializes to that of [16].

We only consider the situation when ξ is invertible. In this case, the expected dimension of $\mathcal{Z}(\xi)$ is 0 but its actual dimension might be strictly larger. Recall from [16] that if $D = M_n(\boldsymbol{k})$, then one may give a definition of a 0-cycle class $[\mathcal{Z}(\xi)]$ as successive intersection of special divisors. When $D \ncong M_n(\boldsymbol{k})$ however, then there are no such divisors. We instead define $[\mathcal{Z}(\xi)]$ through basic uniformization as indicated at the beginning of the introduction.

Our definition of $\mathcal{M}$ is tailored such that the local results of Li–Zhang in [20] (unramified self-dual case), Li–Liu in [19] (even-dimensional exotic smooth case), He–Li–Shi–Yang in [7] (Krämer model case), Cho–He–Zhang in [4] (certain unramified maximal parahoric level cases), and H. Yao in his upcoming work (odd-dimensional exotic smooth case), as well as Y. Liu in [21] and independently Garcia–Sankaran in [6] (archimedean place), give a full description of the arithmetic degree of $[\mathcal{Z}(\xi)]$ in all situations.

Consider the case $\ell = 1$ of §1.1, i.e. $I(s, \chi)$ now denotes the induced representation of $H = U_D(D \oplus D)$. The cited works provide a place-by-place definition of two specific standard sections $\Phi, \Psi \in I(s, \chi)$ whose local Whittaker functions can be used to express the degrees of all $\mathcal{Z}(\xi)$. Using the Siegel–Weil formula (1.4), it is then not difficult to deduce our main result:

Theorem 1.2. *There exists a constant $c > 0$ such that for all positive hermitian invertible elements $\xi \in D$,*

$$\widehat{\deg}\big([\widehat{\mathcal{Z}}(\xi)]\big) = c\, E'_{\xi}(1, 0, \Phi) + c\, E_{\xi}(1, 0, \Psi). \tag{1.5}$$

Assume that D is a division algebra. Then every non-zero hermitian $\xi \in D$ is invertible. Thus the only missing Fourier coefficient in Theorem 1.2 is the 0-th coefficient. The arithmetic degree of $[\mathcal{Z}(0)]$ should essentially be the arithmetic volume of $\mathcal{M}$. If $D = M_n(\boldsymbol{k})$, then a recent result of Bruinier–Howard [1] expresses this volume in terms of logarithmic derivatives of L-functions and it would be interesting to consider this problem also when $D \ncong M_n(\boldsymbol{k})$. Since the proof in [1] relies on the modularity result of [2], and since there are no special divisors on $\mathcal{M}$ when D is non-split, new arguments are needed.

2. Hermitian forms over central simple algebras

2.1. **Hermitian D-modules.** Throughout this section, E/F denotes a quadratic field extension and D/E a central simple algebra. We denote by $\sigma = \sigma_{E/F}$ or by $a \mapsto \bar{a}$ the Galois conjugation of E/F. We write $N_{E/F} : E^\times \to F^\times$ for the norm map and set $E^1 = \ker(N_{E/F})$. We denote by $\mathrm{Trd}, \mathrm{Nrd} : D \to E$ the reduced norm and the reduced trace. They satisfy $\mathrm{Trd}(x^*) = \sigma(\mathrm{Trd}(x))$ and $\mathrm{Nrd}(x^*) = \sigma(\mathrm{Nrd}(x))$.

Definition 2.1. An involution $* : D \to D$ of the second kind (with respect to F) is an F-linear map such that

$$*^2 = \mathrm{id}, \qquad (ab)^* = b^* a^* \text{ for all } a, b \in D, \qquad *|_E = \sigma.$$

For any $\ell \geq 1$, we extend $*$ to the matrix ring $M_\ell(D)$ as

$$(x_{ij})^* = {}^t(x_{ij}^*). \tag{2.1}$$

We call a matrix $\beta \in M_\ell(D)$ hermitian if $\beta^* = \beta$, and we denote these elements by $\mathrm{Herm}_\ell(D)$. We also define $\mathrm{Herm}(D) := \mathrm{Herm}_1(D)$ and $\mathrm{Herm}_\ell^\times(D) := \mathrm{Herm}_\ell(D) \cap GL_\ell(D)$.

Fix an involution $*$ of the second kind on D and let $\dagger$ be any other such involution. Then $\dagger \circ * : D \to D$ is an E-algebra automorphism of D. By Skolem–Noether, this means there exists an element $\beta \in D^\times$ such that $x^\dagger = \beta x^* \beta^{-1}$. As $\dagger$ is an involution, any such β satisfies $\beta^* = \lambda\beta$ for some $\lambda \in E^1$. By Hilbert's Theorem 90, the map $E^\times \to E^1$, $c \mapsto c/\bar{c}$ is surjective. This implies that we may choose such a β with $\beta^* = \beta$. In this way,

$$\mathrm{Herm}^\times(D)/F^\times \xrightarrow{\sim} \{\text{Involutions } \dagger : D \to D \text{ of the second kind}\}. \tag{2.2}$$

Definition 2.2. A hermitian (right) D-module is a finite right D-module V together with a non-degenerate F-bilinear pairing $(\ ,\ ) : V \times V \to D$ such that

$$(y, x) = (x, y)^* \quad \text{and} \quad (xa, yb) = a^*(x, y)b \quad \text{for all } x, y \in V,\ a, b \in D.$$

Note that we do not assume V to be free as D-module. The unitary group of $(V, (\ ,\ ))$ is defined by

$$U(V,\ (\ ,\ )) = \{g \in \mathrm{Aut}_D(V) \mid (gx, gy) = (x, y) \ \text{ for all } x, y \in V\}. \tag{2.3}$$

Assume that V and W are hermitian D-modules. Then there is an adjoint isomorphism

$$\mathrm{Hom}_D(V, W) \xrightarrow{\sim} \mathrm{Hom}_D(W, V), \qquad f \longmapsto f^\dagger \tag{2.4}$$

where $f^\dagger$ is the unique homomorphism that satisfies $(x, fy) = (f^\dagger x, y)$ for all $x \in V$, $y \in W$. As a special case, we may consider the endomorphism ring $\mathrm{End}_D(V)$, which is again a CSA over E. The adjoint map $\dagger : \mathrm{End}_D(V) \to \mathrm{End}_D(V)$ is again an involution of the second kind. In these terms, the unitary group of V may also be described by

$$U(V,\ (\ ,\ )) = \Big\{g \in \mathrm{End}_D(V) \mid g^\dagger g = 1\Big\}. \tag{2.5}$$

It is moreover easily checked that there is a bijection

$$\begin{aligned} \mathrm{Herm}^\times(\mathrm{End}_D(V), \dagger) &\xrightarrow{\sim} \{D\text{-hermitian forms on } V\} \\ \beta &\longmapsto (x, y)_\beta := (x, \beta y). \end{aligned} \tag{2.6}$$

Example 2.3. We write D^ℓ for the right D-module of column vectors of length ℓ. Then $\mathrm{End}_D(V) = M_\ell(D)$ via matrix multiplication from the left. Moreover, there is the standard hermitian form $(x, y) := x^* \cdot y$ on D^ℓ. Its adjoint is the involution $* : M_\ell(D) \to M_\ell(D)$ from (2.1). Identity (2.6) states that the D-valued hermitian forms on D^ℓ are in bijection with $\mathrm{Herm}_\ell(D)$ via

$$(x, y)_\beta := x^* \beta y, \qquad x, y \in V, \quad \beta \in \mathrm{Herm}_\ell(D). \tag{2.7}$$

The unitary group of $(\ ,\ )_\beta$ has the explicit description

$$U\big(D^\ell,\ (\ ,\ )_\beta\big) = \{g \in GL_\ell(D) \mid g^* \beta g = \beta\}. \tag{2.8}$$

Our next aim is to formulate a Morita equivalence for hermitian D-modules. In this way, the theory can always be reduced to hermitian modules over division algebras. First assume that $D = M_m(Q)$ for a CSA Q over E, and that $*$ preserves Q in the sense that $(x_{ij})^* = {}^t(x_{ij}^*)$ where $*$ also denotes an involution on Q.

Lemma 2.4. *There is an equivalence of categories*

$$\{\textit{Hermitian } (Q, *)\textit{-modules}\} \xrightarrow{\sim} \{\textit{Hermitian } (M_m(Q), *)\textit{-modules}\} \tag{2.9}$$

that takes a hermitian Q-module $(V, (\ ,\ ))$ to the right $M_m(Q)$-module of row vectors $V^{(m)}$ with hermitian form

$$(x \otimes e_i, y \otimes e_j) := (x, y)e_{ij}.$$

This equivalence commutes with the adjoint involutions (2.4) *on the two categories. In particular,*

$$U_Q(V) \xrightarrow{\sim} U_{M_m(Q)}(V^{(m)}).$$

Let Q, $D = M_m(Q)$ and $*$ be as before. In general, given an involution of the second kind $\dagger$ on D, it might not be possible to find an isomorphism $(D, \dagger) \cong (D, *)$. For example, consider $M_m(\mathbb{C})$ and let $\dagger$ be the adjoint involution of a hermitian form h on $\mathbb{C}^m$. Then $(M_m(\mathbb{C}), \dagger) \cong (M_m(\mathbb{C}), *)$ if and only if h is definite. For this reason, we now formulate a second equivalence that allows to pass between $*$-hermitian and $\dagger$-hermitian D-modules.

Lemma 2.5. *Let $*$ and $\dagger$ be involutions of the second kind on D. Fix a $*$-hermitian element $\beta \in D^\times$ such that $x^\dagger = \beta x^* \beta^{-1}$. Then there is an equivalence of categories*

$$\begin{aligned} \{*\text{-Hermitian } D\text{-modules}\} &\xrightarrow{\sim} \{\dagger\text{-Hermitian } D\text{-modules}\} \\ (V, (\ ,\ )) &\longmapsto (V, \beta(\ ,\ )). \end{aligned} \tag{2.10}$$

This equivalence commutes with the adjoint involutions (2.4) *on the two categories. In particular,*

$$U_D(V, (\ ,\ )) = U_D(V, \beta(\ ,\ ))$$

as subgroups of $GL_D(V)$.

We end this section with a definition of the Hasse invariant.

Definition 2.6. Let $(D, *)$ be a CSA over E with involution of the second kind and let V be a free hermitian D-module. Choose any basis $V \cong D^\ell$ and let

$$\chi(V) := \mathrm{Nrd}_{M_\ell(D)}\big((e_i, e_j)\big) \in F^\times / N_{E/F}(E^\times). \tag{2.11}$$

Then $\chi(V)$ is independent of the choice of basis and is called the Hasse invariant of V.

2.2. **Lifting forms along the trace.** There is an equivalent perspective on hermitian D-modules. Given $(V, (\ ,\ ))$, consider the F-bilinear form

$$(\ ,\ )_E := \mathrm{Trd} \circ (\ ,\ ).$$

It is an E-valued hermitian form on V that it is compatible with the D-action in the sense that

$$(x, ya)_E = (xa^*, y)_E \quad \text{for all } x, y \in V,\ a \in D. \tag{2.12}$$

Lemma 2.7. *The map $(\ ,\ ) \mapsto (\ ,\ )_E$ defines a bijection between D-valued hermitian forms on V and E-valued non-degenerate hermitian forms that satisfy* (2.12)*.*

Proof. Consider the two E-vector spaces

$$\mathrm{Hom}_D(V, D) \quad \text{and} \quad \mathrm{Hom}_E(V, E).$$

Both are D-left modules; the first via $(a \cdot \varphi)(x) = a\varphi(x)$, the second via $(a \cdot \psi)(x) = \psi(xa)$. The natural map

$$\mathrm{Hom}_D(V, D) \longrightarrow \mathrm{Hom}_E(V, E), \quad \varphi \longmapsto \mathrm{Trd} \circ \varphi \tag{2.13}$$

D-linear because for all $x \in V$,

$$(\mathrm{Trd} \circ (a\varphi))(x) = \mathrm{Trd}(a\varphi(x)) = \mathrm{Trd}(\varphi(x)a) = \mathrm{Trd}(\varphi(xa)) = (a \cdot (\mathrm{Trd} \circ \varphi))(x).$$

Moreover, (2.13) is an isomorphism. Indeed, this can be checked after base change to an algebraic closure where it reduces to the statement that

$$\mathrm{Hom}_{M_n(F)}(F^{(n)}, M_n(F)) \xrightarrow{\sim} \mathrm{Hom}_F(F^{(n)}, F), \quad \varphi \mapsto \mathrm{Tr} \circ \varphi.$$

Now note that a D-valued hermitian form on V is nothing but a D-linear isomorphism

$$\beta : V \xrightarrow{\sim} \mathrm{Hom}_D(V, D) \tag{2.14}$$

such that

$$\beta^\vee : V = \mathrm{Hom}_D(\mathrm{Hom}_D(V, D)) \longrightarrow \mathrm{Hom}_D(V, D)$$

agrees with β. Here, β being D-linear means that $\beta(xa) = a^*\beta(x)$. The lemma follows from these statements. $\square$

We mention a nice application of Lemma 2.7 which however will not be used in the article. Let $(W, (\ ,\ ))$ be a $*$-hermitian left Q-module. We also view it as a right $\mathrm{End}_Q(W)$-module and denote by $f \mapsto f^\dagger$ the adjoint involution. By definition of $\dagger$, the form $(\ ,\ )_E$ satisfies

$$(x, yf)_E = (xf^\dagger, y)_E, \qquad x, y \in W,\ f \in \mathrm{End}_Q(W). \tag{2.15}$$

By Lemma 2.7, there exists a unique lifting $\langle\ ,\ \rangle$ of $(\ ,\ )_E$ to an $\mathrm{End}_Q(W)$-valued $\dagger$-hermitian form. The observation is that this constructions can be extended to generalizes the functors from Lemmas 2.4 and 2.5: Let V be a $*$-hermitian right Q-module. Then $V \otimes_Q W$ carries the E-hermitian form

$$\langle v_1 \otimes w_1,\ v_2 \otimes w_2 \rangle_E := \mathrm{Trd}_{Q/E}\left((v_1, v_2)(w_1, w_2)\right).$$

This form is compatible with $\dagger$ for the right action of $\mathrm{End}_Q(W)$ on $V \otimes_Q W$ in the sense of (2.15). Thus Lemma 2.7 applies and show that there is a unique $\mathrm{End}_Q(W)$-valued hermitian form $\langle\ ,\ \rangle$ on $V \otimes_Q W$ such that

$$\langle\ ,\ \rangle_E = \mathrm{Trd}_{\mathrm{End}_Q(W)/E} \circ \langle\ ,\ \rangle.$$

This defines a functor

$$\{*\text{-Hermitian } Q\text{-modules}\} \longrightarrow \{\dagger\text{-Hermitian } \mathrm{End}_Q(W)\text{-modules}\} \tag{2.16}$$

which is compatible with adjoint involutions. Applying this construction when $W = Q^{(m)}$ recovers the (composition of) the functors of Lemmas 2.4 and 2.5.

2.3. **Local fields.** Assume that E/F is a separable quadratic extension of local fields. Let $(D, *)$ be a CSA over E of degree n with involution of the second kind. A classical Theorem of Landherr, see [27, Theorem 2.4], states that D is isomorphic to $M_n(E)$. Fix an isomorphism $\gamma : D \xrightarrow{\sim} M_n(E)$; this endows $M_n(E)$ with the involution $\dagger = \gamma \circ * \circ \gamma^{-1}$. Fix a hermitian form β on $E^{(n)}$ such that $x^\dagger = \beta x^* \beta^{-1}$. Then Lemmas 2.4 and 2.5 construct an equivalence of categories

$$\begin{aligned} \left\{\begin{matrix}\text{Hermitian } E\text{-vector spaces} \\ \text{of dimension } \ell n\end{matrix}\right\} &\xrightarrow{\sim} \left\{\begin{matrix}\text{Hermitian } D\text{-modules} \\ \text{free of rank } \ell\end{matrix}\right\} \\ V &\longmapsto V \otimes_E (E^{(n)}, \beta). \end{aligned} \tag{2.17}$$

It is immediate from definitions that the Hasse invariants of V (in the usual sense) and of $V \otimes_E (E^{(n)}, \beta)$ (in the sense of Definition 2.6) are related by

$$\chi(V \otimes_E (E^{(n)}, \beta)) = \det(\beta)^\ell \cdot \chi(V). \tag{2.18}$$

Corollary 2.8. *(1) Assume that E/F is a quadratic extension of non-archimedean local fields. Two free hermitian D-modules are isomorphic if and only if they have the same rank and Hasse invariant.*

(2) Consider $D = M_n(\mathbb{C})$ with standard involution $x^* = {}^t\bar{x}$. Two hermitian D-modules are isomorphic if and only if the signatures of their underlying $\mathbb{C}$-valued hermitian forms (Lemma 2.7) have the same signature.

