DERIVED EQUIVALENCE CLASSIFICATION OF THE GENTLE ALGEBRAS ARISING FROM SURFACE TRIANGULATIONS

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Abstract. We provide a derived equivalence classification of the gentle algebras arising from triangulations of marked bordered unpunctured surfaces. Two such algebras are derived equivalent if and only if they have the same derived invariant introduced by Avella-Alaminos and Geiss; this in turn happens if and only if the corresponding quivers with potentials can be connected by a sequence of "good" mutations. Thus there is an effective algorithm that decides whether two such algebras are derived equivalent.

We also show that for any marked bordered unpunctured surface, the gentle algebras arising from its triangulations with the property that their adjacency quiver has maximal number of arrows constitute a complete derived equivalence class of finite-dimensional algebras. Furthermore, any connected gentle algebra whose derived invariant of Avella-Alaminos and Geiss is equal to that of the algebras in this class necessarily belongs to that class.

1. Introduction

Gentle algebras are certain finite-dimensional algebras defined combinatorially in terms of quivers with relations. As shown by Schröer and Zimmermann, the class of gentle algebras is closed under derived equivalence [17]. It is a long standing problem to classify the gentle algebras up to derived equivalence, but only partial results have been achieved so far [2, 3, 7].

In this note we solve the derived equivalence classification problem for the subclass of gentle algebras arising from surface triangulations that has been introduced by Assem, Brüstle, Charbonneau-Jodoin and Plamondon [1]. These algebras are the Jacobian algebras of the quivers with potentials [10] associated to (ideal) triangulations of marked bordered oriented surfaces without punctures [11, 12].

There are two natural approaches to address derived equivalence classification problems of a given collection of algebras arising from some combinatorial data. The *top-down* approach is to divide these algebras into equivalence classes according to some invariants of derived equivalence, so that algebras belonging to different classes are not derived equivalent. The *bottom-up* approach is to systematically construct, based on the combinatorial data, tilting complexes yielding derived equivalences between pairs of these algebras and then to arrange these algebras into groups where any two algebras are related by a sequence of such derived equivalences. To obtain a complete derived

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equivalence classification one has to combine these approaches and hope that the two resulting partitions of the entire collection of algebras coincide.

Indeed, our main result shows that the two approaches can be successfully combined to give a complete derived equivalence classification of the gentle algebras arising from surface triangulations.

For the top-down approach we use the derived invariant for gentle algebras that was developed by Avella-Alaminos and Geiss [3]. It takes the form of a function $\phi_\Lambda : \mathbb{N}^2 \to \mathbb{N}$ that can be effectively computed from the quiver with relations of a gentle algebra Λ. In some cases this invariant is fully capable of distinguishing the derived equivalence classes [2, 3, 7], but in general it is not complete, that is, there exist gentle algebras Λ and Λ' which are not derived equivalent but nevertheless $\phi_\Lambda = \phi_{\Lambda'}$. Therefore one must use other methods in order to establish the derived equivalence of the algebras in question.

For the bottom-up approach we use constructions that are based on good mutations of quivers with potential, which correspond to particular kind of derived equivalences between their corresponding Jacobian algebras. The notion of good mutation has been introduced in our earlier work [14] in relation with assessing the derived equivalence of endomorphism algebras of neighboring cluster-tilting objects in 2-Calabi-Yau categories. It turns out that it plays crucial role in the derived equivalence classification of cluster-tilted algebras of Dynkin types A, D and E as well as affine type $\widetilde{A}$ [4, 5, 6, 8]. For the gentle algebras arising from surface triangulations, good mutations correspond also to certain flips of triangulations.

1.1. **Notions.** In order to state our results more precisely, we introduce some relevant notions. We start by defining good mutations of quivers with potentials. Let (Q, W) be a quiver with potential (QP) without loops and 2-cycles and let $\mathcal{P}(Q, W)$ be its Jacobian algebra. Consider the following complexes of finitely generated right projective $\mathcal{P}(Q, W)$-modules

$$T_k^- = \big(P_k \xrightarrow{f} \bigoplus_{j \to k} P_j\big) \oplus \big(\bigoplus_{i \neq k} P_i\big), \qquad T_k^+ = \big(\bigoplus_{k \to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i \neq k} P_i\big).$$

Here, P_i denotes the projective module corresponding to i spanned by all paths starting at i, the map f (respectively, g) is induced by all the arrows ending (respectively, starting) at k, and the terms P_i for $i \neq k$ lie in degree 0.

