ON DERIVED EQUIVALENCES OF CATEGORIES OF SHEAVES OVER FINITE POSETS

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Abstract. A finite poset X carries a natural structure of a topological space. Fix a field k, and denote by $\mathcal{D}^b(X)$ the bounded derived category of sheaves of finite dimensional k-vector spaces over X. Two posets X and Y are said to be *derived equivalent* if $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ are equivalent as triangulated categories.

We give explicit combinatorial properties of X which are invariant under derived equivalence, among them are the number of points, the $\mathbb{Z}$-congruency class of the incidence matrix, and the Betti numbers. We also show that taking opposites and products preserves derived equivalence.

For any closed subset $Y \subseteq X$, we construct a strongly exceptional collection in $\mathcal{D}^b(X)$ and use it to show an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(A)$ for a finite dimensional algebra A (depending on Y). We give conditions on X and Y under which A becomes an incidence algebra of a poset.

We deduce that a lexicographic sum of a collection of posets along a bipartite graph S is derived equivalent to the lexicographic sum of the same collection along the opposite S^{op}.

This construction produces many new derived equivalences of posets and generalizes other well known ones.

As a corollary we show that the derived equivalence class of an ordinal sum of two posets does not depend on the order of summands. We give an example that this is not true for three summands.

1. Introduction

Over the last years a growing interest in the understanding of derived categories of coherent sheaves over algebraic varieties, and in particular, the question when two varieties have equivalent derived categories of sheaves, has emerged [5].

We investigate a similar question for partially ordered sets (*posets*). A poset X carries a natural structure of a topological space, therefore we can consider the category of sheaves over X with values in an abelian category $\mathcal{A}$.

We focus on the case where $\mathcal{A}$ is the category of finite dimensional vector spaces over a field k, which allows us to identify the category of sheaves with a category of modules over the incidence algebra of X over k, so that tools from the theory of derived equivalence of algebras can be used. However, there is no known algorithm which decides, given two posets, whether their derived categories of sheaves of finite dimensional k-vector spaces are equivalent.

In Section 2, we present in a specific way, appropriate for dealing with posets, the basic notions from sheaf theory that will be used throughout the paper. In Section 3 we discuss combinatorial invariants of derived equivalence, whereas in Section 4 we construct, for any poset X admitting a special structure, new poset derived equivalent to X. This construction is based on the notion of strongly exceptional sequences in triangulated categories and partially generalizes the known constructions of [1, 3].

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2. Preliminaries

2.1. **Finite posets and T_0 spaces.** Throughout this note, the term *poset* will mean a finite partially ordered set. Any poset $(X, \leq)$ carries a structure of a topological space by defining the closed sets to be the subsets $Y \subseteq X$ such that if $y \in Y$ and $y' \leq y$ then $y' \in Y$.

For each $x \in X$, denote by $\{x\}^-$ the closure of $\{x\}$ and by U_x the minimal open subset of X containing x, which equals the intersection of the open sets containing x. Then $\{x\}^- = \{x' \in X : x' \leq x\}$, $U_x = \{x' \in X : x' \geq x\}$ and

$$x \leq x' \iff \{x\}^- \subseteq \{x'\}^- \iff U_{x'} \subseteq U_x$$

If x, y are two distinct points in X, then one of the open sets U_x, U_y does not contain both points, thus X satisfies the T_0 separation property.

Conversely, given a finite T_0 topological space X, let U_x be the intersection of all open sets in X containing $x \in X$. Define a partial order $\leq$ on X by $x \leq x'$ if $U_{x'} \subseteq U_x$.

This leads to an identification of posets with finite T_0 topological spaces. Such spaces have been studied in the past [15, 21], where it turned out that their homotopy and homology properties are more interesting than might seem at first glance. For example, if $\mathcal{K}$ is any finite simplicial complex and X is the T_0 space induced by the partial order on the simplices of $\mathcal{K}$, then there exists a weak homotopy equivalence $|\mathcal{K}| \to X$ [15].

2.2. **Sheaves and diagrams.** Given a poset X, its *Hasse diagram* is a directed graph defined as follows. Its vertices are the elements of X and its directed edges $x \to y$ are the pairs $x < y$ in X such that there is no $z \in X$ with $x < z < y$. The anti-symmetry condition on $\leq$ implies that this graph has no directed cycles.

Let X be a poset and $\mathcal{A}$ be an abelian category. Using the topology on X, we can consider the category of *sheaves over X with values in $\mathcal{A}$*, denoted by $Sh_X \mathcal{A}$ or sometimes $\mathcal{A}^X$.

We note that sheaves over posets were used in [7] for the computation of cohomologies of real subspace arrangements.

Let $\mathcal{F}$ be a sheaf on X. If $x \in X$, let $\mathcal{F}(x)$ be the stalk of $\mathcal{F}$ over x, which equals $\mathcal{F}(U_x)$. The restriction maps $\mathcal{F}(x) = \mathcal{F}(U_x) \to \mathcal{F}(U_{x'}) = \mathcal{F}(x')$ for $x' > x$ give rise to a commutative diagram over the Hasse diagram of X. Conversely, such a diagram $\{F_x\}$ defines a sheaf $\mathcal{F}$ by setting the sections as the inverse limits $\mathcal{F}(U) = \lim_{x \in U} F_x$. Indeed, it is enough to verify the sheaf condition for the sets U_x, which follows from the observation that for any cover $U_x = \bigcup_i U_{z_i}$, one of the z_i equals x.

Thus we may identify $Sh_X\mathcal{A}$ with the category of commutative diagrams over the Hasse diagram of X and interchange the terms sheaf and diagram as appropriate. The latter category can be viewed as the category of functors $X \to \mathcal{A}$ where we consider X as a category whose objects are the points $x \in X$, with unique morphisms $x \to x'$ for $x \leq x'$. Under this identification, the global sections functor $\Gamma(X; -) : \mathcal{A}^X \to \mathcal{A}$ defined as $\Gamma(X; \mathcal{F}) = \mathcal{F}(X)$, coincides with the (inverse) limit functor $\lim_X : \mathcal{A}^X \to \mathcal{A}$.

2.3. **Functors associated with a map $f : X \to Y$.** A map $f : X \to Y$ between two finite posets is continuous if and only if it is *order preserving*, that is, $f(x) \leq f(x')$ for any $x \leq x'$ in X [21, Prop. 7].

A continuous map $f : X \to Y$ gives rise to the functors $f_*, f_! : Sh_X\mathcal{A} \to Sh_Y\mathcal{A}$ and $f^{-1} : Sh_Y\mathcal{A} \to Sh_X\mathcal{A}$ defined, in terms of diagrams, by

$$
\begin{aligned}
(f^{-1}\mathcal{G})(x) &= \mathcal{G}(f(x)) \\
(f_*\mathcal{F})(y) &= \varprojlim\{\mathcal{F}(x) \,:\, f(x) \geq y\} \\
(f_!\mathcal{F})(y) &= \varinjlim\{\mathcal{F}(x) \,:\, f(x) \leq y\}
\end{aligned}
$$

where $x \in X$, $y \in Y$ and $\mathcal{F} \in Sh_X\mathcal{A}$, $\mathcal{G} \in Sh_Y\mathcal{A}$. Viewing X, Y as categories and $\mathcal{F} \in Sh_X\mathcal{A}$ as a functor $\mathcal{F} : X \to \mathcal{A}$, the sheaves $f_*\mathcal{F}$ and $f_!\mathcal{F}$ are the right and left Kan extensions of $\mathcal{F}$ along $f : X \to Y$.

The functors f^{-1}, f_* coincide with the usual ones from sheaf theory. We have the following adjunctions:

$$
\begin{aligned}
&\mathrm{Hom}_{Sh_X\mathcal{A}}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{Sh_Y\mathcal{A}}(\mathcal{G}, f_*\mathcal{F}) \\
&\mathrm{Hom}_{Sh_X\mathcal{A}}(\mathcal{F}, f^{-1}\mathcal{G}) \simeq \mathrm{Hom}_{Sh_Y\mathcal{A}}(f_!\mathcal{F}, \mathcal{G})
\end{aligned}
\tag{2.1}
$$

so that f_* is left exact and $f_!$ is right exact. f^{-1} is exact, as can be seen from its action on the stalks.

If Y is a closed subset of X, we have a *closed embedding* $i : Y \to X$. In this case, i_* is exact. This is because i_* takes a diagram on Y and extends it to X by filling the vertices of $X \setminus Y$ with zeros. Similarly, for an open embedding $j : U \to X$, $j_!$ is exact, as it extends by zeros diagrams on U. Now let $Y \subseteq X$ be closed and $U = X \setminus Y$ its complement. The adjunction morphisms $j_!j^{-1}\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ for the embeddings $i : Y \to X$ and $j : U \to X$ induce a short exact sequence

$$
0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0 \tag{2.2}
$$

for any sheaf $\mathcal{F}$ on X, as can be verified at the stalks.