Proof. (1) follows from the 2.19 together with (2.18) and the classification of hermitian forms over non-archimedean local fields. (2) follows from the equivalence in Lemma 2.4 and the classification of hermitian $\mathbb{C}$-vector spaces by their signature. □

Assume from now on that E is non-archimedean. We are interested in the question of whether there exists a $*$-stable maximal order $O_D \subset D$. This depends on the isomorphism class of $(D, *)$: Via (2.2), the isomorphism classes of pairs $(D, *)$ with $D \cong M_n(E)$ are in bijection with the similitude classes of hermitian E-vector spaces of dimension n.

Lemma 2.9. *Let $\dagger : M_n(E) \to M_n(E)$ be the adjoint involution of a hermitian form β on E^n. Let $\Lambda^\vee$ denote the β-dual of an O_E-lattice $\Lambda \subset E^n$. Then $M_n(E)$ contains a $\dagger$-stable maximal order if and only if there exists an O_E-lattice $\Lambda \subset E^n$ such that $\Lambda^\vee = a\Lambda$ for some $a \in E^\times$.*

Proof. The maximal orders in $M_n(E)$ are in bijection with the homothety classes of O_E-lattices in V via $\Lambda \mapsto \mathrm{End}(\Lambda)$. Moreover, one checks that $\mathrm{End}(\Lambda^\vee) = \mathrm{End}(\Lambda)^\dagger$ which then implies the lemma. □

Finally, assume that there exists a maximal $*$-stable order $O_D \subseteq D$. Then $(O_D, *) \cong (M_n(O_E), \dagger)$ for some involution $\dagger$ on $M_n(O_E)$. It is seen as in the case of fields that there exists a hermitian element $\beta \in GL_n(O_E)$ such that $x^\dagger = \beta x^* \beta^{-1}$. Then Lemmas 2.4 and 2.5 extend verbatim and provide a Morita equivalence

$$\begin{aligned} \{\text{Hermitian } O_E\text{-modules}\} &\xrightarrow{\sim} \{\dagger\text{-Hermitian } M_n(O_E)\text{-modules}\} \\ L &\longmapsto L \otimes_{O_E} (O_E^{(n)}, \beta). \end{aligned} \tag{2.19}$$

2.4. **Global fields.** Assume now that E/F is a quadratic extension of global field. Let $(D, *)$ be a CSA with involution of the second kind over E as before. We state now the main results on the classification of hermitian D-modules. By hyperbolic plane over D, we mean a hermitian D-module that is isomorphic to $(D^{(2)}, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right))$. We call a hermitian D-module isotropic if it admits a direct summand that is a hyperbolic plane. If D is a division algebra, then this condition is equivalent to the unitary group being isotropic as algebraic group over F. The following theorem is due to Landherr [18].

Theorem 2.10 (Hasse principle, [27, §10, Theorems 6.1 and 6.2])**.** *(1) Two hermitian D-modules are isomorphic if and only if for all places v of F, their v-adic completions are isomorphic hermitian D_v-modules.*

(2) Assume that D is a division algebra. Then a hermitian D-module is isotropic if and only if for every place v of F, its v-adic completion is an isotropic hermitian D_v-module.

Definition 2.11. The involution $*$ is called positive if for every archimedean non-split place v of F, there is an isomorphism $(D_v, *) \cong (M_n(\mathbb{C}), x \mapsto {}^t\bar{x})$. Equivalently, for every such place v and every $x \in D_v$, the trace $\mathrm{Trd}(x^*x) \in F_v$ is non-negative.

Theorem 2.12 (Classification, [27, §10, Theorem 6.9]). *Assume that D is a division algebra and that $*$ is positive. Then, for every integer $\ell \geq 1$, for every Hasse invariant $\chi \in F^\times/N_{E/F}(E^\times)$, and for every family of signatures $(p_v, q_v)_v$, $p_v + q_v = \ell$, where v runs through the archimedean non-split places of F, such that for every v*

$$\chi \equiv (-1)^{q_v} \mod F_v^{>0},$$

there exists a hermitian D-module V that is free of rank ℓ with $\chi(V) = \chi$ and $\mathrm{sign}(V_v) = (p_v, q_v)$ for all v. It is unique up to isomorphism by Theorem 2.10.

3. Local reductive dual pairs and local Siegel–Weil formula

3.1. **Local Siegel–Weil formula.** In this section, we assume that F is a local field and that E is an etale quadratic extension of F, i.e., $E = F \times F$ or E is a quadratic field extension of F. We denote by $\epsilon_{E/F} : F^\times \to \{\pm 1\}$ the quadratic character of E/F. Let $(D, *)$ be a central simple algebra over E together with an involution $*$ of the second kind. We assume that $E^{*=\mathrm{id}} = F$ and define $n = \mathrm{rk}_E(D)^{1/2}$. Let ψ be a non-trivial additive character of F, and let $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ and $\psi_D = \psi \circ \mathrm{Tr}_{E/F} \circ \mathrm{Trd}_{D/E}$. We also set $\mathrm{Trd}_{D/F} = \mathrm{Tr}_{E/F} \circ \mathrm{Trd}_{D/E}$.

Let $(V, (\,,\,))$ be a (non-degenerate) free hermitian right D-module of rank m, and let $G = U(V, (\,,\,))$ be its automorphism group as in §2.1, viewed as topological group over F. We also denote this group by $U_D(V)$ for simplicity. Similarly, let $(W, \langle\,,\,\rangle)$ be a (non-degenerate) free skew-hermitian left D-module of rank m' with automorphism group $H = U(W, \langle\,,\,\rangle) = U_D(W)$. Endow $\mathcal{W} = V \otimes_D W$ with the F-symplectic form

$$\ll v_1 \otimes w_1, v_2 \otimes w_2 \gg = \mathrm{Trd}_{D/F}(v_1, v_2)(w_1, w_2)^* \tag{3.1}$$

and let $\mathrm{Sp}(\mathcal{W}) = \mathrm{Sp}(2m'mn^2)$ be the symplectic group of $\mathcal{W}$. The natural actions of G and H on V and W provide a natural map

$$\iota : G \times H \to \mathrm{Sp}(\mathcal{W}), \tag{3.2}$$

which realizes (G, H) as a reductive dual pair in $\mathrm{Sp}(\mathcal{W})$. Let χ be a character of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{mn}$, then there is a Weil representation $\omega = \omega_{V,\chi,\psi}$ of $G \times H$, which can be described explicitly when W is split. Assume from now on that $W = X \oplus Y$ is split with both X and Y totally isotropic, and let P be the stabilizer of X in H. Then the Weil representation is on the space $S(V \otimes_D X)$ of Schwartz functions. To use coordinates, let Write $m' = 2l$, and let $\{e_1, \cdots, e_l, f_1, \cdots f_l\}$ be a standard D-basis of W with gram matrix $J = \left(\begin{smallmatrix} 0 & I_l \\ -I_l & 0 \end{smallmatrix}\right)$ and $e_i \in X, f_j \in Y$. With respect to this standard basis, we have concrete realization (we will also write $U((l,l), J)$ and $U_D((l,l))$ instead of $U_D(X \oplus Y)$ in the following)

$$U_D(l,l) = \{g \in \mathrm{GL}_{2l}(D) : gJg^* = J\}.$$

Moreover, P has the decomposition $P = NM$ with

$$N = \left\{ n(b) = \left(\begin{smallmatrix} I_l & b \\ 0 & I_l \end{smallmatrix}\right) : b^* = b \right\}$$

$$M = \left\{ m(a) = \mathrm{Diag}(a, (a^*)^{-1}) : a \in \mathrm{GL}_l(D) \right\}$$

Here we wrote $(a_{ij})^* = {}^t(a_{ij}^*)$ for matrices $(a_{ij}) \in M_l(D)$ which is the standard extension of $*$ to matrices. It defines an involution of the second kind on $M_l(D)$. Let $I(s, \chi) = \mathrm{Ind}_P^H(\chi|\,|_E^s)$ be the induced representation whose sections (elements) are given by smooth functions Φ on H such that

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|_E^{s+\rho_{n,l}} \tag{3.3}$$

with $\rho_{n,l} = \frac{nl}{2}$. Here we also introduced the notations $\chi(a) = \chi(\mathrm{Nrd}_{M_l(D)/E}(a))$ and $|a|_E = |\mathrm{Nrd}_{M_l(D)/E}(a)|$. For any $T \in \mathrm{Herm}_l(D)$, its T-th Whittaker function (with respect to ψ) is defined to be

$$W_T(h, s, \Phi) = \int_{\mathrm{Herm}_l(D)} \Phi(Jn(b)h, s)\psi_D(-bT)db, \tag{3.4}$$

where db is the self-dual Haar measure on $\mathrm{Herm}_l(D)$ with respect to ψ_D.

We identify $V \otimes_D X$ with V^l via the basis $\{e_1, \cdots, e_l\}$. Then the Weil representation ω of $G \times H$ can be realized on $S(V^l)$ with G acting as

$$\omega(g)\phi(x) = \phi(g^{-1}x), \tag{3.5}$$

and $U_D(l, l)$ acting as

$$\begin{aligned} \omega(n(b))\phi(x) &= \psi_D(b(x, x))\phi(x), \\ \omega(m(a))\phi(x) &= \chi(a)|a|_E^{\frac{mn}{2}}\phi(xa), \\ \omega(J)\phi(x) &= \gamma(V)\int_{V^l}\phi(y)\psi_D((x, y))\,dy, \end{aligned} \tag{3.6}$$

where $\gamma(V)$ is the local Weil index (an 8-th root of unity). Here db is the self-dual Haar measure with respect to ψ_D. Setting $s_0 = \frac{n(m-l)}{2}$, there is an H-linear map—the Rallis map:

$$\lambda : S(V^l) \longrightarrow I(s_0, \chi), \quad \lambda(\phi)(h) = \omega(h)\phi(0). \tag{3.7}$$

This map does not depends on the choice of the basis and only depends on the choice of X (we need X to define P). Let $R(V)$ denote the image of this map. Then it is known that $R(V)$ is the largest quotient of $S(V^l)$ such that G acts trivially. For its properties, such as irreducibility, we refer to [17].

Let $\mathcal{H}_D = D^2$ be the right D-hyperplane with hermitian form $(x, y) = x_1^*y_2 + x_2^*y_1$. For an integer $r \geq 0$, let $V_r = V \oplus \mathcal{H}_D^r$. For a function $\phi \in S(V^l)$, let $\phi_r = \phi \otimes \mathrm{char}(O_D^{2rl}) \in S(V_r^l)$. Denote by $\Phi = \Phi_\phi(h, s)$ the standard section in $I(s, \chi)$ with $\Phi(h, s_0) = \lambda(\phi)(h)$, then $\Phi(h, s_0 + r) = \omega)h)\phi_r(0)$. We denote

$$W_T(h, s, \phi) = W_T(h, s, \Phi).$$

Then we have

$$\begin{aligned} W_T(h, s_0 + r, \phi) &= \int_{\mathrm{Herm}_l(D)} \Phi(h, s_0 + r)\psi_D(-bT)db \\ &= \gamma(V)\int_{\mathrm{Herm}_l(D)}\int_{V_r^l}\omega(h)\phi_r(x)\psi_D(b((x, x) - T))dx\,db \end{aligned} \tag{3.8}$$

Here, dx and db denote the Haar measures that are self-dual with respect to ψ_D.

We view V^l as an affine variety and let V_{reg}^l be the subvariety of vectors $v = (v_1, \cdots, v_l) \in V^l$ such that the D-module generated by $v_1, \cdots, v_l$ has is free of rank l, and let $\mathrm{Herm}_l(D)^\times \subset M_l(D)$ be the set of invertible hermitian matrices of rank l. Then the map

$$\alpha : V_{reg}^l \longrightarrow \mathrm{Herm}_l^\times(D), \quad x \longmapsto (x, x)$$

is a moment map. Choose a translation invariant top differential form ω_T on $\Omega_T = \alpha^{-1}(T)$ and a top translation invariant differential form ω_D on $\mathrm{Herm}_l(D)$ such that $\omega_V = \omega_T \wedge \omega_D \neq 0$ is a top translation invariant differential form on V_{reg}^l. Let $|\omega_T|$ be the associated Haar measure on Ω_T with respect to ω_T and ψ_D (see [10, §9]). Moreover $|\omega_D| = db$ and $|\omega_V| = dx$ are self-dual with respect to ψ_D.

On the other hand, G acts on Ω_T transitively. Choose and fix an $x \in \Omega_T$ (if it exists), and let G_x be its stabilizer of x in G, then

$$G_x\backslash G \xrightarrow{\sim} \Omega_T, \quad g \longmapsto g^{-1}x. \tag{3.9}$$

Notice that G acts on the top differential ω_T via the reduced norm map which has norm 1 Choose a top invariant differential ω_x on G_x so that $\omega_D = \omega_x \wedge \Omega_T$ is an invariant top differential on G (up to a scalar in E^1. So they induce Haar measures with $|\omega_G|/|\omega_x| = |\omega_T|$. The following is an analogue to the local Siegel–Weil formula in [3, Section 2].

Proposition 3.1 (Local Siegel–Weil formula)**.** *Let $T \in \mathrm{Herm}_l(D)^\times$ and $\phi \in S(V^l)$. Then*

$$\gamma(V)^{-1}W_T(h, s_0, \phi) = \int_{\Omega_T} \phi(x)|\omega_T| = \int_{G_x\backslash G} \phi(g^{-1}x)|\omega_G|/|\omega_x|.$$

Proof. The second identity is true by definition. For the first one, we calculate

$$\begin{aligned}
\gamma(V)^{-1}W_T(h, s_0, \phi) &= \int_{\mathrm{Herm}_l(D)} \int_{V^l} \phi(x)\psi_D(b((x, x) - T))|\omega_V||\omega_D| \\
&= \int_{\mathrm{Herm}_l(D)} \int_{\mathrm{Herm}_l(D)} M_\phi(\beta)\psi_D(b(\beta - T))|\omega_D|(\beta)|\omega_D|(b) \\
&= \int_{\mathrm{Herm}_l(D)} \hat{M}_\phi(-b)\psi_D(-bT)|\omega_D|(b) \\
&= \hat{\hat{M}}_\phi(-T) \\
&= M_\phi(T).
\end{aligned}$$

Here, we have written

$$M_\phi(T) = \int_{\Omega_T} \phi(x)|\omega_T|.$$

□

3.2. **Relation with usual unitary dual pairs.** When E is a quadratic field extension of F, then one has $D \cong M_n(E)$ (see §2.3). However, the involution $*$ may not be the standard one $x \mapsto {}^t\bar{x}$. We assume in this subsection that $x^* = {}^t\bar{x}$ and will comment on the general case in §3.3. Then, the dual pair (G, H) and the Rallis map are exactly the same as the dual pair $(U_E(V_0), U_E(ln, ln))$ and its Rallis map, where $V = V_0^{(n)}$ under the Morita equivalence in Lemma 2.4. In particular, for $T \in \mathrm{Herm}_l(D) = \mathrm{Herm}_{ln}(E)$ and $\phi \in S(V^l)$, let ϕ_0 be the corresponding Schwartz function in $S(V_0^{ln})$. Then we have

$$W_T^D(h, s, \phi) = W_T^E(h, s, \phi_0). \tag{3.10}$$

Here, the superscripts D and E indicate the Whittaker functions with respect to D and E, respectively. In particular, the local Siegel–Weil formula becomes a local Siegel-Weil formula with the unitary local Siegel–Weil formula, whose orthogonal analogue is given in [3].

