WS 2024/25

Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Exercises for **Topology I**
Sheet 2

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Exercise 1. Let X be a CW-complex.

1. Consider the equivalence relation on the set X_0 of 0-cells generated by $x \sim y$ whenever x, y are *joined by a* 1-*cell*, i.e. whenever there exists a 1-cell e with $x, y \in \bar{e}$. Show that the map

$$\begin{aligned} X_0/_\sim &\longrightarrow \pi_0(X) \\ x &\longmapsto [x] \end{aligned}$$

   is well-defined and bijective.

2. Show that every path component of X is open in X. Conclude that X is the topological disjoint union (= coproduct) of its path components.

Exercise 2. Let X be a CW-complex with skeleta $X_k \subseteq X$, and let $x_0 \in X_0$.

1. Show that the map $\pi_1(X_k, x_0) \to \pi_1(X, x_0)$ induced by the inclusion is surjective for $k \geq 1$ and bijective for $k \geq 2$.
2. Give an example of a CW-complex X with a chosen basepoint $x_0 \in X_0$ such that $\pi_1(X_1, x_0) \to \pi_1(X, x_0)$ is *not* an isomorphism.
3. Determine the minimum number of cells in any CW-structure on the torus $S^1 \times S^1$.

Exercise 3. Describe a CW-structure on the real projective plane $\mathbb{R}\mathrm{P}^2$ which contains a subcomplex homeomorphic to the *Möbius strip*

$$\overline{M} \coloneqq \big([0,1] \times [0,1]\big)\big/_{(0,t)\sim(1,1-t)}.$$

Does there also exist a CW-structure on $\mathbb{R}\mathrm{P}^2$ which contains a subcomplex homeomorphic to the *open Möbius strip*

$$M \coloneqq \big([0,1] \times (0,1)\big)\big/_{(0,t)\sim(1,1-t)}?$$

Exercise 4. Let A be a topological space.

1. Let $\alpha, \beta \colon \partial D^n \rightrightarrows A$ be two homotopic maps. Show that the spaces

$$A \cup_{\alpha, \partial D^n} D^n \qquad \text{and} \qquad A \cup_{\beta, \partial D^n} D^n$$

   are homotopy equivalent.

$*$2. (*5 bonus points*) Give an example of topological spaces A, X, a subspace $Y \subseteq X$, and homotopic maps $\alpha, \beta \colon Y \rightrightarrows A$ such that

$$A \cup_{\alpha, Y} X \qquad \text{and} \qquad A \cup_{\beta, Y} X$$

   are *not* homotopy equivalent.

Please hand in your solutions by **October 28, 10am** via eCampus.