Definition 1. The QP mutation of (Q, W) at the vertex k is *good* if at least one of the complexes T_k^-, T_k^+ is a tilting complex over $\mathcal{P}(Q, W)$ whose endomorphism algebra is isomorphic to the Jacobian algebra $\mathcal{P}(\mu_k(Q, W))$ of the mutation of (Q, W) at k.

By definition, a good QP mutation at k implies the derived equivalence of the Jacobian algebras $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_k(Q, W))$.

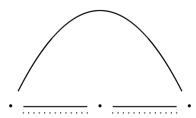
We recall the definition of the gentle algebras arising from surface triangulations, which are our main object of study. A marked bordered oriented surface without punctures is a pair (S, M) where S is a compact connected Riemann surface with non-empty boundary ∂S and $M \subset \partial S$ is a finite set of *marked points* containing at least one point from each connected component of ∂S. The homeomorphism type of (S, M) is determined by the genus g of S, the number $b \geq 1$ of connected components of its boundary and the (positive) numbers $n_1, \dots, n_b$ of marked points on each boundary component.

Let T be a triangulation of (S, M). Fomin, Shapiro and Thurston [11] associate to T its *adjacency quiver* Q_T whose vertices are in bijection with the (internal) arcs of T and for any pair of arcs i, j which are two sides of a triangle τ of T there is an arrow $i \to j$ if j follows i in the clockwise order on the sides of τ induced by the orientation on S. An internal triangle τ whose sides in the clockwise order are the arcs i, j, k thus yields an oriented 3-cycle $i \to j \to k \to i$ in Q_T. The potential W_T on Q_T associated to T by Labardini [12] is the sum of all the oriented 3-cycles corresponding to the internal triangles of T and its Jacobian algebra $\Lambda_T = \mathcal{P}(Q_T, W_T)$ is then the gentle algebra associated to T in [1].

We now define the parameters of a triangulation which encode the topological data relevant to the derived equivalence classification.

Definition 2. Let T be a triangulation of (S, M). A *dome* in T is a triangle containing two boundary segments (necessarily from the same component) as sides.

The following picture explains the name, where a solid line —— denotes an arc and a line ······ denotes a boundary segment.



Thus one can associate to T a sequence $d_1, \dots, d_b$ of b non-negative integers counting the number of domes incident to each boundary component.

Definition 3. Let T be a triangulation of (S, M). The sequence

$$g, b, \{(n_1, d_1), \dots, (n_b, d_b)\}$$

obtained by combining the topological data of (S, M) with the information on the domes of T is called the *parameters* of T.

Note that the sequence $(n_1, d_1), \dots, (n_b, d_b)$ is considered as a multi-set, that is, up to permutation of the indices $1, \dots, b$.

1.2. **Main results.** For a triangulation T, denote by (Q_T, W_T) the corresponding quiver with potential, by $\Lambda_T = \mathcal{P}(Q_T, W_T)$ its Jacobian algebra and by ϕ_{Λ_T} its derived invariant of Avella-Alaminos and Geiss.

Theorem A. *Let T, T' be triangulations of marked bordered unpunctured surfaces. Then the following conditions are equivalent:*

(a) *T and T' have the same parameters;*
(b) *$\phi_{\Lambda_T} = \phi_{\Lambda_{T'}}$;*
(c) *Λ_T and $\Lambda_{T'}$ are derived equivalent;*
(d) *(Q_T, W_T) and $(Q_{T'}, W_{T'})$ are connected by a sequence of good QP mutations.*

Note that the implications (d) $\Rightarrow$ (c) $\Rightarrow$ (b) are always true by definition. In addition, the equivalence of conditions (a) and (b) follows from a recent result of David-Roesler and Schiffler [9], see Section 2.1 below. The crucial part of the proof of the theorem is thus to show the implication (a) $\Rightarrow$ (d). To this end, we characterize the good mutations as those corresponding to flips of triangulations preserving the number of internal triangles, see Section 2.2, and then invoke a suitable connectivity argument, see Section 2.3.