Now assume that instead $E = F \times F$. Then $D = D_0 \times D_0^{\mathrm{op}}$ by the next lemma, where D_0^{op} is the opposite central simple algebra of D. If we further have $D_0 \cong M_n(F)$, then the dual pair (G, H) and the Rallis map are also exactly the same as the degenerate case of the dual pair $(U_E(V_0), U_E(ln, ln))$.

Let $D = D_1 \times D_2$ be the decomposition of D that corresponds to $E = F \times F$. Both D_1 and D_2 are central simple algebras over F. Also let $V = V_1 \oplus V_2$ be the resulting decomposition of V; each V_i is a right D_i-module of rank m. The following is well-known and straightforward.

Lemma 3.2. *Let the notation be as above. Then*
(1) There is a natural isomorphism $\alpha : D_1^{\mathrm{op}} \xrightarrow{\sim} D_2$ *given by*

$$[x, 0]^* = [0, \alpha(x)]. \tag{3.11}$$

Here we write elements of D as $[x, y] \in D_1 \times D_2$.
(2) There is a natural D^{op}-linear isomorphism $\alpha : V_2 \xrightarrow{\sim} V_1^\vee = \mathrm{Hom}_{D_1}(V_1, D_1)$, *given by*

$$((0, v), (w, 0)) = [\alpha(w)(v), 0].$$

We will simply write $(v, w) = \alpha(w)(v) \in D_1$, which can be viewed as the natural form on $V_1 \times V_1^\vee \to D_1$ after identifying V_2 with $V_1^\vee$.
(3) For $g_0 \in \mathrm{GL}_{D_1}(V_1)$, let $\tilde{g}_0 \in \mathrm{GL}_{D_1^{\mathrm{op}}}(V_1^\vee)$ be its dual in the sense

$$(g_0 v, w) = (v, \tilde{g}_0 w), \quad v \in V_1,\ w \in V_1^\vee.$$

Then under the above identification, we have

$$G = \left\{(g_0, \tilde{g}_0^{-1}) :\ g_0 \in \mathrm{GL}_{D_1}(V_1)\right\} \cong \mathrm{GL}_{D_1}(V_1).$$

We set $D_0 = D_1$ and $V_0 = V_1$. Similarly, for a split skew-hermitian space $W = D^{(l)} \oplus D^{(l)}$ as above, we have $W = W_0 \oplus W_0$ with W_0 being a free left D_0-module of rank $2l$. Then we have

$$H = U_D(l, l) = \{(h, (h^*)^{-1}) :\ h \in \mathrm{GL}_{D_0}(W_0) = \mathrm{GL}_{2l}(D_0)\} \cong \mathrm{GL}_{2l}(D_0). \tag{3.12}$$

Under this identification, the standard Siegel parabolic subgroup $P = NM$ becomes

$$N = \{n(b, b^*) :\ b \in M_l(D_0)\} \cong \left\{n(b) = \begin{pmatrix} I_l & b \\ 0 & I_l \end{pmatrix} :\ b \in M_l(D_0)\right\}, \tag{3.13}$$
$$M = \left\{\begin{pmatrix} (a, \tilde{d}^{-1}) & 0 \\ 0 & (\tilde{a}^{-1}, d)) \end{pmatrix}\right\} \cong \{m(a, d) = \mathrm{diag}(a, d) \in \mathrm{GL}_{2l}(D_0) :\ a, d \in \mathrm{GL}_l(D_0)\}.$$

For a character χ of $E^\times$, we write $\chi = (\chi_0, \chi_1)$ for two characters χ_0 and χ_1 of $F^\times$. The induced representation $I(s, \chi_v)$ becomes $\mathrm{Ind}_P^{\mathrm{GL}_{2l}(D_0)}(\chi_0|\ |^s, \chi_1^{-1}|\ |^{-s})$: a section is a smooth function Φ on $\mathrm{GL}_{2l}(D_0)$ such that

$$\Phi(n(b)m(a, d)h, s) = \chi_0(a)|a|_F^{s+\frac{n}{2}}\chi_1^{-1}(d)|d|^{-s-\frac{n}{2}}\Phi(h, s). \tag{3.14}$$

In our case, χ satisfies $\chi|_{F^\times} = 1$, so $\chi_1 = \chi_0^{-1}$ and hence

$$I(s, \chi_v) = \chi_0 \,\mathrm{Ind}_P^{\mathrm{GL}_{2l}(D_0)}(|\ |^s, |\ |^{-s})$$

is simply the twist of $\mathrm{Ind}_P^{\mathrm{GL}_{2l}(D_0)}(|\ |^s, |\ |^{-s})$ by χ_0. Here, we used the notations $\chi_0(a) = \chi_0(\mathrm{Nrd}_{M_l(D_0)/F}(a))$ and $|a|_F = |\,\mathrm{Nrd}_{M_l(D_0)/F}(a)|_F$. The Weil representation on functions $\phi \in S(V^l) = S(V_0^{2l})$ becomes (we write $[x, y] \in V_0^l \otimes V_0^l$)

$$\omega(n(b))\phi([x, y]) = \psi_D(b(x, y))\phi(x),$$
$$\omega(m(a, d))\phi([x, y]) = \chi_0(ad^*)|a(d^*)^{-1}))|^{\frac{mn}{2}}\Phi([xa, y(d^*)^{-1}]), \tag{3.15}$$
$$\omega(J)\phi([x, y]) = \gamma(V)\int_{V_0^{2l}} \phi([u, w])\psi_D((u, y) + (w, x)))du\, dw$$

with $\gamma(V)=1$. Finally, the Rallis map (3.7) works in this case too. The case of $D_0=M_n(F)$ is described in [17, Section 7] since in this case $M_l(D_0)=M_{ln}(F)$.

3.3. **Change of involutions.** For later use in Section 6, we deal with effects of involution change on local Whittaker functions and Rallis map. We assume that D is endowed with two involutions $*$ and $\dagger$ of the second kind over E. Then there is some $\beta\in D^\times$ with $\beta^*=\beta$ and $\dagger=\beta*\beta^{-1}$, i.e., $x^\dagger=\beta x^*\beta^{-1}$. Notice that $\beta^\dagger=\beta$ too. We use superscript $*$ or $\dagger$ to indicate that the objects are with respect to $*$ or $\dagger$. The following lemma is straightforward and the proof is left to the reader.

Lemma 3.3. *The map $T\mapsto\beta T$ gives an isomorphism from $\mathrm{Herm}_l^*(D)$ onto $\mathrm{Herm}_l^\dagger(D)$. So is the map $T\mapsto T\beta^{-1}$.*

Proposition 3.4. *Let $d(\beta)=Diag(\beta I_l,I_l)$. Then we have group isomorphism*

$$\beta: U_D^*(l,l)\cong U_D^\dagger(l,l),\quad \beta(h)=d(\beta)hd(\beta)^{-1}.$$

Moreover, it induces the isomorphism of the induced representations

$$\beta^\vee: I^\dagger(s,\chi)\cong I^*(s,\chi),\quad \Phi^+\mapsto\Phi^*=\Phi^\dagger\circ\beta.$$

Finally, under the isomorphism, we have for $T\in\mathrm{Herm}_l^*(D)$

$$W_T^*(h,s,\Phi^*)=\chi(\beta)|\beta|_E^{s+\rho_{n,l}}W^\dagger_{T\beta^{-1}}(h,s,\Phi^+). \tag{3.16}$$

Proof. We first record some simple formulas, which should be useful in general

$$\begin{aligned}\beta J=J\beta&=d(\beta)Jd(\beta),\\ \beta(n^*(b))&=d(\beta)n^*(b)d(\beta)^{-1}=n^\dagger(\beta b).\\ \beta(m^*(a))&=m^*(\beta a\beta^{-1}),\\ \beta(J)&=m^\dagger(\beta)J\end{aligned}\tag{3.17}$$

Then the first two claims are straightforward. We now verify (3.16). Indeed,

$$\begin{aligned}W_T^*(h,s,\Phi^*)&=\int_{\mathrm{Herm}_l^*(D)}\Phi^*(Jn(b)h,s)\psi_D(-bT)d^*b\\ &=\int_{\beta^{-1}\,\mathrm{Herm}_l^\dagger(D)}\Phi^\dagger(\beta(J)\beta(n(b))\beta(h),s)\psi_D(-bT)d^*b\\ &=\int_{\beta^{-1}\,\mathrm{Herm}_l^\dagger(D)}\chi(\beta)|\beta|_E^{s+\rho_{n,l}}\Phi^\dagger(Jn(\beta b)\psi_D(-bT)d^*b\\ &=\chi(\beta)|\beta|_E^{s+\rho_{n,l}}\int_{\mathrm{Herm}_l^\dagger(D)}\Phi^\dagger(Jn(b)\beta(h))\psi_D(\beta^{-1}bT)d^\dagger b\\ &=\chi(\beta)|\beta|_E^{s+\rho_{n,l}}W^\dagger_{T\beta^{-1}}(\beta(h),s,\Phi^\dagger).\end{aligned}$$

The last identity uses the facts that d^*b and $d^\dagger\beta b$ are self-dual with respect to $*$ and $\dagger$ respectively and that $\psi_D(\beta^{-1}bT)=\psi_D(bT\beta^{-1})$. □

Now we look at how the Schwartz functions change with change of involution. Recall $(v_1,v_2)_\dagger=\beta(v_1,v_2)_*$ and $\langle w_1,w_2\rangle_\dagger=\langle w_1,w_2\rangle_*\beta^{-1}$, where we use subscript to indicate the dependence as superscript has different meaning here. Since the symplectic form on $\mathcal{W}^\dagger = V^\dagger \otimes_D W^\dagger$ is give by

$$\begin{aligned}\ll v_1 \otimes w_1, v_2 \otimes w_2 \gg_\dagger &= \mathrm{tr}_{D/F}\beta(v_1, v_2)_*(\langle w_1, w_2\rangle_* \beta^{-1})^\dagger \\ &= \mathrm{tr}_{D/F}\beta(v_1, v_2)_*(\langle w_1, w_2\rangle_*)^* \beta^{-1} = \ll v_1 \otimes w_1, v_2 \otimes w_2 \gg_*\end{aligned}$$

This implies that the Weil representation of $G \times H$ on $S(V \otimes X)$ does not depend on the choice of involutions. Using a standard D-basis $\{e_1, \cdots, e_l, f_1, \cdots, f_l\}$ of $W = X \oplus Y$ with respect to $*$, we have $S(V \otimes_W X) = S^*(V^l)$, using a standard D-basis $\{\beta e_1, \cdots, \beta e_l, f_1, \cdots, f_l\}$ of $W = X \oplus Y$ with respect to $\dagger$, we have $S(V \otimes_W X) = S^\dagger(V^l)$. Then we have identification

$$S^*(V^l) \cong S^\dagger(V^l), \phi(x) \mapsto \phi_\beta(x) := \phi(x\beta), \tag{3.18}$$

under this identification, we have $\omega(h)\phi \mapsto \omega(\beta(h))\phi_\beta$. This implies that the standard section Φ^* associated to ϕ and the standard section $\Phi^\dagger$ associated to ϕ_β are related by $\Phi^* = \Phi^\dagger \circ \beta$. So Proposition 3.4 implies

Corollary 3.5. *Let the notation be as above, and let $\phi \in S_*(V^l)$. Then we have*

$$W_T^*(h, s, \phi) = \chi(\beta)|\beta|_E^{s+\rho_{n,l}} W_{T\beta^{-1}}(\beta(h), s, \phi_\beta).$$

4. Siegel–Weil formula

Let F be a number field, set $\mathbb{A} = \mathbb{A}_F$, and let E be a quadratic field extension of F. Denote by $\epsilon_{E/F} : \mathbb{A}^\times \to \{\pm 1\}$ the quadratic character of E/F obtained from class field theory. Let D be a central simple algebra over E with involution $*$ of second kind such that $F = E^{*=\mathrm{id}}$. Let $n = \dim_E(D)^{1/2}$ be the degree of D. Fix a non-trivial additive character $\psi = \prod' \psi_v$ of $\mathbb{A}$ that is trivial on F.

4.1. **Induced representation and Eisenstein series.** Similar to §3, let $W = D^{(l)} \otimes D^{(l)}$ be a free left D-module of rank $2l$ with standard split D-hermitian form $\langle\, , \rangle$ with gram matrix J, and let $U_D(W, J) = U_D(l, l)$ be the associated unitary group (viewed as an algebraic group over F) with standard Siegel parabolic subgroup $P = NM$. Let χ be an idele class character of E, and let $I_D(s, \chi)$ be the induced representation, consisting of smooth functions Φ on $H(\mathbb{A})$ such that

$$\Phi(n(b)m(a)h, s) = \chi(a)|a|_E^{s+\rho_{n,l}}\Phi(h, s), \tag{4.1}$$

where $\chi(a) = \chi(\mathrm{Nrd}_{M_l(D)/E}(a))$, $|a|_E = |\,\mathrm{Nrd}_{M_l(D)/E}(a)|_{\mathbb{A}_E}$ and $\rho_{n,l} = \frac{nl}{2}$.

Let $K_f = M_{2l}(\hat{\mathcal{O}}_D) \cap H(\mathbb{A}_f)$, and let K_∞ be a fixed maximal compact subgroup of H_∞. We say that Φ is standard if $\Phi|_{K_f K_\infty}$ is independent of s, and that Φ is factorizable if $\Phi = \otimes \Phi_v$ where v runs through primes of F. We say that Φ is holomorphic if for every $h \in H(\mathbb{A})$, the function $\Phi(h, s)$ is holomorphic in s. Standard sections are holomorphic. For a holomorphic section Φ, we define the Eisenstein series

$$E(h, s, \Phi) = \sum_{\gamma \in P\backslash H} \Phi(\gamma h, s) \tag{4.2}$$

which is absolutely convergent for $\Re(s)$ being big. It has memomorphic continuation to the whole complex s-plane with finitely many poles, and satisfies the functional equation

$$E(h, s, \Phi) = E(h, -s, M(s)\Phi), \tag{4.3}$$

where $M(s) : I(s, \chi) \to I(-s, (\bar{\chi})^{-1})$ is the intertwining operator given by

$$M(s)\Phi(h, s) = \int_{\mathrm{Herm}_n(\mathbb{A}_D)} \Phi(Jn(b)h, s)db \tag{4.4}$$

for the Haar measure db on $\mathrm{Herm}_n(\mathbb{A}_D)$ that is self-dual with respect to ψ_D.