2.4. **Simples, projectives and injectives.** When $f : X \to \bullet$ is the mapping to a point, $f_* = \Gamma(X; -)$ and $f^{-1}(M)$ for an object M of $\mathcal{A}$ gives the *constant sheaf* on X with value M.

Let $x \in X$ and consider the map $i_x : \bullet \to X$ whose image is $\{x\}$. Then $i_x^{-1}(\mathcal{F}) = \mathcal{F}(x)$ is the stalk at x and for an object M of $\mathcal{A}$ we have

$$
(i_{x*}M)(y) = \begin{cases} M & \text{if } y \leq x \\ 0 & \text{otherwise} \end{cases} \qquad (i_{x!}M)(y) = \begin{cases} M & \text{if } y \geq x \\ 0 & \text{otherwise} \end{cases}
$$

with identity arrows between the M-s. The adjunctions (2.1) take the form:

$$\begin{aligned} \mathrm{Hom}_{Sh_X\mathcal{A}}(\mathcal{F}, i_{x*}M) &\simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(x), M) \\ \mathrm{Hom}_{Sh_X\mathcal{A}}(i_{x!}M, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{F}(x)) \end{aligned} \tag{2.3}$$

and we deduce the following lemma:

Lemma 2.1. *If I is injective in $\mathcal{A}$, $i_{x*}I$ is injective in $Sh_X\mathcal{A}$. If P is projective in $\mathcal{A}$, $i_{x!}P$ is projective in $Sh_X\mathcal{A}$.*

Corollary 2.2. *If $\mathcal{A}$ has enough injectives (projectives), so does $Sh_X\mathcal{A}$.*

Proof. The identity maps $\mathcal{F}(x) \xrightarrow{=} \mathcal{F}(x)$ induce, via (2.3), an injection $\mathcal{F} \hookrightarrow \oplus_{x\in X} i_{x*}\mathcal{F}(x)$ and surjection $\oplus_{x\in X} i_{x!}\mathcal{F}(x) \twoheadrightarrow \mathcal{F}$. Now replace each $\mathcal{F}(x)$ by an injective (or projective) cover. $\square$

For a sheaf $\mathcal{F}$, let $\operatorname{supp}\mathcal{F} = \{x \in X : \mathcal{F}(x) \neq 0\}$ be its *support.* We call $\mathcal{F}$ a *stalk sheaf* if its support is a point. For any object M of $\mathcal{A}$ and $x \in X$ there exists a stalk sheaf M_x whose stalk at x equals M. Moreover M_x is simple in $Sh_X\mathcal{A}$ if and only if M is simple in $\mathcal{A}$.

The following lemma is proved by induction on the number of elements $|X|$, using (2.2) and the fact that the partial order on X can be extended to a linear order, i.e. one can write the elements of X in a sequence $x_1, x_2, \dots, x_n$ such that for any $1 \le i, j \le n$, $x_i < x_j$ implies that $i < j$.

Lemma 2.3. *Any sheaf $\mathcal{F}$ on X admits a finite filtration whose quotients are stalk sheaves.*

Denote by $\mathrm{gl.dim}\,\mathcal{A}$ the *global dimension* of an abelian category $\mathcal{A}$. This is the maximal integer n for which there exist objects M, M' of $\mathcal{A}$ with $\mathrm{Ext}^n(M, M') \neq 0$ (and ∞ if there is no such maximal n). Recall that an abelian category is a *finite length* category if every object is of finite length. From Lemma 2.3, we have:

Corollary 2.4. *If $\mathcal{A}$ is a finite length category, so is $Sh_X\mathcal{A}$.*

Definition 2.5. A strictly increasing sequence $x_0 < x_1 < \dots < x_n$ in X is called a *chain of length n*. The *dimension* of X, denoted $\dim X$, is the maximal length of a chain in X.

Proposition 2.6 ([16])**.** $\mathrm{gl.dim}\, Sh_X\mathcal{A} \le \mathrm{gl.dim}\,\mathcal{A} + \dim X$.

The difference $\mathrm{gl.dim}\, Sh_X\mathcal{A} - \mathrm{gl.dim}\,\mathcal{A}$ obviously depends on X, but it may well depend also on $\mathcal{A}$, see the examples in [13, 20].

2.5. **Sheaves of finite-dimensional vector spaces.** Fix a field k and consider the category $\mathcal{A}$ of finite dimensional vector spaces over k. Denote by Sh_X the category $Sh_X\mathcal{A}$ and by $\mathrm{Hom}_X(-,-)$ the morphism spaces $\mathrm{Hom}_{Sh_X}(-,-)$ (We omit the reference to k to emphasize that it is to be fixed throughout).

The *incidence algebra* of X over k, denoted kX, is the algebra spanned by e_{xy} for the pairs $x \le y$ in X, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$.

Lemma 2.7. *The category Sh_X is equivalent to the category of finite dimensional right modules over the incidence algebra kX.*

Proof. The proof is similar to the corresponding fact about representations of a quiver and right modules over its path algebra. Namely, for a sheaf $\mathcal{F}$, consider $M = \oplus_{x \in X} \mathcal{F}(x)$ and let $\iota_x : \mathcal{F}(x) \to M$, $\pi_x : M \to \mathcal{F}(x)$ be the natural maps. Equip M with a structure of a right kX-module by letting the basis elements $e_{xx'}$ for $x \leq x'$ act from the right as the composition $M \xrightarrow{\pi_x} \mathcal{F}(x) \to \mathcal{F}(x') \xrightarrow{\iota_{x'}} M$. Conversely, given a finite dimensional right module M over kX, set $\mathcal{F}(x) = Me_{xx}$ and define the maps $\mathcal{F}(x) \to \mathcal{F}(x')$ using the right multiplication by $e_{xx'}$. □

The one dimensional space k is both simple, projective and injective in the category of k-vector spaces. Applying the results of the previous subsection, we get, for any $x \in X$, sheaves S_x, P_x, I_x which are simple, projective and injective, respectively. Explicitly,

$$S_x(y) = \begin{cases} k & y = x \\ 0 & \text{otherwise} \end{cases}, \quad P_x(y) = \begin{cases} k & y \geq x \\ 0 & \text{otherwise} \end{cases}, \quad I_x(y) = \begin{cases} k & y \leq x \\ 0 & \text{otherwise} \end{cases}$$

By (2.3), for any sheaf $\mathcal{F}$, $\operatorname{Hom}_X(P_x, \mathcal{F}) = \mathcal{F}(x)$ and $\operatorname{Hom}_X(\mathcal{F}, I_x) = \mathcal{F}(x)^\vee$ (the dual space). Since the sets $U_x, \{x\}^-$ are connected, the sheaves P_x, I_x are indecomposable. The sheaves S_x, P_x, I_x form a complete set of representatives of the isomorphism classes of simples, indecomposable projectives and indecomposable injectives (respectively) in kX.

By Corollary 2.2, Sh_X has enough projectives and injectives (note that this can also be deduced by its identification with the category of finite dimensional modules over a finite dimensional algebra). It has finite global dimension, since by Proposition 2.6, $\operatorname{gl.dim} Sh_X \leq \dim X$.

Proposition 2.8. *Sh_X and Sh_Y are equivalent if and only if X and Y are isomorphic (as posets).*

Proof. Since the isomorphism classes of simple objects in Sh_X are in one-to-one correspondence with the elements $x \in X$, and for two such simples S_x, S_y, $\dim_k \operatorname{Ext}^1(S_x, S_y)$ equals 1 if there is a directed edge $x \to y$ in the Hasse diagram of X and 0 otherwise, we see that the Hasse diagram of X, hence X, can be recovered (up to isomorphism) from the category Sh_X. □

2.6. **The derived category of sheaves over a poset.** For a poset X, denote by $\mathcal{D}^b(X)$ the bounded derived category of Sh_X.

If $\mathcal{E}$ is a set of objects of a triangulated category $\mathcal{T}$, we denote by $\langle \mathcal{E} \rangle$ the triangulated subcategory of $\mathcal{T}$ *generated by* $\mathcal{E}$, that is, the minimal triangulated subcategory containing $\mathcal{E}$. We say that $\mathcal{E}$ *generates* $\mathcal{T}$ if $\langle \mathcal{E} \rangle = \mathcal{T}$.

Since Sh_X is of finite global dimension with enough projectives and injectives, $\mathcal{D}^b(X)$ can be identified with the homotopy category of bounded complexes of projectives (or bounded complexes of injectives). Hence the collections $\{P_x\}_{x \in X}$ and $\{I_x\}_{x \in X}$ generate $\mathcal{D}^b(X)$.

