Generalized reflections and derived equivalences of posets

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Posets, diagrams and sheaves

X – *poset* (finite partially ordered set)

$\mathcal{A}$ – abelian category

$\mathcal{A}^X$ – the category of *diagrams* over X with values in $\mathcal{A}$, or *functors* $F : X \to \mathcal{A}$, consisting of:

- An object F_x of $\mathcal{A}$ for each $x \in X$.
- A morphism $r_{xx'} \in \mathrm{Hom}_{\mathcal{A}}(F_x, F_{x'})$ for each $x < x'$.

such that $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x < x' < x''$ (commutativity).

Natural *topology* on X: $U \subseteq X$ is open if $x \in U$, $x \leq x' \Rightarrow x' \in U$

Diagrams can be identified with *sheaves* over X with values in $\mathcal{A}$.

Posets, diagrams and sheaves – Example

Let $X = \{1, 2, 3, 4\}$ with $1 < 2$, $1 < 3$, $1 < 4$, $2 < 4$, $3 < 4$.

A *diagram* over X is shown on the right:

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 4 & \end{tikzcd} \qquad \begin{tikzcd} & F_1 \arrow[dl, "r_{12}"'] \arrow[dr, "r_{13}"] & \\ F_2 \arrow[dr, "r_{24}"'] & & F_3 \arrow[dl, "r_{34}"] \\ & F_4 & \end{tikzcd}$$

$$r_{24} r_{12} = r_{14} = r_{34} r_{13}$$

The *open sets* are

$$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$

Derived categories

$\mathcal{B}$ – abelian category, $\mathcal{C}^b(\mathcal{B})$ – the category of *bounded complexes*

$$K^\bullet = \ldots \xrightarrow{d} K^{-1} \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} \ldots$$

with $K^i \in \mathcal{B}$, $d^2 = 0$ and $K^i = 0$ for $|i| \gg 0$.

A morphism $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if

$$H^i f : H^i K^\bullet \to H^i L^\bullet$$

are isomorphisms for all $i \in \mathbb{Z}$.

The *bounded derived category* $\mathcal{D}^b(\mathcal{B})$ is obtained from $\mathcal{C}^b(\mathcal{B})$ by *localization* with respect to the quasi-isomorphisms (that is, we formally invert all quasi-isomorphisms).

Universal derived equivalence

Two posets X and Y are *universally derived equivalent* ($X \overset{u}{\sim} Y$) if

$$\mathcal{D}^b(\mathcal{A}^X) \simeq \mathcal{D}^b(\mathcal{A}^Y)$$

for any abelian category $\mathcal{A}$.

Fix a field k, and specialize:
$\operatorname{mod} k$ – the category of finite dimensional vector spaces over k.

$(\operatorname{mod} k)^X$ can be identified with the category of finitely generated *right modules* over the *incidence algebra* of X over k.

X and Y are *derived equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(\operatorname{mod} kX) \simeq \mathcal{D}^b(\operatorname{mod} kY)$$

Comments on derived equivalence

No known *algorithm* that decides if $X \sim Y$ (or $X \overset{u}{\sim} Y$).

However, one can use:

• *Invariants* of the derived category;
If $X \sim Y$ then X and Y must have the same invariants.

Examples of invariants are:

- The *number of points* of X.
- The *Euler bilinear form* on X, closely related to the *Möbius function* of X.

• *Constructions*
Start with some “nice” X and get many Y-s with $X \sim Y$.

Known constructions

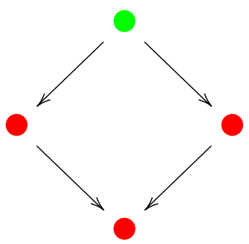
- **BGP Reflection**

When X is a tree and $s \in X$ is a *source* (or a *sink*), invert all arrows from (to) s and get a new tree X' with $X' \sim X$.

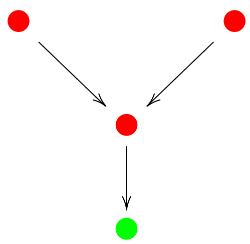
Example.