For a hermitian matrix $T \in \mathrm{Herm}_l(D)$, the D-th Fourier coefficient of the above Eisenstein series is given by

$$E_T(h, s, \Phi) = \int_{[\mathrm{Herm}_l]} E(n(b)h, s, \Phi)\psi_D(-bT)db. \tag{4.5}$$

Here, for an algebraic group G over F, we have used the standard notation $[G] = G(F)\backslash G(\mathbb{A})$. Then one has the Fourier expansion

$$E(h, s, \Phi) = \sum_{T \in \mathrm{Herm}_l(D)} E_T(h, s, \Phi). \tag{4.6}$$

Moreover, when T is invertible, we have

$$E_T(h, s, \Phi) = \prod_{v \le \infty} W_{T,v}(h, s, \Phi) \tag{4.7}$$

where, for each place v of F (set $D_v = F_v \otimes_F D$),

$$W_{T,v}(h, s, \Phi) = \int_{\mathrm{Herm}_v(D_v)} \Phi_v(Jn(b)h, s)\psi_{D,v}(-bT)db \tag{4.8}$$

is the local Whittaker function defined in 3. Note that the intertwining operator from (4.4) factors as $M(s) = \prod_v M_v(s)$ and that

$$M_v(s)\Phi(h, s) = W_{0,v}(h, s, \Phi).$$

4.2. **Reductive dual pair and Rallis' map.** Let V be a hermitian right D-module that is free of rank m; let $G = U_D(V)$ be its unitary group. Note that V is an E-vector space of dimension mn^2. Let $\mathcal{W} = V \otimes_D W$ with F-symplectic form

$$\ll v_1 \otimes w_1, v_2 \otimes w_2 \gg = \mathrm{Trd}_{D/F}(v_1, v_2)(w_1, w_2)^*, \tag{4.9}$$

where $\mathrm{Trd}_{D/F} = \mathrm{tr}_{E/F} \circ \mathrm{Trd}_{D/E}$. Let $\mathrm{Sp}(\mathcal{W}) = \mathrm{Sp}(4lmn^2)$ be the symplectic group of $\mathcal{W}$. The natural actions of G and H on V and W respectively give a natural map

$$\iota : G \times H \to \mathrm{Sp}(\mathcal{W}), \tag{4.10}$$

which realizes (G, H) as reductive dual pair in $\mathrm{Sp}(\mathcal{W})$. Assume from now on that the chosen idele class character χ of $\mathbb{A}_E^\times$ satisfies $\chi|_{\mathbb{A}^\times} = \epsilon_{E/F}^{mn}$. Following the local consideration in §3, there is a Weil representation $\omega = \omega_{V,\chi,\psi} = \bigotimes_v \omega_v$ of $G(\mathbb{A}) \times H(\mathbb{A})$ on $S(V_{\mathbb{A}}^l) = \bigotimes_v S(V_v^l)$. The group $G(\mathbb{A})$ acts on $S(V_{\mathbb{A}}^l)$ linearly:

$$\omega(g)\phi(x) = \phi(g^{-1}x). \tag{4.11}$$

On the other hand the $H(\mathbb{A})$-action has the following properties:

$$\begin{aligned} \omega(n(b))\phi(x) &= \psi_D(b(x, x))\phi(x), \\ \omega(m(a))\phi(x) &= \chi(a)|a|_E^{\frac{mn}{2}} \phi(xa), \\ \omega(J)\phi(x) &= \gamma(V_{\mathbb{A}}) \int_{V_{\mathbb{A}}^l} \phi(y)\psi_D((y, x))\, dy. \end{aligned} \tag{4.12}$$

Here $\gamma(V_{\mathbb{A}}) = \prod_v \gamma(V_v) = 1$. Moreover, we have the Rallis map, which is $H(\mathbb{A})$-equivariant (again $s_0 = n(m-l)/2$):

$$\lambda = \lambda_{\mathbb{A}} : S(V_{\mathbb{A}}^l) \to I(s_0, \chi), \quad \lambda(\phi)(h) = \omega(h)\phi(0). \tag{4.13}$$

Let $\Phi \in I(s, \chi)$ be the associated standard section with $\Phi(h, s_0) = \lambda(\phi)$, and we denote

$$W(h, s, \phi) = E(h, s, \Phi). \tag{4.14}$$

4.3. **Theta integral and Siegel–Weil formula.** Given a Schwartz function $\phi = \bigotimes \phi_v \in S(V_{\mathbb{A}}^l)$, let $\Phi = \bigotimes \Phi_v$ be the standard section in $I_D(s, \chi)$ associated to $\lambda(\phi) = \bigotimes \lambda(\phi_v)$. There are two natural ways to construct an $G(\mathbb{A})$-invariant linear map from $S(V_{\mathbb{A}}^l)$ to automorphic forms on $[H]$. The first way is the theta integral (the Haar measure dg is taken to be the Tamagama measure)

$$I(h, \phi) = \frac{1}{\mathrm{Vol}([G])} \int_{[G]} \theta(g, h, \phi) dg \tag{4.15}$$

if it is absolutely convergent, where

$$\theta(g, h, \phi) = \sum_{x \in V^l} \omega(h)\phi(g^{-1}x)$$

is the theta kernel, a two variable automorphic form on $[G] \times [H]$.

The second way is via the Eisenstein series $E(h, s_0, \phi)$ when the Eisenstein series is holomorphic at $s = s_0$. The Siegel–Weil formula claims that these two constructions are basically the same. The following is a special case of [30, Theorem 5].

Theorem 4.1 ([30, Theorem 5]). *When $m > 2l$, both the theta integral and $E(h, s_0, \phi)$ for all $\phi \in S(V_{\mathbb{A}}^l)$ are absolutely convergent, and they are equal:*

$$I(h, \phi) = E(h, s_0, \phi).$$

When D is a field, Kudla–Rallis [13, 14] extended Weil's theorem (for the dual pair $(O(V), \mathrm{Sp}(2l))$) first to the case when the theta integral is absolutely convergent. In their case, the Eisenstein series has holomorphic continuation at $s = s_0$. Kudla–Rallis also considered the case that not all theta integrals are convergent, introduced a regularization of the theta integral, and proved the so-called first identity. The analog in the unitary case was proved by Ichino [9, 10]. These results have been extended further by a lot of authors, including the second identity. We refer to [5] (the first 7 sections) for a quick survey on this subject. All these results should have extensions to central simple algebras, and it would be interesting to work them out. Here, we are satisfied with the following result which is needed later in our paper.

We need some notation first. Given $T \in \mathrm{Herm}_l^{\times}(D)$ (i.e., $\mathrm{Nrd}(T) \neq 0$), the T-th Fourier coefficient of $I(h, \phi)$ is formally given by

$$\begin{aligned} I_T(h, \phi) &= \int_{[\mathrm{Herm}_l]} I(n(b)h, \phi) \\ &= \frac{1}{2} \int_{[G]} \sum_{x \in V^l, \, (x,x) = T} \omega(h)\phi(g^{-1}x) dg, \end{aligned} \tag{4.16}$$

if there exists some $x \in V^l$ with $(x, x) = T$. Otherwise, $I_T(h, \phi) = 0$. Note that the second expression might still be meaningful even if $I(h, \phi)$ is divergent. We will use the second expression as our definition of $I_T(h, \phi)$ whenever it makes sense. Similarly, we define the T-th Fourier coefficient of the Eisenstein series $E(h, s, \phi)$ via (4.7).

Proposition 4.2. *Assume that $m \geq l$ and $T \in \mathrm{Herm}_l^\times(D)$, and let $\phi \in S(V_{\mathbb{A}}^l)$. Then $E_T(h, s, \phi)$ is holomorphic at $s = s_0$, and $I_T(h, \phi)$ is absolutely convergent. Moreover*

$$E_T(h, s_0, \phi) = \kappa\, I_T(h, \phi).$$

with $\kappa = 2$ or 1 depending on whether $m = l$ or $m > l$. Here $s_0 = \frac{n(m-l)}{2}$.

Proof. We follow the proof of [10, Proposition 6.2] closely.

Step 1. Let $\mathcal{S}$ be a finite set of primes of F containing all infinite primes and such that all our data are 'unramified' at every $v \notin \mathcal{S}$. More precisely, for $v \notin \mathcal{S}$, we demand that E_v/F_v is unramified, $(D_v, *) = (M_n(E_v), x \mapsto {}^t\bar{x})$, $\phi_v = \mathrm{Char}(L_v^l)$ where $L_v \subset V_v$ is a self-dual $M_n(O_{E,v})$-lattice in V_v, and $h_v \in U_D(l,l)(O_{F,v})$. In this case, $G_v = U_D(V_v) = U_E(V_{0,v})$ and $H_v = U(ln, ln)$ are the usual quasi-split unitary groups and [28, Proposition 3.2] (see also [10, (6.2)]) gives

$$E_T(h, s, \phi) = \frac{1}{\prod_{i=1}^{ln} L_F^{\mathcal{S}}(2s + ln - i + 1, \epsilon_{E/F}^{mn+i-1})} \prod_{v \in \mathcal{S}} W_{T,v}(h_v, s, \phi_v).$$

This implies that $E_T(h, s, \phi)$ is holomorphic at $s = s_0$, because the partial L-function $L_F^{\mathcal{S}}(s, \eta)$ is holomorphic (and non-zero) at s_0, and because the local Whittaker functions are holomorphic for holomorphic sections.

Step 2. Applying [23, Proposition 1.2.3] to the exact sequence (the last map is the reduced norm map)

$$1 \longrightarrow \mathrm{SU}_D(V) \longrightarrow U_D(V) \longrightarrow E^1 \longrightarrow 1,$$

and using the fact that $\mathrm{SU}_D(V)$ is simply connected and hence has Tamagawa measure 1, we have that the Tamagawa measure of $U_D(V)$ is given by

$$\tau(U_D(V)) = \tau(E^1) = 2.$$

In particular, it is independent of $\dim V$ or D. One can easily see from definitions (see [23, §1]) that the Artin L-function associated to G is $L(s, G) = L(s, \epsilon_{E/F})$, which is again independent of $\dim V$ or D. Chose and fix one $x \in \Omega_T(F)$ (if such an x does not exist, both sides are zero), and let G_x be the stabilizer of x in G as done locally in (3.9) before. Then G_x is trivial when $l = m$ because we assumed $\mathrm{Nrd}(T) \neq 0$. Moreover, if there exists some $x \in \Omega_T(F)$, then

$$G_x \backslash G \xrightarrow{\sim} \Omega_T, \quad g \longmapsto g^{-1}x.$$

As in the local case, we choose an translation invariant top degree form ω_T on Ω_T, and a top degree right invariant differential forms ω_x on G_x respectively such that $\omega_G = \omega_T \wedge \omega_x$ is a G-invariant top degree differential form on G up to a scalar in E^1. This gives for each place v of F that

$$\int_{G_x(F_v)\backslash G(F_v)} \omega(h_v)\phi_v(g^{-1}x)|\omega_{G,v}|/|\omega_{x,v}| = \int_{\Omega_T(F_v)} \omega(h_v)\phi_v|\omega_{T,v}|.$$

Here $|\omega|$ denotes the local Haar measure associated to the top degree differential form ω. Recall from [23, §1] or [10, §9] that the Tamagawa measure is defined by

$$dg_{\mathbb{A}} = L(1, \epsilon_{E/F})^{-1} \prod_v L_v(1, \epsilon_{E/F})|\omega_{G,v}|$$

Since G acts transitively on $\Omega_T(F)$, we have formally

$$\begin{aligned}I_T(h,\phi) &= \frac{1}{\tau(G)}\int_{G_x(\mathbb{Q})\backslash G(\mathbb{A})}\omega(h)\phi(g^{-1}x)dg\\ &= \frac{\tau(G_x)}{\tau(G)}\frac{L(1,G_x)}{L(1,G)}\prod_v\int_{G_x(F_v)\backslash G(F_v)}\frac{L_v(1,G)}{L_v(1,G_x)}\omega(h)\phi_v(g^{-1}x)|\omega_G|_v/|\omega_x|_v\\ &= \frac{\tau(G_x)}{\tau(G)}\frac{L(1,G_x)}{L(1,G)}\prod_v\lambda_v\int_{y\in\Omega_T(F_v)}\omega(h)\phi_v(y)|\omega_T|_v.\end{aligned}$$

So we obtain

$$I_T(h,\phi)=\begin{cases}\frac{1}{2}\frac{1}{L(1,E/F)}\prod_v\lambda_v\int_{y\in\Omega_T(F_v)}\omega(h)\phi_v(y) & \text{if } m=l,\\ \prod_v\lambda_v\int_{y\in\Omega_T(F_v)}\omega(h)\phi_v(y) & \text{if } m>l,\end{cases}\tag{4.17}$$

whenever that infinite product is absolutely convergent. Here,

$$\lambda_v=\begin{cases}1 & \text{if } m>1 \text{ or } v|\infty\\ L_v(1,\epsilon_{E/F}) & \text{if } m=l \text{ and } v<\infty.\end{cases}$$

Step 3. Let $V^l_{reg}\subset V^l$ denote the subset of tuples $v=(v_1,\cdots,v_l)$ such that the D-module generated by $v_1,\cdots,v_l$ is free of rank l. Then the map

$$\alpha: V^l_{reg}\longrightarrow \mathrm{Herm}_l^\times(D),\quad x\longmapsto (x,x)$$

is a moment map. Let ω_T be as in Step 2. Choose a top degree translation invariant differential form ω_D on $\mathrm{Herm}_l(D)$ such that $\omega_V=\omega_T\wedge\omega_D\neq 0$ is a top degree translation invariant differential form on V^l_{reg}. Let $|\omega_T|=\prod_v|\omega_T|_v$ be the associated Haar measure on $\Omega_T(\mathbb{A})$ with respect to ω_T and ψ_D (see [10, §9]) . Then $|\omega_D|_{\mathbb{A}}$ and $|\omega_V|_{\mathbb{A}}$ are self-dual with respect to ψ_D. Then the local Siegel–Weil formula (Proposition 3.1) gives

$$W_{T,v}(h,s_0,\phi_v)=\gamma(V_v)\int_{y\in\Omega_T(F_v)}\omega(h)\phi_v(y)|\omega_T|_v.\tag{4.18}$$

Note that almost all $\gamma(V_v)=1$ and that $\prod_v\gamma(V_v)=1$. When $m>l$,

$$\prod_v W_{T,v}(h,s_0,\phi_v)=\prod_{v\notin\mathcal{S}}\frac{1}{\prod_{i=1}^{ln}L_v(mn-i+1,\epsilon_{E/F}^{mn+i-1})}\prod_{v\in\mathcal{S}}W_{T,v}(h,s_0,\phi_v)$$

is absolutely convergent. When $m=l$, the same is not true, but

$$\prod_v L_v(1,\epsilon_{E/F})W_{T,v}(h,s_0,\phi_v)=\prod_{v\notin\mathcal{S}}\frac{1}{\prod_{i=1}^{ln-1}L_v(ln-i+1,\epsilon_{E/F}^{ln+i-1})}\prod_{v\in\mathcal{S}}\lambda_vW_{T,v}(h,s_0,\phi_v)$$

is absolutely convergent. So

$$\prod_v\lambda_v\int_{y\in\Omega_T(F_v)}\omega(h)\phi_v(y)|\omega_T|_v=\prod_v\lambda_vW_{T,v}(h,s_0,\phi_v)\tag{4.19}$$

is always absolutely convergent. Combining this with (4.17), we have

$$W_T(h,s_0,\phi)=\kappa\, I_T(h,\phi)$$

as expected.

□

5. Shimura varieties

In this section, we define Shimura varieties for central simple algebras with involution of the second kind. Our main references are [16] and [25], which we adapt from the matrix algebra case to that of general CSAs.

5.1. **Shimura Data.** We begin with the choice of rational PEL data. Let $\boldsymbol{k}/\mathbb{Q}$ be an imaginary-quadratic field and let $\boldsymbol{k} \to \mathbb{C}$ be a fixed embedding. We assume that 2 is split in $\boldsymbol{k}$. We consider a CSA $D/\boldsymbol{k}$ of degree $n = [D : \boldsymbol{k}]$, together with a positive involution $* : D \to D$ of the second kind. Recall that this means that $*$ is $\mathbb{Q}$-linear and satisfies

$$*^2 = 1, \qquad (ab)^* = b^* a^*, \qquad *|_{\boldsymbol{k}} = \sigma_{\boldsymbol{k}/\mathbb{Q}}, \qquad \mathrm{trd}_{D/\mathbb{Q}}(x^* x) > 0 \ \text{ for } x \neq 0.$$

The following convention will be used throughout: Whenever we consider a prime p, then we denote by $\dagger$ the standard involution on $M_n(\boldsymbol{k}_p)$, i.e. $(x_{ij})^\dagger = {}^t(\bar{x}_{ij})$. For example, the condition on $*$ being positive is equivalent to the existence of an isomorphism $(D_\mathbb{R}, *) \cong (M_n(\mathbb{C}), \dagger)$.

Next, let $(V, (\ ,\ ))$ be a hermitian right D-module that is free of rank 1 over D. We also assume V to be of signature $(n-1, 1)$ in the following sense. Choose an isomorphism $(D_\mathbb{R}, *) \cong (M_n(\mathbb{C}), \dagger)$. Then, by the Morita equivalence from Lemma 2.4, there exists an n-dimensional hermitian $\mathbb{C}$-vector space (unique up to isomorphism) such that $V_\mathbb{R} \cong W \otimes_\mathbb{C} \mathbb{C}^{(n)}$. The condition now is that the signature of W is $(n-1, 1)$ in the usual sense. Equivalently, we require that the signature of $(V_\mathbb{R}, (\ ,\ )_{\boldsymbol{k}})$ is $(n^2 - n, n)$. Here, $(\ ,\ )_{\boldsymbol{k}} = \mathrm{Trd}_{D/\boldsymbol{k}} \circ (\ ,\ )$ is the $\boldsymbol{k}$-valued form corresponding to $(\ ,\ )$ as in Lemma 2.7.

