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Exercises for **Topology I**
Sheet 8

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Definition. A topological space X is called *totally disconnected* if it does not contain a connected subspace with more than one point. (For example, every discrete space is totally disconnected, as is $\mathbb{Q}$ with the subspace topology.)

Exercise 1. Let X be a totally disconnected space with underlying set X_0, and let A be an abelian group. Construct an explicit isomorphism $H_0(X, A) \cong \bigoplus_{X_0} A$ and determine $H_n(X, A)$ for all $n > 0$.

Definition. A *double complex* $C_{\bullet,\bullet}$ is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{2,2} & \longrightarrow & C_{2,1} & \longrightarrow & C_{2,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{1,2} & \longrightarrow & C_{1,1} & \longrightarrow & C_{1,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{0,2} & \longrightarrow & C_{0,1} & \longrightarrow & C_{0,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of abelian groups such that all rows and columns are chain complexes. We write $d^h_{p,q}$ for the 'horizontal' differential $C_{p,q} \to C_{p,q-1}$ and $d^v_{p,q}$ for the 'vertical' differential $C_{p,q} \to C_{p-1,q}$.

Exercise 2.

1. Let $C_{\bullet,\bullet}$ be a double complex. Show that $\mathrm{Tot}(C_{\bullet,\bullet})_n := \bigoplus_{p+q=n} C_{p,q}$ becomes a chain complex via the differential given in degree n by

$$C_{p,q} \ni x \mapsto (-1)^p d^h_{p,q}(x) + d^v_{p,q}(x)$$

   (with the convention that $d^h_{p,0}$ and $d^v_{0,q}$ are zero). The chain complex $\mathrm{Tot}(C_{\bullet,\bullet})$ is called the *total complex* of the double complex $C_{\bullet,\bullet}$.

2. Show: if the chain complexes $C_{\bullet,q}$ are exact for all $q > 0$, then the inclusions $C_{p,0} \hookrightarrow \mathrm{Tot}(C_{\bullet,\bullet})_p$ induce isomorphisms $H_n(C_{\bullet,0}) \cong H_n\big(\mathrm{Tot}(C_{\bullet,\bullet})\big)$.

3. Use this to show that if the chain complexes $C_{\bullet,q}$ and $C_{p,\bullet}$ are exact for all $p, q > 0$, then the complexes $C_{\bullet,0}$ and $C_{0,\bullet}$ have isomorphic homology.

please turn over

Definition. Let C, D be chain complexes. We define the *external tensor product* $C \boxtimes D$ as the double complex with $(C \boxtimes D)_{p,q} = C_p \otimes D_q$ and differentials $C_p \otimes d$ and $d \otimes D_q$ (you can convince yourself that this is indeed a double complex). The *tensor product* of C and D is then defined as the total complex $\mathrm{Tot}(C \boxtimes D)$ in the sense of the previous exercise.

Exercise 3. 1. The *interval* is the chain complex I given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\binom{1}{-1}} \mathbb{Z}^2 \longrightarrow 0.$$

Let e_0, e_1 denote the standard basis vectors of $\mathbb{Z}^2 = I_0$. Show that the maps $C_n \to (C \otimes I)_n, c \mapsto c \otimes e_k$ define a chain map $\iota_k \colon C \to C \otimes I$ for $k = 0, 1$.

2. Let $f, g \colon C \rightrightarrows D$ be chain maps. Construct a bijection

$$\{\text{chain homotopies from } f \text{ to } g\} \cong \{\text{chain maps } H \colon C \otimes I \to D \text{ with } H\iota_0 = f, H\iota_1 = g\}.$$

Hint. First construct a chain homotopy from ι_0 to ι_1.

Exercise 4. Recall from the previous sheet the definition of the *nerve* $\mathrm{N}(\mathcal{C})$ of a small category $\mathcal{C}$.

1. Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors and let $\tau \colon F \Rightarrow G$ be a natural transformation. Construct a simplicial homotopy from $\mathrm{N}(F)$ to $\mathrm{N}(G)$.
2. Conclude that if $\mathcal{C}$ has a terminal object (i.e. an object 1 such that $|\mathrm{Hom}(X, 1)| = 1$ for all $X \in \mathcal{C}$), then the identity of $\mathrm{N}(\mathcal{C})$ is simplicially homotopic to a constant map.

Definition. A chain map $f \colon C \to D$ is called a *quasi-isomorphism* if the induced map $H_n(f) \colon H_n(C) \to H_n(D)$ is an isomorphism for every $n \geq 0$. We say that C and D are *quasi-isomorphic* if they can be connected by a zig-zag of quasi-isomorphisms, i.e. there exists a diagram

$$C = C_0 \to C_1 \leftarrow C_2 \to \cdots C_{n-1} \leftarrow C_n = D$$

for some n such that all maps are quasi-isomorphisms.

$*$ **Exercise 5** (*4 + 6 bonus points*). 1. Let C be a chain complex. Show that there exists a quasi-isomorphism $C' \to C$ such that each C'_n is free abelian.

2. Let C be a chain complex. Show that C is quasi-isomorphic to the chain complex

$$\cdots \xrightarrow{0} H_n(C) \xrightarrow{0} H_{n-1}(C) \xrightarrow{0} \cdots \xrightarrow{0} H_1(C) \xrightarrow{0} H_0(C) \longrightarrow 0.$$

Hint. As an intermediate step, show that C is quasi-isomorphic to a direct sum of complexes each of which vanishes outside of two adjacent degrees.

Remark. As a consequence, we can check whether two chain complexes are quasi-isomorphic by computing their homology. Note/recall that on the other hand we can*not* check whether two CW-complexes are weakly equivalent by simply looking at their disembodied homotopy groups, so this is something really special about chain complexes (of abelian groups).