Remark 1. The parameters of a triangulation encode in particular the homeomorphism type of the marked surface. Thus, two algebras arising from triangulations of non-homeomorphic marked unpunctured surfaces are never derived equivalent. When considering the implication (a) $\Rightarrow$ (d), we may therefore assume that T and T' are triangulations of the same marked unpunctured surface.

Remark 2. As special instances of the theorem one recovers the derived equivalence classifications of cluster-tilted algebras of Dynkin type A by Buan and Vatne [8] and those of affine type $\widetilde{A}$ by Bastian [4], corresponding to the cases where the surface S is a disc ($g = 0$ and $b = 1$) or an annulus ($g = 0$ and $b = 2$), respectively.

Remark 3. Since the Jacobian algebras $\Lambda_T = \mathcal{P}(Q_T, W_T)$ are gentle, good QP mutations are Brenner-Butler (co-)tilts. Moreover, it turns out that a QP mutation of (Q_T, W_T) is good if and only if the corresponding Jacobian algebras are derived equivalent, see Section 2.2. Thus, we may formulate the "connectivity" condition (d) in several equivalent ways, as in the next proposition.

Proposition 1. *Let T, T' be triangulations of a marked bordered unpunctured surface (S, M). Then the following conditions are equivalent:*

(d) *(Q_T, W_T) and $(Q_{T'}, W_{T'})$ are connected by a sequence of good QP mutations;*
(d') *(Q_T, W_T) and $(Q_{T'}, W_{T'})$ are connected by a sequence of QP mutations such that all intermediate Jacobian algebras are derived equivalent;*
(d") *Λ_T and $\Lambda_{T'}$ are connected by a sequence of Brenner-Butler tilts and co-tilts such that all intermediate algebras are gentle algebras arising from triangulations of (S, M).*

The implication (c) $\Rightarrow$ (d) in the theorem is remarkable, as it allows to translate the question of derived equivalence to that of the existence of a sequence of basic moves which are much easier to control. This is a highly non-trivial assertion, as demonstrated by the following remarks.

Remark 4. Quivers with potentials have also been assigned to triangulations of marked surfaces *with punctures*, i.e. when the set of marked points is not entirely contained in the boundary of the surface [11, 12]. However, for the corresponding Jacobian algebras, the implication (c) $\Rightarrow$ (d) does not hold in general. Indeed, already for cluster-tilted algebras of Dynkin type D, which arise from triangulations of a disk with one puncture, there are derived equivalent such algebras whose quivers are not connected by a sequence of good mutations [6].

Remark 5. The equivalent implication (c) $\Rightarrow$ (d") shows in particular that any two derived equivalent gentle algebras arising from surface triangulations are related by a sequence of Brenner-Butler tilts or co-tilts. This does not hold for gentle algebras in general. For example, the gentle algebras corresponding to the two quivers below with the relations $\alpha\beta$ and $\beta\gamma$ are derived equivalent but nevertheless cannot be related by a sequence of Brenner-Butler tilts or co-tilts, see also the discussion in [2, §6].

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\alpha"] & \bullet \\ \bullet \arrow[r, shift left, "\beta"] \arrow[r, shift right] & \bullet \arrow[u, "\gamma"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[d, shift left, "\alpha"] \arrow[d, shift right] & \bullet \\ \bullet \arrow[r, "\beta"] & \bullet \arrow[u, shift left] \arrow[u, shift right, "\gamma"'] \end{tikzcd}$$

Remark 6. The cluster-tilted algebras of Dynkin type E, as well as the Jacobian algebras associated to the members of the mutation classes of the exceptional quivers $E_6^{(1,1)}$ and X_6, are neither gentle nor arising from triangulations of (punctured) surfaces. Nevertheless, a statement analogous to the implication (c) $\Rightarrow$ (d) holds for these algebras [5, 13].