Let $Z = \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}(\mathbb{G}_m)$ and let $N_{\boldsymbol{k}/\mathbb{Q}} : Z \to \mathbb{G}_m$ denote the norm. (The group Z can also be viewed as the unitary similitude group of a 1-dimensional hermitian $\boldsymbol{k}$-vector space as in [25].) Let $G = U_D(V)$ be the group of D-linear isometries of V, let $G^\mathbb{Q} = GU_D(V)$ be the group of D-linear similitudes, and let $c : G^\mathbb{Q} \to \mathbb{G}_m$ denote the similitude factor. Our convention is that G and $G^\mathbb{Q}$ act from the left on V. Finally, we consider the product

$$\widetilde{G} = Z \times_{\mathbb{G}_m} G^\mathbb{Q} = \Big\{(z, g) \in Z \times G^\mathbb{Q} \mid N_{\boldsymbol{k}/\mathbb{Q}}(z) = c(g)\Big\}. \tag{5.1}$$

Note that there is an isomorphism

$$\widetilde{G} \xrightarrow{\sim} Z \times G, \quad (z, g) \longmapsto (z, z^{-1} g). \tag{5.2}$$

We can now define our Shimura varieties. Throughout, we identify $\mathbb{R} \otimes_\mathbb{Q} \boldsymbol{k} \xrightarrow{\sim} \mathbb{C}$ along the fixed embedding $\boldsymbol{k} \to \mathbb{C}$. For the group Z, we define $(Z, \{h_Z\})$ by

$$h_Z : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \longrightarrow Z_\mathbb{R}, \quad z \longmapsto z. \tag{5.3}$$

We define the Shimura datum (G, X_G) as follows. Choose isomorphisms $(D_\mathbb{R}, *) \cong (M_n(\mathbb{C}), \dagger)$ and $V_\mathbb{R} = W \otimes_\mathbb{C} \mathbb{C}^{(n)}$ (Morita equivalence). Then $G_\mathbb{R} \cong U(W)$. Choose an orthonormal basis

$$W \xrightarrow{\sim} (\mathbb{C}, N_{\mathbb{C}/\mathbb{R}})^{\oplus(n-1)} \overset{\perp}{\oplus} (\mathbb{C}, -N_{\mathbb{C}/\mathbb{R}})$$

and let X_G be the $G(\mathbb{R})$-conjugacy class of the homomorphism

$$h_G : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \longrightarrow G_\mathbb{R}, \quad z \longmapsto \mathrm{diag}(1^{(n-1)}, \bar{z}/z). \tag{5.4}$$

Finally, we define the Shimura datum $(\widetilde{G}, X)$ where X is the $\widetilde{G}(\mathbb{R})$-conjugacy class of the diagonal homomorphism (h_Z, h_G) with respect to (5.2). Then X is a connected, n-dimensional, hermitian symmetric domain. It can be described intrinsically as the space of negative definite $D_\mathbb{R}$-stable n-dimensional subspaces of $V_\mathbb{R}$.

Definition 5.1. The reflex field of $(Z, \{h_Z\})$ and $(\widetilde{G}, X)$ is $\boldsymbol{k}$. Let $K_Z \subset Z(\mathbb{A}_f)$ as well as $K_G \subset G(\mathbb{A}_f)$ be level subgroups and set $K = K_Z \times K_G$. We denote by

$$S(Z, \{h_Z\})_{K_Z},\quad S(\widetilde{G}, X)_K \quad\longrightarrow\ \operatorname{Spec} \boldsymbol{k}$$

the Shimura varieties of (Z, h_Z) and $(\widetilde{G}, X)$ for the indicated level subgroups.

The results in this article are new when $D \not\cong M_n(\boldsymbol{k})$. In this case, G is anisotropic and hence $S(\widetilde{G}, X)_K$ projective.

5.2. **Moduli Description.** We next give a PEL moduli description of $S(Z, \{h_Z\})_{K_Z}$ and $S(G, X)_K$.

Definition 5.2. Let M_{0,K_Z} be the following stack over $\operatorname{Spec} \boldsymbol{k}$. For every $\boldsymbol{k}$-scheme S, the S-points $M_{0,K_Z}(S)$ are the groupoid of triples (E, ι, η) where
- E/S is an elliptic curve.
- $\iota : \boldsymbol{k} \to \operatorname{End}^0(E)$ is a strict $\boldsymbol{k}$-action in the sense that $\iota(a)$ acts as multiplication with a on $\operatorname{Lie}(E)$, for every $a \in \boldsymbol{k}$.
- η is a K_Z-coset of $\mathbb{A}_{\boldsymbol{k},f}$-linear isomorphisms $\mathbb{A}_{\boldsymbol{k},f} \xrightarrow{\sim} \widehat{V}(E)$.

An isomorphism $(E, \iota, \eta) \xrightarrow{\sim} (E', \iota', \eta')$ is a $\boldsymbol{k}$-linear quasi-isogeny $\gamma : E \to E'$ such that $\gamma\eta = \eta'$.

For K_Z sufficiently small, M_{0,K_Z} is a finite $\boldsymbol{k}$-scheme. The tower $\big(M_{0,K_Z}\big)_{K_Z}$, endowed with the Hecke action of $\mathbb{A}^\times_{\boldsymbol{k},f}$, is isomorphic to the tower $S(Z, \{h_Z\})_{K_Z}$.

Definition 5.3. Let M_K be the following stack over $\operatorname{Spec} \boldsymbol{k}$. For every $\boldsymbol{k}$-scheme S, the S-points $M(S)$ are the groupoid of tuples $(E, \iota_0, \eta_0, A, \iota, \lambda, \eta)$ where
- $(E, \iota_0, \eta_0) \in M_{0,K_Z}(S)$ is as in Definition 5.2.
- A is an abelian scheme over S of dimension n^2.
- $\iota : D \to \operatorname{End}^0(A)$ is a right D-action which satisfies the Kottwitz condition

$$\operatorname{char}(\iota(a) \mid \operatorname{Lie}(A); T) = \operatorname{charred}_{D/\boldsymbol{k}}(a; T)^{n-1}\operatorname{charred}_{D/\boldsymbol{k}}(a^*; T), \quad a \in D.$$

This is meant as an identity of polynomials in $\mathcal{O}_S[T]$.
- $\lambda : A \to A^\vee$ is a quasi-polarization that is compatible with the D-action in the sense that $\iota(a^*) = \lambda^{-1}\iota(a)^\vee\lambda$.
- η is a K_G-level structure, meaning a K_G-coset of isometric, D-linear isomorphisms

$$\eta : V(\mathbb{A}_f) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbb{A}_{\boldsymbol{k},f}}(\widehat{V}(E), \widehat{V}(A)). \tag{5.5}$$

Here, source is endowed with the hermitian form $(\ ,\ )_{\boldsymbol{k}} = \operatorname{Trd}_{D/\boldsymbol{k}} \circ (\ ,\ )$. This will implicitly always be the understood when we consider isometries between a hermitian D-module and a hermitian $\boldsymbol{k}$-vector space (or between their adelic variants). The target in (5.5) is endowed with the following hermitian form. Choosing a trivialization of the Tate twist $\mathbb{A}_f(1)$, the canonical polarization of E and the polarization λ endow $\widehat{V}(E)$ and $\widehat{V}(A)$ with perfect alternating pairings. The Hom-space in (5.5) acquires a hermitian form by

$$(x, y) := x^* \circ y \in \operatorname{End}_{\mathbb{A}_{\boldsymbol{k},f}}(\widehat{V}(E)) = \mathbb{A}_{\boldsymbol{k},f}.$$

Here, $x \mapsto x^*$ denotes the Rosati adjoint map. The form $(\ ,\ )$ is independent of the choice of trivialization of $\mathbb{A}_f(1)$.

An isomorphism

$$(E, \iota_0, \eta_0, A, \iota, \lambda, \eta) \xrightarrow{\sim} (E', \iota_0', \eta_0', A', \iota', \lambda', \eta')$$

in the groupoid $M_K(S)$ is a pair (γ_E, γ_A) that consists of an isomorphism $\gamma_E : (E, \iota_0, \eta_0) \to (E', \iota_0', \eta_0')$ in $M_{0,K_Z}(S)$ and a D-linear quasi-isogeny $\gamma_A : A \to A'$ with the following two additional properties:

- γ_A is a similitude with same similitude factor as γ_E, meaning $\gamma_A^*(\lambda') = \deg(\gamma_E) \cdot \lambda$.
- Then $(\gamma_E, \gamma_A) : \mathrm{Hom}(\widehat{V}(E), \widehat{V}(A)) \xrightarrow{\sim} \mathrm{Hom}(\widehat{V}(E'), \widehat{V}(A'))$ is an isometry and we further require that $(\gamma_E, \gamma_A)\eta = \eta'$.

Proposition 5.4. *For every level $K = K_Z \times K_G \subset \widetilde{G}(\mathbb{A}_f)$, the stack M_K is a Deligne–Mumford stack that is smooth of relative dimension $n-1$ over* $\mathrm{Spec}\,\boldsymbol{k}$. *The tower $(M_K)_K$, endowed with the Hecke action of $\widetilde{G}(\mathbb{A}_f)$, is isomorphic to the tower $S(\widetilde{G}, X)_K$. If $D \not\cong M_n(\boldsymbol{k})$, then M_K is proper.*

Proof. There is a natural construction of isomorphism of the $\mathbb{C}$-points of $(M_K)_K$ and $S(\widetilde{G}, X)_K$ which is the same as that for [25, Theorem 3.5]. The fact that this isomorphism descends to an isomorphism of varieties (i.e. the fact that $(M_K)_K$ is a canonical model of the Shimura variety), is mentioned at the beginning of [25, §3.3]. It holds since the tower $(M_K)_K$ is open and closed in the product of the PEL moduli problems of the Shimura varities for Z and $G^{\mathbb{Q}}$. The latter are canonical models for the Shimura varieties for Z and $G^{\mathbb{Q}}$ by Deligne's definition. □

5.3. **Integral Models.** From now on, we fix $K_Z \subset \mathbb{A}_{\boldsymbol{k},f}^\times$ as the maximal compact subgroup $\widehat{O}_{\boldsymbol{k}}^\times$ and we abbreviate $M_0 := M_{0,K_Z}$. It is well-known that M_0 is isomorphic to the moduli stack of elliptic curves with strict CM by $O_{\boldsymbol{k}}$.

Definition 5.5. Let $\mathcal{M}_0$ be the stack over $\mathrm{Spec}\, O_{\boldsymbol{k}}$ such that $\mathcal{M}_0(S)$ is the groupoid of pairs (E, ι), where

- E is an elliptic curve over S.
- $\iota : O_{\boldsymbol{k}} \to \mathrm{End}(E)$ is a strict $O_{\boldsymbol{k}}$-action in the sense that $\iota(a)$ acts by multiplication with a on $\mathrm{Lie}(E)$, for all $a \in O_{\boldsymbol{k}}$.

An isomorphism in $\mathcal{M}_0(S)$ is an $O_{\boldsymbol{k}}$-linear isomorphism of elliptic curves. Then $\mathcal{M}_0$ is a finite étale Deligne–Mumford stack over $\mathrm{Spec}\, O_{\boldsymbol{k}}$ and an integral model of M_0.

We next impose additional assumptions on $(D, *)$, V and K_G that allow to define an integral model of M_K over $\mathrm{Spec}\, O_{\boldsymbol{k}}$.

- First, we assume that there exists, and fix, a $*$-stable maximal order $O_D \subset D$. This is equivalent to the existence of $*$-stable maximal orders $O_{D,p} \subset D_p$ for all primes p. That last condition is non-trivial only when n is even and p inert in $\boldsymbol{k}$, as explained in Lemma 2.9.
- Second, we fix an O_D-lattice $\Lambda \subset V$. Let $\Lambda_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda$ denote its p-adic completion. We assume that each Λ_p is of one of the following types:
  - If p is split, then we assume that $\Lambda_p = \Lambda_p^\vee$. The dual here and in the following is with respect to $(\ ,\ )_{\boldsymbol{k}}$.
  - If p is non-split, then we assume that Λ_p is a vertex lattice in the sense that

$$\Lambda_p \subseteq \Lambda_p^\vee \subseteq \pi_p^{-1}\Lambda_p. \tag{5.6}$$

Here, $\pi_p \in \boldsymbol{k}_p$ denotes a uniformizer. If p is ramified, then we moreover assume that

$$\Lambda_p = \Lambda_p^\vee \quad \text{or} \quad \begin{cases} \Lambda_p^\vee = \pi_p^{-1}\Lambda_p & n \text{ even} \\ [\pi_p^{-1}\Lambda_p : \Lambda_p^\vee] = n & n \text{ odd.} \end{cases} \tag{5.7}$$

- We define $K_G = \prod_p \mathrm{Stab}(\Lambda_p)$ and set $K_{\widetilde{G}} = K_Z \times K_G$ as before.

Definition 5.6. Let $\mathcal{M}^\bullet$ be the following moduli stack over $\mathrm{Spec}\, O_{\boldsymbol{k}}$. Its S-points $\mathcal{M}^\bullet(S)$ are the groupoid of tuples $(E, \iota_0, A, \iota, \lambda, \mathcal{F})$ where
- $(E, \iota_0) \in \mathcal{M}_0(S)$ is a CM elliptic curve as in Definition 5.5.
- A is an abelian scheme of dimension n^2 over S.
- $\iota : O_D \to \mathrm{End}(A)$ is a right O_D-action that satisfies the Kottwitz condition

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}(A); T) = \mathrm{charred}_{D/\boldsymbol{k}}(a; T)^{n-1} \mathrm{charred}_{D/\boldsymbol{k}}(a^*; T).$$

This is meant as an identity of polynomial functions on $\mathbb{A}^1_S$.
- $\lambda : A \to A^\vee$ is a polarization that is compatible with the O_D-action in the sense that $\lambda^{-1}\iota(a)^\vee\lambda = \iota(a^*)$ for all $a \in O_D$. We furthermore impose the following conditions on its kernel. For every prime p that splits in $\boldsymbol{k}$, the kernel $\ker(\lambda)$ is p-torsion free. For every prime p that is non-split, we instead have

$$\ker(\lambda)[p^\infty] \subseteq A[\pi_p], \quad |\ker(\lambda)[p^\infty]| = |\Lambda_p^\vee/\Lambda_p|. \tag{5.8}$$

Here, $\pi_p \in \boldsymbol{k}_p$ denotes a uniformizer.
- Let $\mathcal{B} \subset \mathrm{Spec}\, O_{\boldsymbol{k}}$ be the set of primes which are ramified in $\boldsymbol{k}$ and satisfy $\Lambda_p \neq \Lambda_p^\vee$. The datum $\mathcal{F} \subset \mathrm{Lie}(A)[\mathcal{B}^{-1}]$ is a so-called Krämer datum which is added when $S[\mathcal{B}^{-1}] \neq \emptyset$: It is an O_D-stable $\mathcal{O}_S$-submodule with the following properties. It is locally a direct summand and its $\mathcal{O}_S$-rank is $n^2 - n$. Furthermore, the $O_{\boldsymbol{k}}$-action on $\mathcal{F}$ via ι equals the action by the natural map $O_{\boldsymbol{k}} \to \mathcal{O}_S$. The $O_{\boldsymbol{k}}$-action on $\mathrm{Lie}(A)/\mathcal{F}$ on the other hand equals the Galois conjugate of the natural action.
- Finally, for every prime p that ramifies in $\boldsymbol{k}$ and such that $\Lambda_p \neq \Lambda_p^\vee$, we assume that the p-divisible group $A[p^\infty]$, restricted to the p-adic completion $\mathrm{Spf}\, O_{\boldsymbol{k},p} \times S$ of S, satisfies the wedge condition and the (refined) spin condition in the sense of Definition 5.9 below.