$\bullet \longleftarrow \bullet \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \longleftarrow \bullet$

- **The square and D_4**



and



are equivalent.

Constructions of derived equivalent posets

Common theme: structured reversal of order relations.

- Generalized reflections (universal derived equivalences)
  - *Flip-Flops*, with application to posets of cluster tilting objects
  - *Generalized BGP reflections*
  - Hybrid construction
- Mirroring with respect to a *bipartite* structure
  - *Mates* of triangular matrix algebras

Flip-Flops

Let $(X, \leq_X)$, $(Y, \leq_Y)$ be posets, $f : X \to Y$ order-preserving.

Define two partial orders $\leq_+^f$, $\leq_-^f$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside X and Y.
- Add the relations

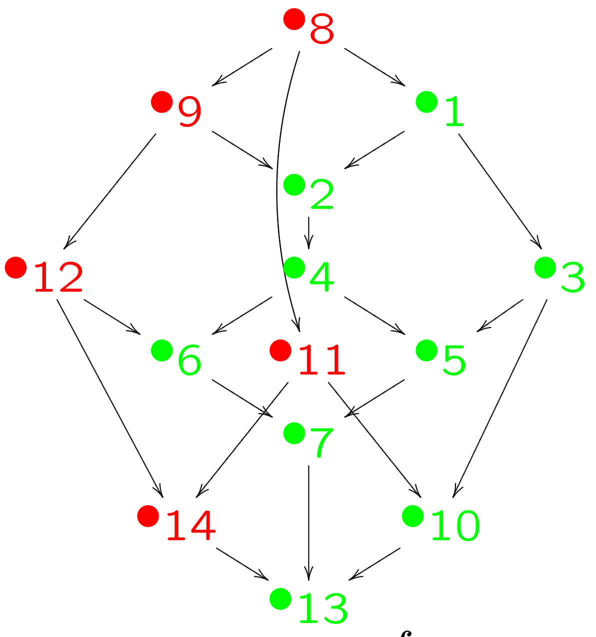
$$x \leq_+^f y \iff f(x) \leq_Y y$$
$$y \leq_-^f x \iff y \leq_Y f(x)$$

for $x \in X$, $y \in Y$.

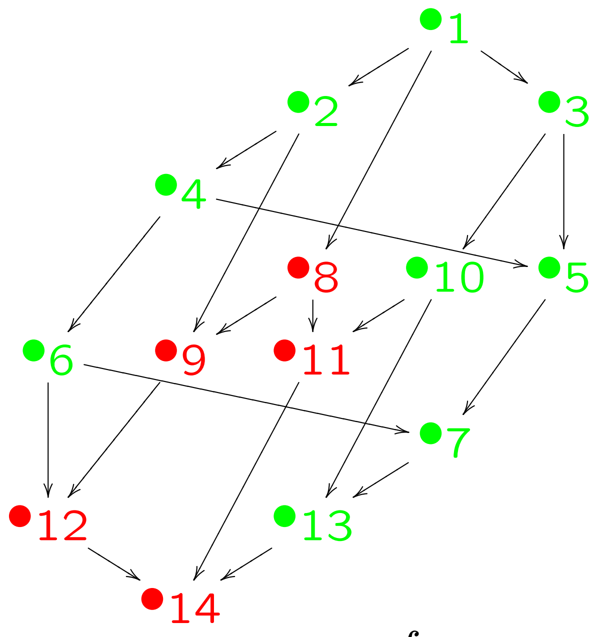
Theorem. $(X \sqcup Y, \leq_+^f) \overset{u}{\sim} (X \sqcup Y, \leq_-^f)$.