For a marked bordered unpunctured surface (S, M), denote by $\mathcal{L}_{(S,M)}$ the family of (isomorphism classes of) gentle algebras arising from the triangulations of (S, M). The family $\mathcal{L}_{(S,M)}$ is finite, but in general it is neither closed under derived equivalence nor all its members are derived equivalent to each other.

Our next result shows that there is always a distinguished subfamily $\mathcal{L}^{\circ}_{(S,M)} \subseteq \mathcal{L}_{(S,M)}$ which constitutes a complete derived equivalence class of finite-dimensional algebras, and moreover it is characterized by its derived invariant of Avella-Alaminos and Geiss.

Definition 4. For a marked bordered unpunctured surface (S, M), let $\mathcal{L}^{\circ}_{(S,M)} \subseteq \mathcal{L}_{(S,M)}$ denote the family of gentle algebras arising from the triangulations of (S, M) having maximal number of domes.

Remark 7. A triangulation T of (S, M) has maximal number of domes if and only if its adjacency quiver Q_T has maximal number of arrows within its mutation class, see Section 2.1.

Theorem B. *Let (S, M) be a marked bordered unpunctured surface. Then the following assertions hold:*

(i) *If $\Lambda, \Lambda' \in \mathcal{L}^{\circ}_{(S,M)}$, then Λ and Λ' are related by a sequence of Brenner-Butler tilts or co-tilts, and in particular they are derived equivalent.*

(ii) *If $\Lambda \in \mathcal{L}^{\circ}_{(S,M)}$ and Λ' is a connected gentle algebra such that $\phi_\Lambda = \phi_{\Lambda'}$, then also $\Lambda' \in \mathcal{L}^{\circ}_{(S,M)}$.*

(iii) *If $\Lambda \in \mathcal{L}^{\circ}_{(S,M)}$ and Λ' is an algebra derived equivalent to Λ', then $\Lambda' \in \mathcal{L}^{\circ}_{(S,M)}$.*

Part (i) of the theorem is a consequence of Theorem A. Part (iii) follows from (ii), since the class of gentle algebras is closed under derived equivalence [17]. The proof of part (ii) is given in Section 2.4.

With the exception of the cases of a disc with 4 or 5 marked points, the subfamily $\mathcal{L}^{\circ}_{(S,M)}$ coincides with $\mathcal{L}_{(S,M)}$ if and only if the set M of marked points contains exactly one point from each boundary component of S, or equivalently, none of the triangulations of (S, M) has any domes, see also [15]. For any pair $(g, b) \neq (0, 1)$ there is, up to homeomorphism, a unique such marked bordered unpunctured surface of genus g with b boundary components. We denote by $\mathcal{Q}_{g,b}$ the set of (right equivalence classes of) quivers with potentials associated with the triangulations of that surface. Applying the above results, we can regard the entire mutation class $\mathcal{Q}_{g,b}$ as a derived equivalence class:

Corollary ([13]). *All the quivers in the mutation class $\mathcal{Q}_{g,b}$ have the same number of arrows and any of their QP mutations is good.*

The corresponding class of (finite-dimensional) Jacobian algebras is closed under derived equivalence, and any two of its members are connected by a sequence of Brenner-Butler tilts or co-tilts, so in particular they are derived equivalent.

2. Outline of the proofs

2.1. **Top-down.** In this section we separate the gentle algebras arising from surface triangulations according to their derived invariant of Avella-Alaminos and Geiss [3]. Recall that for a gentle algebra Λ it is given as a function $\phi_\Lambda : \mathbb{N}^2 \to \mathbb{N}$.

This invariant has been computed by David-Roesler and Schiffler in [9] for a class of algebras called "surface algebras" which contains the gentle algebras arising from surface triangulations.

By abuse of notation, we denote by (n, m) the characteristic function of the singleton $\{(n, m)\} \subset \mathbb{N}^2$, taking the value 1 on (n, m) and zero otherwise. A function $\phi : \mathbb{N}^2 \to \mathbb{N}$ having finite support can thus be written as a formal sum $\phi = \sum_{(n,m)\in\mathbb{N}^2} \phi(n, m)(n, m)$.

