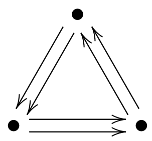
ON CLUSTER ALGEBRAS FROM ONCE PUNCTURED CLOSED SURFACES

SEFI LADKANI

Abstract. We show that many cluster-theoretic properties of the Markov quiver hold also for adjacency quivers of triangulations of once-punctured closed surfaces of arbitrary genus.

Along the way we consider the class $\mathcal{P}$ of quivers introduced by Kontsevich and Soibelman, characterize the mutation-finite quivers that belong to that class and draw some conclusions regarding non-degenerate potentials on them.

The markov quiver Q shown below



has many intriguing properties:

- There are exactly two arrows starting and two arrows ending at any vertex of Q;
- The cluster algebra $\mathcal{A}(Q)$ is not Noetherian [24];
- The upper cluster algebra $\mathcal{U}(Q)$ is not equal to the cluster algebra $\mathcal{A}(Q)$ [3, 24];
- Q has no maximal green mutation sequences [5];
- There is a non-degenerate potential on Q defining a Hom-finite cluster category having cluster-tilting objects that are not reachable from the canonical one [25];
- Q does not belong to the class $\mathcal{P}$ introduced by Kontsevich and Soibelman [18].

The purpose of this note is to demonstrate that this quiver does not comprise a unique, singular, example, but rather it is the simplest member of an infinite family of mutation classes whose quivers share similar properties to the above.

Indeed, recall that Q arises as the adjacency quiver (in the sense of [10]) of any triangulation of a torus with one puncture. It is therefore natural to consider closed surfaces of higher genus. Adjacency quivers of triangulations of closed surfaces with one puncture were studied in our previous work [23] solving a combinatorial problem of classifying the mutation classes of quivers with constant number of arrows. In another work [22] we considered algebraic aspects of triangulations of closed surfaces (with arbitrary number of punctures). By using our previous results and adapting the existing proofs in the literature we can show the following result.

Theorem. *Let Q be an adjacency quiver of a triangulation of a once-punctured closed surface. Then the following assertions hold:*

Date: October 16, 2013.

The author is supported by DFG grant LA 2732/1-1 in the framework of the priority program SPP 1388 “Representation theory”.