Flip-Flop – Example

$8 \mapsto 1 \quad 9 \mapsto 2 \quad 11 \mapsto 10 \quad 12 \mapsto 6 \quad 14 \mapsto 13$



$(X \sqcup Y, \leq_{+}^{f})$



$(X \sqcup Y, \leq_{-}^{f})$

Application – Posets of cluster tilting objects

Q – quiver without oriented cycles, k – field

$\mathcal{T}_Q$ – poset of *cluster tilting objects* in the *cluster category* of kQ [BMRRT, CCS, FZ]

When Q is a Dynkin diagram of type A, D, or E, the poset $\mathcal{T}_Q$ is known as a *Cambrian lattice* [Reading], which is a quotient of the *weak order* on the corresponding Coxeter group.

In type A with the linear orientation, we get the *Tamari lattices*. Their Hasse diagrams are the 1-skeletons of polytopes known as the *Stasheff Associhedra*.

Tamari Lattices for A_1 and A_2

A_1:

$\mathcal{T}_{\bullet}$ $\qquad\qquad$ $(ab)c \longrightarrow a(bc)$

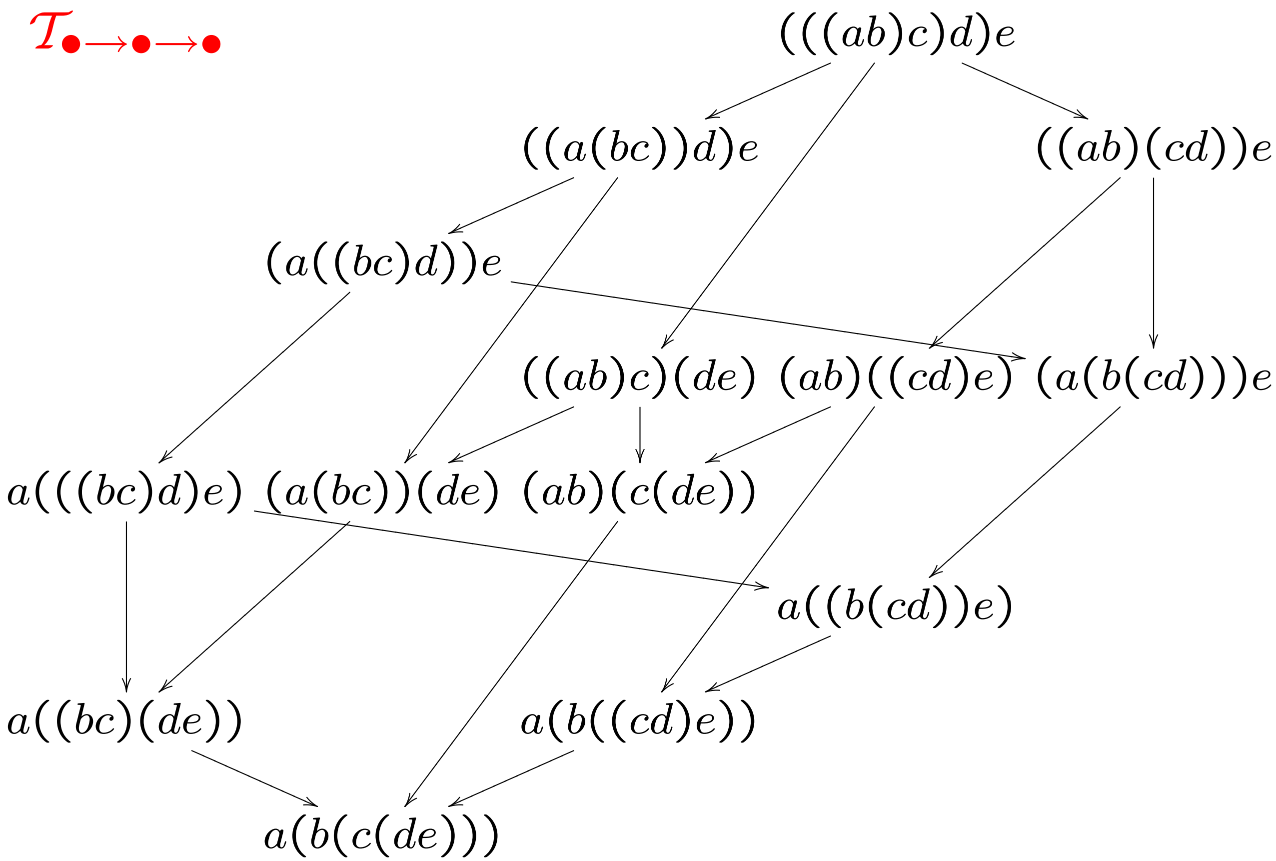
A_2:

$\mathcal{T}_{\bullet\rightarrow\bullet}$

$$\begin{tikzcd}
 & ((ab)c)d \arrow[dl] \arrow[ddr] & \\
(a(bc))d \arrow[dd] & & \\
 & & (ab)(cd) \arrow[ddl] \\
a((bc)d) \arrow[dr] & & \\
 & a(b(cd)) &
\end{tikzcd}$$

Tamari Lattice for A_3

$\mathcal{T}_{\bullet\to\bullet\to\bullet}$



Flip-flops on posets of cluster tilting objects

Q – quiver without oriented cycles.

x – a sink in Q.

Q' – the *BGP reflection* with respect to x.

Theorem. $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.

$$\mathcal{T}_Q \simeq (\mathcal{T}_Q^x \sqcup \mathcal{T}_Q \setminus \mathcal{T}_Q^x, \leq_+^f) \qquad \mathcal{T}_{Q'} \simeq (\mathcal{T}_{Q'}^x \sqcup \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x, \leq_-^{f'})$$

Corollary. If $Q_1 \sim Q_2$ then $\mathcal{T}_{Q_1} \overset{u}{\sim} \mathcal{T}_{Q_2}$.

Generalized BGP reflections

Let $(Y, \leq)$ be poset, $Y_0 \subseteq Y$ a subset with the property

$$[y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \text{ in } Y_0$$

Define two partial orders $\leq_+^{Y_0}$, $\leq_-^{Y_0}$ on $\{*\} \cup Y$ as follows:

- Keep the original partial order inside Y.
- Add the relations

$$* <_+^{Y_0} y \iff \exists y_0 \in Y_0 \text{ with } y_0 \leq y$$
$$y <_-^{Y_0} * \iff \exists y_0 \in Y_0 \text{ with } y \leq y_0$$

for $y \in Y$.

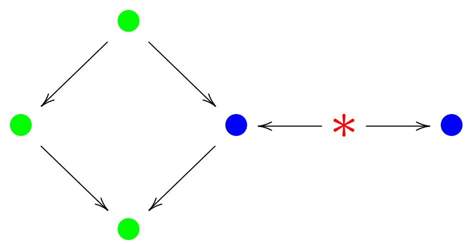
Generalized BGP reflections – continued

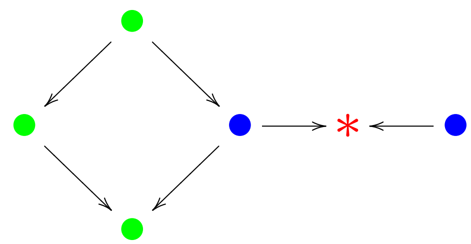
The vertex $*$ is a *source* in the Hasse diagram of $\leq_{+}^{Y_0}$, with arrows ending at the vertices of Y_0.

The Hasse diagram of $\leq_{-}^{Y_0}$ is obtained by reverting the orientations of the arrows from $*$, making it into a *sink*.

Theorem. $(\{*\} \cup Y, \leq_{+}^{Y_0}) \overset{u}{\sim} (\{*\} \cup Y, \leq_{-}^{Y_0})$.

Example.