Proposition 2. *Let T be a triangulation with parameters $g, b, (n_1, d_1), \dots, (n_b, d_b)$ and let Λ_T be the corresponding gentle algebra. Then*

$$\phi_{\Lambda_T} = (n_1 - d_1, n_1 - 2d_1) + (n_2 - d_2, n_2 - 2d_2) + \cdots + (n_b - d_b, n_b - 2d_b) + t(0, 3)$$

where $t = 4(g - 1) + 2b + \sum_{i=1}^{b} d_i$.

Proof. This is a reformulation of [9, Theorem 4.6], once we know that t is the number of internal triangles of T. This is shown in the next lemma below. □

Lemma. *Let T be a triangulation with parameters $g, b, (n_1, d_1), \dots, (n_b, d_b)$. Then:*

(a) *The number of vertices in the quiver Q_T is $6(g - 1) + 3b + \sum_{i=1}^{b} n_i$.*
(b) *The number of arrows in the quiver Q_T is $12(g - 1) + 6b + \sum_{i=1}^{b}(n_i + d_i)$.*
(c) *The number of internal triangles of T is $4(g - 1) + 2b + \sum_{i=1}^{b} d_i$.*

Proof. Let n denote the number of arcs of T, which is also the number of vertices of Q_T. According to [11], it is given by $n = 6(g - 1) + 3b + \sum_{i=1}^{b} n_i$. For $0 \leq j \leq 2$, denote by t_j the number of triangles of T with j sides that are boundary segments. Then t_0 is the number of internal triangles, t_2 is the number of domes and $3t_0 + t_1$ is the number of arrows in the quiver Q_T.

By definition, $t_1 = \sum_{i=1}^{b}(n_i - 2d_i)$ and $t_2 = \sum_{i=1}^{b} d_i$. To determine t_0, let us count in two ways the pairs (γ, τ) there τ is a triangle of T and γ is an arc which is one of its sides. On the one hand, there are $3t_0 + 2t_1 + t_2$ such pairs. On the other hand, since each arc is a side of exactly two triangles, there are $2n$ such pairs. The formula for t_0 now follows from the equality $2n = 3t_0 + 2t_1 + t_2$ by using the above expressions for n, t_1 and t_2. □

Remark 8. Proposition 2 shows that ϕ_{Λ_T} depends only on the parameters of the triangulation T, and that conversely, the parameters of T can be recovered from ϕ_{Λ_T}. This proves the equivalence of conditions (a) and (b) in Theorem A.

Remark 9. We also see that if Λ_T and $\Lambda_{T'}$ are derived equivalent, then T and T' can be viewed as triangulations of the *same* marked bordered unpunctured surface.

2.2. **Bottom-up.** We build derived equivalences by using the combinatorial data underlying the classification problem. Namely, certain flips of triangulations will give rise to derived equivalences of the corresponding gentle algebras.

In view of Remark 9, we may restrict our attention to triangulations of some fixed marked unpunctured surface. Given a triangulation T and an arc k of T, there is another triangulation $\mu_k(T)$ obtained from T by a flip of the arc k. It is known [11] that flips are compatible with quiver mutations, that is, $Q_{\mu_k(T)} \simeq \mu_k(Q_T)$ and moreover the QPs $(Q_{\mu_k(T)}, W_{\mu_k(T)})$ and $\mu_k(Q_T, W_T)$ are right equivalent [12].

Results from our previous works [13, 15] can be used to determine, for any two triangulations connected by a single flip, whether the associated gentle algebras are derived equivalent or not. We summarize them in the following proposition.

Proposition 3. *Let T be a triangulation and k an arc of T. Then all the conditions below are equivalent:*

(i) *The triangulations T and $\mu_k(T)$ have the same parameters;*
(ii) *The triangulations T and $\mu_k(T)$ have the same number of internal triangles;*
(iii) *The quivers Q_T and $\mu_k(Q_T)$ have the same number of arrows;*
(iv) *The in-degree and the out-degree of the vertex k in the quiver Q_T are not both 1;*
(v) *The algebras Λ_T and $\Lambda_{\mu_k(T)}$ are derived equivalent;*
(vi) *The algebras Λ_T and $\Lambda_{\mu_k(T)}$ have the same Cartan determinant;*
(vii) *The mutation at k of (Q_T, W_T) is good.*

2.3. **Connectivity argument.** In order to successfully combine the bottom-up with the top-down approaches and obtain a complete derived equivalence classification, we must show that any two triangulations with the same parameters can be connected by a sequence of flips which do not change these parameters. This is the content of the next proposition.