An isomorphism $(E, \iota_0, A, \iota, \lambda, \mathcal{F}) \to (E', \iota_0', A', \iota', \lambda', \mathcal{F}')$ in this groupoid is a pair (γ_E, γ_A) of isomorphisms $\gamma_E : E \to E'$ and $\gamma_A : A \to A'$ such that γ_E is $O_{\boldsymbol{k}}$-linear, γ_A is O_D-linear, $\gamma_A^*(\lambda') = \lambda$, and $\gamma_A(\mathcal{F}) = \mathcal{F}'$.

Before giving the definition of the wedge condition and the (refined) spin condition in our setting, we explain a general Morita equivalence statement for p-divisible groups. This will also play a central role for the definition of arithmetic special cycles in the next section. First, we define the p-divisible group variant of the above tuples $(A, \iota, \lambda, \mathcal{F})$.

Definition 5.7. Let S be a a scheme over $\mathrm{Spf}\, O_{\boldsymbol{k},p}$.

(1) A (p-divisible) hermitian $O_{\boldsymbol{k},p}$-module (of height $2n$) over a scheme S is a triple (X, ι, λ) where X/S is a p-divisible group of height $2n$, where $\iota : O_{\boldsymbol{k},p} \to \mathrm{End}(X)$ is an $O_{\boldsymbol{k},p}$-action, and where $\lambda : X \to X^\vee$ is a polarization that is compatible with ι in the sense that $\iota(\bar{a}) = \lambda^{-1}\iota(a)^\vee\lambda$ for all $a \in O_{\boldsymbol{k},p}$. We moreover require $\ker(\lambda) \subseteq X[\iota(\pi_p)]$ and $\deg(\ker(\lambda)) = \#(\Lambda_p^\vee/\Lambda_p)/n$.

A Krämer datum for (X, ι, λ) is an $O_{\boldsymbol{k},p}$-stable $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathrm{Lie}(X)$ which is locally a direct summand of rank $n-1$ and which has the property that $O_{\boldsymbol{k},p}$ acts naturally on $\mathcal{F}$ and via Galois conjugation on $\mathrm{Lie}(X)/\mathcal{F}$.
(2) A (p-divisible) hermitian $O_{D,p}$-module (of height $2n^2$) over a scheme S is a triple (X, ι, λ), where X/S is a p-divisible group of height $2n^2$, where $\iota : O_{D,p} \to \mathrm{End}(X)$ is a right action of $O_{D,p}$, and where $\lambda : X \to X^\vee$ is a polarization that is compatible with ι in the sense $\iota(a^*) = \lambda^{-1}\iota(a)^\vee\lambda$ for all $a \in O_{D,p}$. Moreover, we require that $\ker(\lambda) \subseteq X[\iota(\pi_p)]$ and that $\deg(\ker(\lambda)) = \#(\Lambda_p^\vee/\Lambda_p)$.
A Krämer datum for (X, ι, λ) is an $O_{D,p}$-stable $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathrm{Lie}(X)$ which is locally a direct summand of rank $n^2 - n$ and which has the property that $O_{\boldsymbol{k},p}$ acts naturally on $\mathcal{F}$ and via Galois conjugation on $\mathrm{Lie}(X)/\mathcal{F}$.

Construction 5.8. Assume that p is non-split in $\boldsymbol{k}$. Fix an isomorphism $\gamma : O_{D,p} \cong M_n(O_{\boldsymbol{k},p})$ and denote by $*$ the resulting involution on $M_n(O_{\boldsymbol{k},p})$ from the involution $*$ on D. Recall that $\dagger$ denotes the standard involution on $M_n(O_{\boldsymbol{k},p})$. Fix a $\dagger$-hermitian invertible element $\beta \in GL_n(O_{\boldsymbol{k},p})$ with the property that $x^* = \beta^{-1}x^\dagger\beta$ for all $x \in M_n(O_{\boldsymbol{k},p})$. These choices define a Morita equivalence

$$\begin{aligned} \{\text{Hermitian } O_{\boldsymbol{k},p}\text{-modules over } S\} &\xrightarrow{\sim} \{*\text{-Hermitian } O_{D,p}\text{-modules over } S\} \\ (X, \iota, \lambda) &\longmapsto (X, \iota, \lambda) \otimes (O_{\boldsymbol{k},p}^{(n)}, \beta). \end{aligned} \tag{5.9}$$

The image of (X, ι, λ) here is defined as the triple $\big(X^{(n)},\ \iota^{(n)},\ \lambda^{(n)} \circ \iota^{(n)}(\beta)\big)$ where

- $X^{(n)}$ is the n-th power of X viewed as row vectors.
- $\iota^{(n)}$ denotes the natural right action of $M_n(O_{\boldsymbol{k},p})$ on $X^{(n)}$.
- $\lambda^{(n)} : X^{(n)} \xrightarrow{\sim} X^{\vee,(n)} = (X^{(n)})^\vee$ is the diagonal polarization.

Of course, we view this triple as a hermitian $O_{D,p}$-module via γ, even though we have not made this explicit in the notation. We note that if $\mathcal{F}$ is a Krämer datum for (X, ι, λ), then $\mathcal{F}^{(n)}$ is a Krämer datum for $(X, \iota, \lambda) \otimes (O_{\boldsymbol{k},p}^{(n)}, \beta)$ and every Krämer datum for the latter is of this form.

Now assume that p is ramified and that S is a $\mathrm{Spf}\, O_{\boldsymbol{k},p}$-scheme. (By assumption this implies $p \neq 2$.) Let $\pi \in \boldsymbol{k}_p$ be a uniformizer with $\bar{\pi} = -\pi$ and let (X, ι, λ) be a hermitian $O_{\boldsymbol{k},p}$-module of height $2n$ over S. Recall from [24, §6] that (X, ι, λ) is said to satisfy the wedge condition if $\bigwedge^2(\iota(\pi) + \pi)$ acts as zero on $\mathrm{Lie}(X)$. If n is even, then it is said to satisfy the spin condition if $\iota(\pi)$ is non-zero on $\mathrm{Lie}(X)$ in each point of S. If n is odd, then there is the definition of the refined spin condition which is more complicated to state and for which we refer to [24, §7].

Definition 5.9. A hermitian $O_{D,p}$-module (X, ι, λ) of height $2n^2$ over S is said to satisfy the wedge condition and the spin condition (n even), resp. the wedge condition and the refined spin condition (n odd), if it comes under Morita equivalence from a hermitian $O_{\boldsymbol{k},p}$-module that satisfies these two conditions.

It can be shown by a local model argument that this condition is independent of the choices of γ and β in Construction 5.8 (omitted).

Proposition 5.10. *The stack $\mathcal{M}^\bullet$ is a Deligne–Mumford stack that is flat and regular with semi-stable reduction of relative dimension $n-1$ over* $\mathrm{Spec}\, O_{\boldsymbol{k}}$. *It is smooth over all primes p of the following kind:*

- *(Split) p is split in $\boldsymbol{k}$ and $D_p \cong M_n(\boldsymbol{k}_p)$.*
- *(Hyperspecial) p is inert in $\boldsymbol{k}$ and $\Lambda_p^\vee = \Lambda_p$ or $\Lambda_p^\vee = p^{-1}\Lambda_p$.*
- *(Exotic smooth) p is ramified and $\Lambda_p^\vee = \pi_p^{-1}\Lambda_p$ (if n even) or $[\pi_p^{-1}\Lambda_p : \Lambda_p^\vee] = n$ (if n odd).*

If $D \ncong M_n(\boldsymbol{k})$, then $\mathcal{M}^\bullet$ is proper.

Proof. The fact that $\mathcal{M}^\bullet$ is representable by a Deligne–Mumford stack of finite type over $\operatorname{Spec} O_{\boldsymbol{k}}$ is well-known. Its local properties (flatness of relative dimension $n-1$, semi-stable reduction, smoothness over the primes as stated) follow from the standard local model argument: At primes p that split, $\mathcal{M}^\bullet$ is flat of relative dimension $n-1$ and has semi-stable reduction because the parahoric local models of GL_n have this property. At non-split primes, using the Morita equivalence in Construction 5.8, the claim reduces to the properties of the local models as in [25, Theorem 5.4].

It is left to prove the properness of $\mathcal{M}^\bullet$ when $D \ncong M_n(\boldsymbol{k})$. By the semi-stable reduction theorem, and by the fact that the signature and wedge conditions are closed conditions, it suffices to see that there are no semi-abelian varieties B of dimension n^2 and with non-trivial torus part over algebraically closed $O_{\boldsymbol{k}}$-fields k that admit an O_D-action that satisfies those conditions. So assume B/k is semi-abelian with $\dim B = n^2$ and that $\iota : O_D \to \operatorname{End}(B)$ is an O_D-action. Let $B \twoheadrightarrow T$ be the maximal torus quotient of B. Also assume that $D = M_m(D_0)$ for a division algebra D_0 with $d = [D_0 : \boldsymbol{k}] \geq 2$. Then T has rank a multiple of $2md^2$, say $r2md^2$. Moreover, there exists a finite projective $O_{\boldsymbol{k}}$-module Q of $O_{\boldsymbol{k}}$-rank rd^2m such that $T \cong Q \otimes_{\mathbb{Z}} \mathbb{G}_m$ as torus with $O_{\boldsymbol{k}}$-action. If $\operatorname{char}(k) = 0$ or if $\operatorname{char}(k) = p$ with p unramified, then the signature of the $O_{\boldsymbol{k}}$-action on $\operatorname{Lie}(T) = Q \otimes \operatorname{Lie}(\mathbb{G}_m)$ is (rd^2m, rd^2m). If B satisfies the signature $(n-1,1)$-condition in the D-sense, then the $O_{\boldsymbol{k}}$-action on $\operatorname{Lie}(B)$ is of signature (n^2-n, n). The only possibility is $r = 0$ and we are done. If p is ramified and if $\pi \in \boldsymbol{k}_p$ is a uniformizer with $\bar{\pi} = -\pi$, then one similarly has that $\bigwedge^{rd^2m}(\iota(\pi) + \pi \mid \operatorname{Lie}(T)) \neq 0$. If B satisfies the wedge condition, then again the only possibility is $r = 0$ and the proof is complete. $\square$

We next explain how to decompose $\mathcal{M}^\bullet$ according to hermitian D-modules. This is completely parallel to [16, Proposition 2.12]. Let k be an $O_{\boldsymbol{k}}$-field of characteristic $p \geq 0$, and consider a point $(E, \iota_0, A, \iota, \lambda, \mathcal{F}) \in \mathcal{M}^\bullet(k)$. Then

$$\widehat{\Lambda}(E, A)^p := \operatorname{Hom}_{\mathbb{A}^p_{\boldsymbol{k},f}}(\widehat{T}(E)^p, \widehat{T}(A)^p)$$

is a hermitian $\widehat{O}_D$-module. For each prime $\ell \neq p$, the localization $\widehat{\Lambda}(E, A)_\ell$ is of the same type as Λ in (5.6) and (5.7). This follows from the conditions on λ in (5.8). Moreover, the isomorphism class of $\widehat{\Lambda}(E, A)^p$ is locally constant on $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathcal{M}^\bullet$.

Assume next that $k = \mathbb{C}$ and that $\boldsymbol{k} \to \mathbb{C}$ is the fixed inclusion. Then we may pass to analytic spaces and obtain a hermitian O_D-module

$$\Lambda(E, A) := \operatorname{Hom}_{O_{\boldsymbol{k}}}(H_1(E(\mathbb{C}), \mathbb{Z}), H_1(A(\mathbb{C}), \mathbb{Z})).$$

Its signature is $(n-1, 1)$ in the D-sense, which follows from the signature condition in the same way as during the proof of [16, Proposition 2.12]. Moreover, by the Betti–étale comparison,

$$\Lambda(E, A) \otimes_{O_D} \widehat{O}_D \xrightarrow{\sim} \widehat{\Lambda}(E, A).$$

It follows that when k is of characteristic 0, there exists a unique hermitian D-module $V(E, A)$ that is free of rank 1, has signature $(n-1, 1)$, and admits an isometric embedding

$$\widehat{\Lambda}(E, A) \longrightarrow V(E, A) \otimes_D \mathbb{A}_D.$$

This defines a locally constant map

$$\begin{aligned} \boldsymbol{k} \otimes_{O_{\boldsymbol{k}}} \mathcal{M}^{\bullet} \ &\longrightarrow \ \left\{ \begin{matrix} \text{Hermitian } D\text{-modules } V \text{ of rank } 1 \\ \text{s.t. there exists an } O_D\text{-lattice } \Lambda' \subset V \text{ of the same type as } \Lambda \end{matrix} \right\} \\ (E, \iota_0, A, \iota, \lambda, \mathcal{F}) \ &\longmapsto \ V(E, A). \end{aligned} \tag{5.10}$$

Consider now a prime $p > 0$ and denote by $\mathcal{M}^{\bullet}_{(p)} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathcal{M}^{\bullet}$ the localization of $\mathcal{M}^{\bullet}$ at p. Proposition 5.10 established that $\mathcal{M}^{\bullet}_{(p)}$ is flat over $\operatorname{Spec} O_{\boldsymbol{k},(p)}$. Thus the function

$$(E, \iota_0, A, \iota, \lambda, \mathcal{F}) \longmapsto \widehat{\Lambda}(E, A)^p$$

is locally constant on $\mathcal{M}^{\bullet}_{(p)}$ and takes values in hermitian $\widehat{O}^p_D$-modules $\widehat{\Lambda}^p$ that have the property that there exists a hermitian D-module W of rank 1 and signature $(n-1, 1)$ such that there exists an isometric embedding $\widehat{\Lambda}^p \hookrightarrow \mathbb{A}^p_f \otimes_{\mathbb{Q}} W$. By the classification of hermitian D-modules, see Theorem 2.10, any such W is uniquely determined by $\mathbb{A}^p_f \otimes_{\mathbb{Q}} W$. It follows that (5.10) extends to a locally constant function on all of $\mathcal{M}^{\bullet}$.

Definition 5.11. Let $\mathcal{M} := \mathcal{M}^V \subseteq \mathcal{M}^{\bullet}$ denote the fiber above V, our fixed hermitian D-module. By the same arguments as for [16, Proposition 4.4], $\mathcal{M}^V$ is an integral model for $S(\widetilde{G}, X)_K$.

5.4. **Non-archimedean Uniformization.** Let p be a prime that is non-split in $\boldsymbol{k}$. Let $\pi = \pi_p$ be a uniformizer of $O_{\boldsymbol{k},p}$ and let $\mathbb{F}$ denote the algebraic closure of $O_{\boldsymbol{k}}/(\pi)$. Let

$$W = \begin{cases} W(\mathbb{F}) & \text{if } p \text{ inert} \\ O_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W(\mathbb{F}) & \text{if } p \text{ ramified} \end{cases}$$

be its ring of $O_{\boldsymbol{k},p}$-Witt vectors. In particular, there is a natural map $O_{\boldsymbol{k},p} \to W$.

Definition 5.12. Let $(\mathbb{X}, \iota, \lambda)$ be a hermitian $O_{\boldsymbol{k},p}$-module of height $2n$ over $\mathbb{F}$. The RZ space $\mathcal{N} = \mathcal{N}(\mathbb{X}, \iota, \lambda)$ of $(\mathbb{X}, \iota, \lambda)$ is defined as the following functor on the category of $\operatorname{Spf} W$-schemes. Its S-valued points $\mathcal{N}(S)$ are the set of isomorphism classes of tuples

$$(X, \iota, \lambda, \rho) \quad \text{resp.} \quad (X, \iota, \lambda, \rho, \mathcal{F})$$

where (X, ι, λ) is as in Definition 5.7 and where

$$\rho : \overline{S} \times_{\operatorname{Spec} \mathbb{F}} \mathbb{X} \longrightarrow \overline{S} \times_S X$$

is an $O_{\boldsymbol{k},p}$-linear quasi-isogenies that preserves the polarization. We assume that (X, ι, λ) satisfies the wedge and spin condition (n even), resp. the wedge condition and the refined spin condition (n odd). The datum $\mathcal{F}$ is a Krämer datum for (X, ι, λ) which is included in the case that p is ramified and $\Lambda_p = \Lambda_p^{\vee}$.

It is well-known [26] that $\mathcal{N}$ is representable by a formal scheme that is locally formally of finite type over $\operatorname{Spf} W$. It is flat of relative dimension $n-1$ and with semi-stable reduction. If p is inert and Λ_p self-dual or $\Lambda_p^{\vee} = p^{-1}\Lambda_p$, or if p is ramified and Λ_p of exotic smooth type, then $\mathcal{N}$ is formally smooth over $\operatorname{Spf} W$.

