Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 12

Due Date: 23.01.2014

Exercise 1:
Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathcal{O}_X$-modules and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module. Let us further write

$$\mathscr{E}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E},\mathcal{O}_X).$$

(i) Show that $(\mathscr{E}^\vee)^\vee \cong \mathscr{E}$.

(ii) Show that $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E},\mathscr{F}) \cong \mathscr{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{F}$.

(iii) Assume that $\mathscr{F}$ is of finite presentation. Show that the canonical map

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

is an isomorphism.

(iv) Let $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathscr{E}'$ be a locally free $\mathcal{O}_Y$-module of finite rank. Show that there is a canonical and functorial isomorphism

$$f_*(\mathscr{F}\otimes_{\mathcal{O}_X} f^*\mathscr{E}') \cong f_*\mathscr{F}\otimes_{\mathcal{O}_Y}\mathscr{E}'.$$

(Hint: Use the adjointness of f_* and f^* to construct a canonical and functorial map. Then localize on Y to show that this map is an isomorphism.)

Exercise 2:
Let X be a noetherian scheme and let $j:U\hookrightarrow X$ be an open subscheme. Let $\mathscr{F}$ be a coherent $\mathcal{O}_U$-module. We want to show that there is a coherent sheaf $\mathscr{G}$ on X such that $\mathscr{G}|_U\cong\mathscr{F}$. To do so we proceed as follows:

(i) Show that $j_*\mathscr{F}$ is a quasi-coherent sheaf such that $(j_*\mathscr{F})|_U\cong\mathscr{F}$.

(ii) Assume that X is affine and construct a coherent subsheaf $\mathscr{G}\subset j_*\mathscr{F}$ that extends $\mathscr{F}$.

(iii) In the general case use a cover $X=V_1\cup\dots\cup V_n$ of X by finitely many open affine subschemes. Set $X_i=U\cup V_1\cup\dots\cup V_i$ and use (ii) to extend $\mathscr{F}$ successively from W_i to W_{i+1}.

Exercise 3:
Let $i:Z\hookrightarrow X$ be a closed immersion and write $\mathscr{I}=\ker(i^\sharp:\mathcal{O}_X\to i_*\mathcal{O}_Z)$.

(i) Show that i_* and i^* define an equivalence of categories between $(\mathrm{Q}-\mathrm{Coh}_Z)$ and the full subcategory of $(\mathrm{Q}-\mathrm{Coh_X})$ consisting of those objects $\mathscr{F}$ such that $\mathscr{I}\mathscr{F}=0$.

(ii) Assume in addition that X is locally noetherian. Show that i_* and i^* define an equivalence of categories between (Coh_Z) and the full subcategory of $(\mathrm{Coh_X})$ consisting of those objects $\mathscr{F}$ such that $\mathscr{I}\mathscr{F}=0$.

Exercise 4:

Show that the following schemes are not affine by either giving an example of a quasi-coherent sheaf $\mathscr{F}$ that is not generated by its global sections (i.e. the canonical morphism

$$\Gamma(X, \mathscr{F}) \otimes_{\Gamma(X, \mathcal{O}_X)} \mathcal{O}_X \longrightarrow \mathscr{F}$$

is not surjective) or giving an example of a short exact sequence of quasi-coherent sheaves that is not exact on global sections.

(i) $\mathbb{A}_k^n \backslash \{0\}$ for $n \geq 2$ and a field k.

(ii) $\mathbb{P}_k^n$ for $n \geq 1$ and a field k.

(iii) The open subscheme $U = \mathbb{P}_k^n \backslash V_+(T_1, T_2)$ for $n \geq 2$.

(iv) The closed subscheme $V_+(T_0 T_2^2 - T_1^3) \subset \mathbb{P}_k^2$.

(v) The affine line with a double point.