Hybrid construction – setup

$(X, \leq_X)$, $(Y, \leq_Y)$ – posets, $\{Y_x\}_{x \in X}$ – collection of subsets $Y_x \subseteq Y$, with the properties:

- For all $x \in X$,

$$[y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \text{ in } Y_x$$

- For all $x \leq x'$, there exists an isomorphism $\varphi_{x,x'} : Y_x \xrightarrow{\sim} Y_{x'}$ with

$$y \leq_Y \varphi_{x,x'}(y) \qquad \text{for all } y \in Y_x$$

It follows that $\{Y_x\}_{x \in X}$ is a *local system* of subsets of Y:

$$\varphi_{x,x''} = \varphi_{x',x''}\varphi_{x,x'} \qquad \text{for all } x \leq x' \leq x''.$$

Hybrid construction – result

Define two partial orders on $\leq_+$, $\leq_-$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside X and Y.
- Add the relations

$$x \leq_+ y \iff \exists y_x \in Y_x \text{ with } y_x \leq_Y y$$
$$y \leq_- x \iff \exists y_x \in Y_x \text{ with } y \leq_Y y_x$$

  for $x \in X$, $y \in Y$.

Theorem. $(X \sqcup Y, \leq_+) \overset{u}{\sim} (X \sqcup Y, \leq_-)$.

Remarks.

- When $X = \{*\}$, we recover the generalized BGP reflection.
- When $Y_x = \{*\}$ for all $x \in X$, we recover the flip-flop.

Mirroring with respect to a bipartite structure

Let S be *bipartite*. ($S = S_0 \sqcup S_1$ with $s < s' \Rightarrow s \in S_0$ and $s' \in S_1$)

Let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by S.

Define two partial orders $\leq_+$ and $\leq_-$ on $\bigsqcup_{s \in S} X_s$ as follows:

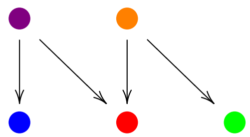
- Keep the original partial order inside each X_s.
- Add the relations

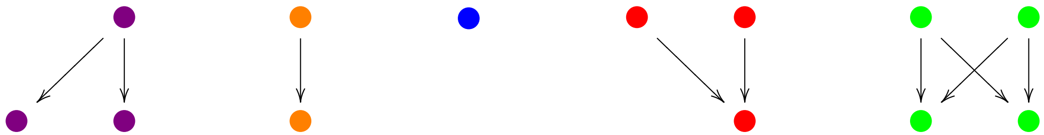
$$x_s <_+ x_t \Longleftrightarrow s < t \Longleftrightarrow x_t <_- x_s$$

  for $x_s \in X_s$, $x_t \in X_t$.

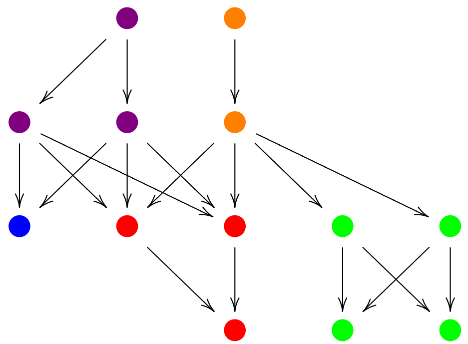
Theorem. $(\bigsqcup_{s \in S} X_s, \leq_+) \overset{u}{\sim} (\bigsqcup_{s \in S} X_s, \leq_-)$.

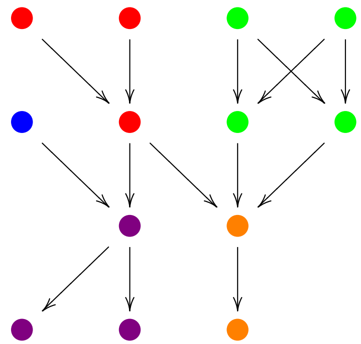
Bipartite structure – example

$S =$ 

$\mathfrak{X} =$ 

$(\bigsqcup_{s\in S} X_s, \leq_+)$





$(\bigsqcup_{s\in S} X_s, \leq_-)$

Mates of triangular matrix algebras

Let k be a field, R and S k-algebras and ${}_RM_S$ bimodule. Consider the *triangular matrix algebras*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \tilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}$$

where $DM = \mathrm{Hom}_k(M, k)$.

Theorem. $\mathcal{D}^b(\mathrm{Mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{Mod}\,\tilde{\Lambda})$, under the assumptions:

- $\dim_k M < \infty$
- $\dim_k S < \infty$, $\mathrm{gl.dim}\, S < \infty$