Homological Properties of Finite Partially Ordered Sets

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by

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Abstract

In this work we study the homological properties of finite partially ordered sets as reflected in their derived categories of diagrams. This subject stands at the junction of the areas of combinatorics, topology, representation theory and homological algebra.

Background

Since their introduction by Verdier and Grothendieck in order to formulate duality in algebraic geometry, triangulated categories in general, and derived categories in particular, have found applications in diverse areas of mathematics and mathematical physics.

Triangulated categories have been successfully used to relate objects of different nature, thus forming bridges between various areas of algebra and geometry. An example is Beilinson's result [6] on the equivalence of the derived category of coherent sheaves over a projective space (which is of commutative nature) and the derived category of finite dimensional modules over a certain finite dimensional, non-commutative, algebra. This result can be seen as a starting point of non-commutative geometry.

Another example, motivated by its applications to physics, is Kontsevich's formulation [56] of the Homological mirror symmetry conjecture as an equivalence between a certain derived category of coherent sheaves over an algebraic variety and a triangulated category of other nature (the Fukaya category).

The question of equivalence of two derived categories arising from objects of the same nature has also attracted a growing interest. For example, the question when two algebraic varieties have equivalent derived categories of sheaves has been recently studied by Bondal and Orlov [11]. Another, earlier, example is Rickard's result [73], characterizing when two rings have equivalent derived categories of modules, in terms of the existence of a so-called tilting complex.

Rickard's result leaves something to be desired, though, as for some pairs of algebras, it is currently notoriously difficult, and sometimes even impossible, to decide whether there exists a tilting complex. Such a difficulty is apparent in the still unsolved Broué's conjecture asserting derived equivalences between certain blocks of group algebras [78].

Figure 1: Hasse diagrams of two non-isomorphic, yet universally derived equivalent posets.

The problem

We investigate similar questions for derived categories arising from finite partially ordered sets. A finite partially ordered set (*poset*) is naturally endowed with a structure of a topological space. The finite spaces obtained in this way are capable of modeling several quantitative topological properties, such as the homology and homotopy groups, of various well-behaved manifolds in Euclidean spaces [66].

A poset can also be considered as a small category, allowing one to form the category of functors $X \to \mathcal{A}$, denoted $\mathcal{A}^X$, from a poset X to an abelian category $\mathcal{A}$. The category $\mathcal{A}^X$, whose objects are also known as *diagrams*, is abelian and can be regarded as a category of sheaves and sometimes of modules, as outlined below.

First, by viewing a poset as a topological space, diagrams can be considered as sheaves over that space. Two recent applications of sheaves over posets include the computation, by Deligne, Goresky and MacPherson [25], of the cohomology of arrangements of subspaces in a real affine space, and the definition, by Karu [49], of intersection cohomology for general polytopes, in order to study their h-vectors.

Second, when $\mathcal{A}$ is the category of finite dimensional vector spaces over a field k, diagrams can also be identified with modules over the *incidence algebra* of X over k.

Definition. We say that two posets X and Y are *universally derived equivalent* if, for any abelian category $\mathcal{A}$, the (bounded) derived categories of diagrams $\mathcal{D}^b(\mathcal{A}^X)$ and $\mathcal{D}^b(\mathcal{A}^Y)$ are equivalent (as triangulated categories).

Two posets X and Y are *derived equivalent* over a field k if their incidence algebras over k are derived equivalent.

Our main research aim is the study of these derived categories of diagrams, and in particular the questions when two posets are (universally) derived equivalent and how the derived equivalence is related to the combinatorial properties of the posets in question.

As diagrams can be identified with sheaves and sometimes even with modules, both geometrical tools from algebraic geometry and algebraic tools from representation theory can be applied when studying the question of derived equivalence.

The derived equivalence relation, either universal or not, is strictly coarser than isomorphism. A simple example demonstrating this is given in Figure 1. However, there is no known algorithm which decides, given two posets, whether they are derived equivalent or not.

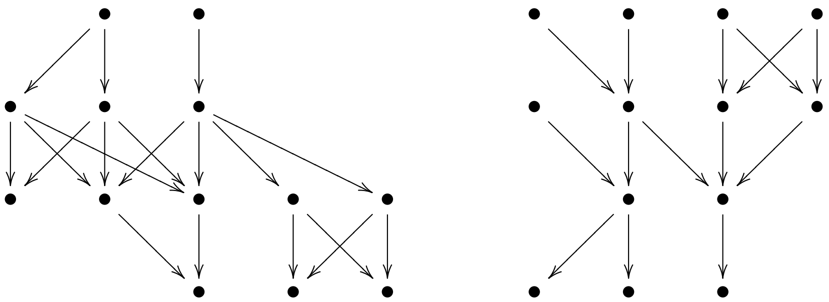


Figure 2: Hasse diagrams of two posets derived equivalent by the bipartite construction.

There are two directions to pursue here. The first is to find invariants of derived equivalence, that is, combinatorial properties of a poset which are shared among all other posets derived equivalent to it. Examples of such invariants are the number of points, the $\mathbb{Z}$-congruency class of the incidence matrix, and the Betti numbers.

The second direction is to systematically construct, given a poset having certain combinatorial structure, new posets that are guaranteed to be derived equivalent to it. An example of an analogous construction in the representation theory of quivers and finite dimensional algebras is the Bernstein-Gelfand-Ponomarev reflection [9].

The results

Constructions of derived equivalences of posets and other objects

We have found several kinds of combinatorial constructions producing derived equivalences and they are described in Part I of this work. The common theme of these constructions is the structured reversal of order relations.

The bipartite construction

This construction is described in Chapter 1, where we show that a poset having a bipartite structure can be mirrored along that structure to obtain a derived equivalent poset. An example is given in Figure 2.

To present this construction more precisely, we introduce the notion of a lexicographic sum of a collection of posets along a poset, which generalizes the known notion of an ordinal sum. If $\{X_s\}_{s \in S}$ is a collection of posets indexed by a poset S, the *lexicographic sum of the X_s along S* is the poset whose underlying set is the disjoint union of the X_s, and two elements are compared first based on the indices of the sets they belong to (using the partial order on S), and when a tie occurs – i.e. they belong to the same set X_s, according to the partial order inside X_s.

A poset S is called *bipartite* if its Hasse diagram is a bipartite directed graph.

Theorem. *Let S be a bipartite poset. Then the lexicographic sum of a collection of posets along S is derived equivalent to the lexicographic sum of the same collection along the opposite poset*

S^{op}.

This construction is inspired by the geometrical viewpoint of diagrams as sheaves, building on the notion of a strongly exceptional collection introduced in the study of derived categories of sheaves over algebraic varieties.

As a corollary, we see that the derived equivalence class of an ordinal sum of any two posets does not depend on the order of summands. However, we gave an example showing that this is not true for three summands.

By using the other, algebraic, viewpoint of diagrams as modules, we have extended the above result to general triangular matrix rings, as described in Chapter 2. Instead of formulating the result in its most generality (see Theorem 2.4.5), we shall demonstrate it in the case of algebras over a field, as expressed in the following theorem.

Theorem. *Let k be a field and let R, S be k-algebras. Assume that (at least) one of R, S is finite dimensional and of finite global dimension. Then for any finite dimensional S-R-bimodule ${}_RM_S$, the triangular matrix algebras*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}$$

are derived equivalent, where the dual $DM = \mathrm{Hom}_k(M, k)$ is viewed as an R-S-bimodule.

Generalized reflections

This construction, described in Chapter 3, produces *universal* derived equivalences that can be considered as generalized reflections, described in very simple, explicit combinatorial terms. This vastly generalizes the well-known Bernstein-Gelfand-Ponomarev reflections for quivers, and has found applications in the representation theory of algebras for the study of the partial orders of tilting modules and cluster-tilting objects over path algebras of quivers, as described later in Part II of this work.

We shall demonstrate this construction in the following situation, called *flip-flop*. For a more general setup, see Section 3.1.1. If X and Y are two posets and $f : X \to Y$ is an order-preserving map, one can consider two partial orders $\leq_+^f$ and $\leq_-^f$ on the disjoint union $X \sqcup Y$, defined as follows. Inside each of the sets X and Y, both orders agree with the original orders, but between them, $x \leq_+^f y$ if $f(x) \leq y$ and $y \leq_-^f x$ if $y \leq f(x)$, where $x \in X$ and $y \in Y$.

Theorem. *The posets $(X \sqcup Y, \leq_+^f)$ and $(X \sqcup Y, \leq_-^f)$ are universally derived equivalent.*

Note that this theorem is true also when the posets X and Y are *infinite*. The main tool used in the construction is the notion of a *formula*, which consists of combinatorial data that produces, simultaneously for any abelian category $\mathcal{A}$, a functor between the categories of complexes of diagrams over two posets Z and Z' with values in $\mathcal{A}$, inducing a triangulated functor between the corresponding derived categories. When Z and Z' have certain combinatorial structure, as for example in the above constructions, we build such functors that are equivalences.

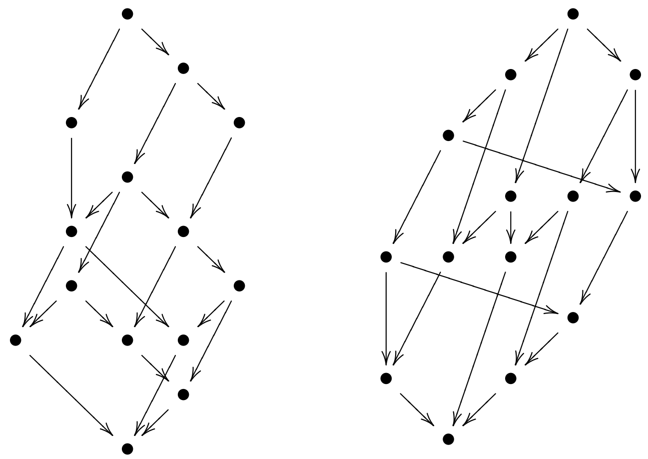


Figure 3: Hasse diagrams of two universally derived equivalent posets of tilting modules of path algebras of quivers whose underlying graph is the Dynkin diagram A_4.

Combinatorial applications for tilting objects

In Part II of this work we demonstrate that the flip-flop constructions appear naturally in combinatorial contexts concerning partial orders of tilting modules and cluster tilting objects arising from path algebras of quivers.

First we show that the posets of tilting modules, in the sense of Riedtmann-Schofield [75], of any two derived equivalent path algebras of quivers without oriented cycles, are always universally derived equivalent. This is described in Chapter 4. An example is given in Figure 3.

Then we show a similar result for the posets of cluster tilting objects, described in Chapter 5. Cluster categories corresponding to quivers without oriented cycles were introduced in [16] as a representation theoretic approach to the cluster algebras introduced and studied by Fomin and Zelevinsky [28]. In these categories one can define, similarly to tilting modules, cluster tilting objects, which correspond to the clusters of the cluster algebra. The set $\mathcal{T}_{\mathcal{C}_Q}$ of cluster tilting objects in the cluster category of a quiver Q without oriented cycles admits a partial order as described in [48].

Theorem. *Let Q and Q' be two quivers without oriented cycles whose path algebras are derived equivalent. Then the posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are universally derived equivalent.*

When the quiver Q is of finite type, the poset $\mathcal{T}_{\mathcal{C}_Q}$ is a Cambrian lattice as introduced by Reading [72], defined as a certain quotient of the weak order on the corresponding Coxeter group. Its Hasse diagram is the 1-skeleton of a polytope known as the corresponding generalized Associhedron.

Piecewise hereditary categories and posets

In Part III of the work we study posets whose categories of diagrams (over a field) are *piecewise hereditary*, that is, derived equivalent to an abelian category of global dimension one. One should think of such categories as the simplest ones after the semi-simple ones. We present three results on such categories and posets.

The first result concerns the global dimension of a piecewise hereditary abelian category. In general, this quantity can be arbitrarily large. However, we show that for a piecewise hereditary category of diagrams over a poset, the global dimension cannot exceed 3. Moreover, we extend this result to categories of modules over sincere algebras and more generally to a wide class of finite length piecewise hereditary categories satisfying certain connectivity conditions expressed via their graphs of indecomposable objects. Note that the bound of 3 is sharp. This result is described in Chapter 6.

Second, we explore the relationships between spectral properties of the Coxeter transformation and positivity properties the Euler form, for finite dimensional algebras which are piecewise hereditary. We show that for such algebras, if the Coxeter transformation is of finite order, then the Euler bilinear form is non-negative. We also demonstrate, through incidence algebras of posets, that the assumption of being piecewise hereditary cannot be omitted. This is done in Chapter 7.

Finally, we give a complete description of all the canonical algebras (which form a special class of piecewise hereditary algebras, introduced by Ringel [76]) that are derived equivalent to incidence algebras of posets. This is expressed in the following theorem, whose proof can be found in Chapter 8.

Theorem. *A canonical algebra of type* $(\mathbf{p}, \boldsymbol{\lambda})$ *over an algebraically closed field is derived equivalent to an incidence algebra of a poset if and only if the number of weights of* $\mathbf{p}$ *is either 2 or 3.*

Some parts of this work have appeared in journal papers; Chapter 1 is based on the paper [59] and Chapters 6, 7 and 8 are based on the papers [57],[60] and [58], respectively.

Contents

Part I

Constructions of Derived Equivalences of Posets and Other Objects

Chapter 1

On Derived Equivalences of Categories of Sheaves over Finite Posets

1.1 Introduction

Since their introduction by Verdier [85] and Grothendieck in order to formulate duality in algebraic geometry, triangulated categories in general, and derived categories in particular, have found applications in diverse areas of mathematics and mathematical physics.

Triangulated categories have been successfully used to relate objects of different nature, thus forming bridges between various areas of algebra and geometry. An example is Beilinson's result [6] on the equivalence of the derived category of coherent sheaves over a projective space (which is of commutative nature) and the derived category of finite dimensional modules over a certain finite dimensional, non-commutative, algebra. This result can be seen as a starting point of non-commutative geometry.

Another example, motivated by its applications to physics, is Kontsevich's formulation [56] of the Homological mirror symmetry conjecture as an equivalence between a certain derived category of coherent sheaves over an algebraic variety and a triangulated category of other nature (the Fukaya category).

The question of equivalence of two derived categories arising from objects of the same nature has also attracted a growing interest. For example, the question when two algebraic varieties have equivalent derived categories of sheaves has been recently studied by Bondal and Orlov [11]. Another, earlier, example is Rickard's result [73], characterizing when two rings have equivalent derived categories of modules, in terms of the existence of a so-called tilting complex.

Rickard's result leaves something to be desired, though, as for some pairs of algebras, it is currently notoriously difficult, and sometimes even impossible, to decide whether there exists a tilting complex. Such a difficulty is apparent in the still unsolved Broué's conjecture asserting derived equivalences between certain blocks of group algebras [78].

We investigate a similar question replacing the algebraic varieties with finite partially ordered sets (*posets*). Since a poset X carries a natural structure of a topological space, one can consider the category of sheaves over X with values in an abelian category $\mathcal{A}$.

In this chapter we focus on the case where $\mathcal{A}$ is the category of finite dimensional vector spaces over a field k, which allows us to identify the category of sheaves with a category of modules over the *incidence algebra* of X over k, so that tools from the theory of derived equivalence of algebras can be used. In Chapter 3 we shall study in greater detail the case where $\mathcal{A}$ is an arbitrary abelian category.

We establish the notations and terminology to be used throughout this work in Section 1.2, where we present in a specific way, appropriate for dealing with posets, the relevant basic notions from sheaf theory.

Fix a field k, and denote by $\mathcal{D}^b(X)$ the bounded derived category of sheaves of finite dimensional k-vector spaces over X. Two posets X and Y are said to be *derived equivalent* if $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ are equivalent as triangulated categories.

The derived equivalence relation is strictly coarser than isomorphism (see Example 1.4.16). However, there is no known algorithm which decides, given two posets, whether their derived categories of sheaves of finite dimensional k-vector spaces are equivalent.

There are two directions to pursue here. The first is to find invariants of derived equivalence, that is, combinatorial properties of a poset which are shared among all other posets derived equivalent to it. Examples of such invariants are the number of points, the $\mathbb{Z}$-congruency class of the incidence matrix, and the Betti numbers. These invariants are discussed in Section 1.3, where we also note that taking opposites and products preserves derived equivalence.

The second direction is to systematically construct, given a poset having certain combinatorial structure, new posets that are guaranteed to be derived equivalent to it. An example of an analogous construction in the representation theory of quivers and finite dimensional algebras is the Bernstein-Gelfand-Ponomarev reflection [9].

In Section 1.4 we present one such construction, the *bipartite construction*, which is based on the notion of strongly exceptional sequences in triangulated categories and partially generalizes the known constructions of [3, 9]. A purely algebraic formulation of the bipartite construction will be given in Chapter 2, where we describe applications to derived equivalences of general triangular matrix rings. Other constructions will be described in Chapter 3.

The bipartite construction produces, for any poset admitting a special structure, new poset derived equivalent to it. In fact, for any closed subset $Y \subseteq X$, we construct a strongly exceptional collection in $\mathcal{D}^b(X)$ and use it to show an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(A)$ for a finite dimensional algebra A which depends on Y. We give conditions on X and Y under which A becomes an incidence algebra of a poset, and deduce that a lexicographic sum of a collection of posets along a bipartite graph S is derived equivalent to the lexicographic sum of the same collection along the opposite S^{op}.

As a corollary, we show that the derived equivalence class of an ordinal sum of two posets does not depend on the order of summands. We give an example that this is not true for three summands.

1.2 Preliminaries

1.2.1 Finite posets and T_0 spaces

Throughout this work, the term *poset* will mean a partially ordered set, which is assumed to be finite unless stated otherwise. Any poset $(X, \leq)$ carries a structure of a topological space by defining the closed sets to be the subsets $Y \subseteq X$ such that if $y \in Y$ and $y' \leq y$ then $y' \in Y$.

For each $x \in X$, denote by $\{x\}^-$ the closure of $\{x\}$ and by U_x the minimal open subset of X containing x, which equals the intersection of the open sets containing x. Then $\{x\}^- = \{x' \in X \,:\, x' \leq x\}$, $U_x = \{x' \in X \,:\, x' \geq x\}$ and

$$x \leq x' \iff \{x\}^- \subseteq \{x'\}^- \iff U_{x'} \subseteq U_x$$

If x, y are two distinct points in X, then one of the open sets U_x, U_y does not contain both points, thus X satisfies the T_0 separation property.

Conversely, given a finite T_0 topological space X, let U_x be the intersection of all open sets in X containing $x \in X$. Define a partial order $\leq$ on X by $x \leq x'$ if $U_{x'} \subseteq U_x$.

This leads to an identification of posets with finite T_0 topological spaces. Such spaces have been studied in the past [66, 82], where it turned out that their homotopy and homology properties are more interesting than might seem at first glance. For example, if $\mathcal{K}$ is any finite simplicial complex and X is the T_0 space induced by the partial order on the simplices of $\mathcal{K}$, then there exists a weak homotopy equivalence $|\mathcal{K}| \to X$ [66].

1.2.2 Sheaves and diagrams

Given a poset X, its *Hasse diagram* is a directed graph defined as follows. Its vertices are the elements of X and its directed edges $x \to y$ are the pairs $x < y$ in X such that there is no $z \in X$ with $x < z < y$. The anti-symmetry condition on $\leq$ implies that this graph has no directed cycles.

Let X be a poset and $\mathcal{A}$ be an abelian category. Using the topology on X, we can consider the category of *sheaves over X with values in $\mathcal{A}$*, denoted by $Sh_X\mathcal{A}$ or sometimes $\mathcal{A}^X$.

We note that sheaves over posets were used in [25] for the computation of cohomologies of real subspace arrangements. In addition, it is of interest to note the relation between (weakly) $\mathcal{K}$-constructible sheaves on a finite simplicial complex $\mathcal{K}$ and sheaves on the poset of simplices of $\mathcal{K}$, see [50, § 8.1].

Let $\mathcal{F}$ be a sheaf on X. If $x \in X$, let $\mathcal{F}(x)$ be the stalk of $\mathcal{F}$ over x, which equals $\mathcal{F}(U_x)$. The restriction maps $\mathcal{F}(x) = \mathcal{F}(U_x) \to \mathcal{F}(U_{x'}) = \mathcal{F}(x')$ for $x' > x$ give rise to a commutative diagram over the Hasse diagram of X. Conversely, such a diagram $\{F_x\}$ defines a sheaf $\mathcal{F}$ by setting the sections as the inverse limits $\mathcal{F}(U) = \lim_{x \in U} F_x$. Indeed, it is enough to verify the sheaf condition for the sets U_x, which follows from the observation that for any cover $U_x = \bigcup_i U_{z_i}$, one of the z_i equals x.

Thus we may identify $Sh_X\mathcal{A}$ with the category of commutative diagrams over the Hasse diagram of X and interchange the terms sheaf and diagram as appropriate. The latter category can be viewed as the category of functors $X \to \mathcal{A}$ where we consider X as a category whose objects are the points $x \in X$, with unique morphisms $x \to x'$ for $x \leq x'$. Under this identification, the global sections functor $\Gamma(X;-) : \mathcal{A}^X \to \mathcal{A}$ defined as $\Gamma(X;\mathcal{F}) = \mathcal{F}(X)$, coincides with the (inverse) limit functor $\lim_X : \mathcal{A}^X \to \mathcal{A}$.

1.2.3 Functors associated with a map $f : X \to Y$

A map $f : X \to Y$ between two finite posets is continuous if and only if it is *order preserving*, that is, $f(x) \leq f(x')$ for any $x \leq x'$ in X [82, Prop. 7].

A continuous map $f : X \to Y$ gives rise to the functors $f_*, f_! : Sh_X\mathcal{A} \to Sh_Y\mathcal{A}$ and $f^{-1} : Sh_Y\mathcal{A} \to Sh_X\mathcal{A}$, defined, in terms of diagrams, by

$$\begin{aligned}(f^{-1}\mathcal{G})(x) &= \mathcal{G}(f(x)) \\ (f_*\mathcal{F})(y) &= \varprojlim\{\mathcal{F}(x) \,:\, f(x) \geq y\} \\ (f_!\mathcal{F})(y) &= \varinjlim\{\mathcal{F}(x) \,:\, f(x) \leq y\}\end{aligned}$$

where $x \in X, y \in Y$ and $\mathcal{F} \in Sh_X\mathcal{A}, \mathcal{G} \in Sh_Y\mathcal{A}$. Viewing X, Y as categories and $\mathcal{F} \in Sh_X\mathcal{A}$ as a functor $\mathcal{F} : X \to \mathcal{A}$, the sheaves $f_*\mathcal{F}$ and $f_!\mathcal{F}$ are the right and left Kan extensions of $\mathcal{F}$ along $f : X \to Y$.

The functors f^{-1}, f_* coincide with the usual ones from sheaf theory. We have the following adjunctions:

$$\begin{aligned}\mathrm{Hom}_{Sh_X\mathcal{A}}(f^{-1}\mathcal{G}, \mathcal{F}) &\simeq \mathrm{Hom}_{Sh_Y\mathcal{A}}(\mathcal{G}, f_*\mathcal{F}) \\ \mathrm{Hom}_{Sh_X\mathcal{A}}(\mathcal{F}, f^{-1}\mathcal{G}) &\simeq \mathrm{Hom}_{Sh_Y\mathcal{A}}(f_!\mathcal{F}, \mathcal{G})\end{aligned} \tag{1.2.1}$$

so that f_* is left exact and $f_!$ is right exact. f^{-1} is exact, as can be seen from its action on the stalks.

If Y is a closed subset of X, we have a *closed embedding* $i : Y \to X$. In this case, i_* is exact. This is because i_* takes a diagram on Y and extends it to X by filling the vertices of $X \setminus Y$ with zeros. Similarly, for an open embedding $j : U \to X$, $j_!$ is exact, as it extends by zeros diagrams on U. Now let $Y \subseteq X$ be closed and $U = X \setminus Y$ its complement. The adjunction morphisms $j_!j^{-1}\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ for the embeddings $i : Y \to X$ and $j : U \to X$ induce a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0 \tag{1.2.2}$$

for any sheaf $\mathcal{F}$ on X, as can be verified at the stalks.

1.2.4 Simples, projectives and injectives

When $f : X \to \bullet$ is the mapping to a point, $f_* = \Gamma(X;-)$, and for an object M of $\mathcal{A}$, $f^{-1}(M)$ is the *constant sheaf* on X with value M.

Let $x \in X$ and consider the map $i_x : \bullet \to X$ whose image is $\{x\}$. Then $i_x^{-1}(\mathcal{F}) = \mathcal{F}(x)$ is the stalk at x and for an object M of $\mathcal{A}$ we have

$$(i_{x*}M)(y) = \begin{cases} M & \text{if } y \leq x \\ 0 & \text{otherwise} \end{cases} \qquad (i_{x!}M)(y) = \begin{cases} M & \text{if } y \geq x \\ 0 & \text{otherwise} \end{cases}$$

with identity arrows between the M-s. The adjunctions (1.2.1) take the form:

$$\begin{aligned} \mathrm{Hom}_{Sh_X\mathcal{A}}(\mathcal{F}, i_{x*}M) &\simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(x), M) \\ \mathrm{Hom}_{Sh_X\mathcal{A}}(i_{x!}M, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{F}(x)) \end{aligned} \tag{1.2.3}$$

and we deduce the following lemma:

Lemma 1.2.1. *If I is injective in $\mathcal{A}$, $i_{x*}I$ is injective in $Sh_X\mathcal{A}$. If P is projective in $\mathcal{A}$, $i_{x!}P$ is projective in $Sh_X\mathcal{A}$.*

Corollary 1.2.2. *If $\mathcal{A}$ has enough injectives (projectives), so does $Sh_X\mathcal{A}$.*

Proof. The identity maps $\mathcal{F}(x) \xrightarrow{=} \mathcal{F}(x)$ induce, via the adjuctions (1.2.3), an injection $\mathcal{F} \hookrightarrow \oplus_{x\in X} i_{x*}\mathcal{F}(x)$ and surjection $\oplus_{x\in X} i_{x!}\mathcal{F}(x) \twoheadrightarrow \mathcal{F}$. Now replace each $\mathcal{F}(x)$ by an injective (or projective) cover. □

For a sheaf $\mathcal{F}$, let $\operatorname{supp}\mathcal{F} = \{x \in X : \mathcal{F}(x) \neq 0\}$ be its *support*. We call $\mathcal{F}$ a *stalk sheaf* if its support is a point. For any object M of $\mathcal{A}$ and $x \in X$ there exists a stalk sheaf M_x whose stalk at x equals M. Moreover M_x is simple in $Sh_X\mathcal{A}$ if and only if M is simple in $\mathcal{A}$.

The following lemma is proved by induction on the number of elements $|X|$, using (1.2.2) and the fact that the partial order on X can be extended to a linear order, i.e. one can write the elements of X in a sequence $x_1, x_2, \dots, x_n$ such that for any $1 \leq i, j \leq n$, $x_i < x_j$ implies that $i < j$.

Lemma 1.2.3. *Any sheaf $\mathcal{F}$ on X admits a finite filtration whose quotients are stalk sheaves.*

Denote by $\mathrm{gl.dim}\,\mathcal{A}$ the *global dimension* of an abelian category $\mathcal{A}$. This is the maximal integer n for which there exist objects M, M' of $\mathcal{A}$ with $\mathrm{Ext}^n(M, M') \neq 0$ (and ∞ if there is no such maximal n). Recall that an abelian category is a *finite length* category if every object is of finite length. From Lemma 1.2.3, we have:

Corollary 1.2.4. *If $\mathcal{A}$ is a finite length category, so is $Sh_X\mathcal{A}$.*

Definition 1.2.5. A strictly increasing sequence $x_0 < x_1 < \cdots < x_n$ in X is called a *chain of length n*. The *dimension* of X, denoted $\dim X$, is the maximal length of a chain in X.

Proposition 1.2.6 ([68])**.** $\mathrm{gl.dim}\,Sh_X\mathcal{A} \leq \mathrm{gl.dim}\,\mathcal{A} + \dim X$.

The difference $\mathrm{gl.dim}\,Sh_X\mathcal{A} - \mathrm{gl.dim}\,\mathcal{A}$ obviously depends on X, but it may well depend also on $\mathcal{A}$, see the examples in [47, 81].

1.2.5 Sheaves of finite-dimensional vector spaces

Fix a field k and consider the category $\mathcal{A}$ of finite dimensional vector spaces over k. Denote by Sh_X the category $Sh_X\mathcal{A}$ and by $\mathrm{Hom}_X(-,-)$ the morphism spaces $\mathrm{Hom}_{Sh_X}(-,-)$ (We omit the reference to k to emphasize that it is to be fixed throughout).

The *incidence algebra* of X over k, denoted kX, is the algebra spanned by e_{xy} for the pairs $x \leq y$ in X, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$.

Lemma 1.2.7. *The category Sh_X is equivalent to the category of finite dimensional right modules over the incidence algebra kX.*

Proof. The proof is similar to the corresponding fact about representations of a quiver and right modules over its path algebra. Namely, for a sheaf $\mathcal{F}$, consider $M = \oplus_{x \in X} \mathcal{F}(x)$ and let $\iota_x : \mathcal{F}(x) \to M$, $\pi_x : M \to \mathcal{F}(x)$ be the natural maps. Equip M with a structure of a right kX-module by letting the basis elements $e_{xx'}$ for $x \leq x'$ act from the right as the composition $M \xrightarrow{\pi_x} \mathcal{F}(x) \to \mathcal{F}(x') \xrightarrow{\iota_{x'}} M$. Conversely, given a finite dimensional right module M over kX, set $\mathcal{F}(x) = Me_{xx}$ and define the maps $\mathcal{F}(x) \to \mathcal{F}(x')$ using the right multiplication by $e_{xx'}$. □

The one dimensional space k is both simple, projective and injective in the category of k-vector spaces. Applying the results of the previous subsection, we get, for any $x \in X$, sheaves S_x, P_x, I_x which are simple, projective and injective, respectively. Explicitly,

$$S_x(y) = \begin{cases} k & y = x \\ 0 & \text{otherwise} \end{cases}, \quad P_x(y) = \begin{cases} k & y \geq x \\ 0 & \text{otherwise} \end{cases}, \quad I_x(y) = \begin{cases} k & y \leq x \\ 0 & \text{otherwise} \end{cases}$$

By (1.2.3), for any sheaf $\mathcal{F}$, $\operatorname{Hom}_X(P_x, \mathcal{F}) = \mathcal{F}(x)$ and $\operatorname{Hom}_X(\mathcal{F}, I_x) = \mathcal{F}(x)^\vee$ (the dual space). Since the sets $U_x, \{x\}^-$ are connected, the sheaves P_x, I_x are indecomposable. The sheaves S_x, P_x, I_x form a complete set of representatives of the isomorphism classes of simples, indecomposable projectives and indecomposable injectives (respectively) in kX.

By Corollary 1.2.2, Sh_X has enough projectives and injectives (note that this can also be deduced by its identification with the category of finite dimensional modules over a finite dimensional algebra). It has finite global dimension, since by Proposition 1.2.6, $\operatorname{gl.dim} Sh_X \leq \dim X$.

Proposition 1.2.8. *Sh_X and Sh_Y are equivalent if and only if X and Y are isomorphic (as posets).*

Proof. Since the isomorphism classes of simple objects in Sh_X are in one-to-one correspondence with the elements $x \in X$, and for two such simples S_x, S_y, $\dim_k \operatorname{Ext}^1(S_x, S_y)$ equals 1 if there is a directed edge $x \to y$ in the Hasse diagram of X and 0 otherwise, we see that the Hasse diagram of X, hence X, can be recovered (up to isomorphism) from the category Sh_X. □

1.2.6 The derived category of sheaves over a poset

For a poset X, denote by $\mathcal{D}^b(X)$ the bounded derived category of Sh_X. For a textbook introduction to derived categories, we refer the reader to [33, Chapter III]. For a quick definition, see Section 3.2.

If $\mathcal{E}$ is a set of objects of a triangulated category $\mathcal{T}$, we denote by $\langle \mathcal{E} \rangle$ the triangulated subcategory of $\mathcal{T}$ *generated by* $\mathcal{E}$, that is, the minimal triangulated subcategory containing $\mathcal{E}$. We say that $\mathcal{E}$ *generates* $\mathcal{T}$ if $\langle \mathcal{E} \rangle = \mathcal{T}$.

Since Sh_X is of finite global dimension with enough projectives and injectives, $\mathcal{D}^b(X)$ can be identified with the homotopy category of bounded complexes of projectives (or bounded complexes of injectives). Hence the collections $\{P_x\}_{x \in X}$ and $\{I_x\}_{x \in X}$ generate $\mathcal{D}^b(X)$.

Lemma 1.2.9. *Let $x, y \in X$ and $i \in \mathbb{Z}$. Then*

$$\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, P_y[i]) = \operatorname{Hom}_{\mathcal{D}^b(X)}(I_x, I_y[i]) = \begin{cases} k & y \leq x \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since P_x is projective, $\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, \mathcal{F}[i]) = 0$ for any sheaf $\mathcal{F}$ and $i \neq 0$. If $x, y \in X$, then

$$\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, P_y) = \operatorname{Hom}_X(P_x, P_y) = P_y(x) = \begin{cases} k & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

The proof for $\{I_x\}_{x \in X}$ is similar. □

For a continuous map $f : X \to Y$, denote by $Rf_*, Lf_!, f^{-1}$ the derived functors of $f_*, f_!, f^{-1}$. The adjunctions (1.2.1) imply that

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}^b(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &\simeq \operatorname{Hom}_{\mathcal{D}^b(Y)}(\mathcal{G}, Rf_*\mathcal{F}) \\ \operatorname{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}, f^{-1}\mathcal{G}) &\simeq \operatorname{Hom}_{\mathcal{D}^b(Y)}(Lf_!\mathcal{F}, \mathcal{G}) \end{aligned} \tag{1.2.4}$$

for $\mathcal{F} \in \mathcal{D}^b(X)$, $\mathcal{G} \in \mathcal{D}^b(Y)$.

Definition 1.2.10. We say that two posets X and Y are *derived equivalent*, denoted $X \sim Y$, if the categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ are equivalent as triangulated categories.

1.3 Combinatorial invariants of derived equivalence

We give a list of combinatorial properties of posets which are preserved under derived equivalence. Most of the properties are deduced from known invariants of derived categories. For the convenience of the reader, we review the relevant definitions.

1.3.1 The number of points and K-groups

Recall that for an abelian category $\mathcal{A}$, the *Grothendieck group* $K_0(\mathcal{A})$ is the quotient of the free abelian group generated by the isomorphism classes $[X]$ of objects X of $\mathcal{A}$ divided by the subgroup generated by the expressions $[X] - [Y] + [Z]$ for all the short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

Similarly, for a triangulated category $\mathcal{T}$, the group $K_0(\mathcal{T})$ is the quotient of the free abelian group on the isomorphism classes of objects of $\mathcal{T}$ divided by its subgroup generated by $[X] - [Y] + [Z]$ for all the triangles $X \to Y \to Z \to X[1]$ in $\mathcal{T}$ (where $[1]$ denotes the shift). The natural inclusion $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{D}^b(\mathcal{A}))$.

Let X be a poset and denote by $|X|$ the number of points of X. Denote by $K_0(X)$ the group $K_0(\mathcal{D}^b(X))$.

Proposition 1.3.1. *$K_0(X)$ is free abelian of rank $|X|$.*

Proof. The set $\{S_x\}_{x \in X}$ forms a complete set of representatives of the isomorphism classes of simple finite dimensional kX-modules, hence it is a $\mathbb{Z}$-basis of $K_0(X)$ (alternatively one could use the filtration of Lemma 1.2.3). $\square$

Corollary 1.3.2. *If $X \sim Y$ then $|X| = |Y|$.*

It is known [26] that rings with equivalent derived categories have the same K-theory. However, higher K-groups do not lead to refined invariants of the number of points.

Proposition 1.3.3. $K_i(Sh_X) \simeq K_i(Sh_\bullet)^{|X|}$ *for $i \geq 0$.*

Proof. Sh_X is a finite length category and by [71, Corollary 1, p. 104],

$$K_i(Sh_X) \simeq \bigoplus_{x \in X} K_i(\mathrm{End}_X(S_x))$$

Clearly, $k = \mathrm{End}_X(S_x)$. $\square$

1.3.2 Connected components

For two additive categories $\mathcal{T}_1$, $\mathcal{T}_2$, consider the category $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ whose objects are pairs (M_1, M_2) and the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{T}}((M_1, M_2), (N_1, N_2)) = \mathrm{Hom}_{\mathcal{T}_1}(M_1, N_1) \times \mathrm{Hom}_{\mathcal{T}_2}(M_2, N_2)$$

$\mathcal{T}_1$, $\mathcal{T}_2$ are embedded in $\mathcal{T}$ via the fully faithful functors $M_1 \mapsto (M_1, 0)$ and $M_2 \mapsto (0, M_2)$. Denoting the images again by $\mathcal{T}_1, \mathcal{T}_2$, we have that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = 0$. In addition, the indecomposables in $\mathcal{T}$ are of the form $(M_1, 0)$ or $(0, M_2)$ for indecomposables $M_1 \in \mathcal{T}_1$, $M_2 \in \mathcal{T}_2$.

An additive category $\mathcal{T}$ is *connected* if for any equivalence $\mathcal{T} \simeq \mathcal{T}_1 \times \mathcal{T}_2$, one of $\mathcal{T}_1$, $\mathcal{T}_2$ is zero.

Definition 1.3.4. A poset X is *connected* if it is connected as a topological space. This is equivalent to the following condition [82, Prop. 5]:

For any $x, y \in X$ there exists a sequence $x = x_0, x_1, \dots, x_n = y$ in X such that for all $0 \leq i < n$, either $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$.

Lemma 1.3.5. *If X is connected then the category $\mathcal{D}^b(X)$ is connected.*

Proof. Let $\mathcal{D}^b(X) \simeq \mathcal{T}_1 \times \mathcal{T}_2$ be an equivalence and consider the indecomposable projectives $\{P_x\}_{x \in X}$. Since each P_x is indecomposable, its image lies in $\mathcal{T}_1$ or in $\mathcal{T}_2$, and we get a partition $X = X_1 \sqcup X_2$.

Assume that X_1 is not empty. Since $\mathrm{Hom}(P_x, P_y) \neq 0$ for all $y \leq x$ and $\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2) = 0$, X_1 must be both open and closed in X, and by connectivity, $X_1 = X$. Moreover, $\{P_x\}_{x \in X}$ generates $\mathcal{D}^b(X)$ as a triangulated category, hence $\mathcal{D}^b(X) \simeq \mathcal{T}_1$ and $\mathcal{T}_2 = 0$. $\square$

Proposition 1.3.6. *Let X and Y be two posets with decompositions*

$$X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_t \qquad\qquad Y = Y_1 \sqcup Y_2 \sqcup \dots \sqcup Y_s$$

into connected components. If $X \sim Y$ then $s = t$ and there exists a permutation π on $\{1, \dots, s\}$ such that $X_i \sim Y_{\pi(i)}$ for all $1 \leq i \leq s$.

Proof. There exists a pair of equivalences

$$\mathcal{D}^b(X_1) \times \cdots \times \mathcal{D}^b(X_t) = \mathcal{D}^b(X) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}^b(Y) = \mathcal{D}^b(Y_1) \times \cdots \times \mathcal{D}^b(Y_s)$$

If $x \in X$, the image $F(P_x)$ is indecomposable in $\mathcal{D}^b(Y)$, hence lands in one of the $\mathcal{D}^b(Y_j)$, and we get a function $f : X \to \{1, \dots, s\}$. For any $x' \leq x$, $\operatorname{Hom}_X(P_x, P_{x'}) \neq 0$, therefore f is constant on the connected components X_i and induces a map $\pi_F : \{1, \dots, t\} \to \{1, \dots, s\}$ via $\pi_F(i) = f(x)$ for $x \in X_i$. Moreover, since $\{P_x\}_{x \in X_i}$ generates $\mathcal{D}^b(X_i)$ as a triangulated category, F restricts to functors $\mathcal{D}^b(X_i) \to \mathcal{D}^b(Y_{\pi_F(i)})$, $1 \leq i \leq t$.

Similarly for G, we obtain a map $\pi_G : \{1, \dots, s\} \to \{1, \dots, t\}$ and functors $\mathcal{D}^b(Y_j) \to \mathcal{D}^b(Y_{\pi_G(j)})$ which are restrictions of G.

For any $1 \leq i \leq t$, the image of $\mathcal{D}^b(X_i)$ under GF lies in $\mathcal{D}^b(X_{\pi_G \pi_F(i)})$. Since GF is isomorphic to the identity functor but on the other hand there are no nonzero maps between $\mathcal{D}^b(X_i)$ and $\mathcal{D}^b(X_{i'})$ for $i \neq i'$ (as we think of X_i as *subsets* of X, not just as abstract sets!), we get that $\pi_G \pi_F(i) = i$ so that $\pi_G \pi_F$ is identity. Similarly, $\pi_F \pi_G$ is identity.

We deduce that $s = t$, π_F and π_G are permutations, and the restrictions of F induce equivalences $\mathcal{D}^b(X_i) \simeq \mathcal{D}^b(Y_{\pi_F(i)})$. □

One can also deduce that the number of connected components is a derived invariant by considering the center $Z(kX)$ of the incidence algebra kX using the fact that derived equivalent algebras have isomorphic centers [73].

Lemma 1.3.7. *$Z(kX) \cong k \times k \times \cdots \times k$ where the number of factors equals the number of connected components of X.*

Proof. Let $c = \sum_{x \leq y} c_{xy} e_{xy} \in Z(kX)$. Comparison of the coefficients of $e_{xx}c$ and ce_{xx} gives $c_{xy} = 0$ for $x \neq y$, thus $c = \sum_x c_x e_{xx}$.

If $x \leq y$ then $c_x e_{xy} = c e_{xy} = e_{xy} c = c_y e_{xy}$, hence $c_x = c_y$ if x, y are in the same connected component. □

1.3.3 The Euler form and Möbius function

Let X be a poset. Since Sh_X has finite global dimension, the expression

$$\langle K, L \rangle_X = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}^b(X)}(K, L[i])$$

is well-defined for $K, L \in \mathcal{D}^b(X)$ and induces a $\mathbb{Z}$-bilinear form on $K_0(X)$, known as the *Euler form.*

Recall that the *incidence matrix* of X, denoted $\mathbf{1}_X$, is the $X \times X$ matrix defined by

$$(\mathbf{1}_X)_{xy} = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise} \end{cases}$$

By extending the partial order on X to a linear order, we can always arrange the elements of X such that the incidence matrix is upper triangular with ones on the diagonal. In particular, $\mathbf{1}_X$ is invertible over $\mathbb{Z}$.

Definition 1.3.8. The *Möbius function* $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x,y) = (\mathbf{1}_X^{-1})_{xy}$.

The following is an immediate consequence of the definition.

Lemma 1.3.9 (Möbius inversion formula)**.** *Let $f : X \to \mathbb{Z}$. Define $g : X \to \mathbb{Z}$ by $g(x) = \sum_{y \geq x} f(y)$. Then $f(x) = \sum_{y \geq x} \mu_X(x,y) g(y)$.*

The Möbius inversion formula can be used to compute the matrix of the Euler form with respect to the basis of simple objects.

Lemma 1.3.10. *$\langle [P_x], [S_y] \rangle_X = \delta_{xy}$ for all $x, y \in X$.*

Proof. Since P_x is projective, $\mathrm{Hom}_{\mathcal{D}^b(X)}(P_x, \mathcal{F}[i]) = 0$ for any sheaf $\mathcal{F}$ and $i \neq 0$. Now by (1.2.3), $\mathrm{Hom}_{\mathcal{D}^b(X)}(P_x, S_y) = \mathrm{Hom}_X(P_x, S_y) = S_y(x)$. $\square$

Proposition 1.3.11. *Let $x, y \in X$. Then $\langle [S_x], [S_y] \rangle_X = \mu_X(x,y)$.*

Proof. Fix y and define $f : X \to \mathbb{Z}$ by $f(x) = \langle [S_x], [S_y] \rangle_X$. Since $[P_x] = \sum_{x' \geq x} [S_{x'}]$, Lemmas 1.3.9 and 1.3.10 imply that

$$f(x) = \sum_{x' \geq x} \mu_X(x,x') \langle [P_{x'}], [S_y] \rangle_X = \mu_X(x,y)$$

$\square$

Definition 1.3.12. Let R be a commutative ring. Two matrices $M_1, M_2 \in \mathrm{GL}_n(R)$ are *congruent* over R if there exists a matrix $P \in \mathrm{GL}_n(R)$ such that $M_2 = P M_1 P^t$.

Note that if M_1, M_2 is a pair of congruent matrices, so are M_1^t, M_2^t and M_1^{-1}, M_2^{-1}. Denote by M^{-t} the inverse of the transpose of M.

Corollary 1.3.13. *If $X \sim Y$ then $\mathbf{1}_X$, $\mathbf{1}_Y$ are congruent over $\mathbb{Z}$.*

Proof. An equivalence $F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induces an isomorphism $[F] : K_0(X) \to K_0(Y)$ which preserves the Euler form. By Proposition 1.3.11, the matrix of the Euler form of $\mathcal{D}^b(X)$ over the basis of simples is $\mathbf{1}_X^{-1}$, hence $[F]^t \mathbf{1}_Y^{-1} [F] = \mathbf{1}_X^{-1}$. $\square$

In practice, testing for congruence over $\mathbb{Z}$ is not an easy task. However, the following necessary condition is often very useful in ruling out congruence.

