WS 2024/25 Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Exercises for **Topology I**
Sheet 10

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

This is the last exercise sheet counting towards admission for the final exam.

Definition. Let $\big((X_i,x_i)\big)_{i\in I}$ be a family of based spaces. The *wedge sum* $\bigvee_{i\in I} X_i$ is the space obtained from the topological disjoint union $\coprod_{i\in I} X_i$ by collapsing the subspace $\coprod_{i\in I}\{x_i\}$ to a single point.

Exercise 1.

1. Let $(X_i)_{i\in I}$ be a family of CW-complexes and pick for each X_i a zero-cell $x_i\in X_i$ as basepoint. Show that for every abelian group A and every $n>0$ the map
$$\bigoplus_{i\in I} H_n(X_i,A)\to H_n\Big(\bigvee_{i\in I} X_i, A\Big)$$
induced by the inclusions $X_i\to\bigvee_{i\in I}X_i$ is an isomorphism.
2. Show that the analogous map
$$\bigoplus_{i\in I} H_0(X_i,A)\to H_0\Big(\bigvee_{i\in I} X_i, A\Big)$$
is surjective and determine its kernel.

Exercise 2. Let (X,x) be a based space.

1. Write $i\colon\nabla^1\to[0,1]$ for the homeomorphism $(t,1-t)\mapsto t$. Show that the map $\pi_1(X,x)\to H_1(X,\mathbb{Z})$ sending the class of a loop $\gamma\colon([0,1],\{0,1\})\to(X,x)$ to the homology class of $[\gamma\circ i]$ is well-defined and a group homomorphism.
2. Let $f_1,\dots,f_n\colon\nabla^1\to X$ be continuous maps with $f_i(0,1)=f_{i+1}(1,0)$ for $i=1,\dots,n-1$. We define $f\colon\nabla^1\to X$ via $f(1-t,t)=f_i(i-nt,nt-i+1)$ for $t\in[(i-1)/n,i/n]$. Show that the formal sum $f_1+\dots+f_n$ in $C(X,\mathbb{Z})_1$ is homologous to f (i.e. the difference between the two is a boundary).
3. Assume now that X is path-connected and choose for every $y\in X$ a path $\omega_y\colon[0,1]\to X$ from the basepoint x to y. For every singular 1-simplex $f\colon\nabla^1\to X$ we consider the loop $h(f)$ at x given by
$$\omega_{f(0,1)}*(f\circ i^{-1})*\overline{\omega_{f(1,0)}}$$
where $*$ denotes concatenation and $\overline{\gamma}$ denotes the reversal of a path γ.
Show that $f\mapsto h(f)$ descends to a group homomorphism $h\colon H_1(X,\mathbb{Z})\to\pi_1(X,x)^{\mathrm{ab}}$ into the abelianization of $\pi_1(X,x)$ (i.e. the quotient by the commutator subgroup).
4. Show that the homomorphism $\pi_1(X,x)\to H_1(X,\mathbb{Z})$ from Part 1 descends to $\pi_1(X,x)^{\mathrm{ab}}$ and that the resulting homomorphism is inverse to the homomorphism h. (In particular, the first homology group of a path-connected space is isomorphic to the abelianization of its fundamental group with respect to an arbitrary basepoint.)

please turn over

Exercise 3. Let $k \in \mathbb{Z}$ and define $\tau_k \colon S^1 \to S^1, z \mapsto z^k$. Show that for every abelian group A the map $\tau_{k*} \colon H_1(S^1, A) \to H_1(S^1, A)$ is multiplication by k. (In particular, τ_k has degree k.)

Hint. First prove the statement for $k = 0, 1$ and then apply an 'additivity' argument with respect to the multiplication in π_1.

Exercise 4. Let $k \in \mathbb{Z}$ and let M_k be the space obtained from S^1 by attaching a 2-cell via the map τ_k from the previous exercise. Compute the homology groups $H_*(M_k, A)$ for every abelian group A.

$*$ **Exercise 5** (*10 bonus points*)**.** Let

$$0 \to A_1 \to A_2 \to A_3 \to 0 \qquad \text{and} \qquad 0 \to A_3 \to A_4 \to A_5 \to 0$$

be short exact sequences of abelian groups. By an exercise from the previous sheet, these induce *Bockstein homomorphisms*

$$\beta \colon H_{n+1}(X, A_3) \to H_n(X, A_1) \qquad \text{and} \qquad \beta' \colon H_{n+1}(X, A_5) \to H_n(X, A_3)$$

for every $n \geq 0$. Prove that the composite

$$H_{n+1}(X, A_5) \xrightarrow{\beta'} H_n(X, A_3) \xrightarrow{\beta} H_{n-1}(X, A_1)$$

is the zero map. (In particular, the $\mathbb{Z}/p$-homology groups of any topological space assemble into a chain complex again with differential given by the Bockstein homomorphisms for $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$.)