Lemma 2.9. *Let $x, y \in X$ and $i \in \mathbb{Z}$. Then*

$$\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, P_y[i]) = \operatorname{Hom}_{\mathcal{D}^b(X)}(I_x, I_y[i]) = \begin{cases} k & y \leq x \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since P_x is projective, $\mathrm{Hom}_{\mathcal{D}^b(X)}(P_x, \mathcal{F}[i]) = 0$ for any sheaf $\mathcal{F}$ and $i \neq 0$. If $x, y \in X$, then

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(P_x, P_y) = \mathrm{Hom}_X(P_x, P_y) = P_y(x) = \begin{cases} k & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

The proof for $\{I_x\}_{x \in X}$ is similar. □

For a continuous map $f : X \to Y$, denote by $Rf_*, Lf_!, f^{-1}$ the derived functors of $f_*, f_!, f^{-1}$. The adjunctions (2.1) imply that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^b(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathcal{D}^b(Y)}(\mathcal{G}, Rf_*\mathcal{F}) \\ \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}, f^{-1}\mathcal{G}) &\simeq \mathrm{Hom}_{\mathcal{D}^b(Y)}(Lf_!\mathcal{F}, \mathcal{G}) \end{aligned} \tag{2.4}$$

for $\mathcal{F} \in \mathcal{D}^b(X)$, $\mathcal{G} \in \mathcal{D}^b(Y)$.

Definition 2.10. We say that two posets X and Y are *derived equivalent*, denoted $X \sim Y$, if the categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ are equivalent as triangulated categories.

3. Combinatorial invariants of derived equivalence

We give a list of combinatorial properties of posets which are preserved under derived equivalence. Most of the properties are deduced from known invariants of derived categories. For the convenience of the reader, we review the relevant definitions.

3.1. **The number of points and K-groups.** Recall that for an abelian category $\mathcal{A}$, the *Grothendieck group* $K_0(\mathcal{A})$ is the quotient of the free abelian group generated by the isomorphism classes $[X]$ of objects X of $\mathcal{A}$ divided by the subgroup generated by the expressions $[X]-[Y]+[Z]$ for all the short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

Similarly, for a triangulated category $\mathcal{T}$, the group $K_0(\mathcal{T})$ is the quotient of the free abelian group on the isomorphism classes of objects of $\mathcal{T}$ divided by its subgroup generated by $[X] - [Y] + [Z]$ for all the triangles $X \to Y \to Z \to X[1]$ in $\mathcal{T}$ (where $[1]$ denotes the shift). The natural inclusion $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{D}^b(\mathcal{A}))$.

Let X be a poset and denote by $|X|$ the number of points of X. Denote by $K_0(X)$ the group $K_0(\mathcal{D}^b(X))$.

Proposition 3.1. *$K_0(X)$ is free abelian of rank $|X|$.*

Proof. The set $\{S_x\}_{x \in X}$ forms a complete set of representatives of the isomorphism classes of simple finite dimensional kX-modules, hence it is a $\mathbb{Z}$-basis of $K_0(X)$ (alternatively one could use the filtration of Lemma 2.3). □

Corollary 3.2. *If $X \sim Y$ then $|X| = |Y|$.*

It is known [8] that rings with equivalent derived categories have the same K-theory. However, higher K-groups do not lead to refined invariants of the number of points.

Proposition 3.3. *$K_i(Sh_X) \simeq K_i(Sh_\bullet)^{|X|}$ for $i \geq 0$.*

Proof. Sh_X is a finite length category and by [17, Corollary 1, p. 104],

$$K_i(Sh_X) \simeq \bigoplus_{x \in X} K_i(\mathrm{End}_X(S_x))$$

Clearly, $k = \mathrm{End}_X(S_x)$. $\square$

3.2. **Connected components.** For two additive categories $\mathcal{T}_1$, $\mathcal{T}_2$, consider the category $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ whose objects are pairs (M_1, M_2) and the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{T}}((M_1, M_2), (N_1, N_2)) = \mathrm{Hom}_{\mathcal{T}_1}(M_1, N_1) \times \mathrm{Hom}_{\mathcal{T}_2}(M_2, N_2)$$

$\mathcal{T}_1$, $\mathcal{T}_2$ are embedded in $\mathcal{T}$ via the fully faithful functors $M_1 \mapsto (M_1, 0)$ and $M_2 \mapsto (0, M_2)$. Denoting the images again by $\mathcal{T}_1, \mathcal{T}_2$, we have that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = 0$. In addition, the indecomposables in $\mathcal{T}$ are of the form $(M_1, 0)$ or $(0, M_2)$ for indecomposables $M_1 \in \mathcal{T}_1$, $M_2 \in \mathcal{T}_2$.

An additive category $\mathcal{T}$ is *connected* if for any equivalence $\mathcal{T} \simeq \mathcal{T}_1 \times \mathcal{T}_2$, one of $\mathcal{T}_1$, $\mathcal{T}_2$ is zero.

Definition 3.4. A poset X is *connected* if it is connected as a topological space. This is equivalent to the following condition [21, Prop. 5]:

For any $x, y \in X$ there exists a sequence $x = x_0, x_1, \dots, x_n = y$ in X such that for all $0 \le i < n$, either $x_i \le x_{i+1}$ or $x_i \ge x_{i+1}$.

Lemma 3.5. *If X is connected then the category $\mathcal{D}^b(X)$ is connected.*

Proof. Let $\mathcal{D}^b(X) \simeq \mathcal{T}_1 \times \mathcal{T}_2$ be an equivalence and consider the indecomposable projectives $\{P_x\}_{x \in X}$. Since each P_x is indecomposable, its image lies in $\mathcal{T}_1$ or in $\mathcal{T}_2$, and we get a partition $X = X_1 \sqcup X_2$.

Assume that X_1 is not empty. Since $\mathrm{Hom}(P_x, P_y) \neq 0$ for all $y \le x$ and $\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2) = 0$, X_1 must be both open and closed in X, and by connectivity, $X_1 = X$. Moreover, $\{P_x\}_{x \in X}$ generates $\mathcal{D}^b(X)$ as a triangulated category, hence $\mathcal{D}^b(X) \simeq \mathcal{T}_1$ and $\mathcal{T}_2 = 0$. $\square$

Proposition 3.6. *Let X and Y be two posets with decompositions*

$$X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_t \qquad\qquad Y = Y_1 \sqcup Y_2 \sqcup \dots \sqcup Y_s$$

into connected components. If $X \sim Y$ then $s = t$ and there exists a permutation π on $\{1, \dots, s\}$ such that $X_i \sim Y_{\pi(i)}$ for all $1 \le i \le s$.

Proof. There exists a pair of equivalences

$$\mathcal{D}^b(X_1) \times \dots \times \mathcal{D}^b(X_t) = \mathcal{D}^b(X) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}^b(Y) = \mathcal{D}^b(Y_1) \times \dots \times \mathcal{D}^b(Y_s)$$

If $x \in X$, the image $F(P_x)$ is indecomposable in $\mathcal{D}^b(Y)$, hence lands in one of the $\mathcal{D}^b(Y_j)$, and we get a function $f : X \to \{1, \dots, s\}$. For any $x' \le x$, $\mathrm{Hom}_X(P_x, P_{x'}) \neq 0$, therefore f is constant on the connected components X_i and induces a map $\pi_F : \{1, \dots, t\} \to \{1, \dots, s\}$ via $\pi_F(i) = f(x)$ for $x \in X_i$. Moreover, since $\{P_x\}_{x \in X_i}$ generates $\mathcal{D}^b(X_i)$ as a triangulated category, F restricts to functors $\mathcal{D}^b(X_i) \to \mathcal{D}^b(Y_{\pi_F(i)})$, $1 \le i \le t$.

Similarly for G, we obtain a map $\pi_G : \{1, \dots, s\} \to \{1, \dots, t\}$ and functors $\mathcal{D}^b(Y_j) \to \mathcal{D}^b(Y_{\pi_G(j)})$ which are restrictions of G.

For any $1 \leq i \leq t$, the image of $\mathcal{D}^b(X_i)$ under GF lies in $\mathcal{D}^b(X_{\pi_G \pi_F(i)})$. Since GF is isomorphic to the identity functor but on the other hand there are no nonzero maps between $\mathcal{D}^b(X_i)$ and $\mathcal{D}^b(X_{i'})$ for $i \neq i'$ (as we think of X_i as *subsets* of X, not just as abstract sets!), we get that $\pi_G \pi_F(i) = i$ so that $\pi_G \pi_F$ is identity. Similarly, $\pi_F \pi_G$ is identity.

We deduce that $s = t$, π_F and π_G are permutations, and the restrictions of F induce equivalences $\mathcal{D}^b(X_i) \simeq \mathcal{D}^b(Y_{\pi_F(i)})$. $\square$

One can also deduce that the number of connected components is a derived invariant by considering the center $Z(kX)$ of the incidence algebra kX using the fact that derived equivalent algebras have isomorphic centers [18].

Lemma 3.7. *$Z(kX) \cong k \times k \times \cdots \times k$ where the number of factors equals the number of connected components of X.*

Proof. Let $c = \sum_{x \leq y} c_{xy} e_{xy} \in Z(kX)$. Comparison of the coefficients of $e_{xx}c$ and ce_{xx} gives $c_{xy} = 0$ for $x \neq y$, thus $c = \sum_x c_x e_{xx}$.