Lemma 1.3.14. *Let $M_1, M_2 \in \mathrm{GL}_n(R)$ be congruent. Then the matrices $M_1 M_1^{-t}$, $M_2 M_2^{-t}$ are conjugate in the group $\mathrm{GL}_n(R)$.*

Proof. If $M_2 = P M_1 P^t$ for some $P \in \mathrm{GL}_n(R)$, then

$$M_2 M_2^{-t} = (P M_1 P^t)(P^{-t} M_1^{-t} P^{-1}) = P M_1 M_1^{-t} P^{-1}$$

$\square$

Corollary 1.3.15. *If $X \sim Y$ then $\mathbf{1}_X \mathbf{1}_X^{-t}$ and $\mathbf{1}_Y \mathbf{1}_Y^{-t}$ are similar over $\mathbb{Z}$. In particular, they are similar over $\mathbb{Q}$ and modulo all primes p.*

Note that $\mathbf{1}_X \mathbf{1}_X^{-t}$ is (up to sign) the *Coxeter matrix* of the algebra kX. It is the image in $K_0(X)$ of the Serre functor on $\mathcal{D}^b(X)$.

1.3.4 Betti numbers and Euler characteristic

The Hochschild cohomology is a known derived invariant of an algebra [35, 74]. For posets, one can compute the Hochschild cohomology as the simplicial cohomology of an appropriate simplicial complex [20, 34]. Thus the simplicial cohomology is a derived invariant, which we relate to the cohomology of the constant sheaf.

For the convenience of the reader, we review the notions of sheaf cohomology, simplicial cohomology and Hochschild cohomology. As before, we keep the field k fixed.

Sheaf cohomology

Recall that the i-th cohomology of a sheaf $\mathcal{F} \in Sh_X$, denoted $\mathrm{H}^i(X;\mathcal{F})$, is the value of the i-th right derived functor of the global sections functor $\Gamma(X;-) : Sh_X \to Sh_\bullet$. Observe that $\Gamma(X;\mathcal{F}) = \mathrm{Hom}_X(k_X, \mathcal{F})$ where k_X is the *constant sheaf* on X, i.e. $k_X(x) = k$ for all $x \in X$ with all morphisms being the identity of k. It follows that $\mathrm{H}^i(X;\mathcal{F}) = \mathrm{Ext}^i_X(k_X, \mathcal{F})$. Specializing this for the particularly interesting cohomologies of the constant sheaf, we get that $\mathrm{H}^i(X;k_X) = \mathrm{Ext}^i_X(k_X, k_X)$.

Simplicial cohomology

Let X be a poset, $p \geq 0$. A *p-dimensional simplex* in X is a chain of length p. Since subsets of chains are again chains, the set of all simplices in X forms a simplicial complex $\mathcal{K}(X)$ [66], known as the *order complex* of X. The i-th simplicial cohomology of X is defined as the i-th simplicial cohomology of $\mathcal{K}(X)$, and we denote it by $\mathrm{H}^i(X)$. The number $\beta^i(X) = \dim_k \mathrm{H}^i(X)$ is the i-th *Betti number* of X.

The simplicial cohomology of X is related to the cohomology of the constant sheaf via appropriate simplicial resolution, which we now describe.

Let I_x be the indecomposable injective corresponding to x. For a simplex σ, set $I_\sigma = I_{\min\sigma}$ where $\min\sigma$ is the minimal element of σ. If $\tau \subseteq \sigma$, then $\min\tau \geq \min\sigma$, hence $\mathrm{Hom}_X(I_\tau, I_\sigma) \simeq k$.

Let $X^{(p)}$ denote the set of p-simplices of X and let $\mathcal{I}^p_X = \oplus_{\sigma\in X^{(p)}} I_\sigma$. For a p-simplex $\sigma = x_0 < x_1 < \cdots < x_p$ and $0 \leq j \leq p$, denote by $\widehat{\sigma}^j$ the $(p-1)$-simplex obtained from σ by deleting the vertex x_j. By considering, for all $\sigma \in X^{(p)}$ and $0 \leq j \leq p$, the map $I_{\widehat{\sigma}^j} \to I_\sigma$ corresponding to $(-1)^j \in k$, we get a map $d^{p-1} : \mathcal{I}^{p-1}_X \to \mathcal{I}^p_X$. The usual sign considerations give $d^p d^{p-1} = 0$.

Lemma 1.3.16. $\mathrm{H}^i(X) = \mathrm{H}^i(\mathrm{Hom}_X(k_X, \mathcal{I}^\bullet_X))$ *for all* $i \geq 0$.

Proof. Indeed, the p-th term is

$$\mathrm{Hom}_X(k_X, \mathcal{I}^p_X) = \oplus_{\sigma\in X^{(p)}} \mathrm{Hom}_X(k_X, I_\sigma) \cong \oplus_{\sigma\in X^{(p)}} k_X(\min\sigma)$$

and can be viewed as the space of functions from $X^{(p)}$ to k. Moreover, the differential is exactly the one used in the definition of simplicial cohomology. □

Lemma 1.3.17. *The complex* $0 \to k_X \to \mathcal{I}_X^0 \xrightarrow{d^0} \mathcal{I}_X^1 \xrightarrow{d^1} \dots$ *is an injective resolution of the constant sheaf* k_X.

Proof. It is enough to check acyclicity at the stalks.

Let $x \in X$. Then $I_\sigma(x) \neq 0$ only if $\min \sigma \geq x$, hence it is enough to consider the p-simplices of U_x, and the complex of stalks at x equals

$$0 \to k \to \mathrm{Hom}_{U_x}(k_{U_x}, \mathcal{I}_{U_x}^0) \to \mathrm{Hom}_{U_x}(k_{U_x}, \mathcal{I}_{U_x}^1) \to \dots$$

The acyclicity of this complex follows by Lemma 1.3.16 with $X = U_x$, using the fact that U_x has x as the unique minimal element, hence $\mathcal{K}(U_x)$ is contractible and $\mathrm{H}^i(U_x) = 0$ for $i > 0$, $\mathrm{H}^0(U_x) = k$. □

Proposition 1.3.18. $\mathrm{H}^i(X; k_X) = \mathrm{H}^i(X)$ *for all* $i \geq 0$.

Proof. Using Lemma 1.3.16 and the injective resolution of Lemma 1.3.17,

$$\mathrm{H}^i(X; k_X) = \mathrm{H}^i(\mathrm{Hom}_X(k_X, \mathcal{I}_X^\bullet)) = \mathrm{H}^i(X)$$

□

Hochschild cohomology

A k-algebra Λ has a natural structure of a Λ-Λ-bimodule, or a $\Lambda \otimes_k \Lambda^{op}$ right module. The group $\mathrm{Ext}^i_{\Lambda\otimes\Lambda^{op}}(\Lambda, \Lambda)$ is called the i*-th Hochschild cohomology* of Λ, and we denote it by $\mathrm{HH}^i(\Lambda)$.

The Hochschild cohomology of incidence algebras of posets was widely studied, see [20, 31, 34]. The following theorem relates the Hochschild cohomology of an incidence algebra of a poset X with its simplicial cohomology.

Theorem 1.3.19 ([20, 34])**.** $\mathrm{HH}^i(kX) = \mathrm{H}^i(X)$ *for all* $i \geq 0$.

Combining this with Proposition 1.3.18, we get:

Corollary 1.3.20. $\mathrm{HH}^i(kX) = \mathrm{H}^i(X; k_X) = \mathrm{Ext}^i_X(k_X, k_X)$ *for all* $i \geq 0$.

Derived invariants

Corollary 1.3.21. *If* $X \sim Y$ *then* $\beta^i(X) = \beta^i(Y)$ *for all* $i \geq 0$.

Proof. Follows from Theorem 1.3.19 and the fact that the Hochschild cohomology of a k-algebra is preserved under derived equivalence [36, 74]. □

The alternating sum $\chi(X) = \sum_{i\geq 0}(-1)^i \beta^i(X)$ is known as the *Euler characteristic* of X.

Corollary 1.3.22. *If* $X \sim Y$ *then* $\chi(X) = \chi(Y)$.

We give two interpretations of $\chi(X)$. First, by Proposition 1.3.18,

$$\chi(X) = \sum_{i \geq 0} (-1)^i \beta^i(X) = \sum_{i \geq 0} \dim_k \operatorname{Hom}_{\mathcal{D}^b(X)}(k_X, k_X[i]) = \langle [k_X], [k_X] \rangle_X$$

where $[k_X]$ is the image of k_X in $K_0(X)$. Since $[k_X] = \sum_{x \in X} [S_x]$,

$$\langle [k_X], [k_X] \rangle = \sum_{x,y \in X} \langle [S_x], [S_y] \rangle_X = \sum_{x,y \in X} \mu_X(x,y)$$

hence $\chi(X)$ is the sum of entries of the matrix $\mathbf{1}_X^{-1}$. We see that not only the $\mathbb{Z}$-congruence class of $\mathbf{1}_X^{-1}$ is preserved by derived equivalence, but also the sum of its entries.

For the second interpretation, changing the order of summation we get

$$\sum_{x,y \in X} \langle [S_x], [S_y] \rangle_X = \sum_{i \geq 0} (-1)^i \sum_{x,y \in X} \dim \operatorname{Ext}_X^i(S_x, S_y)$$

Using the fact that $\dim \operatorname{Ext}^i(S_x, S_y)$ equals δ_{xy} for $i = 0$; counts the number of arrows from x to y in the Hasse diagram of X when $i = 1$; and counts the number of commutativity relations between x and y for $i = 2$, we see that at least when $\operatorname{gl.dim} X \leq 2$, $\chi(X)$ equals the number of points minus the number of arrows in the Hasse diagram plus the number of relations etc.

1.3.5 Operations preserving derived equivalence

We show that derived equivalence is preserved under taking opposites and products.

Definition 1.3.23. The *opposite* of a poset X, denoted by X^{op}, is the poset $(X, \leq^{op})$ with $x \leq^{op} x'$ if and only if $x \geq x'$.

Lemma 1.3.24. *Let $\mathcal{A}$ be an abelian category. Then $Sh_{X^{op}} \mathcal{A} \simeq (Sh_X \mathcal{A}^{op})^{op}$.*

Proof. A sheaf $\mathcal{F}$ over X^{op} with values in $\mathcal{A}$ is defined via compatible $\mathcal{A}$-morphisms between the stalks $\mathcal{F}(y) \to \mathcal{F}(x)$ for $x \leq y$. Viewing these morphisms as $\mathcal{A}^{op}$-morphisms we identify $\mathcal{F}$ with a sheaf over X with values in $\mathcal{A}^{op}$. Since a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is specified via compatible $\mathcal{A}$-morphisms $\mathcal{F}(x) \to \mathcal{G}(x)$, this identification gives an equivalence $Sh_{X^{op}} \mathcal{A} \simeq (Sh_X \mathcal{A}^{op})^{op}$. $\square$

Corollary 1.3.25. *$Sh_{X^{op}}$ is equivalent to $(Sh_X)^{op}$.*

Proof. Let $\mathcal{A}$ be the category of finite dimensional k-vector spaces. Then the functor $V \mapsto V^\vee$ mapping a finite dimensional k-vector space to its dual induces an equivalence $\mathcal{A} \simeq \mathcal{A}^{op}$. $\square$

Proposition 1.3.26. *If $X \sim Y$ then $X^{op} \sim Y^{op}$.*

Proof. It is well known that for an abelian category $\mathcal{A}$, the opposite category $\mathcal{A}^{op}$ is also abelian and $\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{A}^{op})^{op}$ by mapping a complex $K = (K^i)_{i \in \mathbb{Z}}$ over $\mathcal{A}$ to the complex $K^\vee$ over $\mathcal{A}^{op}$ with $(K^\vee)^i = K^{-i}$.

Applying this for $\mathcal{A} = Sh_X$ and using Corollary 1.3.25, we deduce that $\mathcal{D}^b(X^{op}) \simeq \mathcal{D}^b(X)^{op}$. $\square$

Definition 1.3.27. The *product* of two posets X, Y, denoted $X \times Y$, is the poset whose underlying set is $X \times Y$, with $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

Lemma 1.3.28. $k(X \times Y) = kX \otimes_k kY$.

Proof. Observe that the function $kX \otimes_k kY \to k(X \times Y)$ defined by mapping the basis elements $e_{xx'} \otimes e_{yy'}$ to $e_{(x,y)(x',y')}$ where $x \leq x'$ and $y \leq y'$, is an isomorphism of k-algebras. $\square$

Proposition 1.3.29. *If* $X_1 \sim X_2$ *and* $Y_1 \sim Y_2$ *then* $X_1 \times Y_1 \sim X_2 \times Y_2$.

Proof. The claim follows from the previous lemma and the corresponding fact for tensor products of finite dimensional algebras over k, see [74, Lemma 4.3]. $\square$

1.4 Derived equivalences via exceptional collections

1.4.1 Strongly exceptional collections

Let k be a field and let $\mathcal{T}$ be a triangulated k-category.

Definition 1.4.1. A sequence $E_1, \dots, E_n$ of objects of $\mathcal{T}$ is called a *strongly exceptional collection* if

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{T}}(E_s, E_t[i]) = 0 && 1 \leq s, t \leq n\,,\, i \neq 0 \\ &\operatorname{Hom}_{\mathcal{T}}(E_s, E_t) = 0 && 1 \leq s < t \leq n \\ &\operatorname{Hom}_{\mathcal{T}}(E_s, E_s) = k && 1 \leq s \leq n \end{aligned} \tag{1.4.1}$$

An unordered finite collection $\mathcal{E}$ of objects of $\mathcal{T}$ will be called *strongly exceptional* if it can be ordered in a sequence which forms a strongly exceptional collection.

Let $\mathcal{E} = E_1, \dots, E_n$ be a strongly exceptional collection in $\mathcal{T}$, and consider $E = \oplus_{s=1}^{n} E_s$. The conditions (1.4.1) imply that $\operatorname{Hom}_{\mathcal{T}}(E, E[i]) = 0$ for $i \neq 0$ and that $\operatorname{End}_{\mathcal{T}}(E)$ is a finite dimensional k-algebra. If $\mathcal{E}$ generates $\mathcal{T}$, then E is a *tilting object* in $\mathcal{T}$.

For an algebra A over k, denote by $\mathcal{D}^b(A)$ the bounded derived category of complexes of finite dimensional right modules over A. The following result of Bondal shows that the existence of a generating strongly exceptional collection in a derived category leads to derived equivalence with $\mathcal{D}^b(A)$ where A is the endomorphism algebra of the corresponding tilting object.

Theorem 1.4.2 ([12, §6])**.** *Let $\mathcal{A}$ be an abelian category and let $E_1, \dots, E_n$ be a strongly exceptional collection which generates $\mathcal{D}^b(\mathcal{A})$. Set $E = \oplus_{s=1}^{n} E_s$. Then the functor*

$$\mathbf{R}\operatorname{Hom}(E, -) : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\operatorname{End}_{\mathcal{D}^b(\mathcal{A})} E)$$

is a triangulated equivalence.

When $\mathcal{A}$ is a category of finite dimensional modules over a finite dimensional algebra, as in the case of Sh_X, the result of the theorem can also be deduced from Rickard's Morita theory of derived equivalences of algebras [73] (see also [51, (3.2)]) by observing that E is a so-called one-sided tilting complex.

Example 1.4.3. For a poset X, the collection $\{P_x\}_{x\in X}$ (and $\{I_x\}_{x\in X}$) of indecomposable projectives (injectives) is strongly exceptional, generates $\mathcal{D}^b(X)$, and the corresponding endomorphism algebra is isomorphic to the incidence algebra of X (Use Lemma 1.2.9).

1.4.2 A gluing construction

Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be three triangulated categories with triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, shift left, "i_*"] & \mathcal{T} \arrow[l, shift left, "i^{-1}"] \arrow[r, shift left, "j^{-1}"] & \mathcal{T}'' \arrow[l, shift left, "j_!"] \end{tikzcd}$$

Assume that there are adjunctions

$$\mathrm{Hom}_{\mathcal{T}'}(i^{-1}\mathcal{F}, \mathcal{F}') \simeq \mathrm{Hom}_{\mathcal{T}}(\mathcal{F}, i_*\mathcal{F}') \tag{1.4.2}$$

$$\mathrm{Hom}_{\mathcal{T}''}(\mathcal{F}'', j^{-1}\mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{T}}(j_!\mathcal{F}'', \mathcal{F}) \tag{1.4.3}$$

for $\mathcal{F} \in \mathcal{T}$, $\mathcal{F}' \in \mathcal{T}'$, $\mathcal{F}'' \in \mathcal{T}''$. Assume also that $j^{-1}i_* = 0$, $i^{-1}j_! = 0$, $i^{-1}i_* \simeq \mathrm{Id}_{\mathcal{T}'}$ and $j^{-1}j_! \simeq \mathrm{Id}_{\mathcal{T}''}$.

Lemma 1.4.4. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{T}$. Then*

$$\mathrm{Hom}_{\mathcal{T}}(i_*i^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{T}}(\mathcal{F}, i_*i^{-1}\mathcal{G}) \tag{1.4.4}$$

$$\mathrm{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, j_!j^{-1}\mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, \mathcal{G}) \tag{1.4.5}$$

$$\mathrm{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) = 0 \tag{1.4.6}$$

Proof. The claims follow from the adjunctions (1.4.2),(1.4.3) and our additional hypotheses. For example, for the first claim use (1.4.2) and $i^{-1}i_* \simeq \mathrm{Id}_{\mathcal{T}'}$ to get that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}(i_*i^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) &\simeq \mathrm{Hom}_{\mathcal{T}'}(i^{-1}i_*i^{-1}\mathcal{F}, i^{-1}\mathcal{G}) = \\ &= \mathrm{Hom}_{\mathcal{T}'}(i^{-1}\mathcal{F}, i^{-1}\mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{T}}(\mathcal{F}, i_*i^{-1}\mathcal{G}) \end{aligned}$$

□

We apply this for the following situation, cf. [7, §1.4]. Let X be a poset and let $Y \subseteq X$ be a closed subset, $U = X \setminus Y$ its complement. Denote by $i : Y \to X$, $j : U \to X$ the embeddings. Since the functors $i_*, j_!$ are exact, we can consider the functors

$$\begin{aligned} i^{-1} &: \mathcal{D}^b(X) \to \mathcal{D}^b(Y) & \qquad j^{-1} &: \mathcal{D}^b(X) \to \mathcal{D}^b(U) \\ i_* &: \mathcal{D}^b(Y) \to \mathcal{D}^b(X) & \qquad j_! &: \mathcal{D}^b(U) \to \mathcal{D}^b(X) \end{aligned}$$

between the derived categories. Taking $\mathcal{T} = \mathcal{D}^b(X)$, $\mathcal{T}' = \mathcal{D}^b(Y)$ and $\mathcal{T}'' = \mathcal{D}^b(U)$, we see that the above assumptions are satisfied, where the adjunctions (1.4.2), (1.4.3) follow from (1.2.4).

For $y \in Y$ and $u \in U$, let $\widetilde{P}_y = i_*i^{-1}P_y$ and $\widetilde{I}_u = j_!j^{-1}I_u$ be "truncated" versions of the projectives and injectives. Explicitly,

$$\widetilde{P}_y(x) = \begin{cases} k & x \in Y\,,\ y \le x \\ 0 & \text{otherwise} \end{cases} \qquad \widetilde{I}_u(x) = \begin{cases} k & x \in U\,,\ x \le u \\ 0 & \text{otherwise} \end{cases}$$

with identity maps between nonzero stalks.

Proposition 1.4.5. *The collection* $\mathcal{E}_Y = \left\{\widetilde{P}_y\right\}_{y\in Y} \bigcup \left\{\widetilde{I}_u[1]\right\}_{u\in U}$ *is strongly exceptional and generates* $\mathcal{D}^b(X)$.

Proof. Let $y, y' \in Y$. By (1.4.4),

$$\operatorname{Hom}(\widetilde{P}_y, \widetilde{P}_{y'}) \simeq \operatorname{Hom}(P_y, \widetilde{P}_{y'}) = \widetilde{P}_{y'}(y) = \begin{cases} k & \text{if } y' \leq y \\ 0 & \text{otherwise} \end{cases} \tag{1.4.7}$$

and $\operatorname{Hom}(\widetilde{P}_y, \widetilde{P}_{y'}[n]) = 0$ for $n \neq 0$. Similarly, for $u, u' \in U$, by (1.4.5),

$$\operatorname{Hom}(\widetilde{I}_u, \widetilde{I}_{u'}) \simeq \operatorname{Hom}(\widetilde{I}_u, I_{u'}) = \widetilde{I}_u(u') = \begin{cases} k & \text{if } u' \leq u \\ 0 & \text{otherwise} \end{cases} \tag{1.4.8}$$

and $\operatorname{Hom}(\widetilde{I}_u, \widetilde{I}_{u'}[n]) = 0$ for $n \neq 0$.

Let $y \in Y$ and $u \in U$. By (1.4.6), $\operatorname{Hom}(\widetilde{I}_u, \widetilde{P}_y[n]) = 0$ for all $n \in \mathbb{Z}$. Consider now $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n])$. The distinguished triangle $\widetilde{I}_u \to I_u \to i_* i^{-1} I_u \to \widetilde{I}_u[1]$ of (1.2.2) gives rise to a long exact sequence

$$\cdots \to \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u) \to \operatorname{Hom}(\widetilde{P}_y, I_u) \to \operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u) \to \dots \tag{1.4.9}$$

Since I_u is injective, $\operatorname{Hom}(\widetilde{P}_y, I_u[n]) = 0$ for $n \neq 0$ and $\operatorname{Hom}(\widetilde{P}_y, I_u) = \widetilde{P}_y(u) = 0$. Therefore (1.4.9) induces isomorphisms

$$\operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u[n]) \xrightarrow{\simeq} \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n+1]) \tag{1.4.10}$$

for all $n \in \mathbb{Z}$. By (1.4.4),

$$\operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u[n]) = \operatorname{Hom}(P_y, i_* i^{-1} I_u[n]) = \begin{cases} (i_* i^{-1} I_u)(y) & n = 0 \\ 0 & n \neq 0 \end{cases}$$

and $(i_* i^{-1} I_u)(y) = k$ if $y < u$ and 0 otherwise, hence

$$\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) = \begin{cases} k & \text{if } y < u \\ 0 & \text{otherwise} \end{cases} \tag{1.4.11}$$

and $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1+n]) = 0$ for $n \neq 0$.

Note that one can also compute $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n])$ by considering the triangle $j_! j^{-1} P_y \to P_y \to \widetilde{P}_y \to j_! j^{-1} P_y[1]$ and using the induced isomorphisms

$$\operatorname{Hom}(j_! j^{-1} P_y, \widetilde{I}_u[n]) \xrightarrow{\simeq} \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n+1]) \tag{1.4.12}$$

The above calculations show that if we order each of the sets Y and U linearly extending the partial order induced by X and arrange the elements of $\mathcal{E}_Y$ in a sequence by first taking the elements of U and then taking those of Y, we get a strongly exceptional collection.

To prove that $\mathcal{E}_Y$ generates $\mathcal{D}^b(X)$, it is enough to show that every sheaf belongs to the triangulated subcategory generated by $\mathcal{E}_Y$. By (1.2.2), it is enough to verify this for $i_*\mathcal{F}'$ and $j_!\mathcal{F}''$ where $\mathcal{F}' \in Sh_Y$, $\mathcal{F}'' \in Sh_U$. The collection of sheaves $i^{-1}P_y$, being a complete set of indecomposable projectives of Sh_Y, generates $\mathcal{D}^b(Y)$. Similarly, the sheaves $j^{-1}I_u$ form a complete set of indecomposable injectives of Sh_U and generate $\mathcal{D}^b(U)$. Now the result follows by applying the triangulated functors $i_*, j_!$. □

1.4.3 The endomorphism algebras A_Y

Fix a poset X, and let $Y \subseteq X$ be a closed subset. Consider $T_Y = (\oplus_{y\in Y}\widetilde{P}_y) \oplus (\oplus_{u\in U}\widetilde{I}_u)[1]$ and let $A_Y = \operatorname{End}_{\mathcal{D}^b(X)} T_Y$. By Theorem 1.4.2 and Proposition 1.4.5, we have:

Corollary 1.4.6. $\mathcal{D}^b(X) \simeq \mathcal{D}^b(A_Y)$.

Proposition 1.4.7. *The algebra A_Y has as a k-basis the elements*

$$\left\{e_{yy'} \,:\, y \le y'\right\} \cup \left\{e_{u'u} \,:\, u' \le u\right\} \cup \left\{e_{uy} \,:\, y < u\right\}$$

where $y, y' \in Y$, $u', u \in U$. The multiplication is defined by

$$e_{yy'}e_{y'y''} = e_{yy''} \qquad\qquad e_{u''u'}e_{u'u} = e_{u''u}$$

$$e_{uy}e_{yy'} = \begin{cases} e_{uy'} & y' < u \\ 0 & \textit{otherwise} \end{cases} \qquad\qquad e_{u'u}e_{uy} = \begin{cases} e_{u'y} & y < u' \\ 0 & \textit{otherwise} \end{cases}$$

for $y \le y' \le y'' \in Y$, $u'' \le u' \le u \in U$ (all other products are zero).

Proof. For $y \le y' \in Y$, using (1.4.7), choose $e_{yy'} \in \operatorname{Hom}(\widetilde{P}_{y'}, \widetilde{P}_y)$ corresponding to $1 \in \widetilde{P}_y(y')$. In other words, the stalk of the morphism $e_{yy'}$ at y' is the identity map on k. Then $e_{yy'}e_{y'y''} = e_{yy''}$ for $y \le y' \le y'' \in Y$.

Similarly, for $u' \le u \in U$, using (1.4.8), choose $e_{u'u} \in \operatorname{Hom}(\widetilde{I}_u[1], \widetilde{I}_{u'}[1])$ corresponding to $1 \in \widetilde{I}_u(u')$. The stalk of $e_{u'u}$ at u' is the identity map on k and we have $e_{u''u'}e_{u'u} = e_{u''u}$ for all $u'' \le u' \le u \in U$.

Now consider $y \in Y$ and $u \in U$ such that $y < u$. Using the isomorphisms (1.4.10) and (1.4.12), we have

$$\begin{tikzcd}
\operatorname{Hom}(j_!j^{-1}P_y, \widetilde{I}_u) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) & \operatorname{Hom}(\widetilde{P}_y, i_*i^{-1}I_u) \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
(j_!j^{-1}P_y)(u) & & (i_*i^{-1}I_u)(y)
\end{tikzcd}$$

There are unique $e_{uy}, \widetilde{e}_{uy} \in \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1])$ such that the image of e_{uy} in $(j_!j^{-1}P_y)(u)$ equals 1 and the image of $\widetilde{e}_{uy}$ in $(i_*i^{-1}I_u)(y)$ is 1. The formula for $e_{u'u}e_{uy}$ where $u' \le u$ now follows by considering the composition

$$\begin{tikzcd}
\operatorname{Hom}(j_!j^{-1}P_y, \widetilde{I}_u) \arrow[r, "\simeq"] \arrow[d, "e_{u'u}\circ -"'] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) \ni e_{uy} \\
\operatorname{Hom}(j_!j^{-1}P_y, \widetilde{I}_{u'}) \arrow[r, "\simeq"] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_{u'}[1]) \ni e_{u'y}
\end{tikzcd}$$

The formula for $e_{uy}e_{yy'}$ would follow in a similar manner by considering the composition $- \circ e_{yy'}$ once we know that the scalar ratio between $\widetilde{e}_{uy}$ and e_{uy} is *independent* of u and y.

Indeed, replacing the objects $\widetilde{P}_y$ and $\widetilde{I}_u[1]$ by the quasi-isomorphic complexes $(j_! j^{-1} P_y \to P_y)[1]$ and $(I_u \to i_* i^{-1} I_u)[1]$, we see that $\mathrm{Hom}(\widetilde{P}_y, \widetilde{I}_u[1])$ equals the set of morphisms (λ, μ) between the two complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & j_! j^{-1} P_y \arrow[r] \arrow[d, "\lambda"] & P_y \arrow[r] \arrow[d, "\mu"] \arrow[dl, dashed, no head] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I_u \arrow[r] & i_* i^{-1} I_u \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

modulo homotopy. Note that $(\lambda, \mu) \sim (\lambda', \mu')$ if and only if $\lambda - \mu = \lambda' - \mu'$. The morphism e_{uy} corresponds to the pair $(1, 0)$ while $\widetilde{e}_{uy}$ corresponds to $(0, 1)$, hence $\widetilde{e}_{uy} = -e_{uy}$.

It is clear that the elements constructed above form a k-basis of A_Y and satisfy the required relations. $\square$

Example 1.4.8. Let X be the poset with Hasse diagram as in the left picture, and let $Y = \{1\}$. The algebra A_Y is shown in the right picture, as the path algebra of the quiver A_3 modulo the zero relation indicated by the dotted arrow (i.e. the product of $2 \to 3$ and $3 \to 1$ is zero).

$$\begin{tikzcd}
1 \arrow[dr] & & 2 \arrow[dl] \\
& 3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
2 \arrow[r] \arrow[rr, dotted, bend right, no head] & 3 \arrow[r] & 1
\end{tikzcd}$$

Lemma 1.4.9. *Let $X' = U \cup Y$ and define a binary relation $\leq'$ on X' by*

$$u' \leq' u \Leftrightarrow u' \leq u \quad y \leq' y' \Leftrightarrow y \leq y' \quad u <' y \Leftrightarrow y < u \tag{1.4.13}$$

for $u, u' \in U$, $y, y' \in Y$. Then $\leq'$ is a partial order if and only if the following condition holds:

$$\text{Whenever } y \leq y' \in Y,\ u' \leq u \in U \text{ and } y < u, \text{ we have that } y' < u'. \tag{$\star$}$$

When this condition holds, the endomorphism algebra A_Y is isomorphic to the incidence algebra of $(X', \leq')$.

Proof. The first part is clear from the requirement of transitivity of $\leq'$.

The condition $(\star)$ implies that $e_{u'u}e_{uy} = e_{u'y}$ and $e_{uy}e_{yy'} = e_{uy'}$ whenever $u' \leq u$, $y \leq y'$ and $y < u$, so that A_Y is the incidence algebra of $(X', \leq')$. $\square$

1.4.4 Lexicographic sums along bipartite graphs

Definition 1.4.10. Let S be a poset, and let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by the elements of S. The *lexicographic sum of $\mathfrak{X}$ along S*, denoted $\oplus_S \mathfrak{X}$, is the poset $(X, \leq)$ where $X = \coprod_{s \in S} X_s$ is the disjoint union of the X_s and for $x \in X_s$, $y \in X_t$ we have $x \leq y$ if either $s < t$ (in S) or $s = t$ and $x \leq y$ (in X_s).

Example 1.4.11. The usual *ordinal sum* $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ of n posets is the lexicographic sum of $\{X_1, \ldots, X_n\}$ along the chain $1 < 2 < \cdots < n$.

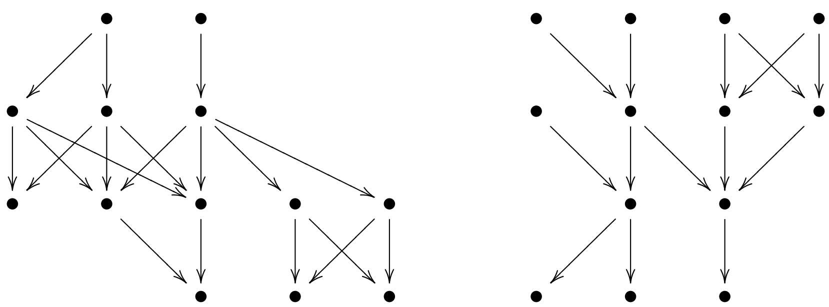
Definition 1.4.12. A poset S is called a *bipartite graph* if it can be written as a disjoint union of two nonempty subsets S_0 and S_1 such that $s < s'$ in S implies that $s \in S_0$ and $s' \in S_1$.

It follows from the definition that the posets S_0, S_1 are *anti-chains*, that is, no two distinct elements in S_0 (or S_1) are comparable.

Example 1.4.13. The left Hasse diagram represents a bipartite poset S. The right one is the Hasse diagram of its opposite S^{op}.

Let $\mathfrak{X} = \{X_1, X_2, X_3, X_4, X_5\}$ be the collection

The graphs shown below are the Hasse diagrams of $\oplus_S \mathfrak{X}$ (left) and $\oplus_{S^{op}} \mathfrak{X}$ (right).



Theorem 1.4.14. *If S is a bipartite graph and $\mathfrak{X} = \{X_s\}_{s \in S}$ is a collection of posets, then $\oplus_S \mathfrak{X} \sim \oplus_{S^{op}} \mathfrak{X}$.*

Proof. Let $S = S_0 \amalg S_1$ be a partition as in the definition of bipartite poset. Let $\mathfrak{X}_0 = \{X_s\}_{s \in S_0}$, $\mathfrak{X}_1 = \{X_s\}_{s \in S_1}$ and let $X = \oplus_S \mathfrak{X}$, $Y = \oplus_{S_0} \mathfrak{X}_0$, $U = \oplus_{S_1} \mathfrak{X}_1$. The sets Y and U can be viewed as disjoint subsets of X with $X = Y \cup U$. Moreover, since there are no relations $s_1 < s_0$ with $s_0 \in S_0$, $s_1 \in S_1$, there are no relations $u < y$ with $y \in Y$, $u \in U$, thus Y is closed and U is open in X. By Corollary 1.4.6, $\mathcal{D}^b(X) \simeq \mathcal{D}^b(A_Y)$ where A_Y is the endomorphism algebra of the direct sum of the strongly exceptional collection of Proposition 1.4.5.

We show that the condition $(\star)$ of Lemma 1.4.9 holds. Indeed, let $y \leq y' \in Y$, $u' \leq u \in U$. There exist $s_0, s_0' \in S_0$, $s_1, s_1' \in S_1$ such that $y \in X_{s_0}$, $y' \in X_{s_0'}$, $u \in X_{s_1}$ and $u' \in X_{s_1'}$. Now, $s_0' = s_0$ and $s_1' = s_1$ since $y \leq y'$, $u' \leq u$ and S_0, S_1 are anti-chains. If $y < u$, then $s_0 < s_1$, hence $y' < u'$ and ($\star$) is satisfied. Therefore A_Y is the incidence algebra of the poset X' defined in (1.4.13).

Since X' is a disjoint union of the posets U and Y with the original order inside each but with reverse order between them, it is easy to see that X' equals the lexicographic sum of $\mathfrak{X}$ along the opposite poset S^{op}. $\square$

Corollary 1.4.15. *Let X, Y be two posets. Then $X \oplus Y \sim Y \oplus X$*

Proof. Take S to be the chain $1 < 2$. $\square$

As special cases, we obtain the following two well known examples.

Example 1.4.16. The following two posets (represented by their Hasse diagrams) are derived equivalent.

The right poset is obtained from the left one by an APR tilt [3], see also [35, (III.2.14)].

Example 1.4.17. The two posets below are derived equivalent.

This is a special case of BGP reflection [9], turning a source into a sink (and vice versa).

Corollary 1.4.18. *Let S be a bipartite graph. Then $S \sim S^{op}$.*

Proof. Take in Theorem 1.4.14 each X_s to be a point. $\square$

Note that the last Corollary can also be deduced from [9] since Sh_S is the category of representations of a quiver without oriented cycles, namely the Hasse diagram of S, and S^{op} is obtained from S by reverting all the arrows.

1.4.5 Ordinal sums of three posets

The result of Corollary 1.4.15 raises the natural question whether the derived equivalence class of an ordinal sum of more than two posets does not depend on the order of the summands. The following proposition shows that it is enough to consider the case of three summands.

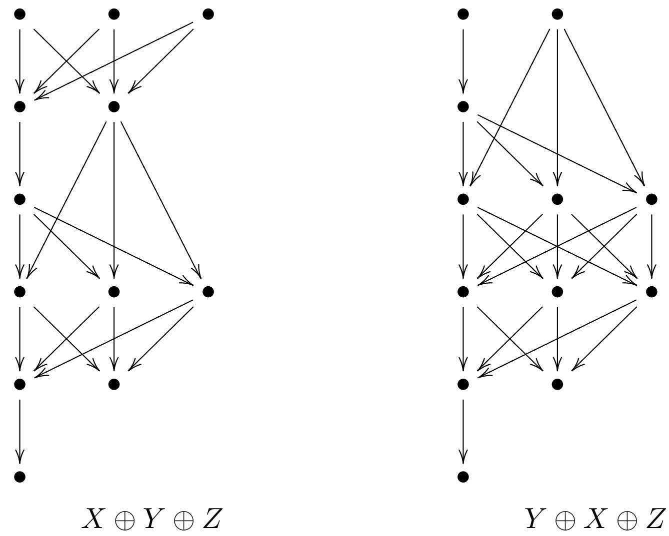


Figure 1.1: Two posets which are not derived equivalent despite their structure as ordinal sums of the same three posets in different orders.

Proposition 1.4.19. *Let $\mathcal{X}$ be a family of posets closed to taking ordinal sums. Assume that for any three posets $X, Y, Z \in \mathcal{X}$,*

$$X \oplus Y \oplus Z \sim Y \oplus X \oplus Z \tag{1.4.14}$$

Then for any $n \geq 1$, $\pi \in S_n$ and $X_1, \dots, X_n \in \mathcal{X}$,

$$X_{\pi(1)} \oplus \dots \oplus X_{\pi(n)} \sim X_1 \oplus \dots \oplus X_n$$

Proof. For $n = 1$ the claim is trivial and for $n = 2$ it is just Corollary 1.4.15. Let $n \geq 3$ and consider the set G_n of permutations in $\pi \in S_n$ such that $X_{\pi(1)} \oplus \dots \oplus X_{\pi(n)} \sim X_1 \oplus \dots \oplus X_n$ for all $X_1, \dots, X_n \in \mathcal{X}$. Then G_n is a subgroup of S_n, and the claim to be proved is that $G_n = S_n$.

Let $X_1, \dots, X_n \in \mathcal{X}$. Taking $X = X_1$ and $Y = X_2 \oplus \dots \oplus X_n$, we see by Corollary 1.4.15 that the cycle $(1\,2\,\dots\,n)$ belongs to G_n. Now take $X = X_1$, $Y = X_2$ and $Z = X_3 \oplus \dots \oplus X_n$. By (1.4.14), $Y \oplus X \oplus Z \sim X \oplus Y \oplus Z$, hence $(1\,2) \in G_n$. The claim now follows since $(1\,2)$ and $(1\,2\,\dots\,n)$ generate S_n. $\square$

We give a counterexample to show that (1.4.14) is false in general.

Example 1.4.20. Let

and let $Z = X \oplus Y$. Then the posets $X \oplus Y \oplus Z$ and $Y \oplus X \oplus Z$, depicted in Figure 1.1, are not derived equivalent since their Euler forms are not equivalent over $\mathbb{Z}$ (they are equivalent over $\mathbb{Q}$, though). This is shown using Corollary 1.3.15 with the prime $p = 11$.

Chapter 2

Derived Equivalences of Triangular Matrix Rings Arising from Extensions of Tilting Modules

A triangular matrix ring Λ is defined by a triplet (R, S, M) where R and S are rings and ${}_RM_S$ is an S-R-bimodule. In the main theorem of this chapter we show that if T_S is a tilting S-module, then under certain homological conditions on the S-module M_S, one can extend T_S to a tilting complex over Λ inducing a derived equivalence between Λ and another triangular matrix ring specified by (S', R, M'), where the ring S' and the R-S'-bimodule M' depend only on M and T_S, and S' is derived equivalent to S. Note that no conditions on the ring R are needed.

These conditions are satisfied when S is an Artin algebra of finite global dimension and M_S is finitely generated. In this case, $(S', R, M') = (S, R, DM)$ where D is the duality on the category of finitely generated S-modules. They are also satisfied when S is arbitrary, M_S has a finite projective resolution and $\mathrm{Ext}_S^n(M_S, S) = 0$ for all $n > 0$. In this case, $(S', R, M') = (S, R, \mathrm{Hom}_S(M, S))$.

2.1 Introduction

Triangular matrix rings and their homological properties have been widely studied, see for example [19, 27, 29, 67, 70]. The question of derived equivalences between different such rings was explored in the special case of one-point extensions of algebras [5]. Another aspect of this question was addressed by considering examples of triangular matrix algebras of a simple form, such as incidence algebras of posets, as we have done in Chapter 1. In this chapter we extend the results of the previous chapter to general triangular matrix rings.

A triangular matrix ring Λ is defined by a triplet (R, S, M) where R and S are rings and ${}_RM_S$ is an S-R-bimodule. The category of (right) Λ-modules can be viewed as a certain gluing of the categories of R-modules and S-modules, specified by four exact functors. This gluing naturally extends to the bounded derived categories. We note the similarity to the classical "recollement" situation, introduced by Beilinson, Bernstein and Deligne [7], involving six functors between three triangulated categories, originally inspired by considering derived categories of sheaves on topological spaces, and later studied for derived categories of modules by Cline, Parshall and Scott [21, 22], see also [55].

In Section 2.2 we show that triangular matrix rings arise naturally as endomorphism rings of certain rigid complexes over abelian categories that are glued from two simpler ones. Here, a complex $T \in \mathcal{D}^b(\mathcal{C})$ is *rigid* if $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{C})}(T, T[n]) = 0$ for all $n \neq 0$, where $\mathcal{D}^b(\mathcal{C})$ denotes the bounded derived category of an abelian category $\mathcal{C}$. Similarly, an object $T \in \mathcal{C}$ is *rigid* if it is rigid as a complex.

Indeed, when $\mathcal{C}$ is glued from the abelian categories $\mathcal{A}$ and $\mathcal{B}$, we construct, for any projective object of $\mathcal{A}$ and a rigid object of $\mathcal{B}$ satisfying some homological conditions, a new rigid complex in $\mathcal{D}^b(\mathcal{C})$ whose endomorphism ring is a triangular matrix ring.

In particular, as demonstrated in Section 2.3, this construction applies for comma categories defined by two abelian categories $\mathcal{A}$, $\mathcal{B}$ and a right exact functor $F : \mathcal{A} \to \mathcal{B}$. In this case, any projective P of $\mathcal{A}$ and a rigid object $T_{\mathcal{B}} \in \mathcal{B}$ satisfying $\mathrm{Ext}^n_{\mathcal{B}}(FP, T_{\mathcal{B}}) = 0$ for all $n > 0$, give rise to a rigid complex T over the comma category, whose endomorphism ring is a triangular matrix ring which can be explicitly computed in terms of P, $T_{\mathcal{B}}$ and FP, as

$$\mathrm{End}_{\mathcal{D}^b(\mathcal{C})}(T) \simeq \begin{pmatrix} \mathrm{End}_{\mathcal{B}}(T_{\mathcal{B}}) & \mathrm{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}}) \\ 0 & \mathrm{End}_{\mathcal{A}}(P) \end{pmatrix}.$$

In Section 2.4 we apply this construction for categories of modules over triangular matrix rings. For a ring Λ, denote by $\mathrm{Mod}\,\Lambda$ the category of all right Λ-modules, and by $\mathcal{D}^b(\Lambda)$ its bounded derived category. Recall that a complex $T \in \mathcal{D}^b(\Lambda)$ is a *tilting complex* if it is rigid and moreover, the smallest full triangulated subcategory of $\mathcal{D}^b(\Lambda)$ containing T and closed under forming direct summands, equals $\mathrm{per}\,\Lambda$, the full subcategory in $\mathcal{D}^b(\Lambda)$ of complexes quasi-isomorphic to *perfect complexes*, that is, bounded complexes of finitely generated projective Λ-modules. If, in addition, $H^n(T) = 0$ for all $n \neq 0$, we call T a *tilting module* and identify it with the module $H^0(T)$.

Two rings Λ and Λ' are *derived equivalent* if $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(\Lambda')$ are equivalent as triangulated categories. By Rickard's Morita theory for derived equivalence [73], this is equivalent to the existence of a tilting complex $T \in \mathcal{D}^b(\Lambda)$ such that $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) = \Lambda'$.

When Λ is a triangular matrix ring defined by two rings R, S and a bimodule ${}_RM_S$, the category $\mathrm{Mod}\,\Lambda$ is the comma category of $\mathrm{Mod}\,R$, $\mathrm{Mod}\,S$ with respect to the functor $- \otimes M : \mathrm{Mod}\,R \to \mathrm{Mod}\,S$. In this case, starting with the projective R-module R and a tilting S-module T_S, the complex T constructed in Section 2.3 is not only rigid, but also a tilting complex, hence we deduce a derived equivalence between Λ and the triangular matrix ring $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, as expressed in the theorem below.

Theorem. *Let R, S be rings and T_S a tilting S-module. Let ${}_RM_S$ be an S-R-bimodule such that as an S-module, $M_S \in \mathrm{per}\,S$ and $\mathrm{Ext}^n_S(M_S, T_S) = 0$ for all $n > 0$. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \widetilde{\Lambda} = \begin{pmatrix} \mathrm{End}_S(T_S) & \mathrm{Hom}_S(M, T_S) \\ 0 & R \end{pmatrix}$$

are derived equivalent.

We note that the assumption that T_S is a tilting module implies that the rings S and $\mathrm{End}_S(T_S)$ are derived equivalent, hence the triangular matrix ring specified by the triplet (R, S, M) is derived equivalent to a one specified by (S', R, M') where S' is derived equivalent to S. We note also that no conditions on the ring R (or on M as a left R-module) are necessary.

