WS 2024/25

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Exercises for **Topology I**
Sheet 7

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Definition. Let $\mathcal{D}$ be a small category. The *nerve of* $\mathcal{D}$ is the simplicial set $\mathrm{N}(\mathcal{D})$ with

$$\mathrm{N}(\mathcal{D})_n = \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}_{[n]}, \mathcal{D})$$

and structure map $\alpha^* \colon \mathrm{N}(\mathcal{D})_n \to \mathrm{N}(\mathcal{D})_m$ for $[m] \to [n]$ given by precomposition with $\mathcal{C}_\alpha \colon \mathcal{C}_{[m]} \to \mathcal{C}_{[n]}$. Here $\mathcal{C}_{(-)} \colon \mathbf{Poset} \to \mathbf{Cat}$ is as on the previous sheet.

Exercise 1. 1. Let $n \geq 0$. Construct a bijection between $\mathrm{N}(\mathcal{D})_n$ and the set N_n of tuples $(\alpha_n, \dots, \alpha_1)$ of composable arrows in $\mathcal{D}$ (i.e. such that the source of α_{i+1} equals the target of α_i); by convention, N_0 is the set of objects of $\mathcal{D}$. What do the face and degeneracy maps d_i^*, s_i^* correspond to under this identification?

2. Upgrade N to a fully faithful functor $\mathbf{Cat} \to \mathbf{SSet}$ from the category of small categories to the category of simplicial sets.

$*$3. (*3 bonus points*) Let X be a simplicial set such that every morphism $\Lambda^n_k \to X$ with $0 < k < n \leq 3$ admits a unique extension to Δ^n. Construct a small category $\mathrm{h}X$ with objects X_0 and with hom sets $\mathrm{Hom}_{\mathrm{h}X}(x, y) \coloneqq \{e \in X_1 : d_1^*(e) = x, d_0^*(e) = y\}$.

$*$4. (*7 bonus points*) Show: a simplicial set X is isomorphic to the nerve of a small category if and only if each $\Lambda^n_k \to X$ with $0 < k < n$ admits a unique extension to Δ^n.

Exercise 2. 1. For a family $(C^i)_{i \in I}$ of chain complexes we define the direct sum $\bigoplus_{i\in I} C^i$ dimensionwise, i.e. $\big(\bigoplus_{i\in I} C^i\big)_n = \bigoplus_{i \in I} C^i_n$ with componentwise differential.

Show that the inclusions $C^i_n \hookrightarrow \bigoplus_{i\in I} C^i_n$ define morphisms of chain complexes $\mathrm{incl}_i \colon C^i \to \bigoplus_{i \in I} C^i$, and that they have the following universal property: for every chain complex D the map

$$\begin{aligned} \mathrm{Hom}\Big(\bigoplus_{i\in I} C^i, D\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}(C^i, D) \\ f &\longmapsto (f \circ \mathrm{incl}_i)_{i\in I} \end{aligned}$$

is bijective.

2. Show that the homomorphism $\bigoplus_{i\in I} H_n(C^i) \to H_n\big(\bigoplus_{i\in I} C^i\big)$ induced by the inclusions $C^i \to \bigoplus_{i\in I} C^i$ is an isomorphism of abelian groups for every $n \geq 0$.

3. Let $\{Y_i\}_{i\in I}$ be a family of simplicial sets and let A be an abelian group. Show that the homomorphism $\bigoplus_{i\in I} C(Y_i, A) \to C\big(\coprod_{i\in I} Y_i, A\big)$ induced by the inclusions is an isomorphism of chain complexes.

4. Let X be a topological space and let $(U_i)_{i\in I}$ be pairwise disjoint open subsets of X with $X = \bigcup_{i\in I} U_i$. Construct an isomorphism of simplicial sets $\coprod_{i\in I} \mathcal{S}(U_i) \to \mathcal{S}(X)$ and use this to construct isomorphisms $\bigoplus_{i\in I} H_n(U_i, A) \cong H_n(X, A)$ for all $n \geq 0$ and all abelian groups A.

please turn over

Exercise 3.

1. Let X be a simplicial set and let $Y \subseteq X$ be a subsimplicial set. Show that there is a unique way to turn the degreewise quotients $(X/Y)_n \coloneqq X_n/Y_n$ into a simplicial set X/Y in such a way that the quotient maps form a morphism of simplicial sets $X \to X/Y$.

2. For $n \geq 1$ we define the *boundary* $\partial\Delta^n \subseteq \Delta^n$ as the subsimplicial set given by the non-surjective maps (you can convince yourself that this indeed a subsimplicial set). Show that

$$H_m(\Delta^n/\partial\Delta^n, A) \cong \begin{cases} A & \text{if } m = 0 \text{ or } m = n \\ 0 & \text{if } 0 < m < n \end{cases}$$

   for every abelian group A, and make the isomorphisms in degrees 0 and n explicit.

   Remark. The quotient $\Delta^n/\partial\Delta^n$ is sometimes called the (standard) *simplicial n-sphere*. We will later see that $H_m(\Delta^n/\partial\Delta^n, A) = 0$ for $m > n$.

Exercise 4. Let

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

be a commutative diagram of groups and homomorphisms such that both rows are exact. Show the following:

1. If β and δ are injective and α is surjective, then γ is injective.

2. If β and δ are surjective and ϵ is injective, then γ is surjective.

3. If β and δ are bijective, α is surjective, and ϵ is injective, then γ is bijective.