We now turn to the basic locus of $\mathcal{M}$ over p, which coincides with the supersingular locus. The general uniformization result of Rapoport–Zink [26] describes the completion $W \widehat{\otimes}_{O_{\boldsymbol{k},p}} \mathcal{M}^{\mathrm{ss}}_p$ of $\mathcal{M}$ along the supersingular locus in terms of a moduli space of hermitian $O_{D,p}$-modules (Definition 5.7). The formulation we give here uses Morita equivalence to translate this into a statement about hermitian $O_{\boldsymbol{k},p}$-modules.

Construction 5.13. Make the following choices.

- Fix a supersingular point $(E, \iota_0, A, \iota, \lambda) \in \mathcal{M}(\mathbb{F})$. If p is of Krämer type, then we also include a Krämer datum for $\mathrm{Lie}(A)$. We will suppress this in the following however; everything works the same when including Krämer data.
- Let $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ be the p-divisible group of (A, ι, λ). It is a hermitian $O_{D,p}$-module in the sense of Definition 5.7.
- Choose an isomorphism $\gamma : O_{D,p} \cong M_n(O_{\boldsymbol{k},p})$ and a hermitian matrix $\beta \in GL_n(O_{\boldsymbol{k},p})$ as in Construction 5.8. Consider a hermitian $O_{\boldsymbol{k},p}$-module (X, ι, λ) of height $2n$ over $\mathbb{F}$ together with an isomorphism

$$(\mathbb{X}, \iota, \lambda) \otimes (O_{\boldsymbol{k},p}^{(n)}, \beta) \xrightarrow{\sim} (\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}}).$$

Let $\mathcal{N} = \mathcal{N}(\mathbb{X}, \iota, \lambda)$ denote the RZ space of $(\mathbb{X}, \iota, \lambda)$.
Composing the general uniformization morphism from [26] with Construction 5.8 defines a morphism

$$Z(\mathbb{A}_f)/K_Z \;\times\; \Big[\mathcal{N} \times G(\mathbb{A}_f^p)/K_G^p\Big] \longrightarrow W \widehat{\otimes}_{O_{\boldsymbol{k},p}} \widehat{\mathcal{M}}_p^{\mathrm{ss}}. \tag{5.11}$$

Our next aim is to describe the automorphism group of the fixed supersingular point. For this, we first give a general definition:

Definition 5.14. The space of special homomorphisms of a point $(E, \iota_0, A, \iota, \lambda, \mathcal{F}) \in \mathcal{M}(S)$ is the $O_{\boldsymbol{k}}$-module

$$L(E, A) := \mathrm{Hom}_{O_{\boldsymbol{k}}}(E, A).$$

It is endowed with the $O_{\boldsymbol{k}}$-hermitian form

$$(x, y)_{\boldsymbol{k}} := x^* \circ y \in \mathrm{End}_{O_{\boldsymbol{k}}}(E). \tag{5.12}$$

It is naturally a right O_D-module by $xa = \iota(a) \circ x$, and this action is compatible with the hermitian form in the sense that $(x, ya) = (xa^*, y)$. We also set

$$V(E, A) = L(E, A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consider again the fixed supersingular point $(E, \iota_0, A, \iota, \lambda) \in \mathcal{M}(\mathbb{F})$. Then A is isogeneous to the n^2-th power of E, so $V(E, A)$ is a $\boldsymbol{k}$-vector space of dimension n^2. It is necessarily free of rank 1 as D-module and the natural map

$$\mathbb{A}_f^p \otimes_{\mathbb{Q}} V(E, A) \xrightarrow{\sim} \widehat{V}(E, A)^p$$

has to be an isomorphism. Moreover, by positivity of the Rosati involution, we know that $V(E, A)$ is a positive definite hermitian D-module. By the definition of $\mathcal{M}$ in terms of (5.10), we obtain that $V(E, A)$ is the unique positive definite hermitian D-module such that $\mathbb{A}_f^p \otimes_{\mathbb{Q}} V(E, A) \xrightarrow{\sim} \mathbb{A}_f^p \otimes_{\mathbb{Q}} V$. (The uniqueness follows from the classification of hermitian D-modules in §2.4.) We denote that D-module by $V^{(p)}$ and call it the p-nearby hermitian D-module of V. The group of quasi-automorphisms of $(E, \iota_0, A, \iota, \lambda)$ is then

$$\widetilde{G}^{(p)}(\mathbb{Q}) = [\boldsymbol{k}^\times \times GU_D^{\mathbb{Q}}(V^{(p)})(\mathbb{Q})]^{N(z)=c(g)}.$$

This group acts on the left hand side of (5.11) by composition in the framing.

Proposition 5.15. *The uniformization morphism* (5.11) *descends to an isomorphism*

$$Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/K_Z \;\times\; G^{(p)}(\mathbb{Q})\backslash \Big[\mathcal{N} \times G(\mathbb{A}_f^p)/K_G^p\Big] \xrightarrow{\sim} W \widehat{\otimes}_{O_{\boldsymbol{k},p}} \mathcal{M}_p^{\mathrm{ss}}. \tag{5.13}$$

Proof. This is the uniformization result of [26] combined with the Morita equivalence (5.9). $\square$

6. Special Cycles

6.1. **Algebraic Cycles.** The hermitian form on $L(E,A)$ defined in (5.12) takes values in $O_{\boldsymbol{k}}$. Let $(\ ,\ )_D$ be its lift along $\mathrm{Trd} : D \to \boldsymbol{k}$ as in Lemma 2.7. It takes values in the inverse different of $D/\boldsymbol{k}$,

$$\partial_{D/\boldsymbol{k}}^{-1} = \{x \in D \mid \mathrm{trd}_{D/\boldsymbol{k}}(xy) \in O_{\boldsymbol{k}} \text{ for all } y \in O_D\}. \tag{6.1}$$

Thus the forms $(\ ,\ )_{\boldsymbol{k}}$ and $(\ ,\ )_D$ are related by the diagram

$$\begin{tikzcd} & \partial_{D/\boldsymbol{k}}^{-1} \arrow[d, "\mathrm{trd}_{D/\boldsymbol{k}}"] \\ L \times L \arrow[ur, "(\ ,\ )_D"] \arrow[r, "(\ ,\ )_{\boldsymbol{k}}"'] & O_{\boldsymbol{k}}. \end{tikzcd} \tag{6.2}$$

Remark 6.1. The inverse different is a two-sided O_D-ideal which may be described as follows. If p is non-split in K, then $\partial_{D/\boldsymbol{k},p}^{-1} = O_{D,p}$. If p is split, then choose a central division algebra Q over $\mathbb{Q}_p$ and an isomorphism

$$O_{D,p} \cong M_m(O_Q) \times M_m(O_Q^{\mathrm{op}}).$$

Let $d = n/m$ be the degree of Q and let $\varpi \in O_Q$ be a uniformizer. Then

$$\partial_{D/\boldsymbol{k},p}^{-1} = M_m(\partial_{Q/\mathbb{Q}_p}^{-1}) \times M_m(\partial_{Q/\mathbb{Q}_p}^{-1})^{\mathrm{op}} \cong \varpi^{-d+1}(M_m(O_Q) \times M_m(O_Q^{\mathrm{op}})). \tag{6.3}$$

Indeed, let $H/\mathbb{Q}_p$ be an unramified field extension of degree d and let $O_H \to O_Q$ be any embedding. Then $O_Q = \bigoplus_{i=0}^{d-1} \varpi^i O_H$ so

$$\mathrm{trd}_{Q/\mathbb{Q}_p}\left(\sum_{i=0}^{d-1} \varpi^i a_i\right) = \mathrm{tr}_{H/\mathbb{Q}_p}(a_0).$$

On the other hand, the trace pairing on O_H is perfect by the unramifiedness of $H/\mathbb{Q}_p$.

Recall that $\mathrm{Herm}(D)$ and $\mathrm{Herm}^\times(D)$ denote the sets of hermitian (invertible) elements in D. We write $\mathrm{Herm}^{>0}(D) \subset \mathrm{Herm}^\times(D)$ for all those elements that are positive definite at infinity.

Definition 6.2. For $\xi \in \mathrm{Herm}(D)$, define $\mathcal{Z}(\xi) \to \mathcal{M}$ as the relative scheme with functor of points

$$\mathcal{Z}(\xi)(S) = \{(E, \iota_0, A, \iota, \lambda, \mathcal{F}, x) \mid x \in L(E,A),\ (x,x)_D = \xi\}.$$

Set $\mathbb{V}_f = \mathbb{A}_f \otimes_{\mathbb{Q}} V$ and let $\mathbb{V}_\infty$ be a positive definite hermitian D_∞-module that is free of rank 1. Then $\mathbb{V} = \mathbb{V}_f \otimes \mathbb{V}_\infty$ is a hermitian right $\mathbb{A}_D$-module that is incoherent, meaning that its Hasse invariant $\chi(\mathbb{V}) \in \mathbb{A}^\times / N_{\boldsymbol{k}/\mathbb{Q}}(\mathbb{A}_{\boldsymbol{k}}^\times)$ does not lie in $\mathbb{Q}^\times / N_{\boldsymbol{k}/\mathbb{Q}}(\boldsymbol{k}^\times)$. The next definition goes back to Kudla [12].

Definition 6.3. The difference set of an element $\xi \in \mathrm{Herm}^\times(D)$ is the set

$$\mathrm{Diff}(\mathbb{V}, \xi) = \{p \le \infty \mid \chi(\mathbb{V}_p) \ne \mathrm{Nrd}(\xi)\}.$$

The incoherence of $\mathbb{V}$ implies that $\mathrm{Diff}(\mathbb{V}, \xi)$ is always non-empty and consists of an odd number of places. Note that if p is non-archimedean, then $\chi(\mathbb{V}_p) = \chi(\xi)$ if and only if $\mathbb{V}_p$ represents ξ.

Proposition 6.4. *Assume that $\xi \in \mathrm{Herm}^\times(D)$ and that $\mathcal{Z}(\xi) \neq \emptyset$. Then $\mathrm{Diff}(\mathbb{V}, \xi)$ consists of a unique non-archimedean non-split prime p, and $\xi > 0$. Moreover, $\mathcal{Z}(\xi)$ is supported over the supersingular locus of $\mathcal{M}_p$.*

Proof. Assume that there exists a field k of characteristic $p \neq 0$ and a tuple $(E, \iota_0, A, \iota, \lambda, \mathcal{F}, x) \in \mathcal{Z}(\xi)(k)$. Then x defines an element $\widehat{x} \in \widehat{V}^p(E, A) = \mathrm{Hom}(\widehat{V}^p(E), \widehat{V}^p(A))$ with $(\widehat{x}, \widehat{x}) = \xi$. Since $\widehat{V}^p(E, A) \cong V \otimes \mathbb{A}_f^p$ by definition of $\mathcal{M}$, it follows that no finite prime $\ell \neq p$ is contained in $\mathrm{Diff}(\mathbb{V}, \xi)$. Moreover, by positivity of the Rosati involution and because ξ is non-singular, it also has the property that $\mathrm{Trd}_{D/K}(a^*\xi a) > 0$ for every $0 \neq a \in D$. Hence ξ is a positive definite element and thus $\infty \notin \mathrm{Diff}(\mathbb{V}, \xi)$. Since we know a priori that $\mathrm{Diff}(\mathbb{V}, \xi)$ has an odd number of elements, this proves $\mathrm{Diff}(\mathbb{V}, \xi) = \{p\}$.

Only non-split places may occur in $\mathrm{Diff}(\mathbb{V}, \xi)$, so p is non-split. Then E is supersingular. Moreover, the homomorphism $x : E \to A$ extends to an O_D-linear map

$$O_D \otimes_{\mathbb{Z}} E \longrightarrow A.$$

Since ξ is non-singular, this map is an isogeny so A is supersingular as well, and the proof is complete. □

We next give a description of $\mathcal{Z}(\xi)$ in terms of uniformization (Proposition 5.15). Assume for this that $\mathrm{Diff}(\mathbb{V}, \xi) = \{p\}$. Fix a supersingular point $(E, \iota_0, A, \iota, \lambda) \in \mathcal{M}(\mathbb{F})$ as well as choices of γ, β and $(\mathbb{X}, \iota, \lambda)$ as in Construction 5.13. Let $\mathbb{E}$ be the p-divisible group of E. We denote by $\mathcal{X}$ the universal hermitian $O_{\boldsymbol{k},p}$-module over $\mathcal{N}$ and by $\mathcal{E}/\operatorname{Spf} W$ the canonical lifting of $\mathbb{E}$.

Definition 6.5. For a tuple $x = (x_1, \dots, x_n) \in \mathrm{Hom}(\mathbb{E}, \mathbb{X})^{(n)}$, denote by $\mathcal{Z}(x) \to \mathcal{N}$ the closed formal subscheme of triples $(X, \iota, \lambda, \rho)$, resp. quadruples $(X, \iota, \lambda, \mathcal{F}, \rho)$ in the Krämer case, that is defined by the condition that the n quasi-homomorphisms $x_1 \circ \rho, \dots, x_n \circ \rho : \mathcal{E} \to \mathcal{X}$ are homomorphisms.

It was explained before Proposition 5.15 that $V(E, A)$ is the p-nearby hermitian D-module $V^{(p)}$ of $\mathbb{V}$. Its p-adic completion agrees with $\mathrm{Hom}_{\boldsymbol{k}}(\mathbb{E}, \mathbb{Y})$ for rank reasons because both $\mathbb{E}$ and $\mathbb{Y}$ are supersingular. Using γ and β, there is an identification as hermitian D_p-modules

$$\mathrm{Hom}_{\boldsymbol{k}_p}(\mathbb{E}, \mathbb{X}) \otimes_{\boldsymbol{k}_p} (\boldsymbol{k}_p^{(n)}, \beta) \xrightarrow{\sim} \mathrm{Hom}_{\boldsymbol{k}_p}(\mathbb{E}, \mathbb{Y}).$$

Under this bijection, elements $x \in \mathrm{Hom}_{\boldsymbol{k}_p}(\mathbb{E}, \mathbb{Y})$ with $(x, x)_D = \xi$ correspond to tuples $(x_1, \dots, x_n) \in \mathrm{Hom}_{\boldsymbol{k}_p}(\mathbb{E}, \mathbb{X})^{(n)}$ such that $(x_i, x_j)_{ij} = \beta\xi$. (Here and below, we use γ to view ξ as element of $M_n(\boldsymbol{k}_p)$.) Note that ξ is $*$-hermitian $\xi^* = \xi$, which implies that $\beta\xi$ is $\dagger$-hermitian.

Fix an element $x \in V^{(p)}$ with $(x, x)_D = \xi$. At p, we view x as an element of $\mathrm{Hom}(\mathbb{E}, \mathbb{X})^{(n)}$ with $(x_i, x_j)_{ij} = \beta\xi$ as just explained.

Proposition 6.6. *The uniformization morphism from* (5.11) *induces an isomorphism*

$$\Pi : Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/K_Z \times \sum_{g^p \in G(\mathbb{A}_f^p)/K^p} 1_{\Lambda^p}(g^{p,-1}x) \cdot \mathcal{Z}(x) \times [g^p K^p] \xrightarrow{\sim} \mathcal{Z}(\xi). \tag{6.4}$$

Note that the universal map $x : O_D \otimes_{\mathbb{Z}} E \to A$ over $\mathcal{Z}(\xi)$ is essentially a quasi-isogeny to the framing object. So it is not surprising that $\mathcal{Z}(\xi)$ embeds into the covering space of (5.11).