The above theorem has two interesting corollaries, corresponding to the cases where T_S is injective or projective.

For the first corollary, let S be an Artin algebra, and let $D : \mathrm{mod}\, S \to \mathrm{mod}\, S^{op}$ denote the duality. When S has finite global dimension, one can take T_S to be the module DS which is then an injective tilting module.

Corollary. *Let R be a ring, S an Artin algebra with* $\mathrm{gl.dim}\, S < \infty$ *and ${}_R M_S$ an S-R-bimodule which is finitely generated as an S-module. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \widetilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}$$

are derived equivalent, where D is the duality on $\mathrm{mod}\, S$.

The ring $\widetilde{\Lambda}$ depends only on R, S and M, hence it may be considered as a derived equivalent *mate* of Λ.

The second corollary of the above theorem is obtained by taking the tilting S-module to be S.

Corollary. *Let R, S be rings and ${}_R M_S$ an S-R-bimodule such that as an S-module, $M_S \in$* $\mathrm{per}\, S$ *and* $\mathrm{Ext}^n_S(M_S, S) = 0$ *for all $n > 0$. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \widetilde{\Lambda} = \begin{pmatrix} S & \mathrm{Hom}_S(M, S) \\ 0 & R \end{pmatrix}$$

are derived equivalent.

This corollary applies to the following situations, listed in descending order of generality; the ring S is *self-injective* (that is, S is injective as a module over itself) and M_S is finitely generated projective; the ring S is semi-simple and M_S is finitely generated; the ring S is a division ring and M is finite dimensional over S. The latter case implies that a triangular matrix ring which is a one-point extension is derived equivalent to the corresponding one-point co-extension.

In Section 2.5 we conclude with three remarks concerning the specific case of finite dimensional triangular matrix algebras over a field.

First, in the case when R and S are finite dimensional algebras over a field and both have finite global dimension, an alternative approach to show the derived equivalence of the triangular matrix algebras specified by (R, S, M) and its mate (S, R, DM) is to prove that the corresponding repetitive algebras are isomorphic and then use Happel's Theorem [35, (II,4.9)]. However, in the case that only one of R and S has finite global dimension, Happel's Theorem cannot be used, but the derived equivalence still holds. Moreover, as we show in Example 2.5.3, there are cases when none of R and S have finite global dimension and the corresponding algebras are not derived equivalent, despite the isomorphism between their repetitive algebras.

Second, one can directly prove, using only matrix calculations, that when at least one of R and S has finite global dimension, the Cartan matrices of the triangular matrix algebra (R, S, M) and its mate are equivalent over $\mathbb{Z}$, a result which is a direct consequence of Theorem 2.4.9.

Third, we note that in contrast to triangular matrix algebras, in the more general situation of trivial extensions of algebras, the mates $A \ltimes M$ and $A \ltimes DM$ for an algebra A and a bimodule ${}_AM_A$, are typically not derived equivalent.

2.2 The gluing construction

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three abelian categories. Similarly to [7, (1.4)], we view $\mathcal{C}$ as glued from $\mathcal{A}$ and $\mathcal{B}$ if there exist certain functors $i^{-1}, i_*, j^{-1}, j_!$ as described below. Note, however, that we start by working at the level of the abelian categories and not their derived categories. In addition, the requirement in [7] of the existence of the additional adjoint functors $i^!, j_*$ is replaced by the orthogonality condition (2.2.6).

Definition 2.2.1. A quadruple of additive functors $i^{-1}, i_*, j^{-1}, j_!$ as in the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "i_*"] & \mathcal{C} \arrow[l, shift left, "i^{-1}"] \arrow[r, shift left, "j^{-1}"] & \mathcal{B} \arrow[l, shift left, "j_!"] \end{tikzcd}$$

is called *gluing data* if it satisfies the four properties below.

Adjunction

i^{-1} is a left adjoint of i_* and j^{-1} is a right adjoint of $j_!$. That is, there exist bi-functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(i^{-1}C, A) \simeq \mathrm{Hom}_{\mathcal{C}}(C, i_*A) \tag{2.2.1}$$

$$\mathrm{Hom}_{\mathcal{B}}(B, j^{-1}C) \simeq \mathrm{Hom}_{\mathcal{C}}(j_!B, C) \tag{2.2.2}$$

for all $A \in \mathrm{ob}\,\mathcal{A}$, $B \in \mathrm{ob}\,\mathcal{B}$, $C \in \mathrm{ob}\,\mathcal{C}$.

Exactness

The functors $i^{-1}, i_*, j^{-1}, j_!$ are exact.

Note that by the adjunctions above, we automatically get that the functors i_*, j^{-1} are left exact while $i^{-1}, j_!$ are right exact.

Extension

For every $C \in \mathrm{ob}\,\mathcal{C}$, the adjunction morphisms $j_!j^{-1}C \to C$ and $C \to i_*i^{-1}C$ give rise to a short exact sequence

$$0 \to j_!j^{-1}C \to C \to i_*i^{-1}C \to 0 \tag{2.2.3}$$

Orthogonality

$$i^{-1}j_! = 0 \qquad\qquad j^{-1}i_* = 0 \tag{2.2.4}$$

$$i^{-1}i_* \simeq \mathrm{Id}_{\mathcal{A}} \qquad\qquad j^{-1}j_! \simeq \mathrm{Id}_{\mathcal{B}} \tag{2.2.5}$$

and in addition,

$$\mathrm{Hom}_{\mathcal{C}}(i_*A, j_!B) = 0 \qquad\qquad \text{for all } A \in \mathrm{ob}\,\mathcal{A},\, B \in \mathrm{ob}\,\mathcal{B} \tag{2.2.6}$$

Using the adjunctions (2.2.1) and (2.2.2), the assumptions of (2.2.4),(2.2.5) can be rephrased as follows. First, the two conditions of (2.2.4) are equivalent to each other and each is equivalent to the condition

$$\mathrm{Hom}_{\mathcal{C}}(j_!B, i_*A) = 0 \qquad\qquad \text{for all } A \in \mathrm{ob}\,\mathcal{A},\, B \in \mathrm{ob}\,\mathcal{B} \tag{2.2.7}$$

Similarly, the conditions in (2.2.5) are equivalent to the requirement that i_* and $j_!$ are fully faithful functors, so that one can think of $\mathcal{A}$ and $\mathcal{B}$ as embedded in $\mathcal{C}$. Moreover, from (2.2.3) and (2.2.7) we see that $(\mathcal{B}, \mathcal{A})$ is a *torsion pair* [41, (I.2)] in $\mathcal{C}$.

Observe also that (2.2.6) could be replaced with the assumption that the functor $(i^{-1}, j^{-1}) : \mathcal{C} \to \mathcal{A} \times \mathcal{B}$ is faithful. Indeed, one implication follows from (2.2.4) and (2.2.5), and the other follows using (2.2.3).

From now on assume that $(i^{-1}, i_*, j^{-1}, j_!)$ form a gluing data.

Lemma 2.2.2. *If P is a projective object of $\mathcal{C}$, then $i^{-1}P$ is projective in $\mathcal{A}$. Similarly, if I an injective object of $\mathcal{C}$, then $j^{-1}I$ is injective in $\mathcal{B}$.*

Proof. A functor which is a left adjoint to an exact functor preserves projectives, while a right adjoint to an exact functor preserves injectives [29, Corollary 1.6]. $\square$

The exact functors $i^{-1}, i_*, j^{-1}, j_!$ give rise to triangulated functors between the corresponding bounded derived categories. We use the same notation for these derived functors:

$$\begin{tikzcd} \mathcal{D}^b(\mathcal{A}) \arrow[r, shift left, "i_*"] & \mathcal{D}^b(\mathcal{C}) \arrow[l, shift left, "i^{-1}"] \arrow[r, shift left, "j^{-1}"] & \mathcal{D}^b(\mathcal{B}) \arrow[l, shift left, "j_!"] \end{tikzcd}$$

Note that adjunctions and orthogonality relations analogous to (2.2.1), (2.2.2), (2.2.4), (2.2.5) (but not (2.2.6)) hold also for the derived functors. In particular, i_* and $j_!$ are fully faithful.

Definition 2.2.3. An object T in an abelian category $\mathcal{A}$ is called *rigid* if $\mathrm{Ext}^n_{\mathcal{A}}(T, T) = 0$ for all $n > 0$.

Proposition 2.2.4. *Let P be a projective object of $\mathcal{C}$ and $T_{\mathcal{B}}$ be a rigid object of $\mathcal{B}$ such that $\mathrm{Ext}^n_{\mathcal{B}}(j^{-1}P, T_{\mathcal{B}}) = 0$ for all $n > 0$. Consider the complex*

$$T = i_*i^{-1}P \oplus j_!T_{\mathcal{B}}[1]$$

in $\mathcal{D}^b(\mathcal{C})$. Then $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(T, T[n]) = 0$ for $n \neq 0$ and

$$\operatorname{End}_{\mathcal{D}^b(\mathcal{C})}(T) \simeq \begin{pmatrix} \operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}}) & \operatorname{Ext}^1_{\mathcal{C}}(i_* i^{-1} P, j_! T_{\mathcal{B}}) \\ 0 & \operatorname{End}_{\mathcal{A}}(i^{-1} P) \end{pmatrix}$$

is a triangular matrix ring.

Proof. Since T has two summands, the space $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(T, T[n])$ decomposes into four parts, which we now consider.

Since i_* is fully faithful and $i^{-1}P$ is projective,

$$\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(i_* i^{-1} P, i_* i^{-1} P[n]) \simeq \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(i^{-1} P, i^{-1} P[n]) \tag{2.2.8}$$

vanishes for $n \neq 0$. Similarly, since $j_!$ is fully faithful and $T_{\mathcal{B}}$ is rigid,

$$\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(j_! T_{\mathcal{B}}, j_! T_{\mathcal{B}}[n]) \simeq \operatorname{Hom}_{\mathcal{D}^b(\mathcal{B})}(T_{\mathcal{B}}, T_{\mathcal{B}}[n])$$

vanishes for $n \neq 0$. Moreover, by orthogonality,

$$\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(j_! T_{\mathcal{B}}, i_* i^{-1} P[n]) = 0$$

for all $n \in \mathbb{Z}$.

It remains to consider $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(i_* i^{-1} P, j_! T_{\mathcal{B}}[n])$ and to prove that it vanishes for $n \neq 1$. Using (2.2.3), we obtain a short exact sequence $0 \to j_! j^{-1} P \to P \to i_* i^{-1} P \to 0$ that induces a long exact sequence, a fragment of which is shown below:

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}^b(\mathcal{B})}(j^{-1} P, T_{\mathcal{B}}[n-1]) \simeq \operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(j_! j^{-1} P[1], j_! T_{\mathcal{B}}[n]) \to \\ \operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(i_* i^{-1} P, j_! T_{\mathcal{B}}[n]) \to \operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(P, j_! T_{\mathcal{B}}[n]). \end{aligned} \tag{2.2.9}$$

Now observe that the right term vanishes for $n \neq 0$ since P is projective, and the left term of (2.2.9) vanishes for $n \neq 1$ by our assumption on the vanishing of $\operatorname{Ext}^\bullet_{\mathcal{B}}(j^{-1} P, T_{\mathcal{B}})$. Therefore

$$\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(i_* i^{-1} P, j_! T_{\mathcal{B}}[n]) = 0$$

for $n \neq 0, 1$. This holds also for $n = 0$ by the assumption (2.2.6).

To complete the proof, note that $\operatorname{Ext}^1_{\mathcal{C}}(i_* i^{-1} P, j_! T_{\mathcal{B}})$ has a natural structure of an $\operatorname{End}_{\mathcal{A}}(i^{-1} P)$-$\operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}})$-bimodule via the identifications

$$\operatorname{End}_{\mathcal{A}}(i^{-1} P) \simeq \operatorname{End}_{\mathcal{C}}(i_* i^{-1} P) \qquad \operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}}) \simeq \operatorname{End}_{\mathcal{C}}(j_! T_{\mathcal{B}})$$

$\square$

Remark 2.2.5. The assumptions in the proposition are always satisfied when P is a projective object of $\mathcal{C}$ and $T_{\mathcal{B}}$ is any injective object of $\mathcal{B}$.

Remark 2.2.6. One can formulate an analogous statement for a rigid object $T_{\mathcal{A}}$ of $\mathcal{A}$ and an injective object I of $\mathcal{C}$.

2.3 Gluing in comma categories

Let $\mathcal{A}, \mathcal{B}$ be categories and $F : \mathcal{A} \to \mathcal{B}$ a functor. The *comma category* with respect to the pair of functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{\mathrm{Id}} \mathcal{B}$ [65, II.6], denoted by $(F \downarrow \mathrm{Id})$, is the category $\mathcal{C}$ whose objects are triples (A, B, f) where $A \in \operatorname{ob}\mathcal{A}$, $B \in \operatorname{ob}\mathcal{B}$ and $f : FA \to B$ is a morphism (in $\mathcal{B}$). The morphisms between objects (A, B, f) and (A', B', f') are all pairs of morphisms $\alpha : A \to A'$, $\beta : B \to B'$ such that the square

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "F\alpha"] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "f'"] & B' \end{tikzcd} \tag{2.3.1}$$

commutes.

Assume in addition that $\mathcal{A}$, $\mathcal{B}$ are abelian and that $F : \mathcal{A} \to \mathcal{B}$ is an additive, right exact functor. In this case, the comma category $\mathcal{C}$ is abelian [29]. Consider the functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "i_*"] & \mathcal{C} \arrow[l, shift left, "i^{-1}"] \arrow[r, shift left, "j^{-1}"] & \mathcal{B} \arrow[l, shift left, "j_!"] \end{tikzcd}$$

defined by

$$\begin{array}{llll} i_*(A) = (A, 0, 0) & i_*(\alpha) = (\alpha, 0) & i^{-1}(A, B, f) = A & i^{-1}(\alpha, \beta) = \alpha \\ j_!(B) = (0, B, 0) & j_!(\beta) = (0, \beta) & j^{-1}(A, B, f) = B & j^{-1}(\alpha, \beta) = \beta \end{array}$$

for objects $A \in \operatorname{ob}\mathcal{A}$, $B \in \operatorname{ob}\mathcal{B}$ and morphisms α, β.

Lemma 2.3.1. *The quadruple $(i^{-1}, i_*, j^{-1}, j_!)$ is a gluing data.*

Proof. We need to verify the four properties of gluing data. The adjunction follows by the commutativity of the diagrams

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "F\alpha"] & B \arrow[d] \\ FA' \arrow[r] & 0 \end{tikzcd} \qquad\qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "f'"] & B' \end{tikzcd}$$

for $\alpha : A \to A'$ and $\beta : B \to B'$.

For exactness, note that kernels and images in $\mathcal{C}$ can be computed componentwise, that is, if $(\alpha, \beta) : (A, B, f) \to (A', B', f')$ is a morphism in $\mathcal{C}$, then $\ker(\alpha, \beta) = (\ker\alpha, \ker\beta, f|_{F(\ker\alpha)})$ and similarly for the image. The extension condition follows from

$$0 \to (0, B, 0) \xrightarrow{(0, 1_B)} (A, B, f) \xrightarrow{(1_A, 0)} (A, 0, 0) \to 0$$

and orthogonality is straightforward. $\square$

One can use the special structure of the comma category $\mathcal{C}$ to define another pair of functors. Let $i_! : \mathcal{A} \to \mathcal{C}$ and $j^\natural : \mathcal{C} \to \mathcal{B}$ be the functors defined by

$$\begin{aligned} i_!(A) &= (A, FA, 1_{FA}) & \qquad i_!(\alpha) &= (\alpha, F\alpha) \\ j^\natural(A, B, f) &= \operatorname{coker} f & \qquad j^\natural(\alpha, \beta) &= \bar{\beta} \end{aligned}$$

where $\bar{\beta} : \operatorname{coker} f \to \operatorname{coker} f'$ is induced from β.

Lemma 2.3.2. *$i_!$ is a left adjoint of i^{-1}, $j^\natural$ is a left adjoint of $j_!$, and*

$$i^{-1} i_! \simeq \mathrm{Id}_{\mathcal{A}} \qquad j^{-1} i_! = F \qquad j^\natural i_! = j^\natural i_* = 0 \qquad j^\natural j_! \simeq \mathrm{Id}_{\mathcal{B}}$$

Proof. The adjunctions follow by considering the commutative diagrams

$$\begin{tikzcd} FA \arrow[r, "1_{FA}"] \arrow[d, "F\alpha"] & FA \arrow[d, "\beta = f' \circ F\alpha"] \\ FA' \arrow[r, "f'"] & B' \end{tikzcd} \qquad\qquad \begin{tikzcd} FA \arrow[r, "f"] \arrow[d] & B \arrow[d, "\beta"] \\ 0 \arrow[r] & B' \end{tikzcd}$$

and noting that the commutativity of the right diagram implies that β factors uniquely through $\operatorname{coker} f$. The other relations are straightforward. $\square$

Remark 2.3.3. The diagram

$$\begin{tikzcd} \mathcal{A} \times \mathcal{B} \arrow[r, shift left, "(i_!, j_!)"] & \mathcal{C} \arrow[l, shift left, "(i^{-1}, j^{-1})"] \arrow[r, shift left, "(i^{-1}, j^\natural)"] & \mathcal{A} \times \mathcal{B} \arrow[l, shift left, "(i_*, j_!)"] \end{tikzcd}$$

is a special case of the one in [29, p. 7], viewing $\mathcal{C}$ as a trivial extension of $\mathcal{A} \times \mathcal{B}$.

Proposition 2.3.4. *Let P be a projective object of $\mathcal{A}$ and $T_{\mathcal{B}}$ be a rigid object of $\mathcal{B}$ such that $\operatorname{Ext}^n_{\mathcal{B}}(FP, T_{\mathcal{B}}) = 0$ for all $n > 0$. Assume that $FP \in \operatorname{ob}(\mathcal{B})$ has a projective resolution in $\mathcal{B}$ and consider $T = (P, 0, 0) \oplus (0, T_{\mathcal{B}}, 0)[1] \in \mathcal{D}^b(\mathcal{C})$. Then $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{C})}(T, T[n]) = 0$ for $n \neq 0$ and*

$$\operatorname{End}_{\mathcal{D}^b(\mathcal{C})}(T) \simeq \begin{pmatrix} \operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}}) & \operatorname{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}}) \\ 0 & \operatorname{End}_{\mathcal{A}}(P) \end{pmatrix},$$

where the bimodule structure on $\operatorname{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}})$ is given by left composition with $\operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}})$ and right composition with $\operatorname{End}_{\mathcal{B}}(FP)$ through the natural map $\operatorname{End}_{\mathcal{A}}(P) \to \operatorname{End}_{\mathcal{B}}(FP)$.

Proof. Since $i_!$ is a left adjoint of an exact functor, it takes projective objects of $\mathcal{A}$ to projective objects of $\mathcal{C}$. Hence $i_!P = (P, FP, 1_{FP})$ is projective and we can apply Proposition 2.2.4 for $i_!P$ and $T_{\mathcal{B}}$. As $(P, 0, 0) = i_* i^{-1} i_! P$ and $(0, T_{\mathcal{B}}, 0) = j_! T_{\mathcal{B}}$, we only need to show the isomorphism

$$\operatorname{Ext}^1_{\mathcal{C}}\left((P, 0, 0), (0, T_{\mathcal{B}}, 0)\right) \simeq \operatorname{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}})$$

as $\operatorname{End}_{\mathcal{A}}(P)$-$\operatorname{End}_{\mathcal{B}}(T_{\mathcal{B}})$-bimodules.

Indeed, let

$$\cdots \to Q^2 \to Q^1 \to FP \to 0 \tag{2.3.2}$$

be a projective resolution of FP. Then $(P, 0, 0)$ is quasi-isomorphic to the complex

$$\cdots \to j_! Q^2 \to j_! Q^1 \to i_! P \to 0 \to \dots$$

whose terms are projective since $j_!$ is a left adjoint of an exact functor. Therefore $\mathrm{Ext}^1_{\mathcal{C}}((P,0,0),(0,T_{\mathcal{B}},0))$ can be identified with the morphisms, up to homotopy, between the complexes

$$\cdots \longrightarrow j_! Q^2 \longrightarrow j_! Q^1 \longrightarrow i_! P \longrightarrow 0 \longrightarrow \cdots \tag{2.3.3}$$

$$\cdots \longrightarrow 0 \longrightarrow j_! T_{\mathcal{B}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

By Lemma 2.3.2, $\mathrm{Hom}_{\mathcal{C}}(i_! P, j_! T_{\mathcal{B}}) = \mathrm{Hom}_{\mathcal{A}}(P, i^{-1} j_! T_{\mathcal{B}}) = 0$, thus any homotopy between these complexes vanishes, and the morphism space equals $\ker(\mathrm{Hom}_{\mathcal{C}}(j_! Q^1, j_! T_{\mathcal{B}}) \to \mathrm{Hom}_{\mathcal{C}}(j_! Q^2, j_! T_{\mathcal{B}}))$. Using the fact that $j_!$ is fully faithful and applying the functor $\mathrm{Hom}_{\mathcal{B}}(-, T_{\mathcal{B}})$ on the exact sequence (2.3.2), we get that the morphism space equals $\mathrm{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}})$, as desired.

Under this identification, the left action of $\mathrm{End}_{\mathcal{B}}(T_{\mathcal{B}}) \simeq \mathrm{End}_{\mathcal{C}}(j_! T_{\mathcal{B}})$ is given by left composition. As for the right action of $\mathrm{End}_{\mathcal{A}}(P)$, observe that any $\alpha \in \mathrm{End}_{\mathcal{A}}(P)$ extends uniquely to an endomorphism in the homotopy category

$$\begin{array}{ccc} (0, FP, 0) & \xrightarrow{(0,1)} & (P, FP, 1_{FP}) \\ \big\downarrow{\scriptstyle (0,F\alpha)} & & \big\downarrow{\scriptstyle (\alpha,F\alpha)} \\ (0, FP, 0) & \xrightarrow{(0,1)} & (P, FP, 1_{FP}) \end{array}$$

which determines a unique endomorphism, in the homotopy category, of the top complex of (2.3.3). $\square$

Remark 2.3.5. When the functor $F : \mathcal{A} \to \mathcal{B}$ admits a right adjoint $G : \mathcal{B} \to \mathcal{A}$, the comma category $(F \downarrow \mathrm{Id})$ is equivalent to the comma category $(\mathrm{Id} \downarrow G)$ corresponding to the pair $\mathcal{A} \xrightarrow{\mathrm{Id}} \mathcal{A} \xleftarrow{G} \mathcal{B}$. In this case, one can define also a right adjoint $i^!$ of i_* and a right adjoint j_* of j^{-1}, and we end up with the eight functors $(i_!, i^{-1}, i_*, i^!)$ and $(j^\natural, j_!, j^{-1}, j_*)$. The bimodule $\mathrm{Hom}_{\mathcal{B}}(FP, T_{\mathcal{B}})$ in Proposition 2.3.4 can then be identified with $\mathrm{Hom}_{\mathcal{A}}(P, GT_{\mathcal{B}})$.

2.4 Application to triangular matrix rings

2.4.1 Triangular matrix rings

Let R and S be rings, and let ${}_R M_S$ be an S-R-bimodule. Let Λ be the *triangular matrix ring*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, s \in S, m \in M \right\} \tag{2.4.1}$$

where the ring structure is induced by the ordinary matrix operations.

For a ring R, denote the category of right R-modules by $\operatorname{Mod} R$. The functor $- \otimes M : \operatorname{Mod} R \to \operatorname{Mod} S$ is additive and right exact, hence the corresponding comma category $(- \otimes M \downarrow \operatorname{Id}_{\operatorname{Mod} R})$ is abelian.

Lemma 2.4.1 ([4, III.2])**.** *The category* $\operatorname{Mod} \Lambda$ *is equivalent to the comma category* $(- \otimes M \downarrow \operatorname{Id}_{\operatorname{Mod} R})$.

Proof. One verifies that by sending a triple $(X_R, Y_S, f : X \otimes M \to Y)$ to the Λ-module $X \oplus Y$ defined by

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} xr & f(x \otimes m) + ys \end{pmatrix} \tag{2.4.2}$$

and sending a morphism $(\alpha, \beta) : (X, Y, f) \to (X', Y', f')$ to $\alpha \oplus \beta : X \oplus Y \to X' \oplus Y'$, we get a functor $(- \otimes M \downarrow \operatorname{Id}_{\operatorname{Mod} R}) \to \operatorname{Mod} \Lambda$ which is an equivalence of categories. $\square$

Corollary 2.4.2. *There exists gluing data*

$$\operatorname{Mod} R \underset{i^{-1}}{\overset{i_*}{\rightleftarrows}} \operatorname{Mod} \Lambda \underset{j_!}{\overset{j^{-1}}{\rightleftarrows}} \operatorname{Mod} S.$$

The functors occurring in Corollary 2.4.2 can be described explicitly. Let

$$e_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (2.4.2), observe that for a Λ-module Z_Λ,

$$(i^{-1} Z)_R = Z e_R \qquad (j^{-1} Z)_S = Z e_S \tag{2.4.3}$$

where r acts on $i^{-1}Z$ via $\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and s acts on $j^{-1}Z$ via $\left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right)$. The morphism $(i^{-1}Z) \otimes M \to j^{-1}Z$ is obtained by considering the actions of $\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right)$, $m \in M$, and the map $Z \mapsto (i^{-1}Z, j^{-1}Z, (i^{-1}Z) \otimes M \to j^{-1}Z)$ defines a functor which is an inverse to the equivalence of categories constructed in the proof of Lemma 2.4.1.

Conversely, for an R-module X_R and S-module Y_S, we have $(i_* X)_\Lambda = X$ and $(j_! Y)_\Lambda = Y$ where $\left(\begin{smallmatrix} r & m \\ 0 & s \end{smallmatrix}\right)$ acts on X via r and on Y via s.

Lemma 2.4.3. *The image of* Λ_Λ *in the comma category equals* $(R, M, 1_M) \oplus (0, S, 0)$.

Proof. Use (2.4.3) and

$$\Lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & M \\ 0 & S \end{pmatrix}.$$

$\square$

Remark 2.4.4. Since $- \otimes M$ admits a right adjoint $\mathrm{Hom}(M, -)$, we are in the situation of Remark 2.3.5 and there are eight functors $(i_!, i^{-1}, i_*, i^!)$ and $(j^\natural, j_!, j^{-1}, j_*)$. For the convenience of the reader, we now describe them as standard functors $\otimes$ and Hom involving idempotents, see also [22, Section 2] and [69, Proposition 2.17].

If A is a ring and $e \in A$ is an idempotent, the functor

$$\mathrm{Hom}_A(eA, -) = - \otimes_A Ae : \mathrm{Mod}\, A \to \mathrm{Mod}\, eAe$$

admits a left adjoint $- \otimes_{eAe} eA$ and a right adjoint $\mathrm{Hom}_{eAe}(Ae, -)$. By taking $A = \Lambda$ and $e = e_R$ we get the three functors $(i_!, i^{-1}, i_*)$. Similarly, $e = e_S$ gives $(j_!, j^{-1}, j_*)$.

In addition, the natural inclusion functor

$$\mathrm{Hom}_{A/AeA}(A/AeA, -) = - \otimes_{A/AeA} A/AeA : \mathrm{Mod}\, A/AeA \to \mathrm{Mod}\, A$$

admits a left adjoint $- \otimes_A A/AeA$ and a right adjoint $\mathrm{Hom}_A(A/AeA, -)$. By taking $A = \Lambda$ and $e = e_R$, observing that $e_S \Lambda e_R = 0$, we get the three functors $(j^\natural, j_!, j^{-1})$. For $e = e_S$ we get $(i^{-1}, i_*, i^!)$.

2.4.2 The main theorem

For a ring Λ, denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of $\mathrm{Mod}\,\Lambda$, and by $\mathrm{per}\,\Lambda$ its full subcategory of complexes quasi-isomorphic to *perfect complexes*, that is, bounded complexes of finitely generated projective Λ-modules.

For a complex $T \in \mathcal{D}^b(\Lambda)$, denote by $\langle \mathrm{add}\, T \rangle$ the smallest full triangulated subcategory of $\mathcal{D}^b(\Lambda)$ containing T and closed under forming direct summands. Recall that T is a *tilting complex* if $\langle \mathrm{add}\, T \rangle = \mathrm{per}\,\Lambda$ and $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[n]) = 0$ for all integers $n \neq 0$. If, moreover, $H^n(T) = 0$ for all $n \neq 0$, we call T a *tilting module* and identify it with the module $H^0(T)$.

Theorem 2.4.5. *Let R, S be rings and T_S a tilting S-module. Let ${}_R M_S$ be an S-R-bimodule such that as an S-module, $M_S \in \mathrm{per}\, S$ and $\mathrm{Ext}^n_S(M_S, T_S) = 0$ for all $n > 0$. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \widetilde{\Lambda} = \begin{pmatrix} \mathrm{End}_S(T_S) & \mathrm{Hom}_S(M, T_S) \\ 0 & R \end{pmatrix}$$

are derived equivalent.

Proof. For simplicity, we shall identify $\mathrm{Mod}\,\Lambda$ with the corresponding comma category. We will show that $T = (R, 0, 0) \oplus (0, T_S, 0)[1]$ is a tilting complex in $\mathcal{D}^b(\Lambda)$ whose endomorphism ring equals $\widetilde{\Lambda}$.

Applying Proposition 2.3.4 for the projective R-module R and the rigid S-module T_S, noting that $FR = M_S$ and $\mathrm{Ext}^n_S(M_S, T_S) = 0$ for $n > 0$, we see that $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[n]) = 0$ for all $n \neq 0$ and moreover $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \simeq \widetilde{\Lambda}$.

It remains to show that $\langle \mathrm{add}\, T \rangle = \mathrm{per}\,\Lambda$. First, we show that $T \in \mathrm{per}\,\Lambda$. Observe that $j_!(\mathrm{per}\, S) \subseteq \mathrm{per}\,\Lambda$, since $j_!$ is an exact functor which takes projectives to projectives and $j_! S = (0, S, 0)$ is a direct summand of Λ. Hence in the short exact sequence

$$0 \to (0, M, 0) \to (R, M, 1_M) \to (R, 0, 0) \to 0, \tag{2.4.4}$$

we have that $(0, M, 0) \in \operatorname{per} \Lambda$ by the assumption that $M_S \in \operatorname{per} S$, and $(R, M, 1_M) \in \operatorname{per} \Lambda$ as a direct summand of Λ. Therefore $(R, 0, 0) \in \operatorname{per} \Lambda$. In addition, $(0, T_S, 0) \in \operatorname{per} \Lambda$ by the assumption $T_S \in \operatorname{per} S$, hence T is isomorphic in $\mathcal{D}^b(\Lambda)$ to a perfect complex.

Second, in order to prove that $\langle \operatorname{add} T \rangle = \operatorname{per} \Lambda$ it is enough to show that $\Lambda \in \langle \operatorname{add} T \rangle$. Indeed, since $(0, T_S, 0)[1]$ is a summand of T, by the exactness of $j_!$ and our assumption that $\langle \operatorname{add} T_S \rangle = \operatorname{per} S$, we have that $(0, S, 0) \in \langle \operatorname{add} T \rangle$ and $(0, M, 0) \in \langle \operatorname{add} T \rangle$. Since $(R, 0, 0)$ is a summand of T, by invoking again the short exact sequence (2.4.4) we see that $(R, M, 1_M) \in \langle \operatorname{add} T \rangle$, hence $\Lambda \in \langle \operatorname{add} T \rangle$.

Therefore T is a tilting complex in $\mathcal{D}^b(\Lambda)$, and by [73] (see also [51, (1.4)]), the rings Λ and $\widetilde{\Lambda} \simeq \operatorname{End}_{\mathcal{D}^b(\Lambda)}(T)$ are derived equivalent. □

Remark 2.4.6. The assumption that T_S is a tilting module implies that the rings S and $\operatorname{End}_S(T_S)$ are derived equivalent.

Remark 2.4.7. When the tilting module T_S is also injective, it is enough to assume that $M_S \in \operatorname{per} S$.

2.4.3 Applications

Let S be an Artin algebra over an Artinian commutative ring k, and let $\operatorname{mod} S$ be the category of finitely generated right S-modules. Let $D : \operatorname{Mod} S \to \operatorname{Mod} S^{op}$ be the functor defined by $D = \operatorname{Hom}_k(-, J)$, where J is an injective envelope of the direct sum of all the non-isomorphic simple modules of k. Recall that D restricts to a duality $D : \operatorname{mod} S \to \operatorname{mod} S^{op}$. Applying it on the bimodule ${}_S S_S$, we get the bimodule ${}_S DS_S = \operatorname{Hom}_k(S, J)$.

Lemma 2.4.8. *Let R be a ring and ${}_R M_S$ a bimodule. Then ${}_S DM_R \simeq \operatorname{Hom}_S({}_R M_S, {}_S DS_S)$ as R-S-bimodules.*

Proof. By the standard adjunctions,

$$\operatorname{Hom}_S({}_R M_S, \operatorname{Hom}_k({}_S S_S, J)) \simeq \operatorname{Hom}_k({}_R M_S \otimes_S S_S, J) = \operatorname{Hom}_k({}_R M_S, J).$$

□

It follows that $D = \operatorname{Hom}_S(-, DS_S)$, hence DS_S is an injective object in $\operatorname{Mod} S$. We denote by $\operatorname{gl.dim} S$ the global dimension of $\operatorname{mod} S$.

Theorem 2.4.9. *Let R be a ring, S an Artin algebra with $\operatorname{gl.dim} S < \infty$ and ${}_R M_S$ an S-R-bimodule which is finitely generated as an S-module. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \widetilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}$$

are derived equivalent, where D is the duality on $\operatorname{mod} S$.

Proof. The module DS_S is injective in $\operatorname{Mod} S$ and any module in $\operatorname{mod} S$ has an injective resolution with terms that are summands of finite direct sums of DS. Since $\operatorname{gl.dim} S < \infty$, such a resolution is finite, hence $\langle \operatorname{add} DS \rangle = \operatorname{per} S$ and $M \in \operatorname{per} S$ for any $M \in \operatorname{mod} S$.

Therefore the assumptions of Theorem 2.4.5 are satisfied for $T_S = DS$ (see also Remark 2.4.7), and it remains to show that $\operatorname{End}_S(T_S) = S$ and $\operatorname{Hom}_S(M, T_S) \simeq {}_S DM_R$ (as bimodules). This follows by the Lemma 2.4.8 applied for the bimodules ${}_S DS_S$ and ${}_R M_S$. □

Remark 2.4.10. Under the assumptions of Theorem 2.4.9, when R is also an Artin k-algebra and k acts centrally on M, the rings Λ and $\widetilde{\Lambda}$ are Artin algebras and the derived equivalence in the theorem implies that $\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \widetilde{\Lambda})$.

Moreover, by using the duality D, one sees that Theorem 2.4.9 is true for two Artin algebras R and S and a bimodule ${}_R M_S$ on which k acts centrally under the weaker assumptions that M is finitely generated over k and at least one of $\operatorname{gl.dim} R$, $\operatorname{gl.dim} S$ is finite.

By taking $T_S = S$ in Theorem 2.4.5, we get the following corollary.

Corollary 2.4.11. *Let R, S be rings and ${}_R M_S$ an S-R-bimodule such that as an S-module, $M_S \in \operatorname{per} S$ and $\operatorname{Ext}^n_S(M_S, S) = 0$ for all $n > 0$. Then the triangular matrix rings*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \widetilde{\Lambda} = \begin{pmatrix} S & \operatorname{Hom}_S(M, S) \\ 0 & R \end{pmatrix}$$

are derived equivalent.

Remark 2.4.12. The conditions of Corollary 2.4.11 hold when the ring S is *self-injective*, that is, S is injective as a (right) module over itself, and ${}_R M_S$ is finitely generated projective as an S-module. In particular, this applies when S is a semi-simple ring and M is finitely generated as an S-module.

Remark 2.4.13. Recall that for a ring R, a division ring S and a bimodule ${}_S N_R$ which is finite dimensional as a left S-vector space, the *one-point extension* $R[N]$ and the *one-point coextension* $[N]R$ of R by N are defined as the triangular matrix rings

$$R[N] = \begin{pmatrix} S & {}_S N_R \\ 0 & R \end{pmatrix} \qquad\qquad [N]R = \begin{pmatrix} R & {}_R DN_S \\ 0 & S \end{pmatrix}.$$

where $D = \operatorname{Hom}_S(-, S)$ is the duality on $\operatorname{mod} S$. By taking $M = DN$ in the preceding remark, we see that the rings $R[N]$ and $[N]R$ are derived equivalent. Compare this with the construction of "reflection with respect to an idempotent" in [83].

2.5 Concluding remarks

2.5.1 Repetitive algebras

In the specific case of Artin algebras, another approach to the connection between a triangular matrix algebra Λ and its mate $\widetilde{\Lambda}$ involves the use of repetitive algebras, as outlined below.

Let Λ be an Artin algebra over a commutative Artinian ring k and let $D : \operatorname{mod} k \to \operatorname{mod} k$ be the duality. Recall that the *repetitive algebra* $\widehat{\Lambda}$ of Λ, introduced in [46], is the algebra (without unit) of matrices of the form

$$\widehat{\Lambda} = \begin{pmatrix} \ddots & D\Lambda_{i-1} & 0 & & \\ 0 & \Lambda_i & D\Lambda_i & 0 & \\ & 0 & \Lambda_{i+1} & D\Lambda_{i+1} & \\ & & 0 & \ddots & \end{pmatrix}$$

where $\Lambda_i = \Lambda$, $D\Lambda_i = D\Lambda$ for $i \in \mathbb{Z}$, and only finite number of entries are nonzero. The multiplication is defined by the canonical maps $\Lambda \otimes_\Lambda D\Lambda \to D\Lambda$, $D\Lambda \otimes_\Lambda \Lambda \to D\Lambda$ induced by the bimodule structure on $D\Lambda$, and the zero map $D\Lambda \otimes_\Lambda D\Lambda \to 0$.

When Λ is a triangular matrix algebra, one can write

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \qquad\qquad D\Lambda = \begin{pmatrix} DR & 0 \\ DM & DS \end{pmatrix}$$

and a direct calculation shows that the maps $\Lambda \otimes D\Lambda \to D\Lambda$ and $D\Lambda \otimes \Lambda \to D\Lambda$ are given by multiplication of the above matrices, under the convention that $M \otimes_S DS \to 0$ and $DR \otimes_R M \to 0$.

As for the mate $\widetilde{\Lambda}$, we have

$$\widetilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix} \qquad\qquad D\widetilde{\Lambda} = \begin{pmatrix} DS & 0 \\ M & DR \end{pmatrix},$$

therefore the repetitive algebras of Λ and its mate $\widetilde{\Lambda}$ have the form

$$\widehat{\Lambda} = \begin{pmatrix} \ddots & DM & DS & & & & \\ & R & M & DR & & & \\ & & S & DM & DS & & \\ & & & R & M & DR & \\ & & & & S & DM & DS \\ & & & & & R & M \\ & & & & & & \ddots \end{pmatrix}, \qquad \widehat{\widetilde{\Lambda}} = \begin{pmatrix} \ddots & M & DR & & & & \\ & S & DM & DS & & & \\ & & R & M & DR & & \\ & & & S & DM & DS & \\ & & & & R & M & DR \\ & & & & & S & DM \\ & & & & & & \ddots \end{pmatrix}$$

and are thus clearly seen to be isomorphic.

When k is a field and *both* algebras R and S have finite global dimension, this can be combined with Happel's Theorem [35, (II.4.9)] to deduce that Λ and its mate $\widetilde{\Lambda}$ are derived equivalent.

Note, however, that for the derived equivalence between Λ and $\widetilde{\Lambda}$ to hold, it is enough to assume that only *one* of R, S has finite global dimension (see Remark 2.4.10).

Moreover, while the repetitive algebras of Λ and $\widetilde{\Lambda}$ are always isomorphic, in the case where none of R, S have finite global dimension, the algebras Λ and $\widetilde{\Lambda}$ may not be derived equivalent, see Example 2.5.3 below.

2.5.2 Grothendieck groups

In this subsection, k denotes an algebraically closed field. Let Λ be a finite dimensional k-algebra and let $P_1, \dots, P_n$ be a complete collection of the non-isomorphic indecomposable projectives in $\operatorname{mod}\Lambda$. The *Cartan matrix* of Λ is the $n \times n$ integer matrix defined by $C_{ij} = \dim_k \operatorname{Hom}(P_i, P_j)$.

The Grothendieck group $K_0(\operatorname{per}\Lambda)$ of the triangulated category $\operatorname{per}\Lambda$ can be viewed as a free abelian group on the generators $[P_1], \dots, [P_n]$, and the Euler form

$$\langle K, L\rangle = \sum_{r\in\mathbb{Z}} (-1)^r \dim_k \operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(K, L[r])$$

on $\operatorname{per}\Lambda$ induces a bilinear form on $K_0(\operatorname{per}\Lambda)$ whose matrix with respect to that basis equals the Cartan matrix.

It is well known that a derived equivalence of two algebras Λ and Λ' induces an equivalence of the triangulated categories $\operatorname{per}\Lambda$ and $\operatorname{per}\Lambda'$, and hence an isometry of their Grothendieck groups preserving the Euler forms. We now consider the consequences of the derived equivalence of Theorem 2.4.9 (when R and S are finite dimensional k-algebras) for the corresponding Grothendieck groups.

For simplicity, assume that Λ is basic. In this case, there exist primitive orthogonal idempotents $\{e_1, \dots, e_n\}$ in Λ such that $P_i \simeq e_i\Lambda$ for $1 \le i \le n$. Therefore by the isomorphisms $\operatorname{Hom}_\Lambda(e_i\Lambda, N) \simeq Ne_i$ of k-spaces for any Λ-module N_Λ, we get that $C_{ij} = \dim_k e_j\Lambda e_i$.

Lemma 2.5.1. *Let R, S be basic, finite dimensional k-algebras, and let ${}_RM_S$ be a finite dimensional S-R-bimodule. Then the Cartan matrix C_Λ of the corresponding triangular matrix algebra Λ is of the form*

$$C_\Lambda = \begin{pmatrix} C_R & 0 \\ C_M & C_S \end{pmatrix}$$

where C_R, C_S are the Cartan matrices of R, S.

Proof. Let $e_1, \dots, e_n$ and $f_1, \dots, f_m$ be complete sets of primitive orthogonal idempotents in R and in S. Let $\bar{e}_i = e_i \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\bar{f}_j = f_j \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\bar{e}_1, \dots, \bar{e}_n, \bar{f}_1, \dots, \bar{f}_m$ is a complete set of primitive orthogonal idempotents of Λ and the result follows by computing the dimensions of $\bar{e}_i\Lambda\bar{e}_{i'}$, $\bar{e}_i\Lambda\bar{f}_j$, $\bar{f}_j\Lambda\bar{e}_i$ and $\bar{f}_j\Lambda\bar{f}_{j'}$. In particular, $(C_M)_{ji} = \dim_k e_iMf_j$. $\square$

Since $\dim_k f_jDMe_i = \dim_k e_iMf_j$, we get by Lemma 2.5.1 that the Cartan matrices of Λ and its mate $\widetilde{\Lambda}$ are

$$C_\Lambda = \begin{pmatrix} C_R & 0 \\ C_M & C_S \end{pmatrix} \qquad\qquad C_{\widetilde{\Lambda}} = \begin{pmatrix} C_S & 0 \\ C_M^t & C_R \end{pmatrix}.$$

When at least one of R and S has finite global dimension, the derived equivalence of Theorem 2.4.9 implies that C_Λ and $C_{\widetilde{\Lambda}}$ represent the same bilinear form, hence they are congruent over $\mathbb{Z}$, that is, there exists an invertible matrix P over $\mathbb{Z}$ such that $P^tC_\Lambda P = C_{\widetilde{\Lambda}}$.

One can also show this congruence directly at the level of matrices, as follows.

Lemma 2.5.2. *Let K be a commutative ring. Let $A \in M_{n\times n}(K)$ be a square matrix, $B \in \mathrm{GL}_m(K)$ an invertible square matrix and $C \in M_{m\times n}(K)$. Then there exists $P \in \mathrm{GL}_{n+m}(K)$ such that*

$$P^t \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} P = \begin{pmatrix} B & 0 \\ C^t & A \end{pmatrix}$$

Proof. Take $P = \begin{pmatrix} 0 & I_n \\ -B^{-1}B^t & -B^{-1}C \end{pmatrix}$. Then

$$\begin{aligned} P^t \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} P &= \begin{pmatrix} 0 & -BB^{-t} \\ I_n & -C^tB^{-t} \end{pmatrix} \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -B^{-1}B^t & -B^{-1}C \end{pmatrix} \\ &= \begin{pmatrix} 0 & -BB^{-t} \\ I_n & -C^tB^{-t} \end{pmatrix} \begin{pmatrix} 0 & A \\ -B^t & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ C^t & A \end{pmatrix} \end{aligned}$$

□

Note that one could also take $P = \begin{pmatrix} -A^{-t}C^t & -A^{-t}A \\ I_m & 0 \end{pmatrix}$, hence it is enough to assume that at least one of A and B is invertible.

The conclusion of the lemma is false if one does not assume that at least one of the matrices A, B is invertible over K. This can be used to construct triplets consisting of two finite dimensional algebras R, S (necessarily of infinite global dimension) and a bimodule M such that the triangular matrix algebra Λ and its mate $\widetilde{\Lambda}$ are not derived equivalent.