If $x \leq y$ then $c_x e_{xy} = c e_{xy} = e_{xy} c = c_y e_{xy}$, hence $c_x = c_y$ if x, y are in the same connected component. $\square$

3.3. **The Euler form and Möbius function.** Let X be a poset. Since Sh_X has finite global dimension, the expression

$$\langle K, L \rangle_X = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}^b(X)}(K, L[i])$$

is well-defined for $K, L \in \mathcal{D}^b(X)$ and induces a $\mathbb{Z}$-bilinear form on $K_0(X)$, known as the *Euler form*.

Recall that the *incidence matrix* of X, denoted $\mathbf{1}_X$, is the $X \times X$ matrix defined by

$$(\mathbf{1}_X)_{xy} = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise} \end{cases}$$

By extending the partial order on X to a linear order, we can always arrange the elements of X such that the incidence matrix is upper triangular with ones on the diagonal. In particular, $\mathbf{1}_X$ is invertible over $\mathbb{Z}$.

Definition 3.8. The *Möbius function* $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x, y) = (\mathbf{1}_X^{-1})_{xy}$.

The following is an immediate consequence of the definition.

Lemma 3.9 (Möbius inversion formula)**.** *Let $f : X \to \mathbb{Z}$. Define $g : X \to \mathbb{Z}$ by $g(x) = \sum_{y \geq x} f(y)$. Then $f(x) = \sum_{y \geq x} \mu_X(x, y) g(y)$.*

The Möbius inversion formula can be used to compute the matrix of the Euler form with respect to the basis of simple objects.

Lemma 3.10. *$\langle [P_x], [S_y] \rangle_X = \delta_{xy}$ for all $x, y \in X$.*

Proof. Since P_x is projective, $\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, \mathcal{F}[i]) = 0$ for any sheaf $\mathcal{F}$ and $i \neq 0$. Now by (2.3), $\operatorname{Hom}_{\mathcal{D}^b(X)}(P_x, S_y) = \operatorname{Hom}_X(P_x, S_y) = S_y(x)$. $\square$

Proposition 3.11. *Let $x, y \in X$. Then $\langle [S_x], [S_y] \rangle_X = \mu_X(x, y)$.*

Proof. Fix y and define $f : X \to \mathbb{Z}$ by $f(x) = \langle [S_x], [S_y] \rangle_X$. Since $[P_x] = \sum_{x' \geq x} [S_{x'}]$, Lemmas 3.9 and 3.10 imply that

$$f(x) = \sum_{x' \geq x} \mu_X(x, x') \langle [P_{x'}], [S_y] \rangle_X = \mu_X(x, y)$$

□

Definition 3.12. Let R be a commutative ring. Two matrices $M_1, M_2 \in \mathrm{GL}_n(R)$ are *congruent* over R if there exists a matrix $P \in \mathrm{GL}_n(R)$ such that $M_2 = P M_1 P^t$.

Note that if M_1, M_2 is a pair of congruent matrices, so are M_1^t, M_2^t and M_1^{-1}, M_2^{-1}. Denote by M^{-t} the inverse of the transpose of M.

Corollary 3.13. *If $X \sim Y$ then $\mathbf{1}_X$, $\mathbf{1}_Y$ are congruent over $\mathbb{Z}$.*

Proof. An equivalence $F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induces an isomorphism $[F] : K_0(X) \to K_0(Y)$ which preserves the Euler form. By Proposition 3.11, the matrix of the Euler form of $\mathcal{D}^b(X)$ over the basis of simples is $\mathbf{1}_X^{-1}$, hence $[F]^t \mathbf{1}_Y^{-1} [F] = \mathbf{1}_X^{-1}$. □

In practice, testing for congruence over $\mathbb{Z}$ is not an easy task. However, the following necessary condition is often very useful in ruling out congruence.

Lemma 3.14. *Let $M_1, M_2 \in \mathrm{GL}_n(R)$ be congruent. Then the matrices $M_1 M_1^{-t}$, $M_2 M_2^{-t}$ are conjugate in the group $\mathrm{GL}_n(R)$.*

Proof. If $M_2 = P M_1 P^t$ for some $P \in \mathrm{GL}_n(R)$, then

$$M_2 M_2^{-t} = (P M_1 P^t)(P^{-t} M_1^{-t} P^{-1}) = P M_1 M_1^{-t} P^{-1}$$

□

Corollary 3.15. *If $X \sim Y$ then $\mathbf{1}_X \mathbf{1}_X^{-t}$ and $\mathbf{1}_Y \mathbf{1}_Y^{-t}$ are similar over $\mathbb{Z}$. In particular, they are similar over $\mathbb{Q}$ and modulo all primes p.*

Note that $\mathbf{1}_X \mathbf{1}_X^{-t}$ is (up to sign) the *Coxeter matrix* of the algebra kX. It is the image in $K_0(X)$ of the Serre functor on $\mathcal{D}^b(X)$.

3.4. **Betti numbers and Euler characteristic.** The Hochschild cohomology is a known derived invariant of an algebra [11, 19]. For posets, one can compute the Hochschild cohomology as the simplicial cohomology of an appropriate simplicial complex [6, 10]. Thus the simplicial cohomology is a derived invariant, which we relate to the cohomology of the constant sheaf.

For the convenience of the reader, we review the notions of sheaf cohomology, simplicial cohomology and Hochschild cohomology. As before, we keep the field k fixed.

3.4.1. *Sheaf cohomology.* Recall that the i-th cohomology of a sheaf $\mathcal{F} \in Sh_X$, denoted $\mathrm{H}^i(X; \mathcal{F})$, is the value of the i-th right derived functor of the global sections functor $\Gamma(X; -) : Sh_X \to Sh_\bullet$. Observe that $\Gamma(X; \mathcal{F}) = \mathrm{Hom}_X(k_X, \mathcal{F})$ where k_X is the *constant sheaf* on X, i.e. $k_X(x) = k$ for all $x \in X$ with all morphisms being the identity of k. It follows that $\mathrm{H}^i(X; \mathcal{F}) = \mathrm{Ext}_X^i(k_X, \mathcal{F})$. Specializing this for the particularly interesting cohomologies of the constant sheaf, we get that $\mathrm{H}^i(X; k_X) = \mathrm{Ext}_X^i(k_X, k_X)$.

3.4.2. *Simplicial cohomology.* Let X be a poset, $p \geq 0$. A *p-dimensional simplex* in X is a chain of length p. Since subsets of chains are again chains, the set of all simplices in X forms a simplicial complex $\mathcal{K}(X)$ [15], known as the *order complex* of X. The i-th simplicial cohomology of X is defined as the i-th simplicial cohomology of $\mathcal{K}(X)$, and we denote it by $\mathrm{H}^i(X)$. The number $\beta^i(X) = \dim_k \mathrm{H}^i(X)$ is the i-th *Betti number* of X.

The simplicial cohomology of X is related to the cohomology of the constant sheaf via appropriate simplicial resolution, which we now describe.

Let I_x be the indecomposable injective corresponding to x. For a simplex σ, set $I_\sigma = I_{\min \sigma}$ where $\min \sigma$ is the minimal element of σ. If $\tau \subseteq \sigma$, then $\min \tau \geq \min \sigma$, hence $\mathrm{Hom}_X(I_\tau, I_\sigma) \simeq k$.

Let $X^{(p)}$ denote the set of p-simplices of X and let $\mathcal{I}_X^p = \oplus_{\sigma \in X^{(p)}} I_\sigma$. For a p-simplex $\sigma = x_0 < x_1 < \dots < x_p$ and $0 \leq j \leq p$, denote by $\widehat{\sigma}^j$ the $(p-1)$-simplex obtained from σ by deleting the vertex x_j. By considering, for all $\sigma \in X^{(p)}$ and $0 \leq j \leq p$, the map $I_{\widehat{\sigma}^j} \to I_\sigma$ corresponding to $(-1)^j \in k$, we get a map $d^{p-1} : \mathcal{I}_X^{p-1} \to \mathcal{I}_X^p$. The usual sign considerations give $d^p d^{p-1} = 0$.

Lemma 3.16. $\mathrm{H}^i(X) = \mathrm{H}^i(\mathrm{Hom}_X(k_X, \mathcal{I}_X^\bullet))$ *for all* $i \geq 0$.

Proof. Indeed, the p-th term is $\mathrm{Hom}_X(k_X, \mathcal{I}_X^p) = \oplus_{\sigma \in X^{(p)}} \mathrm{Hom}_X(k_X, I_\sigma) \cong \oplus_{\sigma \in X^{(p)}} k_X(\min \sigma)$ and can be viewed as the space of functions from $X^{(p)}$ to k. Moreover, the differential is exactly the one used in the definition of simplicial cohomology. □

Lemma 3.17. *The complex* $0 \to k_X \to \mathcal{I}_X^0 \xrightarrow{d^0} \mathcal{I}_X^1 \xrightarrow{d^1} \dots$ *is an injective resolution of the constant sheaf* k_X.