Given $x = (x_1, \dots, x_n) \in \mathrm{Hom}_{\boldsymbol{k}_p}(\mathbb{E}, \mathbb{X})^{(n)}$, there is a more refined definition of the local cycle as

$$[\mathcal{Z}(x)] := [\mathcal{O}_{\mathcal{Z}(x_1)} \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathcal{M}}}{\otimes}} \cdots \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathcal{M}}}{\otimes}} \mathcal{O}_{\mathcal{Z}(x_n)}] \in K_0'(\mathcal{Z}(x)).$$

It lies in the filtration step that is generated by 0-dimensional cycles, i.e. may be represented by a finite sum of skyscraper sheaves. By the main result of [8], the class $[\mathcal{Z}(x)]$ only depends on the lattice $\langle x_1, \dots, x_n \rangle$ generated by x. By (6.4), the only way to define a global class $z(\xi)$ that is compatible with the local theory is to set

$$z(\xi) := \sum_{g^p} 1_{\widehat{\Lambda}^p}(g^{p,-1}x_0)\ \Pi_* z(g^{p,-1}x_0) \in K_0'(\mathcal{Z}(\xi)). \tag{6.5}$$

Remark 6.7. The class $z(\xi)$ is well-defined, meaning it is independent of the choices in Construction 5.13: First, independence of β is clear because the definition of $z(\xi)$ does not refer to β. Second, any two choices of γ are conjugate under $GL_n(O_{\boldsymbol{k},p})$. In terms of $\mathcal{Z}(x)$, this corresponds to a change of basis of the lattice $\langle x_1, \dots, x_n \rangle$ and it is known by Howard's linear invariance [8] as well as its extensions to the ramified and bad reduction places [4, 19] that $[\mathcal{Z}(x)]$ only depends on that lattice. Finally, independence of the base point $(E, \iota_0, A, \iota, \lambda)$ used for the uniformization can be seen directly.

There is also a recent construction of a canonical derived cycle ${}^{\mathbb{L}}\mathcal{Z}(\xi)[\mathfrak{d}^{-1}]$ by Madapusi [22] which applies at unramified places of good reduction. (Here, $\mathfrak{d}$ denotes the complement of those places.) It should be true that the class $[\mathcal{Z}(\xi)]$ can be constructed canonically from ${}^{\mathbb{L}}\mathcal{Z}(\xi)$ whenever $p \nmid \mathfrak{d}$.

Assume that $D \not\cong M_n(\boldsymbol{k})$. Then $\mathcal{M}$ is proper (Proposition 5.4) and we denote by $\widehat{\mathrm{CH}}^n(\mathcal{M})_{\mathbb{Q}}$ the top degree $\mathbb{Q}$-coefficient Chow group of $\mathcal{M}$. There is a natural map

$$F^n K_0'(\mathcal{Z}(\xi)) \longrightarrow \widehat{\mathrm{CH}}^n(\mathcal{M})_{\mathbb{Q}} \tag{6.6}$$

that sends the skyscraper sheaf $\mathcal{O}_P$ of a closed point $P \in \mathcal{Z}(\xi)$ to the class of $(P, 0)$.

Definition 6.8. We denote by $\widehat{c}(\xi) \in \widehat{\mathrm{CH}}^n(\mathcal{M})$ the image of $z(\xi)$ under (6.6).

The following degree formula is completely analogous to that of [16]. Let γ and β be as in Construction 5.13.

Proposition 6.9. *Assume that $\xi \in \mathrm{Herm}^{>0}(D)$ is positive and such that $\mathrm{Diff}(\mathbb{V}, \xi) = \{p\}$ consists of a single prime. The arithmetic degree of $\widehat{c}(\xi)$ is given by*

$$\widehat{\deg}(\widehat{c}(\xi)) = c_K \mathrm{Int}_p(\xi\beta^{-1})\,\mathrm{Orb}^p(\xi, 1_{\Lambda^p}) \log p^{f_p}.$$

Here $f_p = [O_{\boldsymbol{k},p}/\mathfrak{p} : \mathbb{Z}_p/p]$, $c_K = |Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/K_Z|$, and

$$\mathrm{Orb}^p(\xi, 1_{\Lambda^p}) = \sum_{g^p \in G(\mathbb{A}_f^p)/K^p} 1_{\Lambda^p}(g^{p,-1}x) = \frac{1}{\mathrm{Vol}(K^p)} \int_{G(\mathbb{A}_f^p)} 1_{\Lambda^p}(g^{p,-1}x)dg$$

where dg is the Tamagawa measure on $G(\mathbb{A})$ restricted to $G(\mathbb{A}_f^p)$

Proof. This follows from (6.5), noting that $\mathrm{Int}_p(\xi\beta^{-1}) := \deg(z(x))$ is independent of $x \in V_\xi^{(p)}(\mathbb{Q}_p)$ and that the number of contributing summands in (6.5) is counted by $\mathrm{Orb}^p(\xi, 1_{\Lambda^p})$. $\square$

Proposition 6.10. *Let the notation be as above. Then there is an explicit Schwartz function* $\phi_p^{(p)} \in S(V_p^{(p)})$ *such that*

$$\mathrm{Int}_p(\xi\beta^{-1}) \log p^{f_p} = \frac{W'_{\xi,p}(1,0,1_{\Lambda_p}) + W_{\xi,p}(1,0,\phi_p^{(p)}) \log p^{f_p}}{W_{\xi_0,p}(1,0,1_{\Lambda_p})}.$$

Here $\Lambda_p = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_{D_p}$, ξ_0 *is a Gram matrix of* Λ_p *with respect to involution* $*$, *all Whittaker functions are respect to the involution* $*$ *of* D, *and* $f_p = 1$ *or* 2 *depending on whether* $\boldsymbol{k}/Q$ *is ramified or not at* p. *Moreover, when* p *is inert and* Λ_p *is self-dual or when* p *is ramified and we are in exotic smooth case,* $\phi_p^{(p)} = 0$.

Proof. (Sketch) Typically the local arithmetic Siegel-Weil formula is written in terms of local density functions. The relation between local Whittaker functions and local density functions in the usual Hermitian case is given by

$$W_{\xi,p}(1,s,1_L) = \gamma(L)|\det L|_p^n \mathrm{Den}(L, M_\xi, X) \tag{6.7}$$

with $X = (p^{f_p})^{-2s}$ when $s \geq 0$ is an integer. Here M_ξ is a hermitian $\mathcal{O}_{\boldsymbol{k}_p}$-lattice with Gram matrix ξ. Now the local arithmetic Siegel-Weil formula in [7], [19] and [20] together with equivalence give

$$\mathrm{Int}_p(\xi\beta^{-1}) \log p^{f_p} = \frac{W^{\dagger,\prime}_{\xi\beta^{-1},p}(1,0,1_{\Lambda_p}) + W^{\dagger}_{\xi\beta^{-1},p}(1,0,\tilde{\phi}_p^{(p)}) \log p^{f_p}}{W^{\dagger}_{\xi_0\beta^{-1},p}(1,0,1_{\Lambda_p})}.$$

For some Schwartz function $\tilde{\phi}_p^{(p)} \in S^{\dagger}(V_p^{(p)})$. Here the superscript $\dagger$ indicates that the functions are relative to the standard involution $\dagger$ in $M_n(\boldsymbol{k}_p)$, and the local arithmetic intersection $\mathrm{Int}_p(\xi\beta^{-1})$ is in the standard Rapoport-Zink space. Finally $\xi_0\beta^{-1}$ is a Gram matrix of Λ_p with respect to $\dagger$. Let $\phi_p^{(p)} = \tilde{\phi}_{p,\beta^{-1}}^{(p)}$ as in (**??**), and notice that $1_{\Lambda_p,\beta^{-1}} = 1_{\Lambda_p}$ as $\beta \in \mathrm{GL}_n(\mathcal{O}_E)$ and Λ_p is unimodular. So we have by Corollary 3.5

$$\mathrm{Int}_p(\xi\beta^{-1}) \log p^{f_p} = \frac{W'_{\xi,p}(1,0,1_{\Lambda_p}) + W^{\dagger}_{\xi,p}(1,0,\phi_p^{(p)}) \log p^{f_p}}{W_{\xi_0,p}(1,0,1_{\Lambda_p})},$$

as expected.

□

6.2. **Green Currents.** Recall that $X \subseteq \mathbb{P}^1(W_1)$ denotes the hermitian symmetric domain of our Shimura datum. Let $\mathcal{L} \subset V \otimes_{\mathbb{C}} \mathcal{O}_X$ be the universal n-dimensional D-stable totally negative subspace and let

$$p_{\mathcal{L}} : V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{L}$$

denote the orthogonal projection. Given an element $x \in V$ such that $\xi = (x,x)_D \in D_{\mathbb{R}}^{\times}$, the pair $(\mathcal{L}, p_{\mathcal{L}}(x))$ consists of a hermitian rank n vector bundle with nowhere vanishing section. In this situation, Garcia–Sankaran [6] define a Green current for the empty cycle (i.e. a top degree smooth differential form) $\omega(x) \in \Omega^{n,n}(X)$. It is $G(\mathbb{R})$-equivariant in the sense that

$$g^*\omega(x) = \omega(g^{-1}x).$$

Moreover, $\omega(x)$ is integrable and we define

$$\mathrm{Int}_\infty(x) := \int_X \omega(x).$$

In fact, this quantity only depends on $\xi = (x, x)$. For every $\xi \in \mathrm{Herm}^\times(D)$, the sum

$$\widetilde{\omega}(\xi) := \sum_{x \in V_\xi(\mathbb{Q})} \sum_{gK \in G(\mathbb{A}_f)/K} \omega(x) \otimes 1_\Lambda(g^{-1}x) \in \Omega^{n,n}\big(X \times G(\mathbb{A}_f)/K\big) \tag{6.8}$$

is $G(\mathbb{Q})$-equivariant.

Definition 6.11. Let $\omega(\xi) \in \Omega^{n,n}(\mathcal{M}(\mathbb{C}))$ be the descent of $\widetilde{\omega}(\xi)$ along the complex uniformization morphism. Define

$$\widehat{z}(\xi) := (0, \widetilde{\omega}(\xi)) \in \widehat{\mathrm{CH}}^n(\mathcal{M}).$$

By (6.8), we have

$$\deg \widehat{z}(\xi) = \mathrm{Int}_\infty(\xi)\, \mathrm{Orb}(\xi, 1_\Lambda).$$

7. Arithmetic Siegel–Weil formula

In this section, let $(D, *)$ be a central simple algebra over $\boldsymbol{k}$ of degree n and second kind involution $*$ with a $*$-stable maximal order O_D. Let V be a right free $(D, *)$-hermitian module of signature $(n-1, 1)$, and let Λ be an O_D-lattice of V such that for each non-split prime number p, $\Lambda_p^\vee = \Lambda_p$ or $\pi_p^{-1}\Lambda_p$. We assume that $D \neq M_n(\boldsymbol{k})$ as the case $M_n(\boldsymbol{k})$ has already been known as usual unitary Shimura variety of signature $(n-1, 1)$.

Let $\mathbb{V}_f = \mathbb{A}_f \otimes_{\mathbb{Q}} V$, and let $\mathbb{V}_\infty$ be a positive definite hermitian $(D_\infty, *)$-module that is free of rank 1. Then $\mathbb{V} = \mathbb{V}_f \otimes \mathbb{V}_\infty$ is an incoherent hermitian $\mathbb{A} \otimes_{\mathbb{Q}} (D, *)$-module, meaning it does not come by completion from a hermitian D-module. Let $\phi = \otimes' \phi_p \in S(\mathbb{V}_{\mathbb{A}})$ where where ϕ_p is the characteristic function of Λ_p, and $\phi_\infty(x) = e^{-\pi(x,x)} \in S(\mathbb{V}_\infty)$ is the Gaussian. For each finite prime p ramified in $\boldsymbol{k}$ such that $\Lambda_p' = \Lambda_p$, define $\phi^{(p)} = \otimes_q' \phi_q^{(p)} \in S(V_{\mathbb{A}}^{(p)})$ where $\phi_q^{(p)} = \phi_q$ for all $q \neq p$ and $\phi_p^{(p)}$ is the same as in Proposition 6.10. For $z = x + iy \in \mathcal{H}_n$, and $x \in \mathrm{Herm}_n^\times(D)$, we define the associated arithmetic 0-cycle $\widehat{z}(y, \xi)$ as follow. When $|\mathrm{Diff}(\mathbb{V}, \xi)| > 1$, we have $\widehat{z}(y, \xi) = 0$; when $\mathrm{Diff}(\mathbb{V}, \xi) = \{p\}$, we have

$$\widehat{z}(y, \xi) = \begin{cases} (z(\xi), 0) & \mathrm{if p} < \infty, \\ (0, \tilde{\omega}(y, \xi) & \mathrm{if p} = \infty. \end{cases} \tag{7.1}$$

Theorem 7.1. *For every $\xi \in \mathrm{Herm}^\times(D)$, there is the identity*

$$C\widehat{\deg}\widehat{z}(y, \xi) q^\xi = E_\xi'(z, 0, \phi) + \sum_{p < \infty} E_\xi(z, 0, \phi^{(p)}) \log p^{f_p}. \tag{7.2}$$

Here $C \neq 0$ is an absolute constant independent of ξ, $q = e^{2\pi i \mathrm{Tr}(z\xi)}$, and $z = x + iy \in \mathcal{H}_n$.

Proof. The proof is standard now and is a combination of local arithmetic Siegel-Weil formula and counting via the usual Siegel-Weil formula. As $\mathbb{V}$ is incherent, $|\mathrm{Diff}(\mathbb{V}, \xi)| \geq 1$. When $|\mathrm{Diff}(\mathbb{V}, \xi)| > 1$, both sides are equal to 0. Assume that $\mathrm{Diff}(\mathbb{V}, \xi) = \{p\}$, then p is non-split in $\boldsymbol{k}$. When $p < \infty$, one has ξ_∞ is positive definite, and the sum on the right has only one possible non-zero term, which is $E_\xi(z,0,\phi^{(p)})$. So

$$\begin{aligned}
&E'_\xi(z,0,\phi) + \sum_{p<\infty} E_\xi(z,0,\phi^{(p)}) \log p^{f_p} \\
&= \left[W'_{\xi,p}(1,0,\phi_p) + W_{\xi,p}(z,0,\phi^{(p)} \log p^{f_p}\right] \prod_{q\nmid p\infty} W_{\xi,q}(1,s,\phi_q)|_{s=0} W_{\xi,\infty}(z,0,\phi_\infty) \\
&= \left[W'_{\xi,p}(1,0,\phi_p) + W_{\xi,p}(1,0,\phi^{(p)}) \log p^{f_p}\right] \frac{L_p(1,\epsilon_{\boldsymbol{k}/\mathbb{Q}})}{L(1,\epsilon_{\boldsymbol{k}/Q})} \prod_{q\nmid p\infty} \lambda_q W_{\xi,q}(1,0,\phi_q) \cdot W_{\xi,\infty}(z,0,\phi_\infty),
\end{aligned}$$

and by Proposition 6.9

$$\widehat{\deg}\widehat{z}(y,\xi)q^\xi = \widehat{\deg}\widehat{z}(\xi)q^\xi = c_K \mathrm{Int}_p(\xi\beta^{-1}) \,\mathrm{Orb}^p(\xi, 1_{\Lambda^p}) \log p^{f_p} q^\xi.$$

By Siegel-Weil formula (Proposition 4.2) and local Siegel-Weil formula (Proposition 3.1), we have

$$\mathrm{Orb}^p(\xi, 1_{\Lambda^p}) = (\gamma(V_p)\gamma(V_\infty))^{-1} \prod_{q\nmid p\infty} \lambda_q W_{\xi,q}(1,0,\phi_q).$$

Recall (see for example [21, Proposition 4.5] or [6, Proposition 3.2])

$$W_{\xi,\infty}(\tau,0,\phi_\infty) = \gamma(\mathbb{V}_\infty) \frac{(2\pi)^{n^2}}{\Gamma_n(n)} q^\xi.$$

Here

$$\Gamma_n(s) = (2\pi)^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s-j)$$

is the Siegel Gamma function. So the local arithmetic Siegel-Weil formula implies

$$C_p \widehat{\deg}\widehat{z}(y,\xi)q^\xi = E'_\xi(z,0,\phi) + \sum_{p<\infty} E_\xi(z,0,\phi^{(p)}) \log p^{f_p},$$

for some explicit constant $C_p \neq 0$. The same argument also give the same identity for $p=\infty$ with constanct C_∞. Direct calculation shows gives an explicit constant for all C_p, $p \leq \infty$, which is independent of p. This proves the theorem.

□

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