Proposition 4. *Let T, T' be two triangulations of a marked bordered unpunctured surface (S, M) with the same numbers $d_1, d_2, \dots, d_b$ of domes incident to each boundary component of ∂S. Then one can move from T to T' by a sequence of flips without creating or destroying any dome.*

It is a well-known and classical fact that T and T' can be connected by a sequence of flips, but a-priori each of these flips might change the numbers of domes. So the main point is that one can find a sequence of flips while keeping the numbers of domes fixed along the way. This can be done by carefully adapting the combinatorial proof of the classical fact of connectivity by flips given by Mosher in [16, pp. 36–41].

Remark 10. This proposition, together with Proposition 3, implies the equivalence of conditions (a) and (d) in Theorem A, and thus completes its proof.

Remark 11. Similar connectivity results have been obtained for quivers in the mutation class of a Dynkin quiver of type A [8, Lemma 2.3] and for mutation classes of the affine quivers $\tilde{A}$ [4, Lemma 3.9]. These results can be seen as special instances of Proposition 4 formulated in quiver language, corresponding to the cases where the surface S is a disc (type A) or an annulus (type $\tilde{A}$).

2.4. **Gentle algebras with prescribed invariant.** It remains to show part (ii) of Theorem B. Indeed, it is a consequence of the next proposition.

For a function $\phi : \mathbb{N}^2 \to \mathbb{N}$, denote by $\mathcal{L}_\phi$ the family of connected gentle algebras Λ with $\phi_\Lambda = \phi$.

Proposition 5. *Let $b \geq 1$ and $t \geq 0$. Let $m_1, \dots, m_b$ be positive integers and $\varepsilon_1, \dots, \varepsilon_b \in \{0, 1\}$. Consider the function $\phi : \mathbb{N}^2 \to \mathbb{N}$ given by*

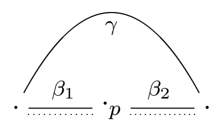
$$\phi = (m_1, \varepsilon_1) + \cdots + (m_b, \varepsilon_b) + t(0, 3). \tag{$\star$}$$

(a) *The family $\mathcal{L}_\phi$ is not empty if and only if there exists an integer $g \geq 0$ such that $t = 4(g-1) + 2b + \sum_{i=1}^{b}(m_i - \varepsilon_i)$.*
(b) *If $\mathcal{L}_\phi$ is not empty, then $\mathcal{L}_\phi = \mathcal{L}^\circ_{(S,M)}$ where (S, M) is the marked bordered unpunctured surface of genus g with b boundary components and $2m_i - \varepsilon_i$ marked points on the i-th component of ∂S.*
(c) *Conversely, let (S, M) be a marked bordered unpunctured surface. Then $\mathcal{L}^\circ_{(S,M)} = \mathcal{L}_\phi$ for the function ϕ given as in $(\star)$ and determined as follows:*
b is the number of components of ∂S, the numbers $m_i \geq 1$ and $\varepsilon_i \in \{0, 1\}$ are uniquely determined by the condition that the number of marked points on the i-th component is $2m_i - \varepsilon_i$, and $t = 4(g-1) + 2b + \sum_{i=1}^{b}(m_i - \varepsilon_i)$ where g is the genus of S.

Proof. Let $\Lambda \in \mathcal{L}_\phi$. Since $\phi_\Lambda(m, \varepsilon)$ vanishes whenever $\varepsilon \geq 2$, any zero-relation $\alpha_1\alpha_2$ in the quiver of Λ must be part of an oriented n-cycle $\alpha_1\alpha_2 \dots \alpha_n$ with full zero-relations $\alpha_i\alpha_{i+1}$ for $1 \leq i < n$ and $\alpha_n\alpha_1$, for some $n \geq 1$. Moreover, we have $n = 3$ since $\phi_\Lambda(0, n)$ vanishes for other values of n.