Example 2.5.3. Let $R = k[x]/(x^2)$, $S = k[y]/(y^3)$ and $M = k$ with x and y acting on k as zero. Then the triangular matrix algebras

$$\Lambda = \begin{pmatrix} k[x]/(x^2) & k \\ 0 & k[y]/(y^3) \end{pmatrix} \qquad \widetilde{\Lambda} = \begin{pmatrix} k[y]/(y^3) & k \\ 0 & k[x]/(x^2) \end{pmatrix}$$

are not derived equivalent, since one can verify that their Cartan matrices

$$C_\Lambda = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad C_{\widetilde{\Lambda}} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

are not congruent over $\mathbb{Z}$. Note that despite the fact that R and S are self-injective, Corollary 2.4.11 cannot be used since M does not have a finite projective resolution.

2.5.3 Trivial extensions

Triangular matrix rings are special cases of trivial extensions [4, p. 78]. Indeed, if R, S are rings and ${}_RM_S$ is a bimodule, the corresponding triangular matrix ring is isomorphic to the trivial extension $A \ltimes M$ where $A = R \times S$ and M is equipped with an A-bimodule structure via $(r, s)m = rm$ and $m(r, s) = ms$.

We remark that even when A is a finite dimensional k-algebra of finite global dimension and M is a finite dimensional A-bimodule, the trivial extension algebras $A \ltimes M$ and $A \ltimes DM$ are generally not derived equivalent, so that the derived equivalence in Theorem 2.4.9 is a special feature of triangular matrix rings.

Example 2.5.4. Let $A = kQ$ where Q is the quiver $\bullet_1 \longrightarrow \bullet_2$ and let M be the kQ-bimodule corresponding to the following commutative diagram of vector spaces

$$\begin{tikzcd} & 0_{(1,1)} \arrow[dr] & \\ 0_{(2,1)} \arrow[ur] \arrow[dr] & & k_{(1,2)} \\ & 0_{(2,2)} \arrow[ur] & \end{tikzcd}$$

Then $A \ltimes M$ is the path algebra of the quiver $\bullet \rightrightarrows \bullet$ while $A \ltimes DM$ is the path algebra of $\bullet \rightleftarrows \bullet$ modulo the compositions of the arrows being zero. These two algebras are not derived equivalent since $\mathrm{gl.dim}(A \ltimes M) = 1$ while $\mathrm{gl.dim}(A \ltimes DM) = \infty$.

Chapter 3

Universal Derived Equivalences of Posets

By using only combinatorial data on two posets X and Y, we construct a set of so-called formulas. A formula produces simultaneously, for any abelian category $\mathcal{A}$, a functor between the categories of complexes of diagrams over X and Y with values in $\mathcal{A}$. This functor induces a triangulated functor between the corresponding derived categories.

This allows us to prove, for pairs X, Y of posets sharing certain common underlying combinatorial structure, that for any abelian category $\mathcal{A}$, regardless of its nature, the categories of diagrams over X and Y with values in $\mathcal{A}$ are derived equivalent.

3.1 Introduction

In Chapter 1 we considered the question when the categories $\mathcal{A}^X$ and $\mathcal{A}^Y$ of diagrams over finite posets X and Y with values in the abelian category $\mathcal{A}$ of finite dimensional vector spaces over a fixed field k, are derived equivalent.

Since in that case the category of diagrams $\mathcal{A}^X$ is equivalent to the category of finitely generated modules over the incidence algebra kX, methods from the theory of derived equivalence of algebras, in particular tilting theory, could be used [35, 73, 74].

Interestingly, in all cases considered, the derived equivalence of two categories of diagrams does not depend on the field k, see for example Theorem 1.4.14. A natural question arises whether there is a general principle which explains this fact and extends to any arbitrary abelian category $\mathcal{A}$.

In this chapter we provide a positive answer in the following sense; we exhibit several constructions of pairs of posets X and Y such that the derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ are equivalent for any abelian category $\mathcal{A}$, regardless of its nature. Such pairs of posets are called *universally derived equivalent*, since the derived equivalence is universal and originates from the combinatorial and topological properties of the posets, rather than the specific abelian categories involved.

Our main tools are the so-called formulas. A formula consists of combinatorial data that produces simultaneously, for any abelian category $\mathcal{A}$, a functor between the categories of complexes of diagrams over X and Y with values in $\mathcal{A}$, which induces a triangulated functor between the corresponding derived categories.

3.1.1 The main construction

Let X and Y be two finite partially ordered sets (*posets*). For $y \in Y$, write $[y, \cdot] = \{y' \in Y : y' \geq y\}$ and $[\cdot, y] = \{y' \in Y : y' \leq y\}$. Let $\{Y_x\}_{x \in X}$ be a collection of subsets of Y indexed by the elements of X, such that

$$[y, \cdot] \cap [y', \cdot] = \phi \quad \text{and} \quad [\cdot, y] \cap [\cdot, y'] = \phi \tag{3.1.1}$$

for any $x \in X$ and $y \neq y'$ in Y_x. Assume in addition that for any $x \leq x'$, there exists an isomorphism $\varphi_{x,x'} : Y_x \xrightarrow{\sim} Y_{x'}$ such that

$$y \leq \varphi_{x,x'}(y) \qquad\qquad \text{for all } y \in Y_x \tag{3.1.2}$$

By (3.1.1), it follows that

$$\varphi_{x,x''} = \varphi_{x',x''}\varphi_{x,x'} \qquad\qquad \text{for all } x \leq x' \leq x''. \tag{3.1.3}$$

Define two partial orders $\leq_+$ and $\leq_-$ on the disjoint union $X \sqcup Y$ as follows. Inside X and Y, the orders $\leq_+$ and $\leq_-$ agree with the original ones, and for $x \in X$ and $y \in Y$ we set

$$\begin{aligned} x \leq_+ y &\Longleftrightarrow \exists\, y_x \in Y_x \text{ with } y_x \leq y \\ y \leq_- x &\Longleftrightarrow \exists\, y_x \in Y_x \text{ with } y \leq y_x \end{aligned} \tag{3.1.4}$$

with no other relations (note that the element y_x is unique by (3.1.1), and that $\leq_+$, $\leq_-$ are partial orders by (3.1.2)).

Theorem 3.1.1. *The two posets $(X \sqcup Y, \leq_+)$ and $(X \sqcup Y, \leq_-)$ are universally derived equivalent.*

The assumption (3.1.1) of the Theorem cannot be dropped, as demonstrated by the following example.

Example 3.1.2. Consider the two posets whose Hasse diagrams are given by

$$\begin{tikzcd} & \bullet_1 \arrow[dl] \arrow[dr] & \\ \bullet_2 \arrow[dr] & & \bullet_3 \arrow[dl] \\ & \bullet_4 & \end{tikzcd} \qquad\qquad \begin{tikzcd} & \bullet_1 & \\ \bullet_2 \arrow[ur] \arrow[dr] & & \bullet_3 \arrow[ul] \arrow[dl] \\ & \bullet_4 & \end{tikzcd}$$

$$(X \sqcup Y, \leq_+) \qquad\qquad\qquad (X \sqcup Y, \leq_-)$$

They can be represented as $(X \sqcup Y, \leq_+)$ and $(X \sqcup Y, \leq_-)$ where $X = \{1\}$, $Y = \{2, 3, 4\}$ and $Y_1 = \{2, 3\} \subset Y$. The categories of diagrams over these two posets are in general not derived equivalent, even for diagrams of vector spaces.

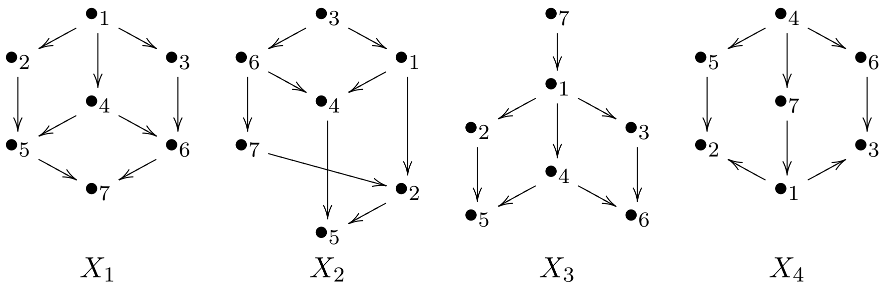


Figure 3.1: Four universally derived equivalent posets

The construction of Theorem 3.1.1 has many interesting consequences, some of them related to ordinal sums and others to generalized BGP reflections [9]. First, consider the case where all the subsets Y_x are single points, that is, there exists a function $f : X \to Y$ with $Y_x = \{f(x)\}$ for all $x \in X$. Then (3.1.1) and (3.1.3) are automatically satisfied and the condition (3.1.2) is equivalent to f being *order preserving*, i.e. $f(x) \leq f(x')$ for $x \leq x'$. Let $\leq_+^f$ and $\leq_-^f$ denote the corresponding orders on $X \sqcup Y$, and note that (3.1.4) takes the simplified form

$$\begin{aligned} x \leq_+^f y &\Longleftrightarrow f(x) \leq y \\ y \leq_-^f x &\Longleftrightarrow y \leq f(x) \end{aligned} \tag{3.1.5}$$

Corollary 3.1.3. *Let $f : X \to Y$ be order preserving. Then the two posets $(X \sqcup Y, \leq_+^f)$ and $(X \sqcup Y, \leq_-^f)$ are universally derived equivalent.*

Example 3.1.4. Consider the four posets X_1, X_2, X_3, X_4 whose Hasse diagrams are drawn in Figure 3.1. For any of the pairs (i, j) where $(i, j) = (1, 2)$, $(1, 3)$ or $(3, 4)$ we find posets X_{ij} and X_{ji} and an order-preserving function $f_{ij} : X_{ij} \to X_{ji}$ such that

$$X_i \simeq (X_{ij} \sqcup X_{ji}, \leq_+^{f_{ij}}) \qquad\qquad X_j \simeq (X_{ij} \sqcup X_{ji}, \leq_-^{f_{ij}})$$

hence X_i and X_j are universally derived equivalent. Indeed, let

$$\begin{aligned} X_{12} &= \{1, 2, 4, 5\} & X_{21} &= \{3, 6, 7\} \\ X_{13} &= \{1, 2, 3, 4, 5, 6\} & X_{31} &= \{7\} \\ X_{34} &= \{1, 2, 3, 7\} & X_{43} &= \{4, 5, 6\} \end{aligned}$$

and define $f_{12} : X_{12} \to X_{21}$, $f_{13} : X_{13} \to X_{31}$ and $f_{34} : X_{34} \to X_{43}$ by

$$\begin{array}{lll} f_{12}(1) = 3 & f_{12}(2) = f_{12}(5) = 7 & f_{12}(4) = 6 \\ f_{13}(1) = \cdots = f_{13}(6) = 7 & & \\ f_{34}(1) = f_{34}(7) = 4 & f_{34}(2) = 5 & f_{34}(3) = 6 \end{array}$$

3.1.2 Applications to ordinal sums

Recall that the *ordinal sum* of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, denoted $P \oplus Q$, is the poset $(P \sqcup Q, \leq)$ where $x \leq y$ if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$ or $x \in P$ and $y \in Q$. Similarly, the *direct sum* $P + Q$ is the poset $(P \sqcup Q, \leq)$ where $x \leq y$ if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$. Note that the direct sum is commutative (up to isomorphism) but the ordinal sum is not. Denote by $\mathbf{1}$ the poset consisting of one element. Taking $Y = \mathbf{1}$ in Corollary 3.1.3, we get the following

Corollary 3.1.5. *For any poset X, the posets $X \oplus \mathbf{1}$ and $\mathbf{1} \oplus X$ are universally derived equivalent.*

In Section 3.4.3 we prove the following additional consequence of Corollary 3.1.3 for ordinal and direct sums.

Corollary 3.1.6. *For any two posets X and Z, the posets $X \oplus \mathbf{1} \oplus Z$ and $\mathbf{1} \oplus (X + Z)$ are universally derived equivalent. Hence the posets $X \oplus \mathbf{1} \oplus Z$ and $Z \oplus \mathbf{1} \oplus X$ are universally derived equivalent.*

The result of Corollary 3.1.6 is no longer true when $\mathbf{1}$ is replaced by an arbitrary poset, even for diagrams of vector spaces, see Example 1.4.20.

By using Theorem 3.1.1, one can generalize Corollary 3.1.5 to ordinal sums with any finite anti-chain.

Corollary 3.1.7. *Let $n\mathbf{1} = \mathbf{1} + \cdots + \mathbf{1}$ be an anti-chain with $n \geq 1$ elements. Then for any poset X, the posets $X \oplus n\mathbf{1}$ and $n\mathbf{1} \oplus X$ are universally derived equivalent.*

Proof. Take $Y = n\mathbf{1}$ and $Y_x = Y$ for all $x \in X$. □

Note that in Chapter 1 we have shown that for arbitrary two posets X and Y, it is true that for any field k, the categories of diagrams of finite dimensional k-vector spaces over $X \oplus Y$ and $Y \oplus X$ are derived equivalent (Corollary 1.4.15).

3.1.3 Generalized BGP reflections

More consequences of Theorem 3.1.1 are obtained by considering the case where $X = \{*\}$ is a single point, that is, there exists a subset $Y_0 \subseteq Y$ such that (3.1.1) holds for any $y \neq y'$ in Y_0. Observe that conditions (3.1.2) and (3.1.3) automatically hold in this case, and the two partial orders on $Y \cup \{*\}$ corresponding to (3.1.4), denoted $\leq_+^{Y_0}$ and $\leq_-^{Y_0}$, are obtained by extending the order on Y according to

$$
\begin{aligned}
* <_+^{Y_0} y &\iff \exists\, y_0 \in Y_0 \text{ with } y_0 \leq y \\
y <_-^{Y_0} * &\iff \exists\, y_0 \in Y_0 \text{ with } y \leq y_0
\end{aligned}
\tag{3.1.6}
$$

Corollary 3.1.8. *Let $Y_0 \subseteq Y$ be a subset satisfying* (3.1.1). *Then the posets $(Y \cup \{*\}, \leq_+^{Y_0})$ and $(Y \cup \{*\}, \leq_-^{Y_0})$ are universally derived equivalent.*

Note that in the Hasse diagram of $\leq_+^{Y_0}$, the vertex $*$ is a source which is connected to the vertices of Y_0, and the Hasse diagram of $\leq_-^{Y_0}$ is obtained by reverting the orientations of the arrows from $*$, making it into a sink. Thus Corollary 3.1.8 can be considered as a generalized BGP reflection principle.

Viewing orientations on (finite) trees as posets by setting $x \leq y$ for two vertices x, y if there exists an oriented path from x to y, and applying a standard combinatorial argument [9], we recover the following corollary, already known for categories of vector spaces over a field.

Corollary 3.1.9. *Any two orientations of a tree are universally derived equivalent.*

3.1.4 Formulas

By using only combinatorial data on two posets X and Y, we construct a set of formulas $\mathcal{F}_X^Y$. A formula $\boldsymbol{\xi}$ produces simultaneously, for any abelian category $\mathcal{A}$, a functor $F_{\boldsymbol{\xi},\mathcal{A}}$ between the categories $C(\mathcal{A}^X)$ and $C(\mathcal{A}^Y)$ of complexes of diagrams over X and Y with values in $\mathcal{A}$. This functor induces a triangulated functor $\widetilde{F}_{\boldsymbol{\xi},\mathcal{A}}$ between the corresponding derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ such that the following diagram is commutative

$$\begin{tikzcd} C(\mathcal{A}^X) \arrow[r, "F_{\boldsymbol{\xi},\mathcal{A}}"] \arrow[d] & C(\mathcal{A}^Y) \arrow[d] \\ D(\mathcal{A}^X) \arrow[r, dotted, "\widetilde{F}_{\boldsymbol{\xi},\mathcal{A}}"] & D(\mathcal{A}^Y) \end{tikzcd}$$

where the vertical arrows are the canonical localizations.

We prove Theorem 3.1.1 by exhibiting a pair of formulas $\boldsymbol{\xi}^+ \in \mathcal{F}_{\leq_+}^{\leq_-}$, $\boldsymbol{\xi}^- \in \mathcal{F}_{\leq_-}^{\leq_+}$ and showing that for any abelian category $\mathcal{A}$, the compositions $\widetilde{F}_{\boldsymbol{\xi}^+,\mathcal{A}}\widetilde{F}_{\boldsymbol{\xi}^-,\mathcal{A}}$ and $\widetilde{F}_{\boldsymbol{\xi}^-,\mathcal{A}}\widetilde{F}_{\boldsymbol{\xi}^+,\mathcal{A}}$ of the corresponding triangulated functors on the derived categories are auto-equivalences, as they are isomorphic to the translations. Hence $\leq_+$ and $\leq_-$ are universally derived equivalent.

3.1.5 A remark on infinite posets

A careful study of the proof of Theorem 3.1.1 shows that the theorem is still true when the posets X and Y are infinite, provided that the posets Y_x are finite for all $x \in X$, as the formulas constructed in the course of the proof involve only a finite number of terms at each point.

It follows that Corollaries 3.1.3, 3.1.5, 3.1.6, 3.1.7 hold for arbitrary (not necessarily finite) posets X, Y and Z, and that Corollary 3.1.8 holds for any poset Y provided that the subset Y_0 is finite.

3.2 Complexes of diagrams

3.2.1 Diagrams and sheaves

Let X be any partially ordered set (not necessarily finite) and let $\mathcal{A}$ be a category. For infinite posets, it is better to replace the notion of a commutative diagram over the Hasse diagram introduced in Section 1.2.2 with the following more general notion.

Definition 3.2.1. A *diagram* (A, r) over X with values in $\mathcal{A}$ consists of the following data:

- For any $x \in X$, an object A_x of $\mathcal{A}$
- For any pair $x \leq x'$, a morphism $r_{xx'} : A_x \to A_{x'}$ (*restriction map*)

subject to the conditions $r_{xx} = \mathrm{id}_{A_x}$ and $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x \leq x' \leq x''$ in X.

A *morphism* $f : (A, r) \to (A', r')$ of diagrams consists of morphisms $f_x : A_x \to A'_x$ for all $x \in X$, such that for any $x \leq x'$, the diagram

$$\begin{tikzcd} A_x \arrow[r, "f_x"] \arrow[d, "r_{xx'}"'] & A'_x \arrow[d, "r'_{xx'}"] \\ A_{x'} \arrow[r, "f_{x'}"'] & A'_{x'} \end{tikzcd}$$

commutes.

Using these definitions, we can speak of the category of diagrams over X with values in $\mathcal{A}$, which will be denoted by $\mathcal{A}^X$.

We can view X as a small category as follows. Its objects are the points $x \in X$, while $\mathrm{Hom}_X(x, x')$ is a one-element set if $x \leq x'$ and empty otherwise. Under this viewpoint, a diagram over X with values in $\mathcal{A}$ becomes a functor $A : X \to \mathcal{A}$ and a morphism of diagrams corresponds to a natural transformation, so that $\mathcal{A}^X$ is naturally identified with the category of functors $X \to \mathcal{A}$. Observe that any functor $F : \mathcal{A} \to \mathcal{A}'$ induces a functor $F^X : \mathcal{A}^X \to \mathcal{A}'^X$ by the composition $F^X(A) = F \circ A$. In terms of diagrams and morphisms, $F^X(A, r) = (FA, Fr)$ where $(FA)_x = F(A_x)$, $(Fr)_{xx'} = F(r_{xx'})$ and $F^X(f)_x = F(f_x)$.

If $\mathcal{A}$ is additive, then $\mathcal{A}^X$ is additive. Assume now that $\mathcal{A}$ is abelian. In this case, $\mathcal{A}^X$ is also abelian, and kernels, images, and quotients can be computed pointwise, that is, if $f : (A, r) \to (A', r')$ is a morphism of diagrams then $(\ker f)_x = \ker f_x$, $(\mathrm{im}\, f)_x = \mathrm{im}\, f_x$, with the restriction maps induced from r, r'. In particular, for any $x \in X$ the evaluation functor $-_x : \mathcal{A}^X \to \mathcal{A}$ taking a diagram (A, r) to A_x and a morphism $f = (f_x)$ to f_x, is exact.

The poset X admits a natural topology, whose open sets are the subsets $U \subseteq X$ with the property that if $x \in U$ and $x \leq x'$ then $x' \in U$. When X is finite, the category of diagrams over X with values in $\mathcal{A}$ can then be naturally identified with the category of sheaves over the topological space X with values in $\mathcal{A}$, see Section 1.2.2.

3.2.2 Complexes and cones

Let $\mathcal{B}$ be an additive category. A *complex* $(K^\bullet, d_K^\bullet)$ over $\mathcal{B}$ consists of objects K^i for $i \in \mathbb{Z}$ with morphisms $d_K^i : K^i \to K^{i+1}$ such that $d_K^{i+1} d_K^i = 0$ for all $i \in \mathbb{Z}$. If $n \in \mathbb{Z}$, the *shift* of $K^\bullet$ by n, denoted $K[n]^\bullet$, is the complex defined by $K[n]^i = K^{i+n}$, $d_{K[n]}^i = (-1)^n d_K^{i+n}$.

Let $(K^\bullet, d_K^\bullet)$, $(L^\bullet, d_L^\bullet)$ be two complexes and $f = (f^i)_{i \in \mathbb{Z}}$ a collection of morphisms $f^i : K^i \to L^i$. If $n \in \mathbb{Z}$, let $f[n] = (f[n]^i)_{i \in \mathbb{Z}}$ with $f[n]^i = f^{i+n}$. Using this notation, the condition that f is a *morphism* of complexes is expressed as $f[1] d_K = d_L f$.

The *cone* of a morphism $f : K^\bullet \to L^\bullet$, denoted $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$, is the complex whose i-th entry equals $K^{i+1} \oplus L^i$, with the differential

$$d(k^{i+1}, l^i) = (-d_K^{i+1}(k^{i+1}), f^{i+1}(k^{i+1}) + d_L^i(l^i))$$

In a more compact form, $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet) = K[1]^\bullet \oplus L^\bullet$ with the differential acting as the matrix

$$\begin{pmatrix} d_K[1] & 0 \\ f[1] & d_L \end{pmatrix}$$

by viewing the entries as column vectors.

When $\mathcal{B}$ is abelian, the *i-th cohomology* of a complex $(K^\bullet, d_K^\bullet)$ is defined by $\mathrm{H}^i(K^\bullet) = \ker d_K^i / \operatorname{im} d_K^{i-1}$. We say that $(K^\bullet, d_K^\bullet)$ is *acyclic* if $\mathrm{H}^i(K^\bullet) = 0$ for all $i \in \mathbb{Z}$. A morphism $f : K^\bullet \to L^\bullet$ induces morphisms $\mathrm{H}^i(f) : \mathrm{H}^i(K^\bullet) \to \mathrm{H}^i(L^\bullet)$. The morphism f is called a *quasi-isomorphism* if $\mathrm{H}^i(f)$ are isomorphisms for all $i \in \mathbb{Z}$.

The following lemma is standard.

Lemma 3.2.2. *$f : K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if the cone $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$ is acyclic.*

Let $C(\mathcal{B})$ denote the category of complexes over $\mathcal{B}$. Denote by $[1] : C(\mathcal{B}) \to C(\mathcal{B})$ the shift functor taking a complex $(K^\bullet, d_K^\bullet)$ to $(K[1]^\bullet, d_{K[1]^\bullet})$ and a morphism f to $f[1]$. Any additive functor $G : \mathcal{B} \to \mathcal{B}'$ induces an additive functor $C(G) : C(\mathcal{B}) \to C(\mathcal{B}')$ by sending a complex $((K^i), (d_K^i))$ to $((G(K^i)), (G(d_K^i)))$ and a morphism (f^i) to $(G(f^i))$.

Lemma 3.2.3. *For any additive category $\mathcal{A}$ and a poset X, there exists an equivalence of categories $\Phi_{X,\mathcal{A}} : C(\mathcal{A}^X) \simeq C(\mathcal{A})^X$ such that for any additive category $\mathcal{A}'$ and an additive functor $F : \mathcal{A} \to \mathcal{A}'$, the diagram*

$$\begin{tikzcd} C(\mathcal{A})^X \arrow[r, "\Phi_{X,\mathcal{A}}", "\sim"'] \arrow[d, "C(F^X)"'] & C(\mathcal{A})^X \arrow[d, "C(F)^X"] \\ C(\mathcal{A}')^X \arrow[r, "\Phi_{X,\mathcal{A}'}", "\sim"'] & C(\mathcal{A}')^X \end{tikzcd} \tag{3.2.1}$$

commutes. In other words, we can identify a complex of diagrams with a diagram of complexes.

Proof. Let $\mathcal{A}$ be additive and let $(K^\bullet, d^\bullet)$ be a complex in $C(\mathcal{A}^X)$. Denote by $d^i : K^i \to K^{i+1}$ the morphisms in $\mathcal{A}^X$ and by $d_x^i : K_x^i \to K_x^{i+1}$ the morphisms on the stalks. Let $r_{xy}^i : K_x^i \to K_y^i$ denote the restriction maps in the diagram K^i.

For a morphism $f : (K^\bullet, d^\bullet) \to (L^\bullet, d^\bullet)$ in $C(\mathcal{A}^X)$, denote by $f^i : K^i \to L^i$ the corresponding morphisms in $\mathcal{A}^X$ and by $f_x^i : K_x^i \to L_x^i$ the morphisms on stalks. Define a functor $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$ by

$$\Phi_{X,\mathcal{A}}(K^\bullet, d^\bullet) = (\{K_x^\bullet\}_{x \in X}, \{r_{xy}\}) \qquad\qquad \Phi_{X,\mathcal{A}}(f) = (f_x)_{x \in X}$$

where $(K_x^\bullet)^i = K_x^i$ with differential $d_x^\bullet = (d_x^i)^i$, $r_{xy} = (r_{xy}^i)^i : K_x^\bullet \to K_y^\bullet$ are the restriction maps, and $f_x = (f_x^i)^i : K_x^\bullet \to L_x^\bullet$.

The commutativity of all squares in the diagram in Figure 3.2 implies that $\Phi_{X,\mathcal{A}}$ is well-defined, induces the required equivalence and that (3.2.1) commutes. □

$$\begin{tikzcd}
 & L_x^i \arrow[rr, "d_x^i"] \arrow[dd, "r_{xy}^i"] & & L_x^{i+1} \arrow[dd, "r_{xy}^{i+1}"] \\
K_x^i \arrow[ur, "f_x^i"] \arrow[rr, "d_x^i"] \arrow[dd, "r_{xy}^i"'] & & K_x^{i+1} \arrow[ur, "f_x^{i+1}"'] \arrow[dd, "r_{xy}^{i+1}"] & \\
 & L_y^i \arrow[rr, "d_y^i"] & & L_y^{i+1} \\
K_y^i \arrow[ur, "f_y^i"] \arrow[rr, "d_y^i"'] & & K_y^{i+1} \arrow[ur, "f_y^{i+1}"'] &
\end{tikzcd}$$

Figure 3.2:

In the sequel, X is a poset, $\mathcal{A}$ is an abelian category and all complexes are in $C(\mathcal{A}^X)$.

Lemma 3.2.4. $\mathrm{H}^i(K^\bullet)_x = \mathrm{H}^i(K_x^\bullet)$

Proof. Kernels and images can be computed pointwise. ☐

Lemma 3.2.5. $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)_x = \mathrm{C}(K_x^\bullet \xrightarrow{f_x} L_x^\bullet)$

Corollary 3.2.6. *Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of diagrams. Then f is a quasi-isomorphism if and only if for every $x \in X$, $f_x : K_x^\bullet \to L_x^\bullet$ is a quasi-isomorphism.*

Proof. Let $x \in X$ and $i \in \mathbb{Z}$. Then by Lemmas 3.2.4 and 3.2.5,

$$\mathrm{H}^i(\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet))_x = \mathrm{H}^i(\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)_x) = \mathrm{H}^i(\mathrm{C}(K_x^\bullet \xrightarrow{f_x} L_x^\bullet))$$

hence $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$ is acyclic if and only if $\mathrm{C}(K_x^\bullet \xrightarrow{f_x} L_x^\bullet)$ are acyclic for every $x \in X$. Using Lemma 3.2.2, we see that f is a quasi-isomorphism if and only if all the f_x are quasi-isomorphisms. ☐

3.2.3 Universal derived equivalence

Recall that the derived category $D(\mathcal{B})$ of an abelian category $\mathcal{B}$ is obtained by formally inverting all the quasi-isomorphisms in $C(\mathcal{B})$. It admits a structure of a triangulated category where the distinguished triangles in $D(\mathcal{B})$ are those isomorphic to $K' \to K \to K'' \to K'[1]$ where $0 \to K' \to K \to K'' \to 0$ is a short exact sequence in $C(\mathcal{B})$.

Definition 3.2.7. Two posets X and Y are *universally derived equivalent* if for any abelian category $\mathcal{A}$, the derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ are equivalent as triangulated categories.

Lemma 3.2.8. *Let X and Y be universally derived equivalent. Then X^{op} and Y^{op} are universally derived equivalent.*

Lemma 3.2.9. *Let X_1, Y_1 and X_2, Y_2 be two pairs of universally derived equivalent posets. Then $X_1 \times X_2$ and $Y_1 \times Y_2$ are universally derived equivalent.*

3.3 Formulas

Throughout this section, the poset X is fixed.

3.3.1 The category $\mathcal{C}_X$

Viewing $X \times \mathbb{Z}$ as a small category with a unique map $(x, m) \to (x', m')$ if $x \le x'$ and $m \le m'$ and no maps otherwise, we can consider the additive category $\widetilde{\mathcal{C}}_X$ whose objects are finite sequences $\{(x_i, m_i)\}_{i=1}^n$ with morphisms $\{(x_i, m_i)\}_{i=1}^n \to \{(x'_j, m'_j)\}_{j=1}^{n'}$ specified by $n' \times n$ integer matrices $(c_{ji})_{i,j}$ satisfying $c_{ji} = 0$ unless $(x_i, m_i) \le (x'_j, m'_j)$. That is, a morphism is a formal $\mathbb{Z}$-linear combination of arrows $(x_i, m_i) \to (x'_j, m'_j)$. Addition and composition of morphisms correspond to the usual addition and multiplication of matrices.

To encode the fact that squares of differentials are zero, we consider a certain quotient of $\widetilde{\mathcal{C}}_X$. Namely, let $\widetilde{\mathcal{I}}_X$ be the ideal in $\widetilde{\mathcal{C}}_X$ generated by all the morphisms $(x, m) \to (x, m+2)$ for $(x, m) \in X \times \mathbb{Z}$ and let $\mathcal{C}_X = \widetilde{\mathcal{C}}_X / \widetilde{\mathcal{I}}_X$ be the quotient. The objects of $\mathcal{C}_X$ are still sequences $\xi = \{(x_i, m_i)\}$ and the morphisms can again be written as integer matrices, albeit not uniquely as we ignore the entries c_{ji} whenever $m'_j - m_i \ge 2$.

Define a translation functor $[1] : \mathcal{C}_X \to \mathcal{C}_X$ as follows. For an object $\xi = \{(x_i, m_i)\}_{i=1}^n$, let $\xi[1] = \{(x_i, m_i + 1)\}_{i=1}^n$. For a morphism $\varphi = (c_{ji}) : \{(x_i, m_i)\} \to \{(x'_j, m'_j)\}$, let $\varphi[1]$ be the morphism $\{(x_i, m_i + 1)\} \to \{(x'_j, m'_j + 1)\}$ specified by the same matrix (c_{ji}).

Let $\mathcal{A}$ be an abelian category. From now on we shall denote a complex in $C(\mathcal{A}^X)$ by K instead of $K^\bullet$, and use Lemma 3.2.3 to identify $C(\mathcal{A}^X)$ with $C(\mathcal{A})^X$. Therefore we may think of K as a diagram of complexes in $C(\mathcal{A})$ and use the notations K_x, d_x, $r_{xx'}$ as in the proof of that lemma.

For two additive categories $\mathcal{B}$ and $\mathcal{B}'$, let $\mathrm{Func}(\mathcal{B}, \mathcal{B}')$ denote the category of additive functors $\mathcal{B} \to \mathcal{B}'$, with natural transformations as morphisms.

Proposition 3.3.1. *There exists a functor $\eta : \mathcal{C}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$ commuting with the translations.*

Proof. An object $\xi = \{(x_i, m_i)\}_{i=1}^n$ defines an additive functor F_ξ from $C(\mathcal{A})^X$ to $C(\mathcal{A})$ by sending $K \in C(\mathcal{A})^X$ and a morphism $f : K \to K'$ to

$$F_\xi(K) = \bigoplus_{i=1}^n K_{x_i}[m_i] \qquad\qquad F_\xi(f) = \bigoplus_{i=1}^n f_{x_i}[m_i] \tag{3.3.1}$$

where the right term is the $n \times n$ diagonal matrix whose (i, i) entry is $f_{x_i}[m_i] : K_{x_i}[m_i] \to K'_{x_i}[m_i]$.

To define η on morphisms $\xi \to \xi'$, consider first the case that $\xi = (x, m)$ and $\xi' = (x', m')$. A morphism $\varphi = (c) : (x, m) \to (x', m')$ in $\mathcal{C}_X$ is specified by an integer c, with $c = 0$ unless $(x, m) \le (x', m')$. Given $K \in C(\mathcal{A})^X$, define a morphism $\eta_\varphi(K) : K_x[m] \to K_{x'}[m']$ by

$$\eta_\varphi(K) = \begin{cases} c \cdot r_{xx'}[m] & \text{if } m' = m \text{ and } x' \ge x \\ c \cdot d_{x'}[m] r_{xx'}[m] & \text{if } m' = m + 1 \text{ and } x' \ge x \\ 0 & \text{otherwise} \end{cases} \tag{3.3.2}$$

Then $\eta_c : F_\xi \to F_{\xi'}$ is a natural transformation since the diagrams

$$\begin{tikzcd} K_x[m] \arrow[r, "r_{xx'}[m]"] \arrow[d, "f_x[m]"'] & K_{x'}[m] \arrow[d, "f_{x'}[m]"] \\ K'_x[m] \arrow[r, "r'_{xx'}[m]"'] & K'_{x'}[m] \end{tikzcd} \qquad \begin{tikzcd} K_x[m] \arrow[r, "d_x[m]"] \arrow[d, "f_x[m]"'] & K_x[m+1] \arrow[d, "f_x[m+1]"] \\ K'_x[m] \arrow[r, "d'_x[m]"'] & K'_x[m+1] \end{tikzcd} \tag{3.3.3}$$

commute.

Let $\varphi' = (c') : (x', m') \to (x'', m'')$ be another morphism in $\mathcal{C}_X$. Then (3.3.2) and the three relations $r_{xx''} = r_{x'x''}r_{xx'}$, $r_{xx'}[1]d_x = d_{x'}r_{xx'}$ and $d_x[1]d_x = 0$, imply that

$$\eta_{\varphi'\varphi}(K) = \eta_{\varphi'}(K)\eta_\varphi(K) \tag{3.3.4}$$

for every $K \in C(\mathcal{A})^X$.

Now for a general morphism $\varphi : \{(x_i, m_i)\}_{i=1}^n \to \{(x'_j, m'_j)\}_{j=1}^{n'}$, define morphisms $\eta_\varphi(K) : \bigoplus_{i=1}^n K_{x_i}[m_i] \to \bigoplus_{j=1}^{n'} K_{x'_j}[m'_j]$ by

$$(\eta_\varphi)_{ji} = \eta_{(c_{ji})} : K_{x_i}[m_i] \to K_{x'_j}[m'_j] \tag{3.3.5}$$

where $\eta_{(c_{ji})}$ is defined by (3.3.2) for $c_{ji} : (x_i, m_i) \to (x'_j, m'_j)$.

It follows from (3.3.3) by linearity that for $f : K \to K'$,

$$F_{\xi'}(f)\eta_\varphi(K) = \eta_\varphi(K')F_\xi(f) \tag{3.3.6}$$

so that $\eta_\varphi : F_\xi \to F_{\xi'}$ is a natural transformation. Linearity also shows that (3.3.4) holds for general morphisms φ, φ'.

Finally, note that by (3.3.1) and (3.3.2),

$$[1] \circ F_\xi = F_\xi \circ [1] = F_{\xi[1]} \qquad\qquad [1] \circ \eta_\varphi = \eta_\varphi \circ [1] = \eta_{\varphi[1]}$$

for any object ξ and morphism φ. $\square$

3.3.2 Formula to a point

So far the differentials on the complexes $F_\xi(K)$ were just the direct sums $\bigoplus_{i=1}^n d_{x_i}[m_i]$. For the applications, more general differentials are needed.

Let $\varphi = (c_{ji}) : \xi \to \xi'$ be a morphism. Define $\varphi^\star : \xi \to \xi'$ by $\varphi^\star = (c^\star_{ji})$ where $c^\star_{ji} = (-1)^{m'_j - m_i} c_{ji}$.

Lemma 3.3.2. *Let $D : \xi \to \xi[1]$ be a morphism and assume that $D^\star[1] \cdot D = 0$ in $\mathcal{C}_X$. Then for any $K \in C(\mathcal{A})^X$, $\eta_D(K)$ is a differential on $F_\xi(K)$.*

Proof. Since $F_{\xi[1]}(K) = F_\xi(K)[1]$, the morphism D induces a map $\eta_D(K) : F_\xi(K) \to F_\xi(K)[1]$. Thinking of $\eta_D(K)$ as a potential differential, observe that

$$\eta_D(K)[1] = \eta_{-D^\star[1]}(K) \tag{3.3.7}$$

Indeed, each component $K_x[m+1] \to K_{x'}[m'+1]$ of $\eta_D(K)[1]$ is obtained from $K_x[m] \to K_{x'}[m']$ by a change of sign. When $m' = m$, changing the sign of a map $r_{xx'}[m]$ leads to the map $-r_{xx'}[m+1]$. When $m' = m+1$, changing the sign of $d_{x'}[m]r_{xx'}[m]$ leads to $d_{x'}[m+1]r_{xx'}[m+1]$, as the sign change is already carried out in the shift of the differential $d_{x'}[m]$. Therefore in both cases a the coefficient c of $(x,m) \to (x',m')$ changes to $-c^\star$.

Now the claim follows from

$$\eta_D(K)[1] \cdot \eta_D(K) = \eta_{-D^\star[1]}(K)\eta_D(K) = \eta_{-D^\star[1]D}(K) = 0$$

□

Definition 3.3.3. A morphism $\varphi = (c) : (x,m) \to (x',m')$ is a *differential* if $m' = m+1$, $x' = x$ and $c = 1$. φ is a *restriction* if $m' = m$ and $x' \geq x$.

A morphism $\varphi : \xi \to \xi'$ is a *restriction* if all its nonzero components are restrictions.

Definition 3.3.4. A *formula to a point* is a pair (ξ, D) where $\xi = \{(x_i, m_i)\}_{i=1}^n$ is an object of $\mathcal{C}_X$ and $D = (D_{ji})_{i,j=1}^n : \xi \to \xi[1]$ is morphism satisfying:

1. $D^\star[1] \cdot D = 0$.
2. $D_{ji} = 0$ for all $i > j$.
3. D_{ii} are differentials for all $1 \leq i \leq n$.

A *morphism of formulas to a point* $\varphi : (\xi, D) \to (\xi', D')$ is a morphism $\varphi : \xi \to \xi'$ in $\mathcal{C}_X$ which is a restriction and satisfies $\varphi[1]D = D'\varphi$.

Denote by $\mathcal{F}_X$ the category of formulas to a point and their morphisms. The translation $[1]$ of $\mathcal{C}_X$ induces a translation $[1]$ on $\mathcal{F}_X$ by $(\xi, D)[1] = (\xi[1], D[1])$ with the same action on morphisms.

Proposition 3.3.5. *There exists a functor* $\eta : \mathcal{F}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$.

Proof. We actually show that the required functor is induced from the functor η of Proposition 3.3.1.

An object (ξ, D) defines an additive functor $F_{\xi,D} : C(\mathcal{A})^X \to C(\mathcal{A})$ by sending $K \in C(\mathcal{A})^X$ and $f : K \to K'$ to

$$F_{\xi,D}(K) = F_\xi(K) \qquad\qquad F_{\xi,D}(f) = F_\xi(f)$$

as in (3.3.1). By Lemma 3.3.2, $\eta_D(K)$ is a differential on $F_\xi(K)$.

Now observe that $F_\xi(f)[1]\eta_D(K) = \eta_D(K')F_\xi(f)$ since $\eta_D : F_\xi \to F_{\xi[1]}$ is a natural transformation. Therefore $F_\xi(f)$ is a morphism of complexes and $F_{\xi,D}$ is a functor.

Let $\varphi : (\xi, D) \to (\xi', D')$ be a morphism in $\mathcal{F}_X$. Since $\varphi : \xi \to \xi'$ in $\mathcal{C}_X$, we have a natural transformation $\eta_\varphi : F_\xi \to F_{\xi'}$. It remains to show that $\eta_\varphi(K)$ is a morphism of complexes. But the commutativity with the differentials $\eta_D(K)$ and $\eta_{D'}(K)$ follows from $\varphi[1]D = D'\varphi$ and the functoriality of η. □

Example 3.3.6 (Zero dimensional chain)**.** Let $x \in X$ and consider $\xi = \{(x, 0)\}$ with $D = (1)$. The functor $F_{(x,0),(1)}$ sends K to the stalk K_x and $f : K \to K'$ to f_x.

Example 3.3.7 (One dimensional chain)**.** Let $x < y$ in X and consider $\xi = \{(x, 1), (y, 0)\}$ with the map $D = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) : \xi \to \xi[1]$. Then for $K \in C(\mathcal{A})^X$ and $f : K \to K'$,

$$F_{\xi,D}(K) = K_x[1] \oplus K_y \qquad\qquad F_{\xi,D}(f) = \begin{pmatrix} f_x[1] & 0 \\ 0 & f_y \end{pmatrix}$$

with the differential

$$\eta_D(K) = \begin{pmatrix} d_x[1] & 0 \\ r_{xy}[1] & d_y \end{pmatrix} : K_x[1] \oplus K_y \to K_x[2] \oplus K_y[1]$$

Since for any object K, $F_{\xi,D}(K) = \mathrm{C}(K_x \xrightarrow{r_{xy}} K_y)$ as complexes, we see that for any $x < y$, the cone $\mathrm{C}(K_x \xrightarrow{r_{xy}} K_y)$ defines a functor $C(\mathcal{A})^X \to C(\mathcal{A})$.

Lemma 3.3.8. *There exists a natural isomorphism* $\varepsilon : [1] \circ \eta \xrightarrow{\simeq} \eta \circ [1]$.

Proof. We first remark that for an object $(\xi, D) \in \mathcal{F}_X$, a morphism φ and $K \in C(\mathcal{A})^X$, $F_{\xi[1],D[1]}(K) = F_{\xi,D}(K[1])$ and $\eta_{\varphi[1]}(K) = \eta_\varphi(K[1])$ so that $(\eta \circ [1])(\xi, D)$ can be viewed as first applying the shift on $C(\mathcal{A})^X$ and then applying $F_{\xi,D}$.

We will construct natural isomorphisms of functors $\varepsilon_{\xi,D} : [1] \circ F_{\xi,D} \xrightarrow{\simeq} F_{\xi,D} \circ [1]$ such that the diagrams

$$\begin{tikzcd} F_{\xi,D}(K)[1] \arrow[r, "\varepsilon_{\xi,D}"] \arrow[d, "{[1]\circ\eta_\varphi}"'] & F_{\xi[1],D[1]}(K) \arrow[d, "\eta_{\varphi[1]}"] \\ F_{\xi',D'}(K)[1] \arrow[r, "\varepsilon_{\xi',D'}"'] & F_{\xi'[1],D'[1]}(K) \end{tikzcd} \tag{3.3.8}$$

commute for all $K \in C(\mathcal{A})^X$.

By (3.3.7), $[1] \circ F_{\xi,D} = F_{\xi[1],-D^\star[1]}$. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$, $D = (D_{ji})_{i,j=1}^n$, and let $I_\xi : \xi \to \xi$ be the morphism defined by the diagonal matrix whose (i, i) entry is $(-1)^{m_i}$. By definition, $D^\star_{ji} = (-1)^{m_j+1-m_i} D_{ji}$, or equivalently $(-1)^{m_j} D_{ji} = -D^\star_{ji}(-1)^{m_i}$ for all i, j, hence $I_\xi[1]D = -D^\star I_\xi$. Therefore $I_\xi[1] : (\xi[1], D[1]) \to (\xi[1], -D^\star[1])$ is an isomorphism in $\mathcal{F}_X$, so we define $\varepsilon_{\xi,D} = \eta_{I_\xi[1]}$.

For the commutativity of (3.3.8), first observe that $[1] \circ \eta_\varphi = \eta_\varphi \circ [1] = \eta_{\varphi[1]}$. Now use the fact that $I_{\xi'}\varphi = \varphi I_\xi$ for any restriction $\varphi : \xi \to \xi'$. □

In the next few lemmas, we fix a formula to a point (ξ, D).