Proof. It is enough to check acyclicity at the stalks.

Let $x \in X$. Then $I_\sigma(x) \neq 0$ only if $\min \sigma \geq x$, hence it is enough to consider the p-simplices of U_x, and the complex of stalks at x equals

$$0 \to k \to \mathrm{Hom}_{U_x}(k_{U_x}, \mathcal{I}_{U_x}^0) \to \mathrm{Hom}_{U_x}(k_{U_x}, \mathcal{I}_{U_x}^1) \to \dots$$

The acyclicity of this complex follows by Lemma 3.16 with $X = U_x$, using the fact that U_x has x as the unique minimal element, hence $\mathcal{K}(U_x)$ is contractible and $\mathrm{H}^i(U_x) = 0$ for $i > 0$, $\mathrm{H}^0(U_x) = k$. □

Proposition 3.18. $\mathrm{H}^i(X; k_X) = \mathrm{H}^i(X)$ *for all* $i \geq 0$.

Proof. Using Lemma 3.16 and the injective resolution of Lemma 3.17,

$$\mathrm{H}^i(X; k_X) = \mathrm{H}^i(\mathrm{Hom}_X(k_X, \mathcal{I}_X^\bullet)) = \mathrm{H}^i(X)$$

□

3.4.3. *Hochschild cohomology.* A k-algebra Λ has a natural structure of a Λ-Λ-bimodule, or a $\Lambda \otimes_k \Lambda^{op}$ right module. The group $\mathrm{Ext}^i_{\Lambda \otimes \Lambda^{op}}(\Lambda, \Lambda)$ is called the *i-th Hochschild cohomology* of Λ, and we denote it by $\mathrm{HH}^i(\Lambda)$.

The Hochschild cohomology of incidence algebras of posets was widely studied, see [6, 9, 10]. The following theorem relates the Hochschild cohomology of an incidence algebra of a poset X with its simplicial cohomology.

Theorem 3.19 ([6, 10])**.** $\mathrm{HH}^i(kX) = \mathrm{H}^i(X)$ *for all* $i \geq 0$.

Combining this with Proposition 3.18, we get:

Corollary 3.20. $\mathrm{HH}^i(kX) = \mathrm{H}^i(X; k_X) = \mathrm{Ext}^i_X(k_X, k_X)$ *for all* $i \geq 0$.

3.4.4. *Derived invariants.*

Corollary 3.21. *If $X \sim Y$ then $\beta^i(X) = \beta^i(Y)$ for all $i \geq 0$.*

Proof. Follows from Theorem 3.19 and the fact that the Hochschild cohomology of a k-algebra is preserved under derived equivalence [12, 19]. □

The alternating sum $\chi(X) = \sum_{i \geq 0} (-1)^i \beta^i(X)$ is known as the *Euler characteristic* of X.

Corollary 3.22. *If $X \sim Y$ then $\chi(X) = \chi(Y)$.*

We give two interpretations of $\chi(X)$. First, by Proposition 3.18,

$$\chi(X) = \sum_{i \geq 0} (-1)^i \beta^i(X) = \sum_{i \geq 0} \dim_k \operatorname{Hom}_{\mathcal{D}^b(X)}(k_X, k_X[i]) = \langle [k_X], [k_X] \rangle_X$$

where $[k_X]$ is the image of k_X in $K_0(X)$. Since $[k_X] = \sum_{x \in X} [S_x]$,

$$\langle [k_X], [k_X] \rangle = \sum_{x,y \in X} \langle [S_x], [S_y] \rangle_X = \sum_{x,y \in X} \mu_X(x, y)$$

hence $\chi(X)$ is the sum of entries of the matrix $\mathbf{1}_X^{-1}$. We see that not only the $\mathbb{Z}$-congruence class of $\mathbf{1}_X^{-1}$ is preserved by derived equivalence, but also the sum of its entries.

For the second interpretation, changing the order of summation we get

$$\sum_{x,y \in X} \langle [S_x], [S_y] \rangle_X = \sum_{i \geq 0} (-1)^i \sum_{x,y \in X} \dim \operatorname{Ext}^i_X(S_x, S_y)$$

Using the fact that $\dim \operatorname{Ext}^i(S_x, S_y)$ equals δ_{xy} for $i = 0$; counts the number of arrows from x to y in the Hasse diagram of X when $i = 1$; and counts the number of commutativity relations between x and y for $i = 2$, we see that at least when $\operatorname{gl.dim} X \leq 2$, $\chi(X)$ equals the number of points minus the number of arrows in the Hasse diagram plus the number of relations etc.

3.5. **Operations preserving derived equivalence.** We show that derived equivalence is preserved under taking opposites and products.

Definition 3.23. The *opposite* of a poset X, denoted by X^{op}, is the poset $(X, \leq^{op})$ with $x \leq^{op} x'$ if and only if $x \geq x'$.

Lemma 3.24. *Let $\mathcal{A}$ be an abelian category. Then $Sh_{X^{op}}\mathcal{A} \simeq (Sh_X \mathcal{A}^{op})^{op}$.*

Proof. A sheaf $\mathcal{F}$ over X^{op} with values in $\mathcal{A}$ is defined via compatible $\mathcal{A}$-morphisms between the stalks $\mathcal{F}(y) \to \mathcal{F}(x)$ for $x \leq y$. Viewing these morphisms as $\mathcal{A}^{op}$-morphisms we identify $\mathcal{F}$ with a sheaf over X with values in $\mathcal{A}^{op}$. Since a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is specified via compatible $\mathcal{A}$-morphisms $\mathcal{F}(x) \to \mathcal{G}(x)$, this identification gives an equivalence $Sh_{X^{op}}\mathcal{A} \simeq (Sh_X \mathcal{A}^{op})^{op}$. □

Corollary 3.25. *$Sh_{X^{op}}$ is equivalent to $(Sh_X)^{op}$.*

Proof. Let $\mathcal{A}$ be the category of finite dimensional k-vector spaces. Then the functor $V \mapsto V^\vee$ mapping a finite dimensional k-vector space to its dual induces an equivalence $\mathcal{A} \simeq \mathcal{A}^{op}$. □

Proposition 3.26. *If $X \sim Y$ then $X^{op} \sim Y^{op}$.*

Proof. It is well known that for an abelian category $\mathcal{A}$, the opposite category $\mathcal{A}^{op}$ is also abelian and $\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{A}^{op})^{op}$ by mapping a complex $K = (K^i)_{i\in\mathbb{Z}}$ over $\mathcal{A}$ to the complex $K^\vee$ over $\mathcal{A}^{op}$ with $(K^\vee)^i = K^{-i}$.

Applying this for $\mathcal{A} = Sh_X$ and using Corollary 3.25, we deduce that $\mathcal{D}^b(X^{op}) \simeq \mathcal{D}^b(X)^{op}$. □

Definition 3.27. The *product* of two posets X, Y, denoted $X \times Y$, is the poset whose underlying set is $X \times Y$, with $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

Lemma 3.28. $k(X \times Y) = kX \otimes_k kY$.

Proof. Observe that the function $kX \otimes_k kY \to k(X \times Y)$ defined by mapping the basis elements $e_{xx'} \otimes e_{yy'}$ to $e_{(x,y)(x',y')}$ where $x \leq x'$ and $y \leq y'$, is an isomorphism of k-algebras. □

Proposition 3.29. *If $X_1 \sim X_2$ and $Y_1 \sim Y_2$ then $X_1 \times Y_1 \sim X_2 \times Y_2$.*

Proof. The claim follows from the previous lemma and the corresponding fact for tensor products of finite dimensional algebras over k, see [19, Lemma 4.3]. □

4. Derived equivalences via exceptional collections

4.1. **Strongly exceptional collections.** Let k be a field and let $\mathcal{T}$ be a triangulated k-category.

Definition 4.1. A sequence $E_1, \dots, E_n$ of objects of $\mathcal{T}$ is called a *strongly exceptional collection* if

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{T}}(E_s, E_t[i]) = 0 && 1 \leq s, t \leq n\,,\ i \neq 0 \\ &\mathrm{Hom}_{\mathcal{T}}(E_s, E_t) = 0 && 1 \leq s < t \leq n \\ &\mathrm{Hom}_{\mathcal{T}}(E_s, E_s) = k && 1 \leq s \leq n \end{aligned} \tag{4.1}$$

An unordered finite collection $\mathcal{E}$ of objects of $\mathcal{T}$ will be called *strongly exceptional* if it can be ordered in a sequence which forms a strongly exceptional collection.