By [1, Prop. 2.8], the algebra Λ arises from a triangulation of a marked unpunctured surface. The rest now follows from Proposition 2, observing that $n_i = 2m_i - \varepsilon_i$ and $d_i = m_i - \varepsilon_i$ it the maximal possible number of domes incident to the i-th boundary component. $\square$

Remark 12. Let (S, M) be a marked bordered unpunctured surface. Assume that there is a boundary component $C \subseteq \partial S$ with boundary segments β_1, β_2 incident to a marked point $p \in M$. Let γ be an arc as in the following picture



giving rise to a triangle Δ whose sides (in clockwise order) are β_1, γ and β_2.

Consider the bordered surface $S' = S \setminus (\Delta \cup \beta_1 \cup \beta_2)$ and let $M' = M \setminus \{p\}$. The corresponding boundary component C' of S' is obtained from C by replacing $\beta_1 \cup \beta_2$ with the arc γ. If (S', M') has non-empty triangulations, there is a one-to-one correspondence between these triangulations and triangulations of (S, M) with dome Δ.

We deduce that when $(g, b) \neq (0, 1)$, any sequence of parameters

$$g, b, \{(n_1, d_1), \dots, (n_b, d_b)\}$$

with $0 \leq d_i \leq \lfloor \frac{n_i}{2} \rfloor$ is realizable by a triangulation of some marked unpunctured surface (S, M) of genus g, and that the sequence $0, 1, \{(n_1, d_1)\}$ with $n_1 \geq 4$ and $2 \leq d_1 \leq \lfloor \frac{n_i}{2} \rfloor$ is realizable by a triangulation of a disc with n_1 marked points.

Let $b \geq 1$ and $t \geq 0$. Consider the function ϕ as in $(\star)$, where $m_1, \dots m_b, \varepsilon_1, \dots, \varepsilon_b$ are integers such that $m_i \geq 1$ and $0 \leq \varepsilon_i \leq m_i$. It follows from the above remark and Proposition 2 that $\mathcal{L}_\phi$ contains a gentle algebra arising from surface triangulation (and in particular, it is non-empty) if $t = 4(g-1) + 2b + \sum_{i=1}^{b}(m_i - \varepsilon_i)$ for some integer $g \geq 0$.

Remark 13. Let T' and T'' be triangulations of marked unpunctured surfaces (S', M'), (S'', M'') with parameters

$$g', b', \left\{(n'_1, d'_1), \dots, (n'_{b'}, d'_{b'})\right\}, \qquad g'', b'', \left\{(n''_1, d''_1), \dots, (n''_{b''}, d''_{b''})\right\},$$

and let $\Lambda_{T'}$, $\Lambda_{T''}$ be the corresponding gentle algebras. The gentle algebra $\Lambda_{T'} \times \Lambda_{T''}$ is not connected, and can be thought as arising from a triangulation of the *non-connected* surface $(S' \cup S'', M' \cup M'')$.

Assume that g' and g'' are not both zero and let Λ_T be the gentle algebra arising from a triangulation T with parameters

$$g' + g'' - 1, b' + b'', \left\{(n'_1, d'_1), \dots, (n'_{b'}, d'_{b'}), (n''_1, d''_1), \dots, (n''_{b''}, d''_{b''})\right\}.$$

We deduce from Proposition 2 that $\phi_{\Lambda_T} = \phi_{\Lambda_{T'}} + \phi_{\Lambda_{T''}}$, hence the algebras Λ_T and $\Lambda_{T'} \times \Lambda_{T''}$ have the same derived invariant of Avella-Alaminos and Geiss. However, they are not derived equivalent, as one is connected and the other is not.

Moreover, observe that $\Lambda_T \in \mathcal{L}^\circ_{(S,M)}$ if and only if $\Lambda_{T'} \in \mathcal{L}^\circ_{(S',M')}$ and $\Lambda_{T''} \in \mathcal{L}^\circ_{(S'',M'')}$, thus the conclusion of Proposition 5 is false if we drop the condition that the algebras in $\mathcal{L}_\phi$ are connected. This justifies our assumption of connectivity throughout the paper.

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