Lemma 3.3.9. *$F_{\xi,D}$ maps short exact sequences to short exact sequences.*

Proof. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$ and let $0 \to K' \xrightarrow{f'} K \xrightarrow{f''} K'' \to 0$ be a short exact sequence. Then $0 \to K'_x \xrightarrow{f'_x} K_x \xrightarrow{f''_x} K''_x \to 0$ is exact for any $x \in X$, hence

$$0 \to \bigoplus_{i=1}^n K'_{x_i}[m_i] \xrightarrow{\bigoplus_{i=1}^n f'_{x_i}[m_i]} \bigoplus_{i=1}^n K_{x_i}[m_i] \xrightarrow{\bigoplus_{i=1}^n f''_{x_i}[m_i]} \bigoplus_{i=1}^n K''_{x_i}[m_i] \to 0$$

is exact. □

By composing with the equivalence $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$, we may view $F_{\xi,D}$ as a functor $C(\mathcal{A}^X) \to C(\mathcal{A})$ between two categories of complexes.

Lemma 3.3.10. *$F_{\xi,D}$ maps quasi-isomorphisms to quasi-isomorphisms.*

Proof. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$. We prove the claim by induction on n. When $n = 1$, we have $\xi = (x, m)$, $F_{\xi,D}(K) = K_x[m]$ and $F_{\xi,D}(f) = f_x[m]$, so that the claim follows from Corollary 3.2.6.

Assume now that $n > 1$, and let $\xi' = \{(x_i, m_i)\}_{i=1}^{n-1}$ and $D' = (D_{ji})_{i,j=1}^{n-1}$ be the corresponding restricted matrix. By the assumption that $D = (D_{ji})$ is lower triangular with ones on the main diagonal, we have that the canonical embedding $\iota_K : K_{x_n}[m_n] \to \bigoplus_{i=1}^n K_{x_i}[m_i]$ and the projection $\pi_K : \bigoplus_{i=1}^n K_{x_i}[m_i] \to \bigoplus_{i=1}^{n-1} K_{x_i}[m_i]$ commute with the differentials, hence there exists a functorial short exact sequence

$$0 \to (K_{x_n}[m_n], d_{x_n}[m_n]) \to (F_{\xi,D}(K), \eta_D(K)) \to (F_{\xi',D'}(K), \eta_{D'}(K)) \to 0 \qquad (3.3.9)$$

Let $f : K \to K'$ be a morphism. The functoriality of (3.3.9) gives rise to the following diagram of long exact sequences in cohomology,

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^i(F_{\xi',D'}(K)) \arrow[r] \arrow[d, "\mathrm{H}^i(F_{\xi',D'}(f))"] & \mathrm{H}^i(K_{x_n}[m_n]) \arrow[r] \arrow[d, "\mathrm{H}^i(f_{x_n}[m_n])"] & \mathrm{H}^i(F_{\xi,D}(K)) = \arrow[d, "\mathrm{H}^i(F_{\xi,D}(f))"] \\
{} \arrow[r] & \mathrm{H}^i(F_{\xi',D'}(K')) \arrow[r] & \mathrm{H}^i(K'_{x_n}[m_n]) \arrow[r] & \mathrm{H}^i(F_{\xi,D}(K')) =
\end{tikzcd}$$

$$\begin{tikzcd}
= \mathrm{H}^i(F_{\xi,D}(K)) \arrow[r] \arrow[d, "\mathrm{H}^i(F_{\xi,D}(f))"] & \mathrm{H}^{i+1}(F_{\xi',D'}(K)) \arrow[r] \arrow[d, "\mathrm{H}^{i+1}(F_{\xi',D'}(f))"] & \mathrm{H}^{i+1}(K_{x_n}[m_n]) \arrow[r] \arrow[d, "\mathrm{H}^{i+1}(f_{x_n}[m_n])"] & {} \\
= \mathrm{H}^i(F_{\xi,D}(K')) \arrow[r] & \mathrm{H}^{i+1}(F_{\xi',D'}(K')) \arrow[r] & \mathrm{H}^{i+1}(K'_{x_n}[m_n]) \arrow[r] & {}
\end{tikzcd}$$

Now assume that $f : K \to K'$ is a quasi-isomorphism. By the induction hypothesis, $f_{x_n}[m_n] : K_{x_n}[m_n] \to K'_{x_n}[m_n]$ and $F_{\xi',D'}(f) : F_{\xi',D'}(K) \to F_{\xi',D'}(K')$ are quasi-isomorphisms, hence by the Five Lemma, $F_{\xi,D}(f)$ is also a quasi-isomorphism. ☐

Corollary 3.3.11. *Let (ξ, D) be a formula to a point. Then $F_{\xi,D}$ induces a triangulated functor $\widetilde{F}_{\xi,D} : D(\mathcal{A}^X) \to D(\mathcal{A})$.*

3.3.3 General formulas

Definition 3.3.12. Let Y be a poset. A *formula from X to Y* is a diagram over Y with values in $\mathcal{F}_X$.

Proposition 3.3.13. *There exists a functor $\eta : \mathcal{F}_X^Y \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A})^Y)$.*

Proof. Let $\eta : \mathcal{F}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$ be the functor of Proposition 3.3.5. Then

$$\eta^Y : \mathcal{F}_X^Y \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))^Y \simeq \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A})^Y)$$

is the required functor. □

Let $\boldsymbol{\xi} \in \mathcal{F}_X$ be a formula.

Lemma 3.3.14. *$F_{\boldsymbol{\xi}}$ maps short exact sequences to short exact sequences.*

Proof. It is enough to consider each component of $F_{\boldsymbol{\xi}}$ separately. The claim now follows from Lemma 3.3.9. □

By composing from the left with the equivalence $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$ and from the right with $\Phi^{-1} : C(\mathcal{A})^Y \to C(\mathcal{A}^Y)$ we may view $F_{\boldsymbol{\xi}}$ as a functor $C(\mathcal{A}^X) \to C(\mathcal{A}^Y)$ between two categories of complexes.

Lemma 3.3.15. *$F_{\boldsymbol{\xi}}$ maps quasi-isomorphisms to quasi-isomorphisms.*

Proof. Let $f : K \to K'$ be a quasi-isomorphism. By Corollary 3.2.6, it is enough to show that each component of $F_{\boldsymbol{\xi}}(f)$ is a quasi-isomorphism in $C(\mathcal{A})$. But this follows from Lemma 3.3.10. □

Corollary 3.3.16. *Let $\boldsymbol{\xi}$ be a formula. Then $F_{\boldsymbol{\xi}}$ induces a triangulated functor $\widetilde{F_{\boldsymbol{\xi}}} : D(\mathcal{A}^X) \to D(\mathcal{A}^Y)$.*

3.4 Applications of formulas

3.4.1 The chain with two elements

As a first application we consider the case where the poset X is a chain of two elements

$$\bullet_1 \longrightarrow \bullet_2$$

We focus on this simple case as the fundamental underlying principle of Theorem 3.1.1 can already be effectively demonstrated in that case.

Let (ξ_1, D_1), (ξ_2, D_2) and (ξ_{12}, D_{12}) be the following three formulas to a point in $\mathcal{F}_{1\to 2}$.

$$\xi_1 = (1,1), D_1 = (1) \qquad \xi_{12} = ((1,1),(2,0)), D_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.4.1}$$
$$\xi_2 = (2,0), D_2 = (1)$$

Let $\mathcal{A}$ be an abelian category and $K = K_1 \xrightarrow{r_{12}} K_2$ be an object of $C(\mathcal{A}^{1\to 2}) \simeq C(\mathcal{A})^{1\to 2}$. In the more familiar notation,

$$F_{\xi_1,D_1}(K) = K_1[1] \qquad F_{\xi_2,D_2}(K) = K_2 \qquad F_{\xi_{12},D_{12}}(K) = \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2)$$

see Examples 3.3.6 and 3.3.7.

The morphisms

$$\varphi_1 = \begin{pmatrix}1 & 0\end{pmatrix} : \xi_{12} \to \xi_1 \qquad\qquad \varphi_2 = \begin{pmatrix}0 \\ 1\end{pmatrix} : \xi_2 \to \xi_{12}$$

are restrictions that satisfy $\varphi_1 D_{12} = D_1 \varphi_1$ and $\varphi_2 D_2 = D_{12}\varphi_2$, hence

$$\boldsymbol{\xi}^- = (\xi_{12}, D_{12}) \xrightarrow{\varphi_1} (\xi_1, D_1) \qquad\qquad \boldsymbol{\xi}^+ = (\xi_2, D_2) \xrightarrow{\varphi_2} (\xi_{12}, D_{12})$$

are diagrams over $1 \to 2$ with values in $\mathcal{F}_{1\to 2}$, thus they define functors $R^-, R^+ : C(\mathcal{A}^{1\to 2}) \to C(\mathcal{A}^{1\to 2})$ inducing triangulated functors $\widetilde{R}^-, \widetilde{R}^+ : D(\mathcal{A}^{1\to 2}) \to D(\mathcal{A}^{1\to 2})$. Their values on objects $K \in C(\mathcal{A}^{1\to 2})$ are

$$\begin{aligned} R^-(K) &= \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2) \xrightarrow{(\,r_{11}[1]\ 0\,)} K_1[1] \\ R^+(K) &= K_2 \xrightarrow{\left(\begin{smallmatrix}0 \\ r_{22}\end{smallmatrix}\right)} \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2) \end{aligned} \tag{3.4.2}$$

Proposition 3.4.1. *There are natural transformations*

$$R^+ \circ R^- \xrightarrow{\varepsilon^{+-}} [1] \xrightarrow{\varepsilon^{-+}} R^- \circ R^+$$

such that $\varepsilon^{+-}(K)$, $\varepsilon^{-+}(K)$ are quasi-isomorphisms for all $K \in C(\mathcal{A}^{1\to 2})$.

Proof. The functors $R^+ \circ R^-$ and $R^- \circ R^+$ correspond to the compositions $\boldsymbol{\xi}^{+-} = \boldsymbol{\xi}^+ \circ (\boldsymbol{\xi}_1^- \to \boldsymbol{\xi}_2^-)$ and $\boldsymbol{\xi}^{-+} = \boldsymbol{\xi}^- \circ (\boldsymbol{\xi}_1^+ \to \boldsymbol{\xi}_2^+)$, given by

$$\begin{aligned} \boldsymbol{\xi}^{+-} &= (\xi_1, D_1) \xrightarrow{\left(\begin{smallmatrix}0 \\ 0 \\ 1\end{smallmatrix}\right)} (\xi_{121}, D_{121}) \\ \boldsymbol{\xi}^{-+} &= (\xi_{212}, D_{212}) \xrightarrow{(\,1\ 0\ 0\,)} (\xi_2[1], D_2[1]) \end{aligned}$$

where

$$\begin{aligned} (\xi_{121}, D_{121}) &= \left(((1,2),(2,1),(1,1)), \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right) \\ (\xi_{212}, D_{212}) &= \left(((2,1),(1,1),(2,0)), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right) \end{aligned} \tag{3.4.3}$$

and the translation $[1]$ corresponds to the diagram

$$\boldsymbol{\nu} = (\xi_1, D_1) \xrightarrow{(\,1\,)} (\xi_2[1], D_2[1])$$

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the morphisms

$$\begin{aligned} &\alpha_1 : (\xi_1, D_1) \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)} (\xi_{212}, D_{212}) && \beta_1 : (\xi_{212}, D_{212}) \xrightarrow{\left(\begin{smallmatrix} 0 & -1 & 0 \end{smallmatrix}\right)} (\xi_1, D_1) \\ &\alpha_2 : (\xi_2[1], D_2[1]) \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)} (\xi_{121}, D_{121}) && \beta_2 : (\xi_{121}, D_{121}) \xrightarrow{\left(\begin{smallmatrix} 0 & 1 & 1 \end{smallmatrix}\right)} (\xi_2[1], D_2[1]) \end{aligned} \tag{3.4.4}$$

The following diagram in $\mathcal{F}_{1\to 2}$

$$\begin{tikzcd} (\xi_1, D_1) \arrow[r, "(1)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)"'] & (\xi_1, D_1) \arrow[r, "\alpha_1"] \arrow[d, "(1)"] & (\xi_{212}, D_{212}) \arrow[d, "(1\ 0\ 0)"] \\ (\xi_{121}, D_{121}) \arrow[r, "\beta_2"'] & (\xi_2[1], D_2[1]) \arrow[r, "(1)"] & (\xi_2[1], D_2[1]) \end{tikzcd}$$

is commutative, hence the horizontal arrows induce morphisms of formulas $\boldsymbol{\xi}^{+-} \to \boldsymbol{\nu}$ and $\boldsymbol{\nu} \to \boldsymbol{\xi}^{-+}$, inducing natural transformations $\varepsilon^{+-} : R^+R^- \to [1]$ and $\varepsilon^{-+} : [1] \to R^-R^+$.

We prove that $\varepsilon^{+-}(K)$ and $\varepsilon^{-+}(K)$ are quasi-isomorphisms for all K by showing that each component is a quasi-isomorphism (see Corollary 3.2.6). Indeed, let $h_1 : \xi_{212} \to \xi_{212}[-1]$ and $h_2 : \xi_{121} \to \xi_{121}[-1]$ be the maps

$$h_1 = h_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\begin{aligned} &\beta_1\alpha_1 = \begin{pmatrix} 1 \end{pmatrix} && \alpha_1\beta_1 + (h_1[1]D_{212} + D^\star_{212}[-1]h_1) = I_3 \\ &\beta_2\alpha_2 = \begin{pmatrix} 1 \end{pmatrix} && \alpha_2\beta_2 + (h_2[1]D_{121} + D^\star_{121}[-1]h_2) = I_3 \end{aligned} \tag{3.4.5}$$

where I_3 is the 3×3 identity matrix, hence $\beta_1\alpha_1$ and $\beta_2\alpha_2$ induce the identities and $\alpha_1\beta_1$, $\alpha_2\beta_2$ induce morphisms $\eta_{\alpha_1\beta_1}(K)$ and $\eta_{\alpha_2\beta_2}(K)$ homotopic to the identities. Therefore $\eta_{\alpha_1}(K)$, $\eta_{\alpha_2}(K)$, $\eta_{\beta_1}(K)$ and $\eta_{\beta_2}(K)$ are quasi-isomorphisms. $\square$

Proposition 3.4.2. *There are natural transformations*

$$R^+ \circ R^+ \xrightarrow{\varepsilon^{++}} R^- \qquad\qquad R^+ \circ [1] \xrightarrow{\varepsilon^{--}} R^- \circ R^-$$

such that $\varepsilon^{++}(K)$, $\varepsilon^{--}(K)$ are quasi-isomorphisms for all $K \in C(\mathcal{A}^{1\to 2})$.

Proof. The functors $R^+ \circ R^+$ and $R^- \circ R^-$ correspond to the compositions $\boldsymbol{\xi}^{++} = \boldsymbol{\xi}^+ \circ (\boldsymbol{\xi}^+_1 \to \boldsymbol{\xi}^+_2)$ and $\boldsymbol{\xi}^{--} = \boldsymbol{\xi}^- \circ (\boldsymbol{\xi}^-_1 \to \boldsymbol{\xi}^-_2)$, given by

$$\begin{aligned} \boldsymbol{\xi}^{++} &= (\xi_{12}, D_{12}) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} (\xi_{212}, D_{212}) \\ \boldsymbol{\xi}^{--} &= (\xi_{121}, D_{121}) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} (\xi_{12}[1], -D^\star_{12}[1]) \end{aligned}$$

where (ξ_{121}, D_{121}) and (ξ_{212}, D_{212}) are as in (3.4.3). The commutative diagrams

$$
\begin{tikzcd}
(\xi_{12}, D_{12}) \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & (\xi_{12}, D_{12}) \arrow[d, "(1\ 0)"] \\
(\xi_{212}, D_{212}) \arrow[r, "-\beta_1"] & (\xi_1[1], D_1[1])
\end{tikzcd}
\qquad
\begin{tikzcd}
(\xi_2[1], D_2[1]) \arrow[r, "\alpha_2"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & (\xi_{121}, D_{121}) \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)"] \\
(\xi_{12}[1], D_{12}[1]) \arrow[r, "\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & (\xi_{12}[1], -D_{12}^{\star}[1])
\end{tikzcd}
$$

where α_2, β_1 are as in (3.4.4), define morphisms of formulas $\boldsymbol{\xi}^{++} \to \boldsymbol{\xi}^{-}$ and $\boldsymbol{\xi}^{+}[1] \to \boldsymbol{\xi}^{--}$, hence natural transformations $\varepsilon^{++} : R^+R^+ \to R^-$ and $\varepsilon^{--} : R^+[1] \to R^-R^-$. Using the homotopies (3.4.5), one proves that $\varepsilon^{++}(K)$ and $\varepsilon^{--}(K)$ are quasi-isomorphisms for all K in the same way as before. □

Corollary 3.4.3. *For any abelian category $\mathcal{A}$, the functors $\widetilde{R}^+$ and $\widetilde{R}^-$ are auto-equivalences of $D(\mathcal{A})^{1\to 2}$ satisfying*

$$\widetilde{R}^+\widetilde{R}^- \simeq [1] \simeq \widetilde{R}^-\widetilde{R}^+ \qquad (\widetilde{R}^+)^2 \simeq \widetilde{R}^- \qquad (\widetilde{R}^-)^2 \simeq \widetilde{R}^+ \circ [1]$$

hence $(\widetilde{R}^+)^3 \simeq [1]$.

3.4.2 Proof of Theorem 3.1.1

Let X and Y be two posets satisfying the assumptions (3.1.1) and (3.1.2), and let $\leq_+$, $\leq_-$ be the partial orders on $X \sqcup Y$ as defined by (3.1.4). We will prove the universal derived equivalence of $\leq_+$ and $\leq_-$ by defining two formulas $\boldsymbol{\xi}^+$, $\boldsymbol{\xi}^-$ that will induce, for any abelian category $\mathcal{A}$, functors

$$R^+ = F_{\boldsymbol{\xi}^+} : C(\mathcal{A})^{\leq_+} \to C(\mathcal{A})^{\leq_-} \qquad R^- = F_{\boldsymbol{\xi}^-} : C(\mathcal{A})^{\leq_-} \to C(\mathcal{A})^{\leq_+}$$

and

$$\widetilde{R}^+ = \widetilde{F}_{\boldsymbol{\xi}^+} : D(\mathcal{A}^{\leq_+}) \to D(\mathcal{A}^{\leq_-}) \qquad \widetilde{R}^- = \widetilde{F}_{\boldsymbol{\xi}^-} : D(\mathcal{A}^{\leq_-}) \to D(\mathcal{A}^{\leq_+})$$

such that $\widetilde{R}^+\widetilde{R}^- \simeq [1]$ and $\widetilde{R}^-\widetilde{R}^+ \simeq [1]$.

Definition of the formulas to points

For $x \in X$ and $y \in Y$, let

$$\xi_x = \big((x,0), (1)\big) \qquad \xi_y = \big((y,0), (1)\big) \qquad \xi_{Y_x} = ((y,0)_{y \in Y_x}, I)$$

where I is the identity matrix. We consider ξ_x, ξ_y and ξ_{Y_x} as formulas either in $\mathcal{F}_{\leq_+}$ or in $\mathcal{F}_{\leq_-}$, as appropriate. If $y \in Y$, define

$$\boldsymbol{\xi}_y^+ = \xi_y \in \mathcal{F}_{\leq_+} \qquad \boldsymbol{\xi}_y^- = \xi_y \in \mathcal{F}_{\leq_-}$$

as in Example 3.3.6. If $x \in X$, let

$$\xi_{x,Y_x} = \big(\xi_x \xrightarrow{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}} \xi_{Y_x}\big) \in \mathcal{F}^{1\to 2}_{\leq_+} \qquad \xi_{Y_x,x} = \big(\xi_{Y_x} \xrightarrow{(1\ 1\ \dots\ 1)} \xi_x\big) \in \mathcal{F}^{1\to 2}_{\leq_-}$$

be formulas to $1 \to 2$ and define

$$\boldsymbol{\xi}^+_x = \xi_{12} \circ \xi_{x,Y_x} \qquad \boldsymbol{\xi}^-_x = \xi_{12} \circ \xi_{Y_x,x}$$

as compositions with the formula ξ_{12} defined in (3.4.1).

In explicit terms, let $K \in C(\mathcal{A})^{\leq_+}$, $L \in C(\mathcal{A})^{\leq_-}$, and denote by $\{r_{xy}\}$ the restriction maps in K and by $\{s_{yx}\}$ the restriction maps in L. For $x \in X$ and $y \in Y_x$, let $\iota_y : K_y \to \bigoplus_{y_x \in Y_x} K_{y_x}$ and $\pi_y : \bigoplus_{y_x \in Y_x} L_{y_x} \to L_y$ be the canonical inclusions and projections. Then

$$R^+(K)_x = \mathrm{C}(K_x \xrightarrow{\sum_{y\in Y_x} \iota_y r_{xy}} \bigoplus_{y \in Y_x} K_y) \qquad R^+(K)_y = K_y$$

$$R^-(L)_x = \mathrm{C}(\bigoplus_{y \in Y_x} L_y \xrightarrow{\sum_{y\in Y_x} s_{yx} \pi_y} L_x) \qquad R^-(L)_y = L_y[1]$$

for $x \in X$, $y \in Y$.

Definition of the restriction maps

We shall denote by ρ^+ the restriction maps between the formulas in R^+ and by ρ^- the maps between those in R^-. We consider several cases, and use the explicit notation.

For $y \leq y'$, define

$$\rho^+_{yy'}(K) = r_{yy'} : K_y \to K_{y'} \qquad \rho^-_{yy'}(L) = s_{yy'}[1] : L_y[1] \to L_{y'}[1]$$

For $x \leq x'$, we use the isomorphism $\varphi_{x,x'} : Y_x \to Y_{x'}$ and the property that $y \leq \varphi_{x,x'}(y)$ for all $y \in Y_x$ to define the diagonal maps

$$\rho^+_{xx'}(K) = r_{xx'}[1] \oplus (\bigoplus_{y \in Y_x} r_{y,\varphi_{xx'}(y)}) : R^+(K)_x \to R^+(K)_{x'}$$

$$\rho^-_{xx'}(L) = (\bigoplus_{y \in Y_x} s_{y,\varphi_{xx'}(y)}[1]) \oplus s_{xx'} : R^-(L)_x \to R^-(L)_{x'}$$

If $y_x \in Y_x$, then by (3.1.4), $y_x \leq_- x$, $x \leq_+ y_x$, and we define

$$\rho^+_{y_x x}(K) = K_{y_x} \xrightarrow{\begin{pmatrix}0\\\iota_{y_x}\end{pmatrix}} \mathrm{C}(K_x \to \bigoplus_{y \in Y_x} K_y)$$

$$\rho^-_{x y_x}(L) = \mathrm{C}(\bigoplus_{y \in Y_x} L_y \to K_x) \xrightarrow{(\pi_{y_x}[1]\ 0)} L_{y_x}[1]$$

Finally, if $y \leq_- x$, by (3.1.1) there exists a *unique* $y_x \in Y_x$ such that $y \leq y_x$ and we set $\rho^+_{yx}(K) = \rho^+_{y_x x}(K)\rho^+_{y y_x}(K)$. Similarly, if $x \leq_+ y$, there exists a unique $y_x \in Y_x$ with $y_x \leq y$, and we set $\rho^-_{xy}(L) = \rho^-_{y_x y}(L)\rho^-_{x y_x}(L)$.

Verification of commutativity

Again there are several cases to consider. First, when $y \leq y' \leq y''$, $\rho^+_{yy''} = \rho^+_{y'y''}\rho^+_{yy'}$ follows from the commutativity of the restrictions $r_{yy''} = r_{y'y''}r_{yy'}$, and similarly for ρ^-.

Let $x \leq x' \leq x''$. Since $\varphi_{xx'} : Y_x \to Y_{x'}$ is an isomorphism and $\varphi_{xx''} = \varphi_{x'x''}\varphi_{xx'}$, we can write

$$\begin{aligned}\rho^+_{x'x''}(K) &= r_{x'x''}[1] \oplus \bigoplus_{y' \in Y_{x'}} r_{y',\varphi_{x'x''}(y')} = r_{x'x''}[1] \oplus \bigoplus_{y \in Y_x} r_{\varphi_{xx'}(y),\varphi_{x'x''}\varphi_{xx'}(y)} \\ &= r_{x'x''}[1] \oplus \bigoplus_{y \in Y_x} r_{\varphi_{xx'}(y),\varphi_{xx''}(y)}\end{aligned}$$

Now $\rho^+_{xx''} = \rho^+_{x'x''}\rho^+_{xx'}$ follows from the commutativity of the restrictions $r_{xx''} = r_{x'x''}r_{xx'}$ and $r_{y,\varphi_{xx''}(y)} = r_{\varphi_{xx'}(y),\varphi_{xx''}(y)}r_{y,\varphi_{xx'}(y)}$. The proof for ρ^- is similar.

If $y' \leq y \leq_- x$, let $y_x, y'_x \in Y_x$ be the elements satisfying $y \leq y_x$, $y' \leq y'_x$. Then $y'_x = y_x$ by uniqueness, since $y' \leq y_x$. Hence

$$\rho^+_{y'x} = \rho^+_{y_xx}\rho^+_{y'y_x} = \rho^+_{y_xx}\rho^+_{yy_x}\rho^+_{y'y} = \rho^+_{yx}\rho^+_{y'y}$$

The proof for ρ^- in the case $x \leq_+ y \leq y'$ is similar.

If $y_x \leq_- x \leq x'$ where $y_x \in Y_x$, then $y_{x'} = \varphi_{xx'}(y_x)$ is the unique element $y_{x'} \in Y_{x'}$ with $y_x \leq y_{x'}$, and

$$\rho^+_{y_xx'} = \rho^+_{\varphi_{xx'}(y_x),x'}\rho^+_{y_x,\varphi_{xx'}(y_x)} = \rho^+_{xx'}\rho^+_{y_xx}$$

by the commutativity of the diagram

$$\begin{tikzcd}
K_{y_x} \arrow[r, "\rho^+_{y_x,x}"] \arrow[d, "r_{y_x,\varphi_{xx'}(y_x)}"'] & \mathrm{C}(K_x \to \bigoplus_{y \in Y_x} K_y) \arrow[d, "\rho^+_{xx'} = r_{xx'}[1] \oplus \bigoplus r_{y,\varphi_{xx'}(y)}"] \\
K_{\varphi_{x,x'}(y_x)} \arrow[r, "\rho^+_{\varphi_{xx'}(y_x),x'}"] & \mathrm{C}(K_{x'} \to \bigoplus_{y' \in Y_{x'}} K_{y'})
\end{tikzcd}$$

Now if $y \leq_- x \leq x'$, let $y_x \in Y_x$ be the element with $y \leq y_x$. Then $y \leq y_x \leq_- x \leq x'$ and commutativity follows from the previous two cases:

$$\rho^+_{yx'} = \rho^+_{y_xx'}\rho^+_{yy_x} = \rho^+_{xx'}\rho^+_{y_xx}\rho^+_{yy_x} = \rho^+_{xx'}\rho^+_{yx}$$

The proof for ρ^- in the cases $x' \leq x \leq_+ y_x$ and $x' \leq x \leq_+ y$ is similar. Here we also use fact that $\varphi_{x'x}$ is an isomorphism to pick $y_{x'} = \varphi^{-1}_{x'x}(y_x)$ as the unique element $y_{x'} \in Y_{x'}$ with $y_{x'} \leq y_x$.

Construction of the natural transformations $R^+R^- \to [1] \to R^-R^+$

Observe that

$$\begin{aligned} (\boldsymbol{\xi}^+\boldsymbol{\xi}^-)_y &= \xi_y[1] & \qquad (\boldsymbol{\xi}^-\boldsymbol{\xi}^+)_y &= \xi_y[1] \\ (\boldsymbol{\xi}^+\boldsymbol{\xi}^-)_x &= \xi_{121} \circ \xi_{Y_x,x} & \qquad (\boldsymbol{\xi}^-\boldsymbol{\xi}^+)_x &= \xi_{212} \circ \xi_{x,Y_x} \end{aligned}$$

where ξ_{121} and ξ_{212} are the formulas defined in (3.4.3).

Let $\boldsymbol{\nu}$ be the formula inducing the translation and define $\varepsilon^{+-} : \boldsymbol{\xi}^{+}\boldsymbol{\xi}^{-} \to \boldsymbol{\nu}$, $\varepsilon^{-+} : \boldsymbol{\nu} \to \boldsymbol{\xi}^{-}\boldsymbol{\xi}^{+}$ by

$$
\begin{aligned}
&\varepsilon_y^{+-} : \xi_y[1] \xrightarrow{(1)} \xi_y[1] \\
&\varepsilon_x^{+-} : \xi_{121} \circ \xi_{Y_x,x} \xrightarrow{\beta_2 \circ \xi_{Y_x,x}} \xi_2[1] \circ \xi_{Y_x,x} = \xi_x[1] \\
&\varepsilon_y^{-+} : \xi_y[1] \xrightarrow{(1)} \xi_y[1] \\
&\varepsilon_x^{-+} : \xi_x[1] = \xi_1 \circ \xi_{x,Y_x} \xrightarrow{\alpha_1 \circ \xi_{x,Y_x}} \xi_{212} \circ \xi_{x,Y_x,x}
\end{aligned}
$$

where ξ_1 and ξ_2 are as in (3.4.1) and α_1 and β_2 are as in Proposition 3.4.1. The proof of that proposition also shows that ε^{+-} and ε^{-+} are morphisms of formulas and induce natural transformations between functors, which are quasi-isomorphisms.

3.4.3 Proof of Corollary 3.1.6

Let X and Z be posets, and let $Y = \mathbf{1} \oplus Z$. Denote by $1 \in Y$ the unique minimal element and consider the map $f : X \to Y$ defined by $f(x) = 1$ for all $x \in X$. Then

$$(X \sqcup Y, \leq_+^f) \simeq X \oplus \mathbf{1} \oplus Z \qquad\qquad (X \sqcup Y, \leq_-^f) \simeq \mathbf{1} \oplus (X + Z)$$

hence by Corollary 3.1.3, $X \oplus \mathbf{1} \oplus Z$ and $\mathbf{1} \oplus (X + Z)$ are universally derived equivalent.

Part II

Combinatorial Applications for Tilting Objects

Chapter 4

Universal Derived Equivalences of Posets of Tilting Modules

We show that for two quivers without oriented cycles related by a BGP reflection, the posets of their tilting modules are related by a simple combinatorial construction, which we call flip-flop.

We deduce that the posets of tilting modules of derived equivalent path algebras of quivers without oriented cycles are universally derived equivalent.

4.1 Introduction

In this chapter we investigate the combinatorial relations between the posets of tilting modules of derived equivalent path algebras. While it is known that these posets are in general not isomorphic, we show that they are related via a sequence of simple combinatorial constructions, which we call flip-flops.

For two partially ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ and an order preserving function $f : X \to Y$, one can define two partial orders $\leq_+^f$ and $\leq_-^f$ on the disjoint union $X \sqcup Y$, by keeping the original partial orders inside X and Y and setting

$$x \leq_+^f y \iff f(x) \leq_Y y$$
$$y \leq_-^f x \iff y \leq_Y f(x)$$

with no other additional order relations. We say that two posets Z and Z' are related via a *flip-flop* if there exist X, Y and $f : X \to Y$ as above such that $Z \simeq (X \sqcup Y, \leq_+^f)$ and $Z' \simeq (X \sqcup Y, \leq_-^f)$.

Throughout this chapter, we fix an algebraically closed field k. Given a (finite) quiver Q without oriented cycles, consider the category of finite-dimensional modules over the path algebra of Q, which is equivalent to the category $\operatorname{rep} Q$ of finite dimensional representations of Q over k, and denote by $\mathcal{T}_Q$ the poset of tilting modules in $\operatorname{rep} Q$ as introduced by [75]. For more information on the partial order on tilting modules see [44], the survey [84] and the references therein.

Let x be a source of Q and let Q' be the quiver obtained from Q by a BGP reflection, that is, by reverting all arrows starting at x. The combinatorial relation between the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ is expressed in the following theorem.

Theorem 4.1.1. *The posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flip.*

In fact, the subset Y in the definition of a flip-flop can be explicitly described as the set of tilting modules containing the simple at x as direct summand, and we show that it is isomorphic as poset to $\mathcal{T}_{Q \setminus \{x\}}$.

While two posets Z and Z' related via a flip-flop are in general not isomorphic, they are *universally derived equivalent* in the following sense; for any abelian category $\mathcal{A}$, the derived categories of the categories of functors $Z \to \mathcal{A}$ and $Z' \to \mathcal{A}$ are equivalent as triangulated categories, see Corollary 3.1.3.

For two quivers without oriented cycles Q and Q', we denote $Q \sim Q'$ if Q' can be obtained from Q by a sequence of BGP reflections (at sources or sinks). It is known that the path algebras of Q and Q' are derived equivalent if and only if $Q \sim Q'$, see [35, (I.5.7)], hence by Corollary 3.1.3 we deduce the following theorem.

Theorem 4.1.2. *Let Q and Q' be two quivers without oriented cycles whose path algebras are derived equivalent. Then the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are universally derived equivalent.*

The chapter is structured as follows. In Section 4.2 we study the structure of the poset $\mathcal{T}_Q$ with regard to a source vertex x, where the main tool is the existence of an exact functor right adjoint to the restriction $\operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\})$. For the convenience of the reader, we record the dual statements for the case of a sink in Section 4.3. Building on these results, we analyze the effect of a BGP reflection in Section 4.4, where a proof of Theorem 4.1.1 is given. We conclude by demonstrating the theorem on a concrete example in Section 4.5.

4.2 Tilting modules with respect to a source

Let Q be a quiver. For a representation M in $\operatorname{rep} Q$, denote by $M(y)$ the vector space corresponding to a vertex y and by $M(y \to y')$ the linear transformation $M(y) \to M(y')$ corresponding to an edge $y \to y'$ in Q.

Let x be a source in the quiver Q, to be fixed throughout this section.

Lemma 4.2.1. *The inclusion $j : Q \setminus \{x\} \to Q$ induces a pair (j^{-1}, j_*) of functors*

$$j^{-1} : \operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\}) \qquad\qquad j_* : \operatorname{rep}(Q \setminus \{x\}) \to \operatorname{rep} Q$$

such that

$$\operatorname{Hom}_{Q\setminus\{x\}}(j^{-1}M, N) \simeq \operatorname{Hom}_Q(M, j_*N) \tag{4.2.1}$$

for all $M \in \operatorname{rep} Q$, $N \in \operatorname{rep}(Q \setminus \{x\})$ (that is, j_* is a right adjoint to j^{-1}).

Proof. We shall write the functors j^{-1} and j_* explicitly. For $M \in \operatorname{rep} Q$, define

$$(j^{-1}M)(y) = M(y) \qquad\qquad (j^{-1}M)(y \to y') = M(y \to y')$$

for any $y \to y'$ in $Q \setminus \{x\}$. For $N \in \operatorname{rep}(Q \setminus \{x\})$, define

$$
\begin{aligned}
(j_*N)(y) &= N(y) & (j_*N)(y \to y') &= N(y \to y') \\
(j_*N)(x) &= \bigoplus_{i=1}^{m} N(y_i) & (j_*N)(x \to y_i) &= (j_*N)(x) \to N(y_i)
\end{aligned}
\tag{4.2.2}
$$

where $y_1, \dots, y_m$ are the endpoints of the arrows starting at x, $(j_*N)(x) \to N(y_i)$ are the natural projections, and y, y' are in $Q \setminus \{x\}$.

Now (4.2.1) follows since the maps $M(y_i) \to N(y_i)$ for $1 \le i \le m$ induce a unique map $M(x) \to N(y_1) \oplus \cdots \oplus N(y_m)$ such that the diagrams

$$
\begin{tikzcd}
M(x) \arrow[r] \arrow[d] & (j_*N)(x) = \bigoplus_{i=1}^{m} N(y_i) \arrow[d] \\
M(y_i) \arrow[r] & N(y_i)
\end{tikzcd}
$$

commute for all $1 \le i \le m$. □

Lemma 4.2.2. *The functor j_* is fully faithful and exact.*

Proof. Observe that $j^{-1}j_*$ is the identity on $\operatorname{rep}(Q \setminus \{x\})$, hence for $N, N' \in \operatorname{rep}(Q \setminus \{x\})$,

$$
\operatorname{Hom}_Q(j_*N, j_*N') \simeq \operatorname{Hom}_{Q\setminus\{x\}}(j^{-1}j_*N, N') = \operatorname{Hom}_{Q\setminus\{x\}}(N, N')
$$

so that j_* is fully faithful. Its exactness follows from (4.2.2). □

Denote by $\mathcal{D}^b(Q)$ the bounded derived category $\mathcal{D}^b(\operatorname{rep} Q)$. The exact functors j^{-1} and j_* induce functors

$$
j^{-1} : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q \setminus \{x\}) \qquad\qquad j_* : \mathcal{D}^b(Q \setminus \{x\}) \to \mathcal{D}^b(Q)
$$

with

$$
\operatorname{Hom}_{\mathcal{D}^b(Q\setminus\{x\})}(j^{-1}M, N) \simeq \operatorname{Hom}_{\mathcal{D}^b(Q)}(M, j_*N) \tag{4.2.3}
$$

for all $M \in \mathcal{D}^b(Q)$, $N \in \mathcal{D}^b(Q \setminus \{x\})$.

Let S_x be the simple (injective) object of $\operatorname{rep} Q$ corresponding to x.

Lemma 4.2.3. *The functor j_* identifies $\operatorname{rep}(Q \setminus \{x\})$ with the right perpendicular subcategory*

$$
S_x^{\perp} = \left\{ M \in \operatorname{rep} Q \ : \ \operatorname{Ext}^i(S_x, M) = 0 \textit{ for all } i \ge 0 \right\} \tag{4.2.4}
$$

of $\operatorname{rep} Q$.

Proof. Observe that $j^{-1}S_x = 0$. Hence by (4.2.3),

$$
\operatorname{Ext}^i_Q(S_x, j_*N) = \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}S_x, N) = 0
$$

for all $N \in \operatorname{rep}(Q \setminus \{x\})$.

Conversely, let M be such that $\operatorname{Ext}^i_Q(S_x, M) = 0$ for $i \geq 0$, and let $\varphi : M \to j_* j^{-1} M$ be the adjunction morphism. From $j^{-1} j_* j^{-1} M = j^{-1} M$ we see that $(\ker \varphi)(y) = 0 = (\operatorname{coker} \varphi)(y)$ for all $y \neq x$.

From $0 \to \ker \varphi \to M$ we get

$$0 \to \operatorname{Hom}_Q(S_x, \ker \varphi) \to \operatorname{Hom}_Q(S_x, M) = 0 \tag{4.2.5}$$

hence $\ker \varphi = 0$. Thus $0 \to M \to j_* j^{-1} M \to \operatorname{coker} \varphi \to 0$ is exact, and from

$$0 = \operatorname{Hom}_Q(S_x, j_* j^{-1} M) \to \operatorname{Hom}_Q(S_x, \operatorname{coker} \varphi) \to \operatorname{Ext}^1_Q(S_x, M) = 0$$

we deduce that $\operatorname{coker} \varphi = 0$, hence $M \simeq j_* j^{-1} M$. □

Lemma 4.2.4. *The functor j_* takes indecomposables of $\operatorname{rep}(Q \setminus \{x\})$ to indecomposables of $\operatorname{rep} Q$.*

Proof. Let N be an indecomposable representation of $Q \setminus \{x\}$, and assume that $j_* N = M_1 \oplus M_2$. Then $N \simeq j^{-1} j_* N = j^{-1} M_1 \oplus j^{-1} M_2$, hence we may assume that $j^{-1} M_2 = 0$.

Thus $M_2 = S_x^n$ for some $n \geq 0$. But $j_* N$ belongs to the right perpendicular subcategory $S_x^{\perp}$ which is closed under direct summands, hence $n = 0$ and $M_2 = 0$. □

Recall that $T \in \operatorname{rep} Q$ is a *tilting module* if $\operatorname{Ext}^i(T, T) = 0$ for all $i > 0$, and the direct summands of T generate $\mathcal{D}^b(Q)$ as a triangulated category. If T is basic, the latter condition can be replaced by the condition that the number of indecomposable summands of T equals the number of vertices of Q.

For a tilting module T, define

$$T^{\perp} = \left\{ M \in \operatorname{rep} Q \,:\, \operatorname{Ext}^i(T, M) = 0 \text{ for all } i > 0 \right\}$$

and set $T \leq T'$ if $T^{\perp} \supseteq T'^{\perp}$. By [44], $T \leq T'$ if and only if $\operatorname{Ext}^i_Q(T, T') = 0$ for all $i > 0$.

Denote by $\mathcal{T}_Q$ the set of basic tilting modules of $\operatorname{rep} Q$, and by $\mathcal{T}_Q^x$ the subset of $\mathcal{T}_Q$ consisting of all tilting modules which have S_x as direct summand.

Lemma 4.2.5. *$\mathcal{T}_Q^x$ is an open subset of $\mathcal{T}_Q$, that is, if $T \in \mathcal{T}_Q^x$ and $T \leq T'$, then $T' \in \mathcal{T}_Q^x$.*

Proof. Let $T \in \mathcal{T}_Q^x$ and $T' \in \mathcal{T}_Q$ such that $T \leq T'$. Then $T' \in T^{\perp}$, and in particular $\operatorname{Ext}^i(S_x, T') = 0$ for $i > 0$. Since S_x is injective, it follows that $\operatorname{Ext}^i(T', S_x) = 0$ for $i > 0$, hence if $T' \notin \mathcal{T}_Q^x$, then $S_x \oplus T'$ would also be a basic tilting module, contradiction to the fact that the number of indecomposable summands of a basic tilting module equals the number of vertices of Q. □

Proposition 4.2.6. *Let T be a tilting module in $\operatorname{rep} Q$. Then $j^{-1} T$ is a tilting module of $\operatorname{rep}(Q \setminus \{x\})$.*

Proof. We consider two cases. First, assume that T contains S_x as direct summand. Write $T = S_x^n \oplus T'$ with $n > 0$, where T' does not have S_x as direct summand. Then $j^{-1}T = j^{-1}T'$ and $T' \in S_x^{\perp}$, hence $j_* j^{-1}T' = T'$ and

$$\begin{aligned}\operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T, j^{-1}T) &= \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T', j^{-1}T') \\ &= \operatorname{Ext}^i_Q(T', j_* j^{-1}T') = \operatorname{Ext}^i_Q(T', T') = 0 \quad (4.2.6)\end{aligned}$$

Now assume that T does not contain S_x as direct summand, and let $\varphi : T \to j_* j^{-1}T$ be the adjunction morphism. Then $\operatorname{Hom}_Q(S_x, T) = 0$ and similarly to (4.2.5), we deduce that $\ker \varphi = 0$. Observe that $\operatorname{coker} \varphi = S_x^n$ for some $n \geq 0$ is injective, hence from the exact sequence $0 \to T \to j_* j^{-1}T \to \operatorname{coker} \varphi \to 0$ we get for $i > 0$,

$$0 = \operatorname{Ext}^i(T, T) \to \operatorname{Ext}^i(T, j_* j^{-1}T) \to \operatorname{Ext}^i(T, \operatorname{coker} \varphi) = 0 \quad (4.2.7)$$

therefore $\operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T, j^{-1}T) = \operatorname{Ext}^i_Q(T, j_* j^{-1}T) = 0$ for $i > 0$.

To show that the direct summands of $j^{-1}T$ generate $\mathcal{D}^b(Q \setminus \{x\})$, it is enough to verify that for any $y \in Q\setminus\{x\}$, the corresponding projective P_y in $\operatorname{rep}(Q\setminus\{x\})$ has a resolution with objects from $\operatorname{add} j^{-1}T$. Indeed, let $y \in Q \setminus \{x\}$ and consider the projective $\widetilde{P}_y$ of $\operatorname{rep} Q$. Applying the exact functor j^{-1} on an $\operatorname{add} T$-resolution of $\widetilde{P}_y$ gives the required $\operatorname{add} j^{-1}T$-resolution of $P_y = j^{-1}\widetilde{P}_y$. □

Note that $j^{-1}T$ may not be basic even if T is basic. Write $\operatorname{basic}(j^{-1}T)$ for the module obtained from $j^{-1}T$ by deleting duplicate direct summands. Then $\operatorname{basic}(j^{-1}T)$ is a basic tilting module with $\operatorname{basic}(j^{-1}T)^{\perp} = (j^{-1}T)^{\perp}$. It follows by the adjunction (4.2.3) that for $N \in \operatorname{rep}(Q \setminus \{x\})$,

$$N \in (j^{-1}T)^{\perp} \iff j_* N \in T^{\perp}$$

Corollary 4.2.7. *The map $\pi_x : T \mapsto \operatorname{basic}(j^{-1}T)$ is an order-preserving function $(\mathcal{T}_Q, \leq) \to (\mathcal{T}_{Q\setminus\{x\}}, \leq)$.*

Proof. Let $T \leq T'$ and consider $N \in (j^{-1}T')^{\perp}$. Then $j_* N \in T'^{\perp} \subseteq T^{\perp}$, hence $N \in (j^{-1}T)^{\perp}$, so that $j^{-1}T \leq j^{-1}T'$. □

Let N, N' be objects of $\operatorname{rep}(Q \setminus \{x\})$ with $\operatorname{Ext}^i_{Q\setminus\{x\}}(N, N') = 0$ for all $i > 0$. By the adjunctions (4.2.3),

$$\begin{aligned}&\operatorname{Ext}^i_Q(j_* N, j_* N') \simeq \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1} j_* N, N') = \operatorname{Ext}^i_{Q\setminus\{x\}}(N, N') = 0 \\ &\operatorname{Ext}^i_Q(S_x, j_* N') \simeq \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1} S_x, N') = 0 \\ &\operatorname{Ext}^i_Q(j_* N, S_x) = 0\end{aligned}$$

where the last equation follows since S_x injective. Hence

$$\operatorname{Ext}^i_Q(S_x \oplus j_* N, S_x \oplus j_* N') = 0 \text{ for all } i > 0 \quad (4.2.8)$$

Corollary 4.2.8. *Let T be a basic tilting module in $\operatorname{rep}(Q \setminus \{x\})$. Then $S_x \oplus j_* T$ is a basic tilting module in $\operatorname{rep} Q$.*

Proof. Indeed, $\mathrm{Ext}^i_Q(S_x \oplus j_*T, S_x \oplus j_*T) = 0$ for $i > 0$, by (4.2.8).