Let $\mathcal{E} = E_1, \dots, E_n$ be a strongly exceptional collection in $\mathcal{T}$, and consider $E = \oplus_{s=1}^n E_s$. The conditions (4.1) imply that $\mathrm{Hom}_{\mathcal{T}}(E, E[i]) = 0$ for $i \neq 0$ and that $\mathrm{End}_{\mathcal{T}}(E)$ is a finite dimensional k-algebra. If $\mathcal{E}$ generates $\mathcal{T}$, then E is a *tilting object* in $\mathcal{T}$.

For an algebra A over k, denote by $\mathcal{D}^b(A)$ the bounded derived category of complexes of finite dimensional right modules over A. The following result of Bondal shows that the existence of a generating strongly exceptional collection in a derived category leads to derived equivalence with $\mathcal{D}^b(A)$ where A is the endomorphism algebra of the corresponding tilting object.

Theorem 4.2 ([4, §6])**.** *Let $\mathcal{A}$ be an abelian category and let $E_1, \dots, E_n$ be a strongly exceptional collection which generates $\mathcal{D}^b(\mathcal{A})$. Set $E = \oplus_{s=1}^n E_s$. Then the functor*

$$\mathbf{R}\mathrm{Hom}(E, -) : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathrm{End}_{\mathcal{D}^b(\mathcal{A})} E)$$

is a triangulated equivalence.

When $\mathcal{A}$ is a category of finite dimensional modules over a finite dimensional algebra, as in the case of Sh_X, the result of the theorem can also be deduced from Rickard's Morita theory of derived equivalences of algebras [18] (see also [14, (3.2)]) by observing that E is a so-called one-sided tilting complex.

Example 4.3. For a poset X, the collection $\{P_x\}_{x\in X}$ (and $\{I_x\}_{x\in X}$) of indecomposable projectives (injectives) is strongly exceptional, generates $\mathcal{D}^b(X)$, and the corresponding endomorphism algebra is isomorphic to the incidence algebra of X (Use Lemma 2.9).

4.2. **A gluing construction.** Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be three triangulated categories with triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, shift left, "i_*"] & \mathcal{T} \arrow[l, shift left, "i^{-1}"] \arrow[r, shift left, "j^{-1}"] & \mathcal{T}'' \arrow[l, shift left, "j_!"] \end{tikzcd}$$

Assume that there are adjunctions

$$\operatorname{Hom}_{\mathcal{T}'}(i^{-1}\mathcal{F}, \mathcal{F}') \simeq \operatorname{Hom}_{\mathcal{T}}(\mathcal{F}, i_*\mathcal{F}') \tag{4.2}$$

$$\operatorname{Hom}_{\mathcal{T}''}(\mathcal{F}'', j^{-1}\mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{T}}(j_!\mathcal{F}'', \mathcal{F}) \tag{4.3}$$

for $\mathcal{F} \in \mathcal{T}$, $\mathcal{F}' \in \mathcal{T}'$, $\mathcal{F}'' \in \mathcal{T}''$. Assume also that $j^{-1}i_* = 0$, $i^{-1}j_! = 0$, $i^{-1}i_* \simeq \mathrm{Id}_{\mathcal{T}'}$ and $j^{-1}j_! \simeq \mathrm{Id}_{\mathcal{T}''}$.

Lemma 4.4. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{T}$. Then*

$$\operatorname{Hom}_{\mathcal{T}}(i_*i^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{T}}(\mathcal{F}, i_*i^{-1}\mathcal{G}) \tag{4.4}$$

$$\operatorname{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, j_!j^{-1}\mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, \mathcal{G}) \tag{4.5}$$

$$\operatorname{Hom}_{\mathcal{T}}(j_!j^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) = 0 \tag{4.6}$$

Proof. The claims follow from the adjunctions (4.2),(4.3) and our additional hypotheses. For example, for the first claim use (4.2) and $i^{-1}i_* \simeq \mathrm{Id}_{\mathcal{T}'}$ to get that

$$\begin{aligned} \operatorname{Hom}_{\mathcal{T}}(i_*i^{-1}\mathcal{F}, i_*i^{-1}\mathcal{G}) &\simeq \operatorname{Hom}_{\mathcal{T}'}(i^{-1}i_*i^{-1}\mathcal{F}, i^{-1}\mathcal{G}) = \\ &= \operatorname{Hom}_{\mathcal{T}'}(i^{-1}\mathcal{F}, i^{-1}\mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{T}}(\mathcal{F}, i_*i^{-1}\mathcal{G}) \end{aligned}$$

□

We apply this for the following situation, cf. [2, §1.4]. Let X be a poset and let $Y \subseteq X$ be a closed subset, $U = X \setminus Y$ its complement. Denote by $i : Y \to X$, $j : U \to X$ the embeddings. Since the functors $i_*, j_!$ are exact, we can consider the functors

$$\begin{aligned} i^{-1} &: \mathcal{D}^b(X) \to \mathcal{D}^b(Y) & \qquad j^{-1} &: \mathcal{D}^b(X) \to \mathcal{D}^b(U) \\ i_* &: \mathcal{D}^b(Y) \to \mathcal{D}^b(X) & \qquad j_! &: \mathcal{D}^b(U) \to \mathcal{D}^b(X) \end{aligned}$$

between the derived categories. Taking $\mathcal{T} = \mathcal{D}^b(X)$, $\mathcal{T}' = \mathcal{D}^b(Y)$ and $\mathcal{T}'' = \mathcal{D}^b(U)$, we see that the above assumptions are satisfied, where the adjunctions (4.2), (4.3) follow from (2.4).

For $y \in Y$ and $u \in U$, let $\widetilde{P}_y = i_* i^{-1} P_y$ and $\widetilde{I}_u = j_! j^{-1} I_u$ be "truncated" versions of the projectives and injectives. Explicitly,

$$\widetilde{P}_y(x) = \begin{cases} k & x \in Y\,,\, y \le x \\ 0 & \text{otherwise} \end{cases} \qquad \widetilde{I}_u(x) = \begin{cases} k & x \in U\,,\, x \le u \\ 0 & \text{otherwise} \end{cases}$$

with identity maps between nonzero stalks.

Proposition 4.5. *The collection* $\mathcal{E}_Y = \left\{\widetilde{P}_y\right\}_{y \in Y} \bigcup \left\{\widetilde{I}_u[1]\right\}_{u \in U}$ *is strongly exceptional and generates* $\mathcal{D}^b(X)$.

Proof. Let $y, y' \in Y$. By (4.4),

$$\operatorname{Hom}(\widetilde{P}_y, \widetilde{P}_{y'}) \simeq \operatorname{Hom}(P_y, \widetilde{P}_{y'}) = \widetilde{P}_{y'}(y) = \begin{cases} k & \text{if } y' \le y \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

and $\operatorname{Hom}(\widetilde{P}_y, \widetilde{P}_{y'}[n]) = 0$ for $n \ne 0$. Similarly, for $u, u' \in U$, by (4.5),

$$\operatorname{Hom}(\widetilde{I}_u, \widetilde{I}_{u'}) \simeq \operatorname{Hom}(\widetilde{I}_u, I_{u'}) = \widetilde{I}_u(u') = \begin{cases} k & \text{if } u' \le u \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

and $\operatorname{Hom}(\widetilde{I}_u, \widetilde{I}_{u'}[n]) = 0$ for $n \ne 0$.

Let $y \in Y$ and $u \in U$. By (4.6), $\operatorname{Hom}(\widetilde{I}_u, \widetilde{P}_y[n]) = 0$ for all $n \in \mathbb{Z}$. Consider now $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n])$. The distinguished triangle $\widetilde{I}_u \to I_u \to i_* i^{-1} I_u \to \widetilde{I}_u[1]$ of (2.2) gives rise to a long exact sequence

$$\dots \to \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u) \to \operatorname{Hom}(\widetilde{P}_y, I_u) \to \operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u) \to \dots \tag{4.9}$$

Since I_u is injective, $\operatorname{Hom}(\widetilde{P}_y, I_u[n]) = 0$ for $n \ne 0$ and $\operatorname{Hom}(\widetilde{P}_y, I_u) = \widetilde{P}_y(u) = 0$. Therefore (4.9) induces isomorphisms

$$\operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u[n]) \xrightarrow{\simeq} \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n+1]) \tag{4.10}$$

for all $n \in \mathbb{Z}$. By (4.4),

$$\operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u[n]) = \operatorname{Hom}(P_y, i_* i^{-1} I_u[n]) = \begin{cases} (i_* i^{-1} I_u)(y) & n = 0 \\ 0 & n \ne 0 \end{cases}$$

and $(i_* i^{-1} I_u)(y) = k$ if $y < u$ and 0 otherwise, hence

$$\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) = \begin{cases} k & \text{if } y < u \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

and $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1+n]) = 0$ for $n \ne 0$.