Let n be the number of vertices of Q. Since T is a basic tilting module for $Q \setminus \{x\}$, it has $n-1$ indecomposable summands, hence by Lemmas 4.2.3 and 4.2.4, j_*T decomposes into $n-1$ indecomposable summands. It follows that $S_x \oplus j_*T$ is a tilting module. $\square$

Corollary 4.2.9. *The map $\iota_x : T \mapsto S_x \oplus j_*T$ is an order preserving function $(\mathcal{T}_{Q\setminus\{x\}}, \leq) \to (\mathcal{T}^x_Q, \leq)$.*

Proof. Let $T \leq T'$ in $\mathcal{T}_{Q\setminus\{x\}}$. Then $\mathrm{Ext}^i_{Q\setminus\{x\}}(T, T') = 0$ for all $i > 0$ and the claim follows from (4.2.8). $\square$

Proposition 4.2.10. *We have*

$$\pi_x \iota_x(T) = T$$

for all $T \in \mathcal{T}_{Q\setminus\{x\}}$. In addition,

$$T \leq \iota_x \pi_x(T)$$

for all $T \in \mathcal{T}_Q$, with equality if and only if $T \in \mathcal{T}^x_Q$.

In particular we see that ι_x induces a retract $\iota_x\pi_x$ of $\mathcal{T}_Q$ onto $\mathcal{T}^x_Q$ and an isomorphism of posets between $\mathcal{T}_{Q\setminus\{x\}}$ and $\mathcal{T}^x_Q$.

Proof. If $T \in \mathcal{T}_{Q\setminus\{x\}}$, then $j^{-1}(S_x \oplus j_*T) = j^{-1}j_*T = T$, hence $\pi_x\iota_x(T) = \mathrm{basic}(T) = T$.

Let $T \in \mathcal{T}_Q$. Then $\mathrm{Ext}^i_Q(T, S_x) = 0$ for $i > 0$. Moreover, by the argument in the proof of Proposition 4.2.6 (see (4.2.6) and (4.2.7)), $\mathrm{Ext}^i_Q(T, j_*j^{-1}T) = 0$. It follows that $S_x \oplus j_*j^{-1}T \in T^\perp$, thus $T \leq \iota_x\pi_x(T)$.

If $T = \iota_x\pi_x(T)$, then obviously T has S_x as summand, so that $T \in \mathcal{T}^x_Q$. Conversely, if $T \in \mathcal{T}^x_Q$, then $T = S_x \oplus T'$ with $T' \in S_x^\perp$, and by Lemma 4.2.3, $T' = j_*j^{-1}T'$, hence $\iota_x\pi_x(T) = S_x \oplus j_*j^{-1}T' = S_x \oplus T' = T$. $\square$

Corollary 4.2.11. *Let $X = \mathcal{T}_Q \setminus \mathcal{T}^x_Q$ and $Y = \mathcal{T}^x_Q$. Define $f : X \to Y$ by $f = \iota_x\pi_x$. Then $\mathcal{T}_Q \simeq (X \sqcup Y, \leq^f_+)$.*

Proof. Let $T \in X$ and $T' \in Y$. If $T \leq T'$, then by the previous proposition,

$$f(T) = \iota_x\pi_x(T) \leq \iota_x\pi_x(T') = T'$$

hence $T \leq T'$ in $\mathcal{T}_Q$ if and only if $f(T) \leq T'$ in $\mathcal{T}^x_Q$. $\square$

4.3 Tilting modules with respect to a sink

Now let Q' be the quiver obtained from Q by reflection at the source x. For the convenience of the reader, we record, without proofs, the analogous (dual) results for this case.

Lemma 4.3.1. *The inclusion $i : Q \setminus \{x\} \to Q'$ induces a pair $(i_!, i^{-1})$ of functors*

$$i^{-1} : \operatorname{rep} Q' \to \operatorname{rep}(Q \setminus \{x\}) \qquad\qquad i_! : \operatorname{rep}(Q \setminus \{x\}) \to \operatorname{rep} Q'$$

such that

$$\operatorname{Hom}_{\operatorname{rep}(Q\setminus\{x\})}(N, i^{-1}M) \simeq \operatorname{Hom}_{\operatorname{rep} Q}(i_!N, M)$$

for all $M \in \operatorname{rep} Q$, $N \in \operatorname{rep}(Q \setminus \{x\})$ (that is, $i_!$ is a left adjoint to i^{-1}).

Proof. For $M \in \operatorname{rep} Q'$, define

$$(i^{-1}M)(y) = M(y) \qquad\qquad (i^{-1}M)(y \to y') = M(y \to y')$$

for any $y \to y'$ in $Q \setminus \{x\}$. For $N \in \operatorname{rep}(Q \setminus \{x\})$, define

$$\begin{aligned} (i_!N)(y) &= N(y) & \qquad (i_!N)(y \to y') &= N(y \to y') \\ (i_!N)(x) &= \bigoplus_{l=1}^{m} N(y_l) & \qquad (i_!N)(y_l \to x) &= N(y_l) \to (i_!N)(x) \end{aligned}$$

where $y_1, \dots, y_m$ are the starting points of the arrows ending at x, $N(y_l) \to (i_!N)(x)$ are the natural inclusions, and y, y' are in $Q \setminus \{x\}$. □

Lemma 4.3.2. *The functor $i_!$ is fully faithful and exact.*

Let S'_x be the simple (projective) object of $\operatorname{rep} Q'$ corresponding to x.

Lemma 4.3.3. *The functor $i_!$ identifies $\operatorname{rep}(Q \setminus \{x\})$ with the left perpendicular subcategory*

$$^{\perp}S'_x = \left\{M \in \operatorname{rep} Q' \,:\, \operatorname{Ext}^i(M, S'_x) = 0 \textit{ for all } i \geq 0\right\}$$

of $\operatorname{rep} Q'$.

Lemma 4.3.4. *The functor $i_!$ takes indecomposables of $\operatorname{rep}(Q \setminus \{x\})$ to indecomposables of $\operatorname{rep} Q'$.*

Denote by $\mathcal{T}^x_{Q'}$ the subset of $\mathcal{T}_{Q'}$ consisting of all tilting modules which have S'_x as direct summand.

Lemma 4.3.5. *$\mathcal{T}^x_{Q'}$ is a closed subset of $\mathcal{T}_{Q'}$, that is, if $T \in \mathcal{T}^x_{Q'}$ and $T' \leq T$, then $T' \in \mathcal{T}^x_{Q'}$.*

Proposition 4.3.6. *Let T be a tilting module in $\operatorname{rep} Q'$. Then $i^{-1}T$ is a tilting module of $\operatorname{rep}(Q \setminus \{x\})$.*

Corollary 4.3.7. *The map $\pi'_x : T \mapsto \operatorname{basic}(i^{-1}T)$ is an order-preserving function $(\mathcal{T}_{Q'}, \leq) \to (\mathcal{T}_{Q\setminus\{x\}}, \leq)$.*

Lemma 4.3.8. *Let T be a basic tilting module in $\operatorname{rep}(Q \setminus \{x\})$. Then $S'_x \oplus i_!T$ is a basic tilting module of $\operatorname{rep} Q'$.*

Corollary 4.3.9. *The map $\iota'_x : T \mapsto S'_x \oplus i_! T$ is an order preserving function $(\mathcal{T}_{Q\setminus\{x\}}, \leq) \to (\mathcal{T}^x_{Q'}, \leq)$.*

Proposition 4.3.10. *We have*

$$\pi'_x \iota'_x(T) = T$$

for all $T \in \mathcal{T}_{Q\setminus\{x\}}$. In addition,

$$T \geq \iota'_x \pi'_x(T)$$

for all $T \in \mathcal{T}_{Q'}$, with equality if and only if $T \in \mathcal{T}^x_{Q'}$.

Corollary 4.3.11. *Let $X' = \mathcal{T}_{Q'} \setminus \mathcal{T}^x_{Q'}$ and $Y' = \mathcal{T}^x_{Q'}$. Define $f' : X' \to Y'$ by $f' = \iota'_x \pi'_x$. Then $\mathcal{T}_{Q'} \simeq (X' \sqcup Y', \leq^{f'}_{-})$.*

4.4 Tilting modules with respect to reflection

Let $F : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q')$ be the BGP reflection defined by the source x. For the convenience of the reader, we describe F explicitly following [33, (IV.4, Exercise 6)] (see also Chapter 3).

Observe that a complex of representations of Q can be described as a collection of complexes K_y of finite-dimensional vector spaces for the vertices y of Q, together with morphisms $K_y \to K_{y'}$ for the arrows $y \to y'$ in Q. Given such data, let $y_1, \dots, y_m$ be the endpoints of the arrows of Q starting at x, and define a collection $\{K'_y\}$ of complexes by

$$\begin{aligned} K'_x &= \mathrm{Cone}\Big(K_x \to \bigoplus_{i=1}^m K_{y_i}\Big) && && (4.4.1) \\ K'_y &= K_y && y \in Q \setminus \{x\} \end{aligned}$$

with the morphisms $K'_y \to K'_{y'}$ identical to $K_y \to K_{y'}$ for $y \to y'$ in $Q \setminus \{x\}$, and the natural inclusions $K'_{y_i} = K_{y_i} \to \mathrm{Cone}(K_x \to \bigoplus K_{y_j}) = K'_x$ for the reversed arrows $y_i \to x$ in Q'.

This definition can be naturally extended to give a functor $\widetilde{F}$ from the category of complexes over $\operatorname{rep} Q$ to the complexes over $\operatorname{rep} Q'$, which induces the triangulated equivalence F. The action of F on complexes is given, up to quasi-isomorphism, by (4.4.1).

Lemma 4.4.1 ([9]). *F induces a bijection between the indecomposables of $\operatorname{rep} Q$ other than S_x and the indecomposables of $\operatorname{rep} Q'$ other than S'_x.*

Proof. If M is an indecomposable of $\operatorname{rep} Q$, then FM is indecomposable of $\mathcal{D}^b(Q')$ since F is a triangulated equivalence.

Now let $M \neq S_x$ be an indecomposable of $\operatorname{rep} Q$. The map $M(x) \to \bigoplus_{i=1}^m M(y_i)$ must be injective, otherwise one could decompose $M = S_x^n \oplus N$ for some $n > 0$ and N. Using (4.4.1) we see that FM is quasi-isomorphic to the stalk complex supported on degree 0 that can be identified with $M' \in \operatorname{rep} Q'$, given by

$$\begin{aligned} M'(x) &= \mathrm{coker}\Big(M(x) \to \bigoplus_{i=1}^m M(y_i)\Big) \\ M'(y) &= M(y) && y \in Q \setminus \{x\} && (4.4.2) \end{aligned}$$

□

Note also that from (4.4.1) it follows that $FS_x = S'_x[1]$.

Corollary 4.4.2. *$j^{-1}T = i^{-1}FT$ for all $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$.*

Proof. This follows from (4.4.2), since T does not have S_x as summand. □

Corollary 4.4.3. *F induces an isomorphism of posets $\rho : \mathcal{T}_Q \setminus \mathcal{T}_Q^x \to \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x$.*

Proof. For $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$, define $\rho(T) = FT$. Observe that if T has n indecomposable summands, so does FT. Moreover, if $T, T' \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$, then $\operatorname{Ext}^i_{Q'}(FT, FT') \simeq \operatorname{Ext}^i_Q(T, T')$, hence $\rho(T) \in \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x$ and $\rho(T) \leq \rho(T')$ if $T \leq T'$. □

Corollary 4.4.4. *We have a commutative diagram*

$$\begin{tikzcd}
 & \mathcal{T}_Q \setminus \mathcal{T}_Q^x \arrow[rr, "\rho", "\simeq"'] \arrow[dl, "f"'] \arrow[dr, "\pi_x"] & & \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x \arrow[dl, "\pi'_x"'] \arrow[dr, "f'"] & \\
\mathcal{T}_Q^x & & \mathcal{T}_{Q \setminus \{x\}} \arrow[ll, "\iota_x"', "\simeq"] \arrow[rr, "\iota'_x", "\simeq"'] & & \mathcal{T}_{Q'}^x
\end{tikzcd}$$

Proof. We have to show the commutativity of the middle triangle, that is, $\pi_x = \pi'_x \rho$. Indeed, let $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$. Then $\pi_x(T) = \operatorname{basic}(j^{-1}T)$, $\pi'_x \rho(T) = \operatorname{basic}(i^{-1}FT)$ and the claim follows from Corollary 4.4.2. □

Theorem 4.4.5. *The posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.*

Proof. Use Corollaries 4.2.11, 4.3.11 and 4.4.4. □

4.5 Example

Consider the following two quivers Q and Q' whose underlying graph is the Dynkin diagram A_4. The quiver Q' is obtained from Q by reflection at the source 4.

$$Q : \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longleftarrow \bullet_4 \qquad\qquad Q' : \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4$$

For $1 \leq i \leq j \leq 4$, denote by ij the indecomposable representation of Q (or Q') supported on the vertices $i, i+1, \dots, j$.

Figure 4.1 shows the Hasse diagrams of the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$, where we used bold font to indicate the tilting modules containing the simple 44 as summand. The subsets $\mathcal{T}_Q^4$ and $\mathcal{T}_{Q'}^4$ of tilting modules containing 44 are isomorphic to the poset of tilting modules of the quiver A_3 with the linear orientation.

Note that $\mathcal{T}_Q$ was computed in [75, Example 3.2], while $\mathcal{T}_{Q'}$ is a Tamari lattice and the underlying graph of its Hasse diagram is the 1-skeleton of the Stasheff associhedron of dimension 3, see [15, 18].

Figure 4.2 shows the values of the functions π_4 and π'_4 on $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$, respectively. The functions $f : \mathcal{T}_Q \setminus \mathcal{T}_Q^4 \to \mathcal{T}_Q^4$ and $f' : \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^4 \to \mathcal{T}_{Q'}^4$ can then be easily computed.

Finally, the isomorphism $\rho : \mathcal{T}_Q \setminus \mathcal{T}_Q^4 \to \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^4$ is induced by the BGP reflection at the vertex 4, whose effect on the indecomposables (excluding 44) is given by

$$11 \leftrightarrow 11 \qquad 12 \leftrightarrow 12 \qquad 13 \leftrightarrow 14 \qquad 22 \leftrightarrow 22 \qquad 23 \leftrightarrow 24 \qquad 33 \leftrightarrow 34.$$

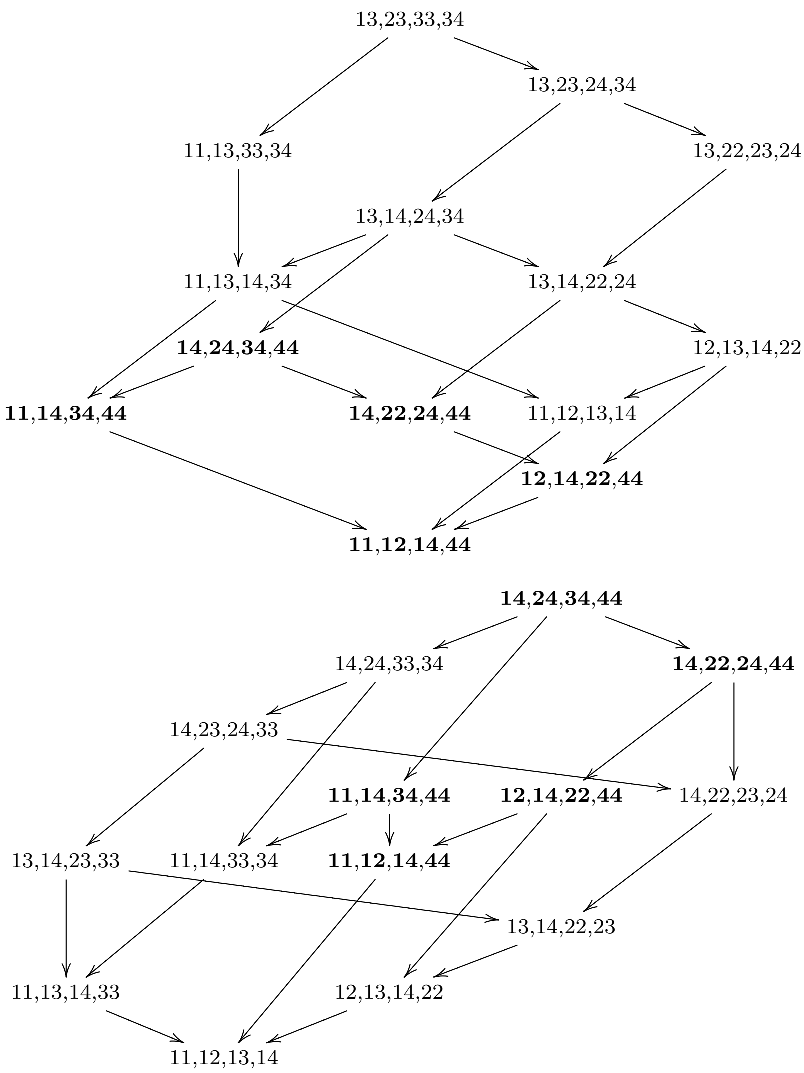


Figure 4.1: Hasse diagrams of the posets $\mathcal{T}_Q$ (top) and $\mathcal{T}_{Q'}$ (bottom).

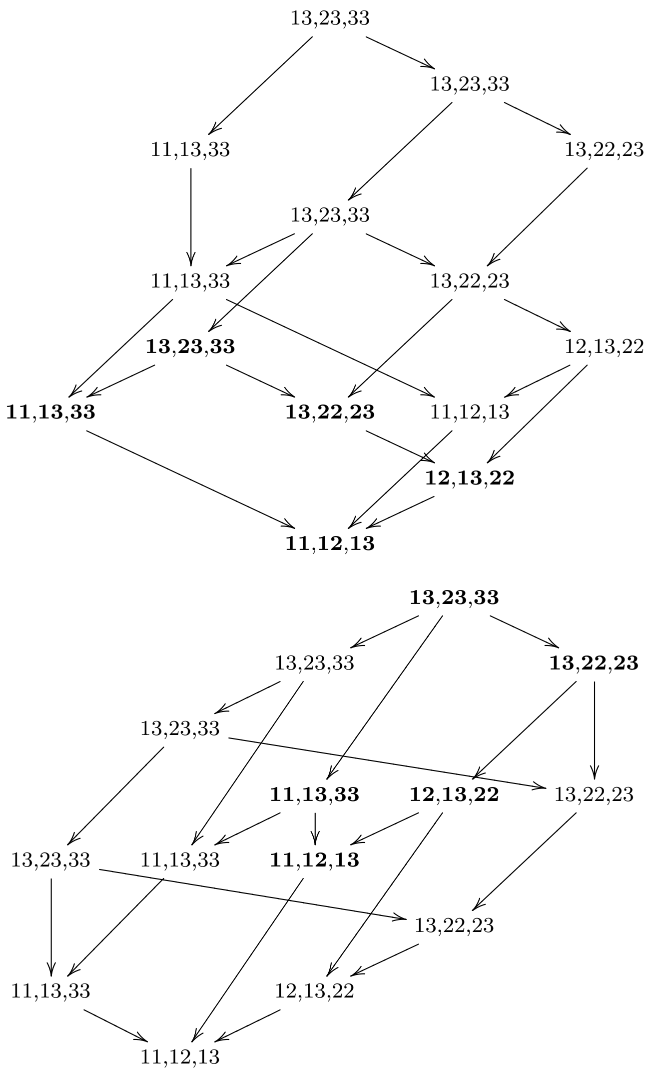


Figure 4.2: The functions π_4, π'_4 on $\mathcal{T}_Q$, $\mathcal{T}_{Q'}$.

Chapter 5

Universal Derived Equivalences of Posets of Cluster Tilting Objects

We show that for two quivers without oriented cycles related by a BGP reflection, the posets of their cluster tilting objects are related by a simple combinatorial construction, which we call a flip-flop.

We deduce that the posets of cluster tilting objects of derived equivalent path algebras of quivers without oriented cycles are universally derived equivalent. In particular, all Cambrian lattices corresponding to the various orientations of the same Dynkin diagram are universally derived equivalent.

5.1 Introduction

In this chapter we investigate the combinatorial relations between the posets of cluster tilting objects of derived equivalent path algebras, continuing our investigation in Chapter 4 of the posets of tilting modules of such algebras.

Throughout this chapter, we fix an algebraically closed field k. Let Q be a finite quiver without oriented cycles and let $\operatorname{rep} Q$ denote the category of finite dimensional representations of Q over k. The associated cluster category $\mathcal{C}_Q$ was introduced in [16] (and in [17] for the A_n case) as a representation theoretic approach to the cluster algebras introduced and studied by Fomin and Zelevinsky [28]. It is defined as the orbit category [52] of the bounded derived category $\mathcal{D}^b(Q)$ of $\operatorname{rep} Q$ by the functor $S \cdot [-2]$ where $S : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q)$ is the Serre functor and $[1]$ is the suspension. The indecomposables of $\mathcal{C}_Q$ can be represented by the indecomposables of $\mathcal{D}^b(Q)$ in the fundamental domain of $S \cdot [-2]$, hence $\operatorname{ind} \mathcal{C}_Q = \operatorname{ind} \operatorname{rep} Q \cup \{P_y[1] \,:\, y \in Q\}$ where P_y are the indecomposable projectives in $\operatorname{rep} Q$.

Cluster tilting theory was investigated in [16]. A basic object $T \in \mathcal{C}_Q$ is a *cluster tilting object* if $\operatorname{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$ and T is maximal with respect to this property, or equivalently, the number of indecomposable summands of T equals the number of vertices of Q. If $T = M \oplus U$ is cluster tilting and M is indecomposable, then there exist a unique indecomposable $M' \neq M$ such that $T' = M' \oplus U$ is cluster tilting. T' is called the *mutation* of T with respect to M.

Denote by $\mathcal{T}_{\mathcal{C}_Q}$ the set of all cluster tilting objects. In [48], a partial order on $\mathcal{T}_{\mathcal{C}_Q}$, extending the partial order on tilting modules introduced in [75], is defined by $T \le T'$ if $\operatorname{fac} T \supseteq \operatorname{fac} T'$. Here, for $M \in \operatorname{rep} Q$, $\operatorname{fac} M$ denotes the full subcategory of $\operatorname{rep} Q$ consisting of all quotients of finite sums of copies of M, and for $T \in \mathcal{T}_{\mathcal{C}_Q}$, $\operatorname{fac} T = \operatorname{fac} \widehat{T}$ where $\widehat{T} \in \operatorname{rep} Q$ is the sum of all indecomposable summands of T which are not shifted projectives.

As shown in [48], the map $T \mapsto \operatorname{fac} T$ induces an order preserving bijection between $(\mathcal{T}_{\mathcal{C}_Q}, \le)$ and the set of finitely generated torsion classes in $\operatorname{rep} Q$ ordered by reverse inclusion. Moreover, it is also shown that when Q is Dynkin, $(\mathcal{T}_{\mathcal{C}_Q}, \le)$ is isomorphic to the corresponding Cambrian lattice defined in [72] as a certain lattice quotient of the weak order on the Coxeter group associated with Q.

For two partially ordered sets $(X, \le_X)$, $(Y, \le_Y)$ and an order preserving function $f : X \to Y$, define two partial orders $\le_+^f$ and $\le_-^f$ on the disjoint union $X \sqcup Y$ by keeping the original partial orders inside X and Y and setting

$$x \le_+^f y \Longleftrightarrow f(x) \le_Y y \qquad\qquad y \le_-^f x \Longleftrightarrow y \le_Y f(x)$$

with no other additional order relations. We say that two posets Z and Z' are related via a *flip-flop* if there exist X, Y and $f : X \to Y$ as above such that $Z \simeq (X \sqcup Y, \le_+^f)$ and $Z' \simeq (X \sqcup Y, \le_-^f)$.

Let x be a sink of Q and let Q' be the quiver obtained from Q by a BGP reflection [9] at x, that is, by reverting all the arrows ending at x. Our main result is the following.

Theorem 5.1.1. *The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are related via a flip-flop.*

We give a brief outline of the proof. Let $\mathcal{T}^x_{\mathcal{C}_Q}$ denote the subset of cluster tilting objects containing the simple projective S_x at x as direct summand. Given $T \in \mathcal{T}^x_{\mathcal{C}_Q}$, let $f(T)$ be the mutation of T with respect to S_x. In Section 5.2 we prove that the function $f : \mathcal{T}^x_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ is order preserving and moreover

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}^x_{\mathcal{C}_Q} \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}), \le_+^f\right) \tag{5.1.1}$$

Similarly, let $\mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$ be the subset of cluster tilting objects containing the shifted projective $P'_x[1]$ at x as direct summand. Given $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$, let $g(T)$ be the mutation of T with respect to $P'_x[1]$. In Section 5.3 we prove that the function $g : \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}} \to \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$ is order preserving and moreover

$$\mathcal{T}_{\mathcal{C}_{Q'}} \simeq \left(\mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}} \sqcup (\mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}), \le_-^g\right) \tag{5.1.2}$$

In Section 5.4 we relate the two isomorphisms given in (5.1.1) and (5.1.2) by considering, following [87], the action of the BGP reflection functor on the cluster tilting objects. We prove the existence of the following commutative diagram with horizontal isomorphisms of posets

$$\begin{tikzcd}
\mathcal{T}^x_{\mathcal{C}_Q} \arrow[r, "\simeq"] \arrow[d, "f"] & \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}} \arrow[d, "g"] \\
\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q} \arrow[r, "\simeq"] & \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}
\end{tikzcd}$$

from which we deduce Theorem 5.1.1. An example demonstrating the theorem and its proof is given in Section 5.5.

In Theorem 4.1.1 we have shown a result analogous to Theorem 5.1.1 for the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ of tilting modules, following a similar strategy of proof. However, there are some important differences.

First, the situation in the cluster tilting case is asymmetric, as the partition (5.1.1) for a sink involves the subset of cluster tilting objects containing the corresponding simple, while the corresponding partition of (5.1.2) at a source involves the subset of cluster tilting objects containing the shifted projective. In contrast, both partitions for the tilting case involve the subset of tilting modules containing the simple, either at a source or sink. This asymmetry is inherent in the proof of (5.1.2), which is not the dual of that of (5.1.1), and also in the analysis of the effect of the BGP reflection functor.

Second, in the cluster tilting case, the order preserving maps occurring in the flip-flop construction are from the set containing the simple (or shifted projective) to its complement, while in the tilting case, they are in the opposite direction, into the set containing the simple. As a consequence, a partition with respect to a sink in the cluster tilting case yields an order of the form $\leq_+$, while for the tilting case it gives $\leq_-$.

While two posets Z and Z' related via a flip-flop are in general not isomorphic, they are *universally derived equivalent* in the following sense; for any abelian category $\mathcal{A}$, the derived categories of the categories of functors $Z \to \mathcal{A}$ and $Z' \to \mathcal{A}$ are equivalent as triangulated categories, see Corollary 3.1.3.

It is known [35, (I.5.7)] that the path algebras of two quivers Q, Q' without oriented cycles are derived equivalent if and only if Q' can be obtained from Q by a sequence of BGP reflections (at sources or sinks). We therefore deduce the following theorem.

Theorem 5.1.2. *Let Q and Q' be two quivers without oriented cycles whose path algebras are derived equivalent. Then the posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are universally derived equivalent.*

Since for a Dynkin quiver Q, the poset $\mathcal{T}_{\mathcal{C}_Q}$ is isomorphic to the corresponding Cambrian lattice, the above theorem can be restated as follows.

Corollary 5.1.3. *All Cambrian lattices corresponding to the various orientations of the same Dynkin diagram are universally derived equivalent.*

In particular, the incidence algebras of the Cambrian lattices corresponding to the various orientations the same Dynkin diagram are derived equivalent, as the universal derived equivalence of two finite posets implies the derived equivalence of their incidence algebras.

5.2 Cluster tilting objects containing P_x

Let $x \in Q$ be a vertex, and denote by $\mathcal{T}^x_{\mathcal{C}_Q}$ the subset of cluster tilting objects containing P_x as direct summand.

Lemma 5.2.1. *Let $M \in \operatorname{rep} Q$. Then $P_x \in \operatorname{fac} M$ if and only if M contains P_x as a direct summand.*

Proof. Assume that $P_x \in \operatorname{fac} M$, and let $q : M^n \twoheadrightarrow P_x$ be a surjection, for some $n \geq 1$. Since P_x is projective, there exists $j : P_x \to M^n$ such that $qj = 1_{P_x}$. Let $N = \operatorname{im} j = \operatorname{im} jq$. As $(jq)^2 = jq$, we deduce that N is a direct summand of M^n and that $j : P_x \to N$ is an isomorphism. Since P_x is indecomposable, it is also a summand of M. $\square$

Corollary 5.2.2. *Let $T \in \mathcal{T}_{\mathcal{C}_Q}$. Then $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ if and only if $P_x \in \operatorname{fac} T$.*

Corollary 5.2.3. *If $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \leq T$, then $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$,*

Proof. Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q}$. If $T' \leq T$, then $P_x \in \operatorname{fac} T \subseteq \operatorname{fac} T'$, hence $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$. $\square$

Define a map $f : \mathcal{T}^x_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ as follows. Given $T \in \mathcal{T}^x_{\mathcal{C}_Q}$, write $T = P_x \oplus U$ and set $f(T) = M \oplus U$ where M is the unique other indecomposable complement of U such that $M \oplus U$ is a cluster tilting object.

Recall that for a tilting module $T \in \operatorname{rep} Q$, we have $\operatorname{fac} T = T^\perp$ where

$$T^\perp = \left\{ M \in \operatorname{rep} Q \ : \ \operatorname{Ext}^1_Q(T, M) = 0 \right\}$$

Lemma 5.2.4. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$. Then $f(T) > T$.*

Proof. One could deduce the claim from Lemma 2.32 of [48]. Instead, we shall give a direct proof. Write $T = P_x \oplus U$ and $f(T) = M \oplus U$. If M is a shifted projective, the claim is clear. Otherwise, by deleting the vertices of Q corresponding to the shifted projective summands of U, we may and will assume that $P_x \oplus U$ and $M \oplus U$ are tilting modules. Therefore

$$\operatorname{fac}(P_x \oplus U) = (P_x \oplus U)^\perp = U^\perp$$

where the last equality follows since P_x is projective. As $M \in U^\perp$, we get that $M \in \operatorname{fac}(P_x \oplus U)$, hence $\operatorname{fac}(M \oplus U) \subseteq \operatorname{fac}(P_x \oplus U)$. $\square$

For the rest of this section, *we assume that the vertex x is a sink in Q*. In this case, $P_x = S_x$ and $\operatorname{ind}\operatorname{fac} S_x = \{S_x\}$. Moreover, $S_x \notin \operatorname{fac} M$ for any other indecomposable $M \neq S_x$, since $\operatorname{Hom}_Q(M, S_x) = 0$.

Lemma 5.2.5. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$. Then $\operatorname{ind}\operatorname{fac} f(T) = \operatorname{ind}\operatorname{fac} T \setminus \{S_x\}$.*

Proof. Write $T = S_x \oplus U$ and $f(T) = M \oplus U$. By the preceding remarks,

$$\operatorname{ind}\operatorname{fac} T = \operatorname{ind}\operatorname{fac}(S_x \oplus U) = \operatorname{ind}\operatorname{fac} S_x \cup \operatorname{ind}\operatorname{fac} U$$

is a disjoint union $\{S_x\} \amalg \operatorname{ind}\operatorname{fac} U$. By Lemma 5.2.4,

$$\operatorname{ind}\operatorname{fac} f(T) = \operatorname{ind}\operatorname{fac}(M \oplus U) \subseteq \operatorname{ind}\operatorname{fac}(S_x \oplus U) = \{S_x\} \amalg \operatorname{ind}\operatorname{fac} U,$$

therefore $\operatorname{ind}\operatorname{fac}(M \oplus U) = \operatorname{ind}\operatorname{fac} U$, as $S_x \notin \operatorname{fac} M$. $\square$

Corollary 5.2.6. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ be such that $T' > T$. Then $T' \geq f(T)$.*

Proof. By assumption, $\operatorname{fac} T' \subseteq \operatorname{fac} T$. Moreover, $S_x \notin \operatorname{fac} T'$, since $T' \notin \mathcal{T}^x_{\mathcal{C}_Q}$. Hence by Lemma 5.2.5, $\operatorname{ind} \operatorname{fac} T' \subseteq \operatorname{ind} \operatorname{fac} f(T)$, thus $T' \geq f(T)$. □

Corollary 5.2.7. *The map $f : \mathcal{T}^x_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ is order preserving and*

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}^x_{\mathcal{C}_Q} \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}), \leq^f_+\right)$$

Proof. If $T, T' \in \mathcal{T}^x_{\mathcal{C}_Q}$ are such that $T' \geq T$, then by Lemma 5.2.4, $f(T') > T' \geq T$, hence by Corollary 5.2.6, $f(T') \geq f(T)$, therefore f is order preserving. The other assertion follows from Corollaries 5.2.3, 5.2.6 and Lemma 5.2.4. □

5.3 Cluster tilting objects containing $P_x[1]$

For $M \in \operatorname{rep} Q$ and $y \in Q$, let $M(y)$ denote the vector space corresponding to y, and let $\operatorname{supp} M = \{y \in Q : M(y) \neq 0\}$ be the *support* of M.

Let $x \in Q$ be a vertex, and denote by $\mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ the subset of cluster tilting objects containing the shifted indecomposable projective $P_x[1]$ as direct summand.

Lemma 5.3.1. *If $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ and $T' \geq T$, then $T' \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$.*

Proof. Since T contains $P_x[1]$ as summand, we have $\operatorname{Ext}^1_{\mathcal{C}_Q}(P_x[1], T) = 0$, that is, $\operatorname{Hom}_Q(P_x, \widehat{T}) = 0$, or equivalently $x \notin \operatorname{supp} \widehat{T}$.

Now let $T' \geq T$. Then all the modules in $\operatorname{fac} T' \subseteq \operatorname{fac} T$ are not supported on x, and in particular $\operatorname{Hom}_Q(P_x, \widehat{T'}) = 0$, thus $\operatorname{Ext}^1_{\mathcal{C}_Q}(P_x[1], T') = 0$. The maximality of T' implies that it contains $P_x[1]$ as summand. □

Similarly to the previous section, define a map $g : \mathcal{T}^{x[1]}_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ as follows. Given $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$, write $T = P_x[1] \oplus U$ and set $g(T) = M \oplus U$ where M is the unique other indecomposable complement of U such that $M \oplus U$ is a cluster tilting object.

Lemma 5.3.2. *Let $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$. Then $g(T) < T$.*

Proof. This is obvious. Indeed, write $T = P_x[1] \oplus U$ and $g(T) = M \oplus U$. Then $\operatorname{fac} g(T) = \operatorname{fac}(M \oplus U) \supseteq \operatorname{fac} U = \operatorname{fac} T$. □

For the rest of this section, *we assume that the vertex x is a source*. In this case, for any module $M \in \operatorname{rep} Q$, we have that $S_x \in \operatorname{fac} M$ if and only if M is supported at x. Therefore we deduce the following lemma, which can be viewed as an analogue of Corollary 5.2.2.

Lemma 5.3.3. *Let $T \in \mathcal{T}_{\mathcal{C}_Q}$. Then $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ if and only if $S_x \notin \operatorname{fac} T$.*

Recall that a basic module $U \in \operatorname{rep} Q$ is an *almost complete tilting module* if $\operatorname{Ext}^1_Q(U, U) = 0$ and the number of indecomposable summands of U equals the number of vertices of Q less one. A *complement* to U is an indecomposable M such that $M \oplus U$ is a tilting module. It is known [43] that an almost complete tilting module U has at most two complements, and exactly two if and only if U is sincere, that is, $\operatorname{supp} U = Q$.

Proposition 5.3.4. *Let U be an almost complete tilting module of $\operatorname{rep} Q$ not supported on x, and let M be its unique indecomposable complement to a tilting module. Let $\mathcal{X}$ be a torsion class in $\operatorname{rep} Q$ satisfying $\operatorname{fac} U \subseteq \mathcal{X}$ and $S_x \in \mathcal{X}$. Then $M \in \mathcal{X}$.*

Proof. The natural inclusion $j : Q \setminus \{x\} \to Q$ induces a pair $(j_!, j^{-1})$ of exact functors

$$j^{-1} : \operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\}) \qquad\qquad j_! : \operatorname{rep}(Q \setminus \{x\}) \to \operatorname{rep} Q$$

where j^{-1} is the natural restriction and $j_!$ is its left adjoint, defined as the extension of a representation of $Q \setminus \{x\}$ by zero at x.

Now $j_! j^{-1} U \simeq U$ since U is not supported on x. By adjunction and exactness,

$$\operatorname{Ext}^1_{Q \setminus \{x\}}(j^{-1}U, j^{-1}U) \simeq \operatorname{Ext}^1_Q(j_! j^{-1}U, U) = \operatorname{Ext}^1_Q(U, U)$$

thus $j^{-1}U$ is a (basic) tilting module of $\operatorname{rep}(Q \setminus \{x\})$. However, by Proposition 4.2.6, $j^{-1}(M \oplus U)$ is also a tilting module of $\operatorname{rep}(Q \setminus \{x\})$, but not necessarily basic. It follows that $j^{-1}M \in \operatorname{add} j^{-1}U$, hence $j_! j^{-1}M \in \operatorname{add} j_! j^{-1}U = \operatorname{add} U$.

The adjunction morphism $j_! j^{-1}M \to M$ is injective, and we have an exact sequence

$$0 \to j_! j^{-1} M \to M \to S_x^n \to 0$$

for some $n \geq 0$. Now $S_x \in \mathcal{X}$ by assumption and $j_! j^{-1}M \in \operatorname{add} U \subseteq \mathcal{X}$, hence $M \in \mathcal{X}$ as $\mathcal{X}$ is closed under extensions. $\square$

Corollary 5.3.5. *Let $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ be such that $T' < T$. Then $T' \leq g(T)$.*

Proof. Write $T = P_x[1] \oplus U$ and $g(T) = M \oplus U$. The assumptions on T' imply that $S_x \in \operatorname{fac} T'$ and $\operatorname{fac} U = \operatorname{fac} T \subseteq \operatorname{fac} T'$.

By deleting the vertices of Q corresponding to the shifted projective summands of U, we may and will assume that $M \oplus U$ is a tilting module, so that U is an almost complete tilting module. Applying Proposition 5.3.4 for $\mathcal{X} = \operatorname{fac} T'$, we deduce that $M \in \operatorname{fac} T'$, hence $\operatorname{fac} g(T) = \operatorname{fac}(M \oplus U) \subseteq \operatorname{fac} T'$. $\square$

Corollary 5.3.6. *The map $g : \mathcal{T}^{x[1]}_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ is order preserving and*

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}^{x[1]}_{\mathcal{C}_Q} \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}), \leq^g_-\right)$$

Proof. The claim follows from Lemmas 5.3.1, 5.3.2 and Corollary 5.3.5 as in the proof of Corollary 5.2.7. $\square$

5.4 The effect of a BGP reflection

Let Q be a quiver without oriented cycles and let x be a sink. Let $y_1, \dots, y_m$ be the endpoints of the arrows ending at x, and denote by Q' the quiver obtained from Q by reflection at x. For a vertex $y \in Q$, denote by S_y, S'_y the simple modules corresponding to y in $\operatorname{rep} Q$, $\operatorname{rep} Q'$ and by P_y, P'_y their projective covers.

The categories $\operatorname{rep} Q$ and $\operatorname{rep} Q'$ are related by the BGP reflection functors, introduced in [9]. We recollect here the basic facts on these functors that will be needed in the sequel.

The BGP reflection functors are the functors

$$F^+ : \operatorname{rep} Q \to \operatorname{rep} Q' \qquad\qquad F^- : \operatorname{rep} Q' \to \operatorname{rep} Q$$

defined by

$$(F^+M)(x) = \ker\Big(\bigoplus_{i=1}^{m} M(y_i) \to M(x)\Big) \qquad (F^+M)(y) = M(y) \tag{5.4.1}$$

$$(F^-M')(x) = \operatorname{coker}\Big(M'(x) \to \bigoplus_{i=1}^{m} M'(y_i)\Big) \qquad (F^-M')(y) = M'(y)$$

for $M \in \operatorname{rep} Q$, $M' \in \operatorname{rep} Q'$ and $y \in Q \setminus \{x\}$, where the maps $(F^+M)(x) \to (F^+M)(y_i)$ and $(F^-M)(y_i) \to (F^-M)(x)$ are induced by the natural projection and inclusion.

It is clear that F^+ is left exact and F^- is right exact. The classical right derived functor of F^+ takes the form

$$(R^1F^+M)(x) = \operatorname{coker}\Big(\bigoplus_{i=1}^{m} M(y_i) \to M(x)\Big) \qquad (R^1F^+M)(y) = 0 \tag{5.4.2}$$

hence R^1F^+ vanishes for modules not containing S_x as direct summand.

The total derived functors

$$RF^+ : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q') \qquad\qquad LF^- : \mathcal{D}^b(Q') \to \mathcal{D}^b(Q)$$

are triangulated equivalences, and their effect on the corresponding cluster categories has been analyzed in [87], where it is shown that RF^+ induces a triangulated equivalence $\mathcal{C}_Q \xrightarrow{\simeq} \mathcal{C}_{Q'}$ whose action on the indecomposables of $\mathcal{C}_Q$ is given by

$$S_x \mapsto P'_x[1] \qquad M \mapsto F^+M \qquad P_x[1] \mapsto S'_x \qquad P_y[1] \mapsto P'_y[1] \tag{5.4.3}$$

with an inverse given by

$$S'_x \mapsto P_x[1] \qquad M' \mapsto F^-M' \qquad P'_x[1] \mapsto S_x \qquad P'_y[1] \mapsto P_y[1] \tag{5.4.4}$$

for $M \neq S_x$, $M' \neq S'_x$ and $y \in Q \setminus \{x\}$. Moreover, this equivalence induces a bijection $\rho : \mathcal{T}_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_{Q'}}$ preserving the mutation graph [87, Proposition 3.2].

Lemma 5.4.1. *Let $T, T' \in \mathcal{T}_{\mathcal{C}_Q}$. If $\rho(T) \leq \rho(T')$, then $T \leq T'$.*

Proof. By (5.4.4), $\operatorname{fac} T = \operatorname{fac} F^-\widehat{\rho(T)}$ if $P'_x[1]$ is not a summand of $\rho(T)$, and $\operatorname{fac} T = \operatorname{fac}(S_x \oplus F^-\widehat{\rho(T)})$ if $P'_x[1]$ is a summand of $\rho(T)$. Note that by Lemma 5.3.1, the latter case implies that $P'_x[1]$ is also a summand of $\rho(T')$, hence in any case it is enough to verify that if $M, N \in \operatorname{rep} Q'$ satisfy $\operatorname{fac} N \subseteq \operatorname{fac} M$, then $\operatorname{fac} F^-N \subseteq \operatorname{fac} F^-M$. Indeed, since F^- is right exact, it takes an exact sequence $M^n \to N \to 0$ to an exact sequence $(F^-M)^n \to F^-N \to 0$. □

Proposition 5.4.2. *ρ induces an isomorphism of posets $\mathcal{T}^x_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$.*

Proof. Note that by (5.4.3), $\rho(\mathcal{T}^x_{\mathcal{C}_Q}) = \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$. In view of Lemma 5.4.1, it remains to show that if $T, T' \in \mathcal{T}^x_{\mathcal{C}_Q}$ satisfy $T \leq T'$, then $\rho(T) \leq \rho(T')$.

Write $\widehat{T} = S_x \oplus U$ and $\widehat{T'} = S_x \oplus U'$. Then $\operatorname{fac}\rho(T) = \operatorname{fac} F^+U$ and $\operatorname{fac}\rho(T') = \operatorname{fac} F^+U'$, and we need to show that $F^+U' \in \operatorname{fac} F^+U$.

Indeed, since $\operatorname{fac} T' \subseteq \operatorname{fac} T$, the proof of Lemma 5.2.5 shows that $U' \in \operatorname{fac} U$, hence there exists a short exact sequence

$$0 \to K \to U^n \xrightarrow{\varphi} U' \to 0$$

for some $n > 0$ and $K \in \operatorname{rep} Q$. Applying $\operatorname{Hom}_Q(-, S_x)$ to this sequence, noting that $\operatorname{Ext}^1_Q(U', S_x) = 0$ since $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$, we get that $\operatorname{Hom}_Q(U^n, S_x) \to \operatorname{Hom}_Q(K, S_x)$ is surjective, hence K does not contain S_x as summand (otherwise U^n would contain S_x as summand). Therefore the exact sequence

$$F^+U^n \to F^+U' \to R^1F^+K = 0$$

shows that $F^+U' \in \operatorname{fac} F^+U$. □

Proposition 5.4.3. *ρ induces an isomorphism of posets $\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$.*

Proof. For a representation $M \in \operatorname{rep} Q$, let Q_M and Q'_M be the subquivers of Q and Q' obtained by deleting the vertices outside $\operatorname{supp} M \cup \{x\}$. The quivers Q'_M and Q_M are related via a BGP reflection at x, and we denote by $F^+_{Q_M} : \operatorname{rep} Q_M \to \operatorname{rep} Q'_M$ the corresponding reflection functor. The restriction functors $i^{-1} : \operatorname{rep} Q \to \operatorname{rep} Q_M$ and $j^{-1} : \operatorname{rep} Q' \to \operatorname{rep} Q'_M$ induced by the natural embeddings $i : Q_M \to Q$ and $j : Q'_M \to Q'$ satisfy

$$j^{-1}F^+M = F^+_{Q_M} i^{-1}M,$$

as can be easily verified using (5.4.1).

As in the proof of Proposition 5.4.2, it is enough to show that if $T, T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ satisfy $T \leq T'$, then $\rho(T) \leq \rho(T')$. In view of the preceding paragraph, we may assume that $Q = \operatorname{supp} \widehat{T} \cup \{x\}$.

We consider two cases. First, assume that $x \in \operatorname{supp} \widehat{T}$. Then $T = \widehat{T}$ is a tilting module, $\rho(T) = F^+T$ and $\operatorname{fac}\rho(T') = \operatorname{fac} F^+\widehat{T'}$ or $\operatorname{fac}\rho(T') = \operatorname{fac}(S'_x \oplus F^+\widehat{T'})$ according to whether $x \in \operatorname{supp} \widehat{T'}$ or not, hence it is enough to show that $S'_x \oplus F^+\widehat{T'} \in \operatorname{fac} F^+T$.

By assumption, $\widehat{T'} \in \operatorname{fac} T = T^{\perp}$. Since T does not contain S_x as summand, F^+T is a tilting module and $F^+\widehat{T'} \in (F^+T)^{\perp} = \operatorname{fac} F^+T$ (see Corollary 4.4.3). Moreover, $S'_x \in \operatorname{fac} F^+T$, as F^+T is sincere.

For the second case, assume that $x \notin \operatorname{supp} \widehat{T}$. Then $T = P_x[1] \oplus \widehat{T}$ and by Lemma 5.3.1, $T' = P_x[1] \oplus \widehat{T'} \oplus P[1]$ where P is a sum of projectives other than P_x. By (5.4.3), $\rho(T) = S'_x \oplus F^+\widehat{T}$ and $\rho(T') = S'_x \oplus F^+\widehat{T'} \oplus P'[1]$, hence it is enough to show that $F^+\widehat{T'} \in \operatorname{fac}(S'_x \oplus F^+\widehat{T})$.

Indeed, since $\widehat{T'} \in \operatorname{fac} \widehat{T}$, there exists a short exact sequence

$$0 \to K \to \widehat{T}^n \to \widehat{T'} \to 0$$

for some $n > 0$ and $K \in \operatorname{rep} Q$. Applying the functor F^+, noting that $\widehat{T}$ does not contain S_x as summand, we get

$$0 \to F^+K \to F^+\widehat{T}^n \to F^+\widehat{T}' \to R^1F^+K \to R^1F^+\widehat{T}^n = 0.$$

By (5.4.2), $R^1F^+K = {S'_x}^{n'}$ for some $n' \geq 0$, hence $F^+\widehat{T}'$ is an extension of ${S'_x}^{n'}$ with a quotient of $F^+\widehat{T}^n$. The result now follows, as $\operatorname{fac}(S'_x \oplus F^+\widehat{T})$ is closed under extensions. □

Corollary 5.4.4. *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{T}^x_{\mathcal{C}_Q} \arrow[rr, "\rho", "\simeq"'] \arrow[d, "f"] & & \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}} \arrow[d, "g"] \\
\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q} \arrow[rr, "\rho", "\simeq"'] & & \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}
\end{tikzcd}$$

Proof. By Propositions 5.4.2 and 5.4.3, ρ induces the two horizontal isomorphisms. For $T \in \mathcal{T}^x_{\mathcal{C}_Q}$, $f(T)$ is defined as the mutation of T with respect to S_x and $g(\rho(T))$ is defined as the mutation of $\rho(T)$ with respect to $P'_x[1]$, which is, by (5.4.3), the image of S_x under the triangulated equivalence $\mathcal{C}_Q \to \mathcal{C}_{Q'}$. Therefore the commutativity of the diagram follows by the fact that ρ preserves the mutation graph [87, Proposition 3.2]. □

Theorem 5.4.5. *The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are related via a flip-flop.*

Proof. Use Corollaries 5.2.7, 5.3.6 and 5.4.4. □

5.5 Example

Consider the following two quivers Q and Q' whose underlying graph is the Dynkin diagram A_3. The quiver Q' is obtained from Q by a BGP reflection at the sink 3.

$$Q : \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \qquad\qquad Q' : \bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3$$

We denote the indecomposables of the cluster categories $\mathcal{C}_Q$ and $\mathcal{C}_{Q'}$ by specifying their dimension vectors. These consist of the positive roots of A_3, which correspond to the indecomposable representations of the quivers, together with the negative simple roots $-e_1, -e_2, -e_3$ which correspond to the shifted projectives.

Figure 5.1 shows the Hasse diagrams of the posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$, where we used bold font to indicate the subsets $\mathcal{T}^3_{\mathcal{C}_Q}$ and $\mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ of cluster tilting objects containing the simple S_3 and the shifted projective $P'_3[1]$ as summand, respectively.

The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are Cambrian lattices, and can be realized as sublattices of the weak order on the group of permutations on 4 letters, see [72, Section 6]. Moreover, the underlying graph of their Hasse diagrams is the 1-skeleton of the three-dimensional Stasheff associhedron. $\mathcal{T}_{\mathcal{C}_Q}$ is a Tamari lattice, corresponding to the linear orientation on A_3.

The BGP reflection at the vertex 3, whose action on the dimension vectors is given by

$$v \mapsto \begin{cases} v & \text{if } v \in \{-e_1, -e_2\} \\ s_3(v) & \text{otherwise} \end{cases}$$

where s_3 is the linear transformation specified by

$$s_3(v) = v \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

induces isomorphisms $\mathcal{T}^3_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ and $\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^3_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ compatible with the mutations at S_3 and $P'_3[1]$.

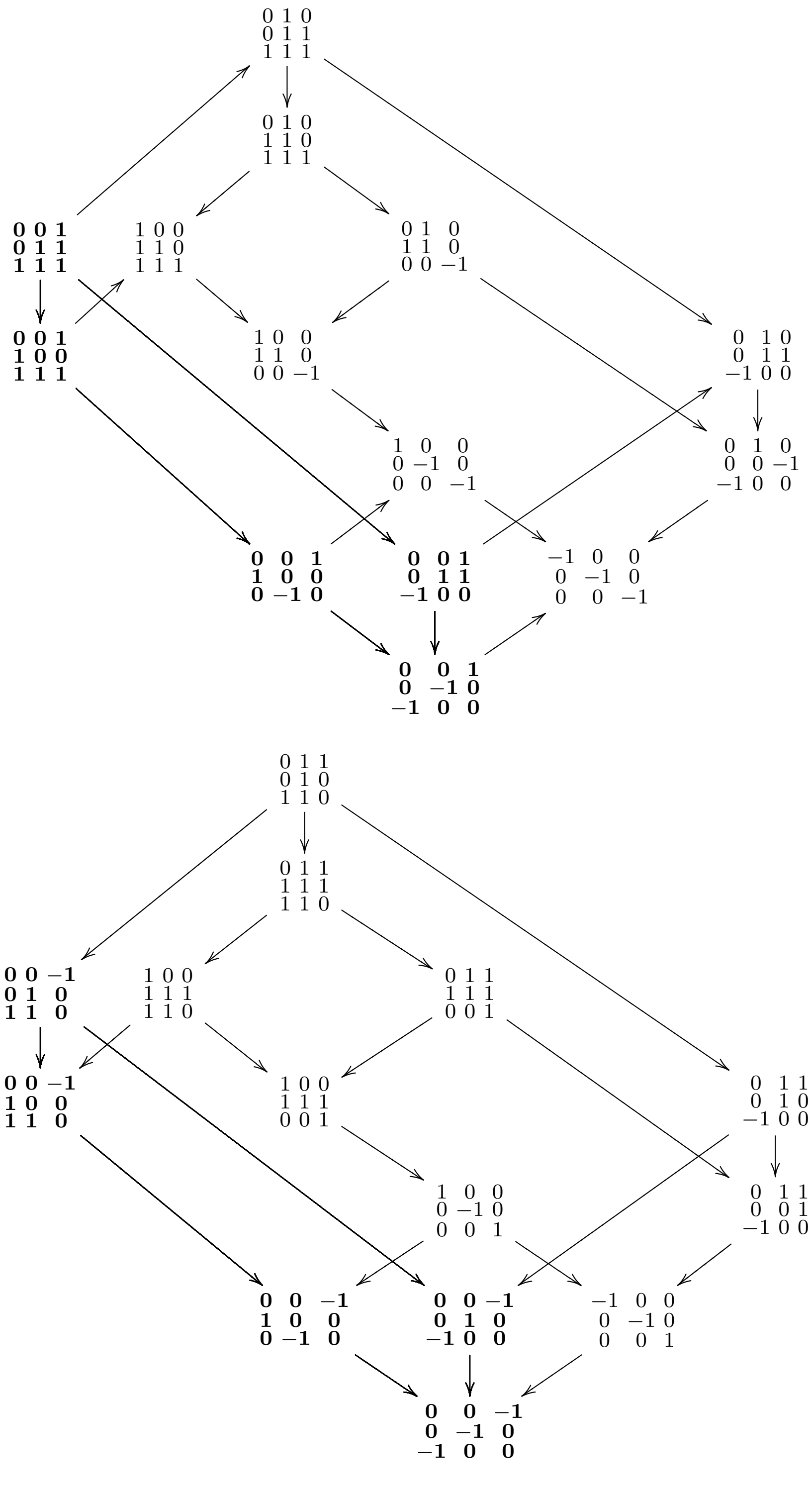


Figure 5.1: Hasse diagrams of the posets $\mathcal{T}_{\mathcal{C}_Q}$ (top) and $\mathcal{T}_{\mathcal{C}_{Q'}}$ (bottom).

Part III

Piecewise Hereditary Categories and Posets

Chapter 6

Bounds on the Global Dimension of Certain Piecewise Hereditary Categories

We give bounds on the global dimension of a finite length, piecewise hereditary category in terms of quantitative connectivity properties of its graph of indecomposables.

We use this to show that the global dimension of a finite dimensional, piecewise hereditary algebra A cannot exceed 3 if A is an incidence algebra of a finite poset or more generally, a sincere algebra. This bound is tight.

6.1 Introduction

Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{D}^b(\mathcal{A})$ its bounded derived category. $\mathcal{A}$ is called *piecewise hereditary* if there exist an abelian hereditary category $\mathcal{H}$ and a triangulated equivalence $\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{H})$. Piecewise hereditary categories of modules over finite dimensional algebras have been studied in the past, especially in the context of tilting theory, see [35, 40, 42].

It is known [40, (1.2)] that if $\mathcal{A}$ is a finite length, piecewise hereditary category with n non-isomorphic simple objects, then its global dimension satisfies $\operatorname{gl.dim} \mathcal{A} \leq n$. Moreover, this bound is almost sharp, as there are examples [54] where $\mathcal{A}$ has n simples and $\operatorname{gl.dim} \mathcal{A} = n-1$.

In this chapter we show how rather simple arguments can yield effective bounds on the global dimension of such a category $\mathcal{A}$, in terms of quantitative connectivity conditions on the graph of its indecomposables, regardless of the number of simple objects.

Let $G(\mathcal{A})$ be the directed graph whose vertices are the isomorphism classes of indecomposables of $\mathcal{A}$, where two vertices Q, Q' are joined by an edge $Q \to Q'$ if $\operatorname{Hom}_{\mathcal{A}}(Q, Q') \neq 0$.

Let $r \geq 1$ and let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{r-1})$ be a sequence in $\{+1, -1\}^r$. An *ε-path* from Q to Q' is a sequence of vertices $Q_0 = Q, Q_1, \dots, Q_r = Q'$ such that $Q_i \to Q_{i+1}$ in $G(\mathcal{A})$ if $\varepsilon_i = +1$ and $Q_{i+1} \to Q_i$ if $\varepsilon_i = -1$.

For an object Q of $\mathcal{A}$, let $\operatorname{pd}_{\mathcal{A}} Q = \sup\{d : \operatorname{Ext}^d_{\mathcal{A}}(Q, Q') \neq 0 \text{ for some } Q'\}$ and $\operatorname{id}_{\mathcal{A}} Q = \sup\{d : \operatorname{Ext}^d_{\mathcal{A}}(Q', Q) \neq 0 \text{ for some } Q'\}$ be the projective and injective dimensions of Q, so that gl.dim $\mathcal{A} = \sup_Q \operatorname{pd}_{\mathcal{A}} Q$.

Theorem 6.1.1. *Let $\mathcal{A}$ be a finite length, piecewise hereditary category. Assume that there exist $r \geq 1$, $\varepsilon \in \{1, -1\}^r$ and an indecomposable Q_0 such that for any indecomposable Q there exists an ε-path from Q_0 to Q.*

Then gl.dim $\mathcal{A} \leq r+1$ *and* $\operatorname{pd}_{\mathcal{A}} Q + \operatorname{id}_{\mathcal{A}} Q \leq r+2$ *for any indecomposable Q.*

We give two applications of this result for finite dimensional algebras.

Let A be a finite dimensional algebra over a field k, and denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules. Recall that a module M in $\operatorname{mod} A$ is *sincere* if all the simple modules occur as composition factors of M. The algebra A is called sincere if there exists a sincere indecomposable module.

Corollary 6.1.2. *Let A be a finite dimensional, piecewise hereditary, sincere algebra. Then* gl.dim $A \leq 3$ *and* $\operatorname{pd} Q + \operatorname{id} Q \leq 4$ *for any indecomposable module Q in* $\operatorname{mod} A$.

Let X be a finite partially ordered set (*poset*) and let k be a field. Recall from Section 1.2.5 that the *incidence algebra kX* is the k-algebra spanned by the elements e_{xy} for the pairs $x \leq y$ in X, with the multiplication defined by setting $e_{xy}e_{y'z} = e_{xz}$ when $y = y'$ and zero otherwise.

Corollary 6.1.3. *Let X be a finite poset. If the incidence algebra kX is piecewise hereditary, then* gl.dim $kX \leq 3$ *and* $\operatorname{pd} Q + \operatorname{id} Q \leq 4$ *for any indecomposable kX-module Q.*

The bounds in Corollaries 6.1.2 and 6.1.3 are sharp, see Examples 6.3.2 and 6.3.3.

The chapter is organized as follows. In Section 6.2 we give the proofs of the above results. Examples demonstrating various aspects of these results are given in Section 6.3.

6.2 The proofs

6.2.1 Preliminaries

Let $\mathcal{A}$ be an abelian category. If X is an object of $\mathcal{A}$, denote by $X[n]$ the complex in $\mathcal{D}^b(\mathcal{A})$ with X at position $-n$ and 0 elsewhere. Denote by $\operatorname{ind} \mathcal{A}$, $\operatorname{ind} \mathcal{D}^b(\mathcal{A})$ the sets of isomorphism classes of indecomposable objects of $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$, respectively. The map $X \mapsto X[0]$ is a fully faithful functor $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ which induces an embedding $\operatorname{ind} \mathcal{A} \hookrightarrow \operatorname{ind} \mathcal{D}^b(\mathcal{A})$.

Assume that there exists a triangulated equivalence $F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{H})$ with $\mathcal{H}$ hereditary. Then F induces a bijection $\operatorname{ind} \mathcal{D}^b(\mathcal{A}) \simeq \operatorname{ind} \mathcal{D}^b(\mathcal{H})$, and we denote by $\varphi_F : \operatorname{ind} \mathcal{A} \to \operatorname{ind} \mathcal{H} \times \mathbb{Z}$ the composition

$$\operatorname{ind} \mathcal{A} \hookrightarrow \operatorname{ind} \mathcal{D}^b(\mathcal{A}) \xrightarrow{\sim} \operatorname{ind} \mathcal{D}^b(\mathcal{H}) = \operatorname{ind} \mathcal{H} \times \mathbb{Z}$$

where the last equality follows from [53, (2.5)].

If Q is an indecomposable of $\mathcal{A}$, write $\varphi_F(Q) = (f_F(Q), n_F(Q))$ where $f_F(Q) \in \operatorname{ind} \mathcal{H}$ and $n_F(Q) \in \mathbb{Z}$, so that $F(Q[0]) \simeq f_F(Q)[n_F(Q)]$ in $\mathcal{D}^b(\mathcal{H})$. From now on we fix the equivalence F, and omit the subscript F.

Lemma 6.2.1. *The map $f : \operatorname{ind} \mathcal{A} \to \operatorname{ind} \mathcal{H}$ is one-to-one.*

Proof. If Q, Q' are two indecomposables of $\mathcal{A}$ such that $f(Q), f(Q')$ are isomorphic in $\mathcal{H}$, then $Q[n(Q') - n(Q)] \simeq Q'[0]$ in $\mathcal{D}^b(\mathcal{A})$, hence $n(Q) = n(Q')$, and $Q \simeq Q'$ in $\mathcal{A}$. $\square$

As a corollary, note that if A and H are two finite dimensional algebras such that $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{D}^b(\operatorname{mod} H)$ and H is hereditary, then the representation type of H dominates that of A.

We recall the following three results, which were introduced in [35, (IV,1)] when $\mathcal{H}$ is the category of representations of a quiver.

Lemma 6.2.2. *Let Q, Q' be two indecomposables of $\mathcal{A}$, Then*

$$\operatorname{Ext}^i_{\mathcal{A}}(Q, Q') \simeq \operatorname{Ext}^{i+n(Q')-n(Q)}_{\mathcal{H}}(f(Q), f(Q'))$$

Corollary 6.2.3. *Let Q, Q' be two indecomposables of $\mathcal{A}$ with $\operatorname{Hom}_{\mathcal{A}}(Q, Q') \neq 0$. Then $n(Q') - n(Q) \in \{0, 1\}$.*

Lemma 6.2.4. *Assume that $\mathcal{A}$ is of finite length and there exist integers n_0, d such that $n_0 \leq n(P) < n_0 + d$ for every indecomposable P of $\mathcal{A}$.*

If Q is indecomposable, then $\operatorname{pd}_{\mathcal{A}} Q \leq n(Q) - n_0 + 1$ and $\operatorname{id}_{\mathcal{A}} Q \leq n_0 + d - n(Q)$. In particular, $\operatorname{gl.dim} \mathcal{A} \leq d$.

Proof. See [35, IV, p.158] or [40, (1.2)]. $\square$

6.2.2 Proof of Theorem 6.1.1

Let $r \geq 1$, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{r-1})$ and Q_0 be as in the Theorem. Denote by r_+ the number of positive ε_i, and by r_- the number of negative ones. Let $F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{H})$ be a triangulated equivalence and write $f = f_F$, $n = n_F$.

Let Q be any indecomposable of $\mathcal{A}$. By assumption, there exists an ε-path $Q_0, Q_1, \ldots, Q_r = Q$, so by Corollary 6.2.3, $n(Q_{i+1}) - n(Q_i) \in \{0, \varepsilon_i\}$ for all $0 \leq i < r$. It follows that $n(Q) - n(Q_0) = \sum_{i=0}^{r-1} \alpha_i \varepsilon_i$ for some $\alpha_i \in \{0, 1\}$, hence

$$n(Q_0) - r_- \leq n(Q) \leq n(Q_0) + r_+$$

and the result follows from Lemma 6.2.4 with $d = r + 1$ and $n_0 = n(Q_0) - r_-$.

6.2.3 Variations and comments

Remark 6.2.5. The assumption in Theorem 6.1.1 that any indecomposable Q is the end of an ε-path from Q_0 can replaced by the weaker assumption that any *simple* object is the end of such a path.

Proof. Assume that $\varepsilon_{r-1} = 1$ and let Q be indecomposable. Since Q has finite length, we can find a simple object S with $g : S \hookrightarrow Q$. Let $Q_0, Q_1, \ldots, Q_{r-1}, S$ be an ε-path from Q_0 to S with $f_{r-1} : Q_{r-1} \twoheadrightarrow S$. Replacing S by Q and f_{r-1} by $g f_{r-1} \neq 0$ gives an ε-path from Q_0 to Q.

The case $\varepsilon_{r-1} = -1$ is similar. $\square$

Remark 6.2.6. Let $\widetilde{G}(\mathcal{A})$ be the undirected graph obtained from $G(\mathcal{A})$ by forgetting the directions of the edges. The *distance* between two indecomposables Q and Q', denoted $d(Q, Q')$, is defined as the length of the shortest path in $\widetilde{G}(\mathcal{A})$ between them (or $+\infty$ if there is no such path).

The same proof gives that $|n(Q) - n(Q')| \leq d(Q, Q')$ for any two indecomposables Q and Q'. Let $d = \sup_{Q,Q'} d(Q, Q')$ be the *diameter* of $\widetilde{G}(\mathcal{A})$. When $d < \infty$, $\inf_Q n(Q)$ and $\sup_Q n(Q)$ are finite, and by Lemma 6.2.4 $\operatorname{gl.dim} \mathcal{A} \leq d + 1$ and $\operatorname{pd}_{\mathcal{A}} Q + \operatorname{id}_{\mathcal{A}} Q \leq d + 2$ for any indecomposable Q.

Remark 6.2.7. The conclusion of Theorem 6.1.1 (or Remark 6.2.6) is still true under the slightly weaker assumption that $\mathcal{A}$ is a finite length, piecewise hereditary category and $\mathcal{A} = \oplus_{i=1}^{s} \mathcal{A}_i$ is a direct sum of abelian full subcategories such that each graph $G(\mathcal{A}_i)$ satisfies the corresponding connectivity condition.

6.2.4 Proof of Corollary 6.1.2

Let A be sincere, and let $S_1, \dots, S_n$ be the representatives of the isomorphism classes of simple modules in $\operatorname{mod} A$. Let $P_1, \dots, P_n$ be the corresponding indecomposable projectives and finally let M be an indecomposable, sincere module.

Take $r = 2$ and $\varepsilon = (-1, +1)$. Now observe that any simple S_i is the end of an ε-path from M, as we have a path of nonzero morphisms $M \leftarrow P_i \twoheadrightarrow S_i$ since M is sincere. The result now follows by Theorem 6.1.1 and Remark 6.2.5.

6.2.5 Proof of Corollary 6.1.3

Let X be a poset and k a field. A *k-diagram* $\mathcal{F}$ is the data consisting of finite dimensional k-vector spaces $\mathcal{F}(x)$ for $x \in X$, together with linear transformations $r_{xx'} : \mathcal{F}(x) \to \mathcal{F}(x')$ for all $x \leq x'$, satisfying the conditions $r_{xx} = 1_{\mathcal{F}(x)}$ and $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x \leq x' \leq x''$, see Section 3.2.1.

The category of finite dimensional right modules over kX can be identified with the category of k-diagrams over X, see Lemma 1.2.7. A complete set of representatives of isomorphism classes of simple modules over kX is given by the diagrams S_x for $x \in X$, defined by

$$S_x(y) = \begin{cases} k & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

with $r_{yy'} = 0$ for all $y < y'$. A module $\mathcal{F}$ is sincere if and only if $\mathcal{F}(x) \neq 0$ for all $x \in X$.

The poset X is *connected* if for any $x, y \in X$ there exists a sequence $x = x_0, x_1, \dots, x_n = y$ such that for all $0 \leq i < n$ either $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$.

Lemma 6.2.8. *If X is connected then the incidence algebra kX is sincere.*

Proof. Let k_X be the diagram defined by $k_X(x) = k$ for all $x \in X$ and $r_{xx'} = 1_k$ for all $x \leq x'$. Obviously k_X is sincere. Moreover, k_X is indecomposable by a standard connectivity argument; if $k_X = \mathcal{F} \oplus \mathcal{F}'$, write $V = \{x \in X : \mathcal{F}(x) \neq 0\}$ and assume that V not empty. If $x \in V$ and

$x < y$, then $y \in V$, otherwise we would get a zero map $k \oplus 0 \to 0 \oplus k$ and not an identity map. Similarly, if $y < x$ then $y \in V$. By connectivity, $V = X$ and $\mathcal{F} = k_X$. $\square$

If X is connected, Corollary 6.1.3 now follows from Corollary 6.1.2 and Lemma 6.2.8. For general X, observe that if $\{X_i\}_{i=1}^s$ are the connected components of X, then the category $\operatorname{mod} kX$ decomposes as the direct sum of the categories $\operatorname{mod} kX_i$, and the result follows from Remark 6.2.7.

Corollary 6.2.9. *Let X and Y be posets such that $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$ and* $\operatorname{gl.dim} kY > 3$. *Then kX is not piecewise hereditary.*

6.3 Examples

We give a few examples that demonstrate various aspects of global dimensions of piecewise hereditary algebras. In these examples, k denotes a field and all posets are represented by their Hasse diagrams.

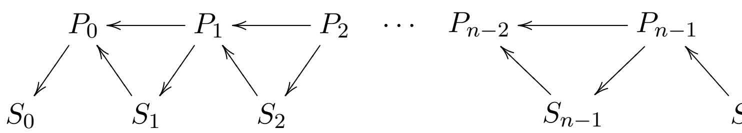
Example 6.3.1 ([54])**.** Let $n \geq 2$, $Q^{(n)}$ the quiver

$$0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_n} n$$

and $I^{(n)}$ be the ideal (in the path algebra $kQ^{(n)}$) generated by the paths $\alpha_i \alpha_{i+1}$ for $1 \leq i < n$. By [35, (IV, 6.7)], the algebra $A^{(n)} = kQ^{(n)}/I^{(n)}$ is piecewise hereditary of Dynkin type A_{n+1}.

For a vertex $0 \leq i \leq n$, let S_i, P_i, I_i be the simple, indecomposable projective and indecomposable injective corresponding to i. Then one has $P_n = S_n$, $I_0 = S_0$ and for $0 \leq i < n$, $P_i = I_{i+1}$ with a short exact sequence $0 \to S_{i+1} \to P_i \to S_i \to 0$.

The graph $G(\operatorname{mod} A^{(n)})$ is shown below (ignoring the self loops around each vertex).

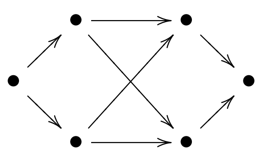


Regarding dimensions, we have $\operatorname{pd} S_i = n-i$, $\operatorname{id} S_i = i$ for $0 \leq i \leq n$, and $\operatorname{pd} P_i = \operatorname{id} P_i = 0$ for $0 \leq i < n$, so that $\operatorname{gl.dim} A^{(n)} = n$ and $\operatorname{pd} Q + \operatorname{id} Q \leq n$ for every indecomposable Q. The diameter of $\widetilde{G}(\operatorname{mod} A^{(n)})$ is $n+1$.

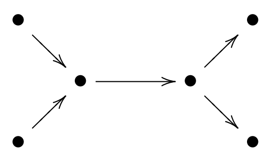
The following two examples show that the bounds given in Corollary 6.1.3 are sharp.

Example 6.3.2. *A poset X with kX piecewise hereditary and* $\operatorname{gl.dim} kX = 3$.

Let X, Y be the two posets:



X

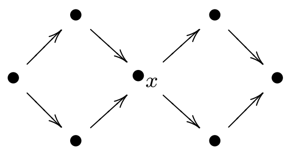


Y

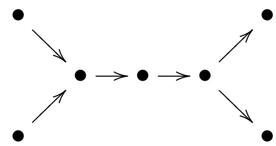
Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\operatorname{gl.dim} kX = 3$, $\operatorname{gl.dim} kY = 1$.

Example 6.3.3. *A poset X with kX piecewise hereditary and an indecomposable $\mathcal{F}$ such that* $\operatorname{pd}_{kX} \mathcal{F} + \operatorname{id}_{kX} \mathcal{F} = 4$.

Let X, Y be the following two posets:



X



Y

Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\operatorname{gl.dim} kX = 2$, $\operatorname{gl.dim} kY = 1$ and for the simple S_x we have $\operatorname{pd}_{kX} S_x = \operatorname{id}_{kX} S_x = 2$.

We conclude by giving two examples of posets whose incidence algebras are not piecewise hereditary.

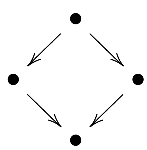
Example 6.3.4. *A product of two trees whose incidence algebra is not piecewise hereditary.*

By specifying an orientation ω on the edges of a (finite) tree T, one gets a finite quiver without oriented cycles whose path algebra is isomorphic to the incidence algebra of the poset $X_{T,\omega}$ defined on the set of vertices of T by saying that $x \leq y$ for two vertices x and y if there is an oriented path from x to y.

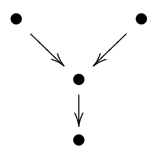
A poset of the form $X_{T,\omega}$ is called a *tree*. Equivalently, a poset is a tree if and only if the underlying graph of its Hasse diagram is a tree. Obviously, $\operatorname{gl.dim} kX_{T,\omega} = 1$, so that $kX_{T,\omega}$ is trivially piecewise hereditary. Moreover, while the poset $X_{T,\omega}$ may depend on the orientation ω chosen, its derived equivalence class depends only on T.

Given two posets X and Y, their *product*, denoted $X \times Y$, is the poset whose underlying set is $X \times Y$ and $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$ where $x, x' \in X$ and $y, y' \in Y$ (cf. Definition 1.3.27). It may happen that the incidence algebra of a product of two trees, although not being hereditary, is piecewise hereditary. Two notable examples are the product of the Dynkin types $A_2 \times A_2$, which is piecewise hereditary of type D_4, and the product $A_2 \times A_3$ which is piecewise hereditary of type E_6.

Consider $X = A_2 \times A_2$ and $Y = D_4$ with the orientations given below.



X



Y

Then $\operatorname{gl.dim} kX = 2$, $\operatorname{gl.dim} kY = 1$ and $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, hence $\mathcal{D}^b(k(X \times X)) \simeq \mathcal{D}^b(k(Y \times Y))$. But $\operatorname{gl.dim} k(X \times X) = 4$, so by Corollary 6.2.9, $Y \times Y$ is a product of two trees of type D_4 whose incidence algebra is not piecewise hereditary.

Example 6.3.5. *The converse to Corollary 6.1.3 is false.*

Let X be the poset

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[dr] & \bullet \\
\bullet \arrow[r] \arrow[ur] & \bullet \arrow[r] \arrow[ur] & \bullet
\end{tikzcd}$$

Then $\operatorname{gl.dim} kX = 2$, hence $\operatorname{pd}_{kX} \mathcal{F} \leq 2$, $\operatorname{id}_{kX} \mathcal{F} \leq 2$ for any indecomposable $\mathcal{F}$, so that X satisfies the conclusion of Corollary 6.1.3. However, kX is not piecewise hereditary since $\operatorname{Ext}^2_X(k_X, k_X) = k$ does not vanish (see [35, (IV, 1.9)]). Note that X is the smallest poset whose incidence algebra is not piecewise hereditary.

Chapter 7

On the Periodicity of Coxeter Transformations and the Non-negativity of Their Euler Forms

We show that for piecewise hereditary algebras, the periodicity of the Coxeter transformation implies the non-negativity of the Euler form. Contrary to previous assumptions, the condition of piecewise heredity cannot be omitted, even for triangular algebras, as demonstrated by incidence algebras of posets.

We also give a simple, direct proof, that certain products of reflections, defined for any square matrix A with 2 on its main diagonal, and in particular the Coxeter transformation corresponding to a generalized Cartan matrix, can be expressed as $-A_+^{-1}A_-^t$, where A_+, A_- are closely associated with the upper and lower triangular parts of A.

7.1 Introduction

Let V be a free abelian group of finite rank and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on V. The *Coxeter transformation* $\Phi : V \to V$ corresponding to $\langle \cdot, \cdot \rangle$ is defined via the equation $\langle x, y \rangle = - \langle y, \Phi x \rangle$ for $x, y \in V$ [61].

In this chapter we study the relations between positivity properties of the form $\langle \cdot, \cdot \rangle$ and periodicity properties of its Coxeter transformation Φ. Recall that $\langle \cdot, \cdot \rangle$ is *positive* if $\langle x, x \rangle > 0$ for all $0 \neq x \in V$, *non-negative* if $\langle x, x \rangle \geq 0$ for all $x \in V$ and *indefinite* otherwise. The transformation Φ is *periodic* if Φ^m equals the identity I for some integer $m \geq 1$ and *weakly periodic* [79] if $(\Phi^m - I)^n = 0$ for some integers $m, n \geq 1$.

Implications in one direction are given in the paper [79], where linear algebra techniques are used to show that the Coxeter matrix Φ is periodic if $\langle \cdot, \cdot \rangle$ is positive and weakly periodic if $\langle \cdot, \cdot \rangle$ is non-negative. It is much harder to establish implications in the other direction. As already noted in [79], even if Φ is periodic, $\langle \cdot, \cdot \rangle$ may be indefinite, so additional constraints are needed.

An alternative definition of the Coxeter matrix is as a certain product of reflections defined by a generalized Cartan matrix [1, 77], whereas the definition given above is $-C^{-1}C^t$ where C

is the matrix of the bilinear form.

We claim similarly to [23], and give a simple, direct proof, that for any square matrix A with 2 on its main diagonal, the product of the n reflections it defines can be expressed as $-A_+^{-1}A_-^t$ where A_+, A_- are closely associated with the upper and lower triangular parts of A, see Section 7.2. This claim can be generalized to products in arbitrary order, and no other conditions on A, such as being generalized Cartan, bipartite [1] or symmetric [45], are needed. In particular, when $\langle \cdot, \cdot \rangle$ is triangular, Φ can be written as a product of the reflections defined by the symmetrization of $\langle \cdot, \cdot \rangle$.

Further connections between periodicity and non-negativity are achieved when we restrict ourselves to pairs $(V, \langle \cdot, \cdot \rangle)$ for which there exists a finite dimensional k-algebra Λ over an algebraically closed field k, having finite global dimension, such that $V \cong K_0(\operatorname{mod} \Lambda)$ and $\langle \cdot, \cdot \rangle$ coincides, under that isomorphism, with the Euler form $\langle \cdot, \cdot \rangle_\Lambda$ of Λ. Here $\operatorname{mod} \Lambda$ denotes the category of finite dimensional right Λ-modules. Since $\operatorname{gl.dim} \Lambda < \infty$, the form $\langle \cdot, \cdot \rangle_\Lambda$ is non-degenerate, hence its Coxeter transformation Φ_Λ is well-defined and coincides with the image in $K_0(\operatorname{mod} \Lambda)$ of the Auslander-Reiten translation on the bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$.

In Section 7.3 we show that if Λ is *piecewise hereditary*, i.e. its bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$ is equivalent as a triangulated category to $\mathcal{D}^b(\mathcal{H})$ for a hereditary abelian category $\mathcal{H}$, then the periodicity of Φ_Λ implies the non-negativity of $\langle \cdot, \cdot \rangle_\Lambda$.

In that Section, we also show that when Λ is an incidence algebra of a poset X, the Euler form $\langle \cdot, \cdot \rangle_\Lambda$ and its Coxeter transformation Φ_Λ can be explicitly described in terms of the combinatorics of X.

Previously, [24] claimed that the condition of $\langle \cdot, \cdot \rangle$ being *triangular*, that is, its matrix with respect to some basis of V is upper triangular with ones on the main diagonal, is enough for the periodicity of Φ to imply the non-negativity of $\langle \cdot, \cdot \rangle$. We find however examples of incidence algebras of posets negating this claim, see Section 7.4.

7.2 Coxeter transformations of bilinear forms

7.2.1 The definition of the Coxeter matrix

Let V be a free abelian group of finite rank and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on V. Recall that $\langle \cdot, \cdot \rangle$ is *positive* if $\langle v, v \rangle > 0$ for all $0 \neq v \in V$, *non-negative* if $\langle v, v \rangle \geq 0$ for all $v \in V$ and *indefinite* otherwise. The *Coxeter transformation* $\Phi : V \to V$ corresponding to $\langle \cdot, \cdot \rangle$ is defined via the equation $\langle v, w \rangle = - \langle w, \Phi v \rangle$ for all $v, w \in V$ [61].

We consider the elements of $\mathbb{Z}^n$ as column vectors, and denote by M^t the transpose of a matrix M. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$. By choosing a $\mathbb{Z}$-basis $v_1, \dots, v_n$ of V, we may identify V with $\mathbb{Z}^n$ and $\langle \cdot, \cdot \rangle$ with the form $\langle \cdot, \cdot \rangle_C$ defined by $\langle x, y \rangle_C = x^t C y$ where $C \in \mathrm{GL}_n(\mathbb{Z})$ is the matrix whose entries are $C_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$. In other words, $\langle v_i, v_j \rangle = \langle e_i, e_j \rangle_C$. Under this identification, the matrix of Φ is $-C^{-1}C^t$, hence we define the *Coxeter matrix* Φ_C of a matrix $C \in \mathrm{GL}_n(\mathbb{Z})$ to be $\Phi_C = -C^{-1}C^t$.

Note that $v = -C^{-1}C^t v$ if and only if $(C + C^t)v = 0$, hence the geometric multiplicity of the eigenvalue 1 in Φ_C equals the dimension of the radical of the symmetrized bilinear form $C + C^t$.

Definition 7.2.1. A matrix $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ is *periodic* if $\Phi^m = I$ for some $m \geq 1$. Φ is *weakly periodic* if for some $m \geq 1$, $\Phi^m - I$ is nilpotent.

Definition 7.2.2. A matrix $C \in \mathrm{GL}_n(\mathbb{Z})$ is *unitriangular* if C is upper triangular and $C_{ii} = 1$ for $1 \leq i \leq n$.

Relations between the positivity of the bilinear form $\langle \cdot, \cdot \rangle_C$ and the periodicity of Φ_C have been studied in [24, 79] and are summarized as follows:

Theorem 7.2.3. *Let $C \in \mathrm{GL}_n(\mathbb{Z})$. Then:*

1. *[79, (2.8)] Φ_C is periodic if $\langle \cdot, \cdot \rangle_C$ is positive.*
2. *[79, (3.4)] Φ_C is weakly periodic if $\langle \cdot, \cdot \rangle_C$ is non-negative.*

However, [79, (3.8)] is an example of a matrix whose Coxeter matrix is periodic but the corresponding bilinear form is indefinite.

7.2.2 Alternative definition as a product of reflections

Following [1, 8, 77], we review an alternative definition of the Coxeter matrix as a product of reflections.

Let A be an $n \times n$ matrix with integer entries satisfying

$$A_{ii} = 2 \qquad 1 \leq i \leq n \tag{A1}$$

$$A_{ij} = 0 \qquad \text{if and only if } A_{ij} = 0, 1 \leq i, j \leq n \tag{A2}$$

The *primitive graph* of A (cf. [8]) is an undirected graph with n vertices, where two vertices $i \neq j$ are connected by an edge if $A_{ij} \neq 0$. The matrix A is *indecomposable* if its primitive graph is connected.

Define *reflections* $r_1, \dots, r_n$ by

$$r_i(e_j) = e_j - A_{ij} e_i \qquad 1 \leq j \leq n \tag{7.2.1}$$

In other words, r_i is the matrix obtained from the identity matrix by subtracting the i-th row of A. Denote by I the $n \times n$ identity matrix.

Lemma 7.2.4. *Let A be a matrix satisfying* (A1).

a. *$r_i^2 = I$ for $1 \leq i \leq n$.*

b. *If A satisfies also* (A2)*, then $r_i r_j = r_j r_i$ for any two non-adjacent vertices i, j on the primitive graph of A.*

Proof. Since $A_{ii} = 2$, we have $r_i(e_i) = -e_i$, thus

$$r_i^2(e_t) = r_i(e_t - A_{it} e_i) = e_t - A_{it} e_i - A_{it} r_i(e_i) = e_t$$

for all $1 \leq t \leq n$, and the first assertion is proved.

If $A_{ij} = 0$ then $r_i(e_j) = e_j$. The assumptions on A imply that if i, j are not adjacent, then $r_i(e_j) = e_j$ and $r_j(e_i) = e_i$. Therefore, if $1 \leq t \leq n$,

$$r_i r_j(e_t) = r_i(e_t - A_{jt}e_j) = e_t - A_{it}e_i - A_{jt}e_j$$

is symmetric in i and j, hence $r_i r_j = r_j r_i$. □

Consider the following two additional properties:

$$A_{ij} \leq 0 \qquad \text{for all } i \neq j \tag{A3}$$

$$\text{The primitive graph of } A \text{ is bipartite} \tag{A4}$$

Definition 7.2.5. A matrix A is a *generalized Cartan matrix* if it satisfies (A1), (A2) and (A3). A matrix A is *bipartite* if it satisfies (A1), (A2) and (A4).

For a generalized Cartan matrix A and a permutation π of $\{1, 2, \dots, n\}$, a *Coxeter transformation* is defined in [77] by $\Phi(A, \pi) = r_{\pi(1)} r_{\pi(2)} \cdots r_{\pi(n)}$. For a bipartite matrix A, let $\Sigma_1 \amalg \Sigma_2$ be a corresponding partition of $\{1, 2, \dots, n\}$ and consider $R_A = R_1 R_2$ where $R_k = \prod_{i \in \Sigma_k} r_i$, $k = 1, 2$, see [8]. Note that by Lemma 7.2.4, the matrices R_k do not depend on the order of reflections within each product. Note also that R_A equals $\Phi(A, \pi)$ for a suitable π.

Recall that the *spectrum* of a square matrix Φ with complex entries, denoted $\operatorname{spec}(\Phi)$, is the set of (complex) roots of the characteristic polynomial of Φ. Let $\rho(\Phi) = \max\{|\lambda| : \lambda \in \operatorname{spec}(\Phi)\}$ be the *spectral radius* of Φ.

We recall two results on the spectrum of Coxeter transformations corresponding to generalized Cartan and bipartite matrices.

Theorem 7.2.6 ([77]). *Let A be an indecomposable generalized Cartan matrix, $\pi \in S_n$. If A is not of finite or affine type, then $\rho(\Phi(A, \pi)) > 1$.*

Theorem 7.2.7 ([1, p. 63],[8, p. 344]). *Let A be a bipartite matrix.*

a. $\lambda^2 \in \operatorname{spec}(R_A)$ *if and only if* $\lambda + 2 + \lambda^{-1} \in \operatorname{spec}(A)$.

b. If A is also symmetric, then $\operatorname{spec}(R_A) \subset S^1 \cup \mathbb{R}$.

7.2.3 Linking the two definitions

Let R be any commutative ring with 1 and let $e_1, \dots, e_n$ be a basis of a free R-module of rank n. Let A be an $n \times n$ matrix with entries in R satisfying (A1) (where 2 means $1 + 1$), and define the reflections $r_1, \dots, r_n$ as in (7.2.1). When we want to stress the dependence of the reflections on A, we shall use the notation $r_1^A, \dots, r_n^A$.

Lemma 7.2.8. *Let $1 \leq s \leq n$. Then for every $1 \leq t \leq n$,*

$$(r_1 \cdots r_s)(e_t) = e_t + \sum_{k=1}^{s} (-1)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}$$

Proof. By induction on s, the case $s = 1$ being just the definition of r_1, and for the induction step, expand $r_{s+1}(e_t)$ as $e_t - A_{s+1,t}e_{s+1}$ and use the hypothesis for s.

$$\begin{aligned}
&(r_1 \cdots r_s r_{s+1})(e_t) = (r_1 \cdots r_s)(e_t) - A_{s+1,t}(r_1 \cdots r_s)(e_{s+1}) \\
&= e_t - A_{s+1,t}e_{s+1} + \sum_{k=1}^{s}(-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1} \\
&\quad + \sum_{k=1}^{s}(-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k, s+1} A_{s+1,t} e_{i_1} \\
&= e_t - \sum_{k=1}^{s+1}(-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le s+1} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}
\end{aligned}$$

□

Define two $n \times n$ matrices A_+ and A_- by

$$(A_+)_{ij} = \begin{cases} A_{ij} & i < j \\ 1 & i = j \\ 0 & i > j \end{cases} \qquad\qquad (A_-)_{ij} = \begin{cases} A_{ji} & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}$$

Then $A = A_+ + A_-^t$, and one can think of A_+, A_- as the upper and lower triangular parts of A. The matrices A_+ and A_- are invertible since $A_+ - I$ and $A_- - I$ are nilpotent. Note that A is symmetric if and only if $A_+ = A_-$

Theorem 7.2.9. *If A satisfies* (A1)*, then $r_1^A r_2^A \cdots r_n^A = -A_+^{-1} A_-^t$.*

Proof. By Lemma 7.2.8 with $s = n$,

$$(r_1 \cdots r_n)(e_t) = e_t + \sum_{k=1}^{n}(-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}$$

This can be written in matrix form, using the definition of A_+, as follows:

$$\begin{aligned}
r_1 \cdots r_n &= I + \sum_{k=1}^{n}(-1)^k (A_+ - I)^{k-1} A \\
&= I - (I - (A_+ - I) + (A_+ - I)^2 - \ldots)A \\
&= I - A_+^{-1}(A_+ + A_-^t) = -A_+^{-1} A_-^t
\end{aligned}$$

□

Remark 7.2.10. Theorem 7.2.9 is still true when we drop the condition (A1) and slightly change the definition of A_-, by $(A_-)_{ii} = A_{ii} - 1$ for $1 \le i \le n$. However, in that case the matrices r_i are no longer reflections.