Note that one can also compute $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n])$ by considering the triangle $j_! j^{-1} P_y \to P_y \to \widetilde{P}_y \to j_! j^{-1} P_y[1]$ and using the induced isomorphisms

$$\operatorname{Hom}(j_! j^{-1} P_y, \widetilde{I}_u[n]) \xrightarrow{\simeq} \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[n+1]) \tag{4.12}$$

The above calculations show that if we order each of the sets Y and U linearly extending the partial order induced by X and arrange the elements of $\mathcal{E}_Y$ in a sequence by first taking the elements of U and then taking those of Y, we get a strongly exceptional collection.

To prove that $\mathcal{E}_Y$ generates $\mathcal{D}^b(X)$, it is enough to show that every sheaf belongs to the triangulated subcategory generated by $\mathcal{E}_Y$. By (2.2), it is enough to verify this for $i_*\mathcal{F}'$ and $j_!\mathcal{F}''$ where $\mathcal{F}' \in Sh_Y$, $\mathcal{F}'' \in Sh_U$. The collection of sheaves $i^{-1}P_y$, being a complete set of indecomposable projectives of Sh_Y, generates $\mathcal{D}^b(Y)$. Similarly, the sheaves $j^{-1}I_u$ form a complete set of indecomposable injectives of Sh_U and generate $\mathcal{D}^b(U)$. Now the result follows by applying the triangulated functors $i_*, j_!$. $\square$

4.3. **The endomorphism algebras A_Y.** Fix a poset X, and let $Y \subseteq X$ be a closed subset. Consider $T_Y = (\oplus_{y \in Y} \widetilde{P}_y) \oplus (\oplus_{u \in U} \widetilde{I}_u)[1]$ and let $A_Y = \operatorname{End}_{\mathcal{D}^b(X)} T_Y$. By Theorem 4.2 and Proposition 4.5, we have:

Corollary 4.6. *$\mathcal{D}^b(X) \simeq \mathcal{D}^b(A_Y)$.*

Proposition 4.7. *The algebra A_Y has as a k-basis the elements*

$$\left\{e_{yy'} \,:\, y \leq y'\right\} \cup \left\{e_{u'u} \,:\, u' \leq u\right\} \cup \left\{e_{uy} \,:\, y < u\right\}$$

where $y, y' \in Y$, $u', u \in U$. The multiplication is defined by

$$e_{yy'} e_{y'y''} = e_{yy''} \qquad\qquad e_{u''u'} e_{u'u} = e_{u''u}$$

$$e_{uy} e_{yy'} = \begin{cases} e_{uy'} & y' < u \\ 0 & \textit{otherwise} \end{cases} \qquad\qquad e_{u'u} e_{uy} = \begin{cases} e_{u'y} & y < u' \\ 0 & \textit{otherwise} \end{cases}$$

for $y \leq y' \leq y'' \in Y$, $u'' \leq u' \leq u \in U$ (all other products are zero).

Proof. For $y \leq y' \in Y$, using (4.7), choose $e_{yy'} \in \operatorname{Hom}(\widetilde{P}_{y'}, \widetilde{P}_y)$ corresponding to $1 \in \widetilde{P}_y(y')$. In other words, the stalk of the morphism $e_{yy'}$ at y' is the identity map on k. Then $e_{yy'} e_{y'y''} = e_{yy''}$ for $y \leq y' \leq y'' \in Y$.

Similarly, for $u' \leq u \in U$, using (4.8), choose $e_{u'u} \in \operatorname{Hom}(\widetilde{I}_u[1], \widetilde{I}_{u'}[1])$ corresponding to $1 \in \widetilde{I}_u(u')$. The stalk of $e_{u'u}$ at u' is the identity map on k and we have $e_{u''u'} e_{u'u} = e_{u''u}$ for all $u'' \leq u' \leq u \in U$.

Now consider $y \in Y$ and $u \in U$ such that $y < u$. Using the isomorphisms (4.10) and (4.12), we have

$$\begin{tikzcd}
\operatorname{Hom}(j_! j^{-1} P_y, \widetilde{I}_u) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) & \operatorname{Hom}(\widetilde{P}_y, i_* i^{-1} I_u) \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
(j_! j^{-1} P_y)(u) & & (i_* i^{-1} I_u)(y)
\end{tikzcd}$$

There are unique $e_{uy}, \widetilde{e}_{uy} \in \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1])$ such that the image of e_{uy} in $(j_! j^{-1} P_y)(u)$ equals 1 and the image of $\widetilde{e}_{uy}$ in $(i_* i^{-1} I_u)(y)$ is 1. The formula for $e_{u'u} e_{uy}$ where $u' \leq u$ now follows by considering the composition

$$\begin{tikzcd}
\operatorname{Hom}(j_! j^{-1} P_y, \widetilde{I}_u) \arrow[r, "\simeq"] \arrow[d, "e_{u'u} \circ -"'] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1]) \ni e_{uy} \\
\operatorname{Hom}(j_! j^{-1} P_y, \widetilde{I}_{u'}) \arrow[r, "\simeq"] & \operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_{u'}[1]) \ni e_{u'y}
\end{tikzcd}$$

The formula for $e_{uy} e_{yy'}$ would follow in a similar manner by considering the composition $- \circ e_{yy'}$ once we know that the scalar ratio between $\widetilde{e}_{uy}$ and e_{uy} is *independent* of u and y.

Indeed, replacing the objects $\widetilde{P}_y$ and $\widetilde{I}_u[1]$ by the quasi-isomorphic complexes $(j_!j^{-1}P_y \to P_y)[1]$ and $(I_u \to i_*i^{-1}I_u)[1]$, we see that $\operatorname{Hom}(\widetilde{P}_y, \widetilde{I}_u[1])$ equals the set of morphisms (λ, μ) between the two complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & j_!j^{-1}P_y \arrow[r] \arrow[d, "\lambda"] & P_y \arrow[r] \arrow[d, "\mu"] \arrow[dl, dashed, no head] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I_u \arrow[r] & i_*i^{-1}I_u \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

modulo homotopy. Note that $(\lambda, \mu) \sim (\lambda', \mu')$ if and only if $\lambda - \mu = \lambda' - \mu'$. The morphism e_{uy} corresponds to the pair $(1, 0)$ while $\widetilde{e}_{uy}$ corresponds to $(0, 1)$, hence $\widetilde{e}_{uy} = -e_{uy}$.

It is clear that the elements constructed above form a k-basis of A_Y and satisfy the required relations. $\square$

Example 4.8. Let X be the poset with Hasse diagram as in the left picture, and let $Y = \{1\}$. The algebra A_Y is shown in the right picture, as the path algebra of the quiver A_3 modulo the zero relation indicated by the dotted arrow (i.e. the product of $2 \to 3$ and $3 \to 1$ is zero).

$$\begin{tikzcd}
1 \arrow[dr] & & 2 \arrow[dl] \\
& 3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
2 \arrow[r] \arrow[rr, bend right, dotted, no head] & 3 \arrow[r] & 1
\end{tikzcd}$$

Lemma 4.9. *Let $X' = U \cup Y$ and define a binary relation $\leq'$ on X' by*

$$u' \leq' u \Leftrightarrow u' \leq u \quad y \leq' y' \Leftrightarrow y \leq y' \quad u <' y \Leftrightarrow y < u \tag{4.13}$$

for $u, u' \in U$, $y, y' \in Y$. Then $\leq'$ is a partial order if and only if the following condition holds:

($\star$) *Whenever $y \leq y' \in Y$, $u' \leq u \in U$ and $y < u$, we have that $y' < u'$.*

When this condition holds, the endomorphism algebra A_Y is isomorphic to the incidence algebra of $(X', \leq')$.

Proof. The first part is clear from the requirement of transitivity of $\leq'$.

The condition ($\star$) implies that $e_{u'u}e_{uy} = e_{u'y}$ and $e_{uy}e_{yy'} = e_{uy'}$ whenever $u' \leq u$, $y \leq y'$ and $y < u$, so that A_Y is the incidence algebra of $(X', \leq')$. $\square$

4.4. **Lexicographic sums along bipartite graphs.**

Definition 4.10. Let S be a poset, and let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by the elements of S. The *lexicographic sum of $\mathfrak{X}$ along S*, denoted $\oplus_S \mathfrak{X}$, is the poset $(X, \leq)$ where $X = \coprod_{s \in S} X_s$ is the disjoint union of the X_s and for $x \in X_s$, $y \in X_t$ we have $x \leq y$ if either $s < t$ (in S) or $s = t$ and $x \leq y$ (in X_s).

Example 4.11. The usual *ordinal sum* $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ of n posets is the lexicographic sum of $\{X_1, \ldots, X_n\}$ along the chain $1 < 2 < \cdots < n$.

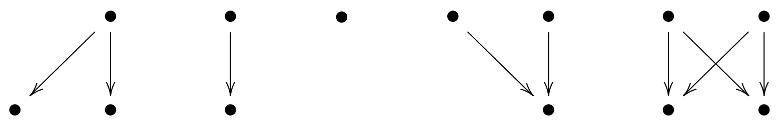
Definition 4.12. A poset S is called a *bipartite graph* if it can be written as a disjoint union of two nonempty subsets S_0 and S_1 such that $s < s'$ in S implies that $s \in S_0$ and $s' \in S_1$.