Theorem 7.2.9 provides a link between the definition of the Coxeter matrix as a specific automorphism of the bilinear form and its definition as a product of n reflections, as shown by the following corollary.

Corollary 7.2.11. *Let $C \in \mathrm{GL}_n(\mathbb{Z})$ be a unitriangular matrix. Then $\Phi_C = \Phi(C + C^t, id)$, that is, $\Phi_C = r_1^A r_2^A \cdots r_n^A$ for $A = C + C^t$.*

In fact, this corollary is proved in [45] for the case where Φ_C is a Coxeter element in an arbitrary Coxeter group of finite rank represented as a group of linear transformations on a real inner product space, so that the Cartan matrix A is symmetric.

Proof. Apply Theorem 7.2.9 for the matrix $A = C + C^t$, which satisfies (A1), (A2) and $A_+ = A_- = C$. $\square$

Denote by S_n the group of permutations on $\{1, 2, \dots, n\}$ and let $\pi \in S_n$. One could deduce a generalized version of Theorem 7.2.9 for the product of the n reflections in an arbitrary order by proving an analogue of Lemma 7.2.8 for arbitrary π. Instead, we will derive the generalized version from the original one using permutation matrices.

Define the permutation matrix P_π by $P_\pi(e_i) = e_{\pi(i)}$ for all $1 \le i \le n$. Note that $P_\pi^{-1} = P_\pi^t$. Given a matrix A, let A_π denote the matrix $P_\pi^{-1} A P_\pi$, so that $(A_\pi)_{ij} = A_{\pi(i)\pi(j)}$. Obviously, if A satisfies (A1), so does A_π.

Lemma 7.2.12. *Let $1 \le i \le n$. Then $r_i^{A_\pi} = P_\pi^{-1} r_{\pi(i)}^A P_\pi$.*

Proof. For all $1 \le t \le n$,

$$\left(P_\pi^{-1} r_{\pi(i)}^A P_\pi\right)(e_t) = P_\pi^{-1}\left(e_{\pi(t)} - A_{\pi(i)\pi(t)} e_{\pi(i)}\right) = e_t - A_{\pi(i)\pi(t)} e_i = r_i^{A_\pi}(e_t)$$

$\square$

Define two $n \times n$ matrices $A_{\pi,+}$ and $A_{\pi,-}$ by

$$(A_{\pi,+})_{ij} = \begin{cases} A_{ij} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases} \qquad (A_{\pi,-})_{ij} = \begin{cases} A_{ji} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Direct calculation shows that $A_{\pi,+} = P_\pi (A_\pi)_+ P_\pi^{-1}$, $A_{\pi,-} = P_\pi (A_\pi)_- P_\pi^t$ and $A = A_{\pi,+} + A_{\pi,-}^t$.

Corollary 7.2.13. *Let A satisfy* (A1) *and let $\pi \in S_n$. Then*

$$r_{\pi(1)}^A r_{\pi(2)}^A \dots r_{\pi(n)}^A = -A_{\pi,+}^{-1} A_{\pi,-}^t$$

Proof. By Lemma 7.2.12 and Theorem 7.2.9 applied for A_π,

$$\begin{aligned} r_{\pi(1)}^A r_{\pi(2)}^A \dots r_{\pi(n)}^A &= P_\pi \left(r_1^{A_\pi} r_2^{A_\pi} \dots r_n^{A_\pi}\right) P_\pi^{-1} = -P_\pi (A_\pi)_+^{-1} (A_\pi)_-^t P_\pi^t \\ &= -\left(P_\pi (A_\pi)_+^{-1} P_\pi^{-1}\right)\left(P_\pi (A_\pi)_-^t P_\pi^t\right) = -A_{\pi,+}^{-1} A_{\pi,-}^t \end{aligned}$$

$\square$

7.3 Periodicity and non-negativity for piecewise hereditary algebras and posets

Let k be a field, and let $\mathcal{A}$ be an abelian k-category of finite global dimension with finite dimensional Ext-spaces. Denote by $\mathcal{D}^b(\mathcal{A})$ its bounded derived category and by $K_0(\mathcal{A})$ its Grothendieck group. The expression

$$\langle X, Y \rangle_{\mathcal{A}} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[i])$$

is well-defined for $X, Y \in \mathcal{D}^b(\mathcal{A})$ and induces a $\mathbb{Z}$-bilinear form on $K_0(\mathcal{A})$, known as the *Euler form*. When $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is non-degenerate, the unique transformation $\Phi_{\mathcal{A}} : K_0(\mathcal{A}) \to K_0(\mathcal{A})$ satisfying $\langle x, y \rangle_{\mathcal{A}} = -\langle y, \Phi_{\mathcal{A}} x \rangle_{\mathcal{A}}$ for all $x, y \in K_0(\mathcal{A})$ is called the *Coxeter transformation* of $\mathcal{A}$. For more details we refer the reader to [62].

Two such abelian k-categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *derived equivalent* if there exists a triangulated equivalence $F : \mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{B})$. In this case, the forms $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ are equivalent over $\mathbb{Z}$, hence the positivity properties of the Euler form and the periodicity properties of the Coxeter transformation are invariants of derived equivalence.

Let Λ be a finite dimensional algebra of finite global dimension over an algebraically closed field k, and consider the k-category $\operatorname{mod} \Lambda$ of finitely generated right modules over Λ. Denote by $\mathcal{D}^b(\Lambda)$ its bounded derived category, by $K_0(\Lambda)$ its Grothendieck group and by $\langle \cdot, \cdot \rangle_{\Lambda}$ the Euler form. Then $K_0(\Lambda)$ is free of finite rank, with a $\mathbb{Z}$-basis consisting of the representatives of the isomorphism classes of simple modules in $\operatorname{mod} \Lambda$. The form $\langle \cdot, \cdot \rangle_{\Lambda}$ is non-degenerate, and its Coxeter transformation Φ_{Λ} coincides with the linear map on $K_0(\Lambda)$ induced by the Auslander-Reiten translation on $\mathcal{D}^b(\Lambda)$. For more details see [35, (III.1)], [76, (2.4)] or [62].

7.3.1 Path algebras of quivers without oriented cycles

The first example of algebras Λ for which the connection between the positivity of $\langle \cdot, \cdot \rangle_{\Lambda}$ and the periodicity of Φ_{Λ} is completely understood is the class of path algebras of quivers without oriented cycles, or more generally hereditary algebras, see [62, Theorem 18.5]. We briefly review the main results.

A (finite) *quiver* Q is a directed graph with a finite number of vertices and edges. The *underlying graph* of Q is the undirected graph obtained from Q by forgetting the orientations of the edges. An *oriented cycle* is a nontrivial path in Q starting and ending at the same vertex. The *path algebra* kQ is the algebra over k having as a k-basis the set of all (oriented) paths in Q; the product of two paths is their composition, if defined, and zero otherwise.

When Q has no oriented cycles, the path algebra kQ is hereditary and finite-dimensional. Denote by $\langle \cdot, \cdot \rangle_Q$ its Euler form and by Φ_Q its Coxeter transformation. The matrix of $\langle \cdot, \cdot \rangle_Q$ with respect to the basis of simple modules is unitriangular, and its symmetrization is generalized Cartan. The relations between the periodicity of Φ_Q and the positivity of $\langle \cdot, \cdot \rangle_Q$ are summarized in the following well-known proposition, see [1, 9, 14, 77] and [76, (1.2)].

Proposition 7.3.1. *Let Q be a connected quiver without oriented cycles. Then:*

a. Φ_Q *is periodic if and only if* $\langle\cdot,\cdot\rangle_Q$ *is positive, equivalently the underlying graph of* Q *is a Dynkin diagram of type* A*,* D *or* E*.*

b. Φ_Q *is weakly periodic if and only if* $\langle\cdot,\cdot\rangle_Q$ *is non-negative, equivalently the underlying graph of* Q *is a Dynkin diagram or an extended Dynkin diagram of type* $\tilde{A}$*,* $\tilde{D}$ *or* $\tilde{E}$*.*

7.3.2 Canonical algebras

Another interesting class of algebras for which the connection between non-negativity and periodicity is established are the canonical algebras, introduced in [76].

The Grothendieck group and the Euler form of canonical algebras were thoroughly studied in [61]. If Λ is canonical of type $(\mathbf{p},\boldsymbol{\lambda})$ where $\mathbf{p} = (p_1,\dots,p_t)$ and $\boldsymbol{\lambda} = (\lambda_3,\dots,\lambda_t)$ is a sequence of pairwise distinct elements of $k\setminus\{0\}$, then the rank of $K_0(\Lambda)$ is $\sum_{i=1}^t p_i-(t-2)$ and the characteristic polynomial of the Coxeter transformation Φ_Λ equals $(T-1)^2\prod_{i=1}^t \frac{T^{p_i}-1}{T-1}$ [61, Prop. 7.8)]. In particular, $\rho(\Phi_\Lambda)=1$ and the eigenvalues of Φ_Λ are roots of unity, hence Φ_Λ is weakly periodic.

The following proposition follows from [61, Prop. 10.3], see also [63].

Proposition 7.3.2. *Let* Λ *be a canonical algebra of type* $(\mathbf{p},\boldsymbol{\lambda})$*. If* Φ_Λ *is periodic then* $\mathbf{p}$ *is one of* $(2,3,6)$*,* $(2,4,4)$*,* $(3,3,3)$ *or* $(2,2,2,2)$*. In any of these cases,* $\langle\cdot,\cdot\rangle_\Lambda$ *is non-negative.*

7.3.3 Extending to piecewise hereditary algebras

We extend the results of the previous sections to the class of all piecewise hereditary algebras.

Definition 7.3.3. An algebra Λ over k is *piecewise hereditary* if there exist a hereditary abelian category $\mathcal{H}$ and a triangulated equivalence $\mathcal{D}^b(\Lambda)\simeq\mathcal{D}^b(\mathcal{H})$.

Theorem 7.3.4. *Let* k *be an algebraically closed field and let* Λ *be a finite dimensional piecewise hereditary* k*-algebra. If* Φ_Λ *is periodic, then* $\langle\cdot,\cdot\rangle_\Lambda$ *is non-negative.*

Proof. By definition, there exists a hereditary category $\mathcal{H}$ and an equivalence of triangulated categories $F:\mathcal{D}^b(\Lambda)\simeq\mathcal{D}^b(\mathcal{H})$. By the invariance under derived equivalence, it is enough to prove the theorem for $\Phi_\mathcal{H}$ and $\langle\cdot,\cdot\rangle_\mathcal{H}$. Moreover, we can assume that $\mathcal{H}$ is connected.

Now $\mathcal{H}$ is an Ext-finite k-category and $F(\Lambda_\Lambda)$ is a tilting complex in $\mathcal{D}^b(\mathcal{H})$, so by [39, Theorem 1.7], $\mathcal{H}$ admits a *tilting object*, that is, an object T with $\mathrm{Ext}^1_\mathcal{H}(T,T)=0$ such that for any object X of $\mathcal{H}$, the condition $\mathrm{Hom}_\mathcal{H}(T,X)=0=\mathrm{Ext}^1_\mathcal{H}(T,X)$ implies that $X=0$.

By the classification of hereditary connected Ext-finite k-categories with tilting object up to derived equivalence over an algebraically closed field [38], $\mathcal{H}$ is derived equivalent to $\mathrm{mod}\,H$ for a finite dimensional hereditary algebra H or to $\mathrm{mod}\,\Lambda$ for a canonical algebra Λ. Again by invariance under derived equivalence we may assume that $\mathcal{H}=\mathrm{mod}\,H$ or $\mathcal{H}=\mathrm{mod}\,\Lambda$.

For $\mathcal{H}=\mathrm{mod}\,H$, we can replace H by a path algebra of a finite connected quiver without oriented cycles, and then use Proposition 7.3.1. For $\mathcal{H}=\mathrm{mod}\,\Lambda$, the result follows from Proposition 7.3.2. $\square$

7.3.4 Incidence algebras of posets

Let X be a finite partially ordered set (*poset*) and let k be a field. We recollect the basic facts on incidence algebras of posets and their Euler forms, and refer the reader to Chapter 1, especially Sections 1.2.5 and 1.3.3, for more details.

The *incidence algebra* kX is the k-algebra spanned by elements e_{xy} for the pairs $x \leq y$ in X, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$. Finite dimensional right modules over kX can be identified with commutative diagrams of finite dimensional k-vector spaces over the *Hasse diagram* of X which is the directed graph whose vertices are the points of X, with an arrow from x to y if $x < y$ and there is no $z \in X$ with $x < z < y$.

The algebra kX is of finite global dimension, hence its Euler form, denoted $\langle \cdot, \cdot \rangle_X$, is well-defined and non-degenerate. Denote by C_X, Φ_X the matrices of $\langle \cdot, \cdot \rangle_X$ and its Coxeter transformation with respect to the basis of simple kX-modules.

The *incidence matrix* of X, denoted $\mathbf{1}_X$, is the $X \times X$ matrix defined by

$$(\mathbf{1}_X)_{xy} = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise} \end{cases}$$

By extending the partial order on X to a linear order, we can always arrange the elements of X such that the incidence matrix is unitriangular. In particular, $\mathbf{1}_X$ is invertible over $\mathbb{Z}$. Recall that the *Möbius function* $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x, y) = (\mathbf{1}_X^{-1})_{xy}$.

Lemma 7.3.5. $C_X = \mathbf{1}_X^{-1}$.

Proof. Use Proposition 1.3.11. □

Lemma 7.3.6. *Let $x, y \in X$. Then $(\Phi_X)_{xy} = -\sum_{z\,:\,z \geq x} \mu_X(y, z)$.*

Proof. Since $\Phi_X = -C_X^{-1}C_X^t = -\mathbf{1}_X\mathbf{1}_X^{-t}$,

$$(\Phi_X)_{xy} = -\sum_{z \in X} (\mathbf{1}_X)_{xz}(\mathbf{1}_X^{-1})_{yz} = -\sum_{z\,:\,z \geq x} \mu_X(y, z)$$

□

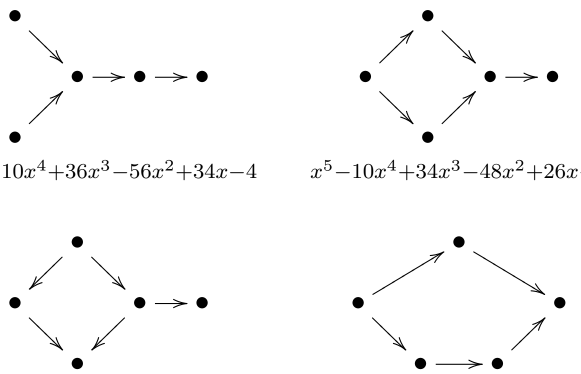
When the Hasse diagram of X has the property that any two vertices x, y are connected by at most one directed path, the Möbius function takes a very simple form, namely

$$\mu_X(x, y) = \begin{cases} 1 & y = x \\ -1 & x \to y \text{ is an edge in the Hasse diagram} \\ 0 & \text{otherwise} \end{cases}$$

In this case, Lemma 7.3.6 coincides with Proposition 3.1 of [10], taking the Hasse diagram as the quiver.

Lemma 7.3.7. *If X and Y are posets, then*

$$C_{X \times Y} = C_X \otimes C_Y \qquad \Phi_{X \times Y} = -\Phi_X \otimes \Phi_Y$$



$x^5-10x^4+36x^3-56x^2+34x-4$ $x^5-10x^4+34x^3-48x^2+26x-4$

$x^5-10x^4+34x^3-48x^2+27x-4$ $x^5-10x^4+34x^3-46x^2+24x-4$

Figure 7.1: Derived equivalent posets with different spectra of the corresponding symmetrized bilinear forms.

Proof. Observe that $\mathbf{1}_{X\times Y} = \mathbf{1}_X \otimes \mathbf{1}_Y$. $\square$

Corollary 7.3.8. *Let X, Y be posets with periodic Coxeter matrices. Then $X \times Y$ has also periodic Coxeter matrix.*

Since non-negativity of forms is not preserved under tensor products, Corollary 7.3.8 can be used to construct posets with periodic Coxeter matrix but with indefinite Euler form, see Example 7.4.4.

7.4 Examples

For a poset X, let C_X, Φ_X be as in the previous section. In particular we may assume that C_X is unitriangular. The symmetrization $A_X = C_X + C_X^t$ satisfies (A1) and (A2), but in general it is not bipartite nor generalized Cartan.

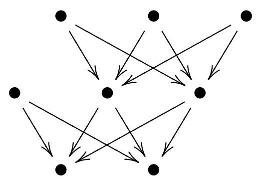
7.4.1 Spectral properties of Φ_X

Example 7.4.1. *The spectrum of Φ_X does not determine that of A_X (Compare with Theorem 7.2.7a).*

The four posets whose Hasse diagrams are depicted in Figure 7.1 are derived equivalent (as they are all piecewise hereditary of type D_5), hence their Coxeter matrices are similar and have the same spectrum, namely the roots of the characteristic polynomial $x^5 + x^4 + x + 1$. However, the spectra of the corresponding symmetrized forms are different. Figure 7.1 also shows for each poset X the characteristic polynomial of the matrix of its symmetrized form.

Example 7.4.2. *A poset X with $\operatorname{spec} \Phi_X \not\subseteq S^1 \cup \mathbb{R}$ (Compare with Theorem 7.2.7b).*

Let X be the following poset.



The characteristic polynomial of Φ_X is $(x+1)^4(x^4 - 2x^3 + 6x^2 - 2x + 1)$, whose roots, besides -1, are $z, \bar{z}, z^{-1}, \bar{z}^{-1}$ with $\Re z = \frac{1+\sqrt{2\sqrt{3}-3}}{2}$ and $|z|^2 = \frac{1+\sqrt{2\sqrt{3}-3}}{2-\sqrt{3}} - 1$. These four roots are neither real nor on the unit circle.

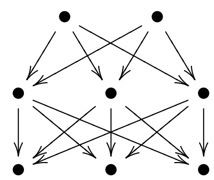
An example of similar spectral behavior for path algebra of a quiver is given in [62, Example 18.1].

Note that for all posets X with 7 elements or less, $\mathrm{spec}(\Phi_X) \subseteq S^1 \cup \mathbb{R}$. This was verified using the database [30] and the MAGMA software package [13].

7.4.2 Counterexamples to [24, Prop. 1.2]

We give two examples of posets showing that in general, for triangular algebras, the periodicity of the Coxeter transformation (and even of the Auslander-Reiten translation up to a shift) does not imply the non-negativity of the Euler form.

Example 7.4.3. Consider the poset X with the following Hasse diagram.



Then $\Phi_X^6 = I$ but $v^t C_X v = -1$ for $v = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\end{pmatrix}^t$ (the vertices are ordered in layers from top to bottom).

Example 7.4.4. Let $X = A_3 \times D_4$ with the following orientations:

$$\begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \\ & 3 \arrow[d] & \\ & 4 & \end{tikzcd}$$

The Hasse diagram of X is given by

$$
\begin{tikzcd}
1,1 \arrow[dr] \arrow[drr] & & 1,2 \arrow[dl] \arrow[drr] & & 2,1 \arrow[dll] \arrow[dr] & & 2,2 \arrow[dll] \arrow[dl] \\
& 1,3 \arrow[d] \arrow[drr] & 3,1 \arrow[dr] & & 3,2 \arrow[dl] & 2,3 \arrow[d] \arrow[dll] & \\
& 1,4 \arrow[drr] & & 3,3 \arrow[d] & & 2,4 \arrow[dll] & \\
& & & 3,4 & & &
\end{tikzcd}
$$

so that X contains the following wild quiver as a subposet.

$$
\begin{tikzcd}
1,3 \arrow[drr] & 3,1 \arrow[dr] & & 3,2 \arrow[dl] & 2,3 \arrow[dll] \\
& & 3,3 \arrow[d] & & \\
& & 3,4 & &
\end{tikzcd}
$$

It follows [64] that kX is not of finite representation type, hence by [86, Theorem 6] the form $\langle \cdot, \cdot \rangle_X$ is not weakly positive, that is, there exists a vector $v \neq 0$ with non-negative coordinates such that $\langle v, v \rangle_X \leq 0$.

Moreover, we can exhibit a non-negative vector v such that $v^t C_X v = -1$, namely $v = (v_x)_{x \in X}$ where the integers v_x are placed at the vertices as in the following picture:

$$
\begin{tikzcd}
0 \arrow[dr] \arrow[drr] & & 0 \arrow[dl] \arrow[drr] & & 0 \arrow[dll] \arrow[dr] & & 0 \arrow[dll] \arrow[dl] \\
& 1 \arrow[d] \arrow[drr] & 1 \arrow[dr] & & 1 \arrow[dl] & 1 \arrow[d] \arrow[dll] & \\
& 1 \arrow[drr] & & 2 \arrow[d] & & 1 \arrow[dll] & \\
& & & 1 & & &
\end{tikzcd}
$$

On the other hand, the Coxeter matrices of the quivers A_3 and D_4 are periodic, their orders are 4 and 6 respectively. By Corollary 7.3.8, the Coxeter matrix of X is periodic of order 12.

Contrary to Example 7.4.3, one can show that not only the image Φ_X of the Auslander-Reiten translation $\tau_X : \mathcal{D}^b(kX) \to \mathcal{D}^b(kX)$ in the Grothendieck group is periodic, but also that actually $\tau_X^e \simeq [d]$ for some integers $d, e \geq 1$.

Chapter 8

Which Canonical Algebras are Derived Equivalent to Incidence Algebras of Posets?

This chapter concerns the characterization of the canonical algebras over an algebraically closed field that are derived equivalent to incidence algebras of finite partially ordered sets (posets), expressed in the following theorem.

Theorem. *Let Λ be a canonical algebra of type $(\mathbf{p}, \boldsymbol{\lambda})$ over an algebraically closed field. Then Λ is derived equivalent to an incidence algebra of a poset if and only if the number of weights of $\mathbf{p}$ is either 2 or 3.*

This theorem can be interpreted both geometrically and algebraically. From a geometric viewpoint, by considering modules over incidence algebras as sheaves over finite spaces (Section 1.2.5) and using the derived equivalence between the categories of modules over a canonical algebra and coherent sheaves over a weighted projective line [32], we are able to obtain explicit derived equivalences between the categories of sheaves of finite dimensional vector spaces over certain finite T_0 topological spaces and the categories of coherent sheaves over certain weighted projective lines.

From an algebraic viewpoint, in an attempt to classify all piecewise hereditary incidence algebras over an algebraically closed field, one first asks which types of piecewise hereditary categories can actually occur. Happel's classification [38] tells us that we only need to consider the canonical algebras and path algebras of quivers. For the canonical algebras the theorem above gives a complete answer, while for path algebras, see the remarks in Section 8.2.2.

We finally note that for the constructions of incidence algebras derived equivalent to canonical algebras, the assumption that the base field is algebraically closed can be omitted.

8.0 Notations

The canonical algebras were introduced in [76]. Let k be a field, $\mathbf{p} = (p_1, \dots, p_t)$ be a sequence of $t \geq 2$ positive integers (*weights*), and $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$ be a sequence of pairwise distinct elements of $k\setminus\{0\}$. The *canonical algebra of type* $(\mathbf{p},\boldsymbol{\lambda})$, is the algebra $\Lambda(\mathbf{p},\boldsymbol{\lambda}) = kQ/I$ where Q is the quiver

$$\begin{tikzcd}
 & \bullet_{1,1} \arrow[r, "x_1"] & \bullet_{1,2} \arrow[r, "x_1"] & \cdots \arrow[r] & \bullet_{1,p_1-1} \arrow[dr, "x_1"] & \\
\bullet_0 \arrow[ur, "x_1"] \arrow[r, "x_2"] \arrow[ddr, "x_t"'] & \bullet_{2,1} \arrow[r, "x_2"] & \bullet_{2,2} \arrow[r, "x_2"] & \cdots \arrow[r] & \bullet_{2,p_2-1} \arrow[r, "x_2"] & \bullet_\omega \\
 & \cdots & \cdots & \cdots & \cdots & \\
 & \bullet_{t,1} \arrow[r, "x_t"] & \bullet_{t,2} \arrow[r, "x_t"] & \cdots \arrow[r] & \bullet_{t,p_t-1} \arrow[uur, "x_t"'] &
\end{tikzcd}$$

and I is the ideal in the path algebra kQ generated by the following linear combinations of paths from 0 to ω:

$$I = \langle x_i^{p_i} - x_2^{p_2} + \lambda_i x_1^{p_1} \ : \ 3 \leq i \leq t \rangle$$

As noted in [76], as long as $t \geq 3$, one can omit weights equal to 1, and when $t = 2$, the ideal I vanishes and the canonical algebra is equal to the path algebra of Q. In the latter case, one usually writes only the weights greater than 1, in particular the algebra of type $()$ equals the path algebra of the Kronecker quiver. Hence when speaking on the *number* of weights, we shall always mean the number of p_i with $p_i \geq 2$.

Let X be a finite partially ordered set (*poset*). The *incidence algebra* kX is the k-algebra spanned by the elements e_{xy} for the pairs $x \leq y$ in X, with the multiplication defined by setting $e_{xy}e_{y'z} = e_{xz}$ if $y = y'$ and $e_{xy}e_{y'z} = 0$ otherwise.

A *k-diagram* $\mathcal{F}$ over X consists of finite dimensional vector spaces $\mathcal{F}(x)$ for $x \in X$, together with linear transformations $r_{xx'} : \mathcal{F}(x) \to \mathcal{F}(x')$ for all $x \leq x'$, satisfying the conditions $r_{xx} = 1_{\mathcal{F}(x)}$ and $r_{xx''} = r_{x'x''}r_{xx'}$ for all $x \leq x' \leq x''$. The category of finite dimensional right modules over kX is equivalent to the category of k-diagrams over X, see Section 1.2.5.

For a finite-dimensional algebra Λ over k, we denote by $\mathcal{C}^b(\Lambda)$ the category of bounded complexes of (right) finite-dimensional Λ-modules, and by $\mathcal{D}^b(\Lambda)$ the bounded derived category. The algebra Λ is *piecewise hereditary* if $\mathcal{D}^b(\Lambda)$ is equivalent as triangulated category to $\mathcal{D}^b(\mathcal{H})$ for a hereditary k-category $\mathcal{H}$.

8.1 The necessity of the condition $t \leq 3$ in the Theorem

For a k-algebra Λ, denote by $\mathrm{HH}^i(\Lambda)$ the *i-th Hochschild cohomology* of Λ, which equals $\mathrm{Ext}^i_{\Lambda\otimes\Lambda^{op}}(\Lambda,\Lambda)$, where Λ is considered as a Λ-Λ-bimodule in the natural way.

Proposition 8.1.1. *Let X be a poset such that kX is piecewise hereditary. Then $\mathrm{HH}^i(kX) = 0$ for any $i > 1$.*

Proof. First, since for any two finite-dimensional k-algebras Λ_1, Λ_2 we have $\mathrm{HH}^i(\Lambda_1 \oplus \Lambda_2) = \mathrm{HH}^i(\Lambda_1) \oplus \mathrm{HH}^i(\Lambda_2)$, we may assume that X is connected, as the decomposition of X into connected components $X = \sqcup_{i=1}^r X_i$ induces a decomposition of the incidence algebra $kX = \bigoplus_{i=1}^r kX_i$.

Let k_X be the *constant diagram* on X, defined by $k_X(x) = k$ for all $x \in X$ with all maps being the identity on k. By Corollary 1.3.20, $\mathrm{HH}^i(kX) = \mathrm{Ext}^i_X(k_X, k_X)$.

Since X is connected, k_X is indecomposable, and by [35, IV (1.9)], the groups $\mathrm{Ext}^i_X(k_X, k_X)$ vanish for $i > 1$, hence $\mathrm{HH}^i(kX) = 0$ for $i > 1$. □

Corollary 8.1.2. *Let k be algebraically closed and let Λ be a canonical algebra over k of type $(\mathbf{p}, \boldsymbol{\lambda})$ where $\mathbf{p} = (p_1, \dots, p_t)$. If $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(kX)$ for some poset X, then $t \leq 3$.*

Proof. Assume that $t \geq 4$. Then Λ is not of domestic type and by [37, Theorem 2.4], $\dim_k \mathrm{HH}^2(kX) = t - 3$, a contradiction to Proposition 8.1.1. Therefore $t \leq 3$. □

8.2 Constructions of posets from canonical algebras

8.2.1 The case $t = 3$

Recall that when $t = 3$, the canonical algebra $\Lambda(\mathbf{p}, \boldsymbol{\lambda})$ is independent of the parameter λ_3, so we may assume that $\lambda_3 = 1$, and denote the algebra by $\Lambda(\mathbf{p})$. Let $\mathbf{p} = (p_1, p_2, p_3)$ be a triplet of weights with $2 \leq p_1 \leq p_2 \leq p_3$. Attach to $\mathbf{p}$ a poset $X_{\mathbf{p}}$ whose Hasse diagram is shown in Figure 8.1. Explicitly, use the Hasse diagram of (8.2.1) if $p_1 > 2$, (8.2.2) if $p_1 = 2$ and $p_2 > 2$, (8.2.3) if $p_2 = 2$ and $p_3 > 2$ and (8.2.4) if $p_3 = 2$.

Theorem 8.2.1. *Let k be a field and let $\mathbf{p} = (p_1, p_2, p_3)$. Then $\Lambda(\mathbf{p})$ is derived equivalent to the incidence algebra of $X_{\mathbf{p}}$.*

Proof. The idea of the proof relies on the notion of a formula introduced in Chapter 3, and we refer to that chapter for more details. We shall first construct a functor $F : \mathcal{C}^b(kX_{\mathbf{p}}) \to \mathcal{C}^b(\Lambda(\mathbf{p}))$ that induces a triangulated functor $\widetilde{F} : \mathcal{D}^b(kX_{\mathbf{p}}) \to \mathcal{D}^b(\Lambda(\mathbf{p}))$, and then prove that $\widetilde{F}$ is an equivalence.

Note that similarly to the identification in Lemma 3.2.3 of complexes of diagrams with diagrams of complexes, we may identify complexes of modules over the canonical algebra with a (non-commutative) diagram of complexes of vector spaces satisfying the canonical algebra relations.

For a morphism $f : K \to L$ of complexes $K = (K^i, d^i_K)$, $L = (L^i, d^i_L)$ of vector spaces, denote by $\mathrm{C}(K \xrightarrow{f} L)$ the *cone* of f, cf. Section 3.2.2. Recall that $\mathrm{C}(K \to L)^i = K^{i+1} \oplus L^i$, with the differential acting as the matrix

$$\begin{pmatrix} -d^{i+1}_K & 0 \\ f^{i+1} & d^i_L \end{pmatrix}$$

by viewing the terms as column vectors of length two. Denote by $[1]$ the shift operator, that is, $K[1]^i = K^{i+1}$ with $d^i_{K[1]} = -d^{i+1}_K$.

We will demonstrate the construction of $\widetilde{F}$ for the posets of type (8.2.1), the other cases being similar. Let K_0, K_ω and $K_{i,j}$ for $1 \leq i \leq 3$, $1 \leq j < p_i$ be complexes in a commutative

$$
\begin{tikzcd}
 & \bullet_{1,1} \arrow[r] & \cdots \arrow[r] & \bullet_{1,p_1-2} \arrow[r] \arrow[ddr] & \bullet_{1,p_1-1} \arrow[dr] & \\
\bullet_0 \arrow[ur] \arrow[r] \arrow[dr] & \bullet_{2,1} \arrow[r] & \cdots \arrow[r] & \bullet_{2,p_2-2} \arrow[r] \arrow[ur] & \bullet_{2,p_2-1} \arrow[r] & \bullet_\omega \\
 & \bullet_{3,1} \arrow[r] & \cdots \arrow[r] & \bullet_{3,p_3-2} \arrow[r] \arrow[ur] & \bullet_{3,p_3-1} \arrow[ur] &
\end{tikzcd}
\qquad 3 \leq p_1 \leq p_2 \leq p_3 \tag{8.2.1}
$$

$$
\begin{tikzcd}
 & & & & \bullet_{1,1} \arrow[dr] & \\
\bullet_0 \arrow[r] \arrow[dr] & \bullet_{2,1} \arrow[r] & \cdots \arrow[r] & \bullet_{2,p_2-2} \arrow[r] \arrow[ur] & \bullet_{2,p_2-1} \arrow[r] & \bullet_\omega \\
 & \bullet_{3,1} \arrow[r] & \cdots \arrow[r] & \bullet_{3,p_3-2} \arrow[r] \arrow[ur] & \bullet_{3,p_3-1} \arrow[ur] &
\end{tikzcd}
\qquad p_1 = 2 < p_2 \leq p_3 \tag{8.2.2}
$$

$$
\begin{tikzcd}
 & & & & \bullet_{1,1} \arrow[dr] & \\
\bullet_0 \arrow[urrrr] \arrow[dr] & & & & \bullet_{2,1} \arrow[r] & \bullet_\omega \\
 & \bullet_{3,1} \arrow[r] & \cdots \arrow[r] & \bullet_{3,p_3-2} \arrow[r] \arrow[ur] & \bullet_{3,p_3-1} \arrow[ur] &
\end{tikzcd}
\qquad p_1 = p_2 = 2 < p_3 \tag{8.2.3}
$$

$$
\begin{tikzcd}
 & \bullet_{1,1} \arrow[dr] & \\
\bullet_0 \arrow[ur] \arrow[r] \arrow[dr] & \bullet_{2,1} \arrow[r] & \bullet_\omega \\
 & \bullet_{3,1} \arrow[ur] &
\end{tikzcd}
\qquad p_1 = p_2 = p_3 = 2 \tag{8.2.4}
$$

Figure 8.1: Hasse diagrams of posets derived equivalent to canonical algebras of type (p_1, p_2, p_3).

diagram

$$\begin{tikzcd}
 & K_{1,1} \arrow[r] & \cdots \arrow[r] & K_{1,p_1-2} \arrow[r, "x_{11}"] \arrow[ddr, "x_{13}"] & K_{1,p_1-1} \arrow[dr, "y_1"] & \\
K_0 \arrow[ur] \arrow[r] \arrow[dr] & K_{2,1} \arrow[r] & \cdots \arrow[r] & K_{2,p_2-2} \arrow[ur, "x_{21}"] \arrow[r, "x_{22}"'] & K_{2,p_2-1} \arrow[r, "y_2"] & K_\omega \\
 & K_{3,1} \arrow[r] & \cdots \arrow[r] & K_{3,p_3-2} \arrow[ur, "x_{32}"'] \arrow[r, "x_{33}"'] & K_{3,p_3-1} \arrow[ur, "y_3"'] &
\end{tikzcd}$$

Let $L_0 = K_0$ and $L_{i,j} = K_{i,j}$ for $1 \leq i \leq 3$ and $1 \leq j < p_i - 1$. Define

$$\begin{aligned}
L_{1,p_1-1} &= \mathrm{C}(K_{1,p_1-1} \oplus K_{3,p_3-1} \xrightarrow{(\,y_1\ y_3\,)} K_\omega)[-1] \\
L_{2,p_2-1} &= \mathrm{C}(K_{2,p_2-1} \oplus K_{1,p_1-1} \xrightarrow{(\,y_2\ y_1\,)} K_\omega)[-1] \\
L_{3,p_3-1} &= \mathrm{C}(K_{3,p_3-1} \oplus K_{2,p_2-1} \xrightarrow{(\,y_3\ y_2\,)} K_\omega)[-1] \\
L_\omega &= \mathrm{C}(K_{1,p_1-1} \oplus K_{2,p_2-1} \oplus K_{3,p_3-1} \xrightarrow{(\,y_1\ y_2\ y_3\,)} K_\omega)[-1]
\end{aligned}$$

with the three maps

$$\begin{aligned}
L_{1,p_1-2} &\xrightarrow{(\,x_{11}\ -x_{13}\ 0\,)^T} L_{1,p_1-1} \\
L_{2,p_2-2} &\xrightarrow{(\,-x_{22}\ x_{21}\ 0\,)^T} L_{2,p_2-1} \\
L_{3,p_3-2} &\xrightarrow{(\,x_{33}\ -x_{32}\ 0\,)^T} L_{3,p_3-1}
\end{aligned}$$

and the three maps from L_{i,p_i-1} to L_ω being the canonical embeddings.

Then the following is a (non-commutative) diagram of complexes

$$\begin{tikzcd}
 & L_{1,1} \arrow[r] & \cdots \arrow[r] & L_{1,p_1-2} \arrow[r] & L_{1,p_1-1} \arrow[dr] & \\
L_0 \arrow[ur] \arrow[r] \arrow[dr] & L_{2,1} \arrow[r] & \cdots \arrow[r] & L_{2,p_2-2} \arrow[r] & L_{2,p_2-1} \arrow[r] & L_\omega \\
 & L_{3,1} \arrow[r] & \cdots \arrow[r] & L_{3,p_3-2} \arrow[r] & L_{3,p_3-1} \arrow[ur] &
\end{tikzcd}$$

that satisfies the canonical algebra relation, and we get the required functor F which induces, by the general considerations in Section 3.3, the functor $\widetilde{F}$.

To prove that $\widetilde{F}$ is an equivalence, we use Beilinson's Lemma [6, Lemma 1] and verify that for any two simple objects S_x, S_y (where $x, y \in X$) and $i \in \mathbb{Z}$, the functor $\widetilde{F}$ satisfies

$$\mathrm{Hom}_{\mathcal{D}^b(kX_{\mathbf{p}})}(S_x, S_y[i]) \simeq \mathrm{Hom}_{\mathcal{D}^b(\Lambda(\mathbf{p}))}(\widetilde{F}S_x, \widetilde{F}S_y[i])$$

and moreover the images $\widetilde{F}S_x$ generate $\mathcal{D}^b(\Lambda(\mathbf{p}))$ as a triangulated category.

We omit the details of this verification. However, we just mention that $\widetilde{F}S_0$ and $\widetilde{F}S_{i,j}$, for $1 \leq i \leq 3$ and $1 \leq j < p_i - 1$, are the corresponding simple $\Lambda(\mathbf{p})$-modules, while $\widetilde{F}S_{1,p_1-1}$, $\widetilde{F}S_{2,p_2-1}$, $\widetilde{F}S_{3,p_3-1}$ and $\widetilde{F}S_\omega$ are given by

$$\begin{tikzcd}
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[dr, "1"] & \\
0 \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"'] & k \\
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"'] & k \\
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[ur, "1"'] &
\end{tikzcd}$$

$\widetilde{F}S_{1,p_1-1}$ $\qquad$ $\widetilde{F}S_{2,p_2-1}$

$$\begin{tikzcd}
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[dr, "1"] & \\
0 \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & k \\
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[ur, "1"'] &
\end{tikzcd}
\qquad
\left(\begin{tikzcd}
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[dr, "1"] & \\
0 \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"'] & k \\
 & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[ur, "1"'] &
\end{tikzcd}\right)[-1]$$

$\widetilde{F}S_{3,p_3-1}$ $\qquad$ $\widetilde{F}S_\omega$

$\square$

Example 8.2.2 ([62, Example 18.6.2])**.** Let A_2 be the quiver $\bullet_1 \longrightarrow \bullet_2$ and let $X = A_2 \times A_2 \times A_2$. Then the incidence algebra of X is derived equivalent to the canonical algebra of type $(3, 3, 3)$.

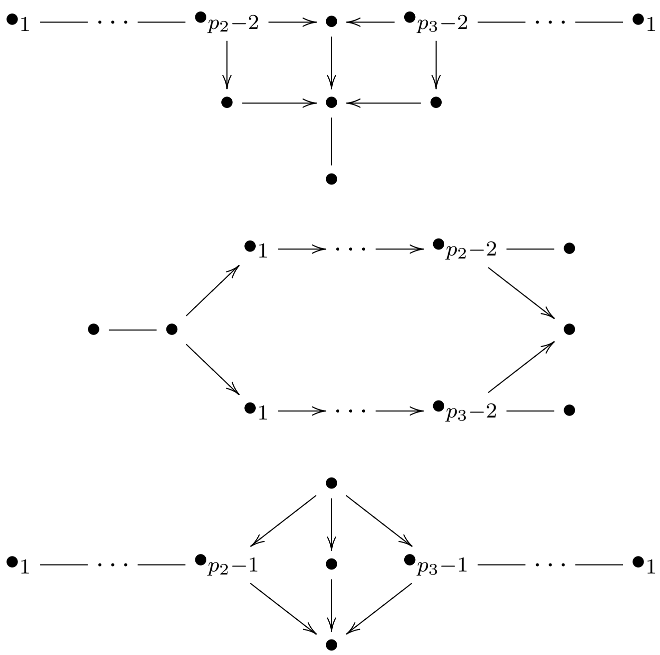
Remark 8.2.3. Observe that ω is the unique maximal element in the posets whose Hasse diagrams are given in (8.2.3) and (8.2.4). Hence by taking $Y = \{\omega\}$ in Corollary 1.4.15, we recover the fact that the canonical algebra of type $(2, 2, p)$ is derived equivalent to the path algebra of the extended Dynkin quiver $\widetilde{D}_{p+2}$.

Remark 8.2.4. Similar applications of Theorem 3.1.1 and its corollaries for the posets in (8.2.2) show that the canonical algebra of type $(2, p_2, p_3)$ is derived equivalent to the incidence algebras of the posets whose Hasse diagrams are given in Figure 8.2, where edges without arrows can be oriented arbitrarily.

8.2.2 Remarks on path algebras

Let Q be a finite quiver without oriented cycles. The set of its vertices Q_0 has a natural partial order defined by $x \leq y$ for two vertices $x, y \in Q_0$ if there exists an oriented path from x to y. When Q has the property that any two vertices are connected by at most one oriented path, the path algebra kQ is isomorphic to the incidence algebra of the poset $(Q_0, \leq)$. A partial converse is given by the following lemma.

Lemma 8.2.5 ([68, Theorem 4.2])**.** *Let X be a poset. Then* $\operatorname{gl.dim} kX \leq 1$ *if and only if any two points in the Hasse diagram of X are connected by at most one path.*



$p_2, p_3 \geq 3$

$p_2, p_3 \geq 3$

$p_2, p_3 \geq 2$

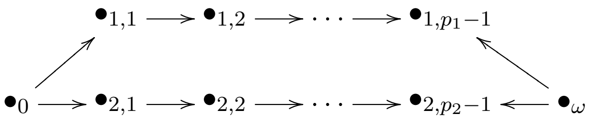
Figure 8.2: More posets whose incidence algebras are derived equivalent to the canonical algebra of type $(2, p_2, p_3)$.

For two quivers Q and Q', we denote $Q \sim Q'$ if Q' can be obtained from Q by applying a sequence of BGP reflections (at sources or sinks), see [35, (I.5.7)]. Since BGP reflections preserve the derived equivalence class, we conclude that if Q is a quiver such that $Q \sim Q'$ for a quiver Q' having the property that any two vertices are connected by at most one oriented path, then the path algebra kQ is derived equivalent to an incidence algebra of a poset.

8.2.3 The case $t = 2$

When the number of weights is at most 2, the corresponding canonical algebra is a path algebra of a quiver, however there are two distinct paths from the source 0 to the sink ω.

When the weight type is (p_1, p_2) (with $p_1, p_2 \geq 2$), we can overcome this problem by applying a BGP reflection at the sink ω. The resulting quiver is shown below,



$2 \leq p_1, p_2$

and its path algebra is an incidence algebra derived equivalent to the canonical algebra of type (p_1, p_2).

In the remaining case, where the weight type is either (p) or $()$, the corresponding canonical algebra equals the path algebra of the quiver $\widetilde{A}_{1,p}$ drawn in Figure 8.3, and we show the following.

$$\begin{tikzcd}
\bullet_0 \arrow[rrrrr] \arrow[dr] & & & & & \bullet_p \\
& \bullet_1 \arrow[r] & \bullet_2 \arrow[r] & \cdots \arrow[r] & \bullet_{p-1} \arrow[ur] &
\end{tikzcd}$$

Figure 8.3: The quiver $\widetilde{A}_{1,p}$ – not derived equivalent to any incidence algebra.

Proposition 8.2.6. *Let $p \geq 1$ and let k be algebraically closed. Then there is no poset whose incidence algebra is derived equivalent to the path algebra of the quiver $\widetilde{A}_{1,p}$.*

Proof. Assume that there exists a poset X such that kX is derived equivalent to the path algebra of $\widetilde{A}_{1,p}$.

If $\mathrm{gl.dim}\, kX \leq 1$, then by Lemma 8.2.5, the algebra kX equals the path algebra of its Hasse diagram Q, thus by [35, (I.5.7)], Q is obtained from $\widetilde{A}_{1,p}$ by a sequence of BGP reflections. But this is impossible since the only possible reflections are at 0 and p, and they give quivers isomorphic to $\widetilde{A}_{1,p}$. However, $\widetilde{A}_{1,p}$ is not the Hasse diagram of any poset.

Hence $\mathrm{gl.dim}\, kX \geq 2$. By Lemma 8.2.5, there exists at least one commutativity relation in the quiver Q, so that kX is not a gentle algebra. This is again impossible since the path algebra of $\widetilde{A}_{1,p}$ is gentle and the property of an algebra being gentle is invariant under derived equivalence [80]. Alternatively, one can use the explicit characterization in [2] of iterated tilted algebras of type $\tilde{A}_n$. $\square$

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