It follows from the definition that the posets S_0, S_1 are *anti-chains*, that is, no two distinct elements in S_0 (or S_1) are comparable.

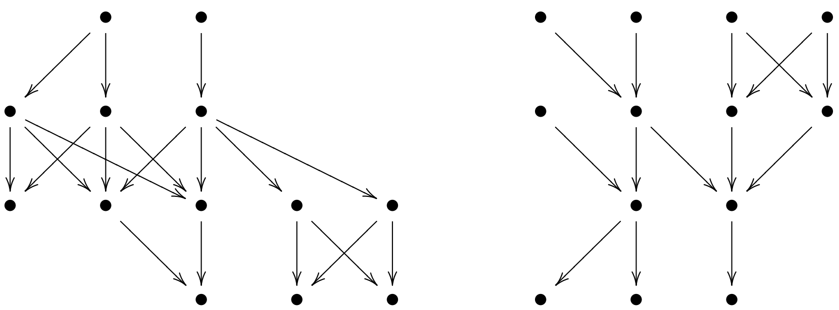
Example 4.13. The left Hasse diagram represents a bipartite poset S. The right one is the Hasse diagram of its opposite S^{op}.

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & 2 \arrow[d] \arrow[dr] & \\ 3 & 4 & 5 \end{tikzcd} \qquad \begin{tikzcd} 3 \arrow[d] & 4 \arrow[dl] \arrow[d] & 5 \arrow[dl] \\ 1 & 2 & \end{tikzcd}$$

Let $\mathfrak{X} = \{X_1, X_2, X_3, X_4, X_5\}$ be the collection



The graphs shown below are the Hasse diagrams of $\oplus_S \mathfrak{X}$ (left) and $\oplus_{S^{op}} \mathfrak{X}$ (right).



Theorem 4.14. *If S is a bipartite graph and $\mathfrak{X} = \{X_s\}_{s\in S}$ is a collection of posets, then $\oplus_S \mathfrak{X} \sim \oplus_{S^{op}} \mathfrak{X}$.*

Proof. Let $S = S_0 \amalg S_1$ be a partition as in the definition of bipartite poset. Let $\mathfrak{X}_0 = \{X_s\}_{s\in S_0}$, $\mathfrak{X}_1 = \{X_s\}_{s\in S_1}$ and let $X = \oplus_S \mathfrak{X}$, $Y = \oplus_{S_0} \mathfrak{X}_0$, $U = \oplus_{S_1} \mathfrak{X}_1$. The sets Y and U can be viewed as disjoint subsets of X with $X = Y \cup U$. Moreover, since there are no relations $s_1 < s_0$ with $s_0 \in S_0$, $s_1 \in S_1$, there are no relations $u < y$ with $y \in Y$, $u \in U$, thus Y is closed and U is open in X. By Corollary 4.6, $\mathcal{D}^b(X) \simeq \mathcal{D}^b(A_Y)$ where A_Y is the endomorphism algebra of the direct sum of the strongly exceptional collection of Proposition 4.5.

We show that the condition $(\star)$ of Lemma 4.9 holds. Indeed, let $y \le y' \in Y$, $u' \le u \in U$. There exist $s_0, s_0' \in S_0$, $s_1, s_1' \in S_1$ such that $y \in X_{s_0}$, $y' \in X_{s_0'}$, $u \in X_{s_1}$ and $u' \in X_{s_1'}$. Now, $s_0' = s_0$ and $s_1' = s_1$ since $y \le y'$, $u' \le u$ and S_0, S_1 are anti-chains. If $y < u$, then $s_0 < s_1$, hence $y' < u'$ and $(\star)$ is satisfied. Therefore A_Y is the incidence algebra of the poset X' defined in (4.13).

Since X' is a disjoint union of the posets U and Y with the original order inside each but with reverse order between them, it is easy to see that X' equals the lexicographic sum of $\mathfrak{X}$ along the opposite poset S^{op}. $\square$

Corollary 4.15. *Let X, Y be two posets. Then $X \oplus Y \sim Y \oplus X$*

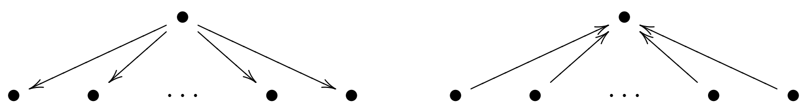
Proof. Take S to be the chain $1 < 2$. $\square$

As special cases, we obtain the following two well known examples.

Example 4.16. The following two posets (represented by their Hasse diagrams) are derived equivalent.

The right poset is obtained from the left one by an APR tilt [1], see also [11, (III.2.14)].

Example 4.17. The two posets below are derived equivalent.



This is a special case of BGP reflection [3], turning a source into a sink (and vice versa).

Corollary 4.18. *Let S be a bipartite graph. Then $S \sim S^{op}$.*

Proof. Take in Theorem 4.14 each X_s to be a point. □

Note that the last Corollary can also be deduced from [3] since Sh_S is the category of representations of a quiver without oriented cycles, namely the Hasse diagram of S, and S^{op} is obtained from S by reverting all the arrows.

4.5. **Ordinal sums of three posets.** The result of Corollary 4.15 raises the natural question whether the derived equivalence class of an ordinal sum of more than two posets does not depend on the order of the summands. The following proposition shows that it is enough to consider the case of three summands.

Proposition 4.19. *Let $\mathcal{X}$ be a family of posets closed to taking ordinal sums. Assume that for any three posets $X, Y, Z \in \mathcal{X}$,*

$$X \oplus Y \oplus Z \sim Y \oplus X \oplus Z \tag{4.14}$$

Then for any $n \geq 1$, $\pi \in S_n$ and $X_1, \dots, X_n \in \mathcal{X}$,

$$X_{\pi(1)} \oplus \cdots \oplus X_{\pi(n)} \sim X_1 \oplus \cdots \oplus X_n$$

Proof. For $n = 1$ the claim is trivial and for $n = 2$ it is just Corollary 4.15. Let $n \geq 3$ and consider the set G_n of permutations in $\pi \in S_n$ such that $X_{\pi(1)} \oplus \cdots \oplus X_{\pi(n)} \sim X_1 \oplus \cdots \oplus X_n$ for all $X_1, \dots, X_n \in \mathcal{X}$. Then G_n is a subgroup of S_n, and the claim to be proved is that $G_n = S_n$.

Let $X_1, \dots, X_n \in \mathcal{X}$. Taking $X = X_1$ and $Y = X_2 \oplus \cdots \oplus X_n$, we see by Corollary 4.15 that the cycle $(1\,2\,\dots\,n)$ belongs to G_n. Now take $X = X_1$, $Y = X_2$ and $Z = X_3 \oplus \cdots \oplus X_n$. By (4.14), $Y \oplus X \oplus Z \sim X \oplus Y \oplus Z$, hence $(1\,2) \in G_n$. The claim now follows since $(1\,2)$ and $(1\,2\,\dots\,n)$ generate S_n. □

We give a counterexample to show that (4.14) is false in general.

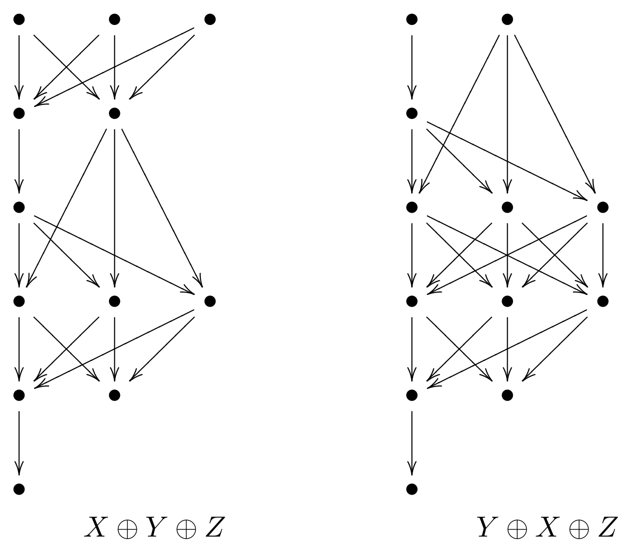


FIGURE 1. Two posets which are not derived equivalent despite their structure as ordinal sums of the same three posets in different orders.

Example 4.20. Let

$$X = \bullet \qquad \bullet \qquad \bullet \qquad\qquad Y = \begin{tikzcd} \bullet \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

and let $Z = X \oplus Y$. Then the posets $X \oplus Y \oplus Z$ and $Y \oplus X \oplus Z$, depicted in Figure 1, are not derived equivalent since their Euler forms are not equivalent over $\mathbb{Z}$ (they are equivalent over $\mathbb{Q}$, though). This is shown using Corollary 3.15 with the prime $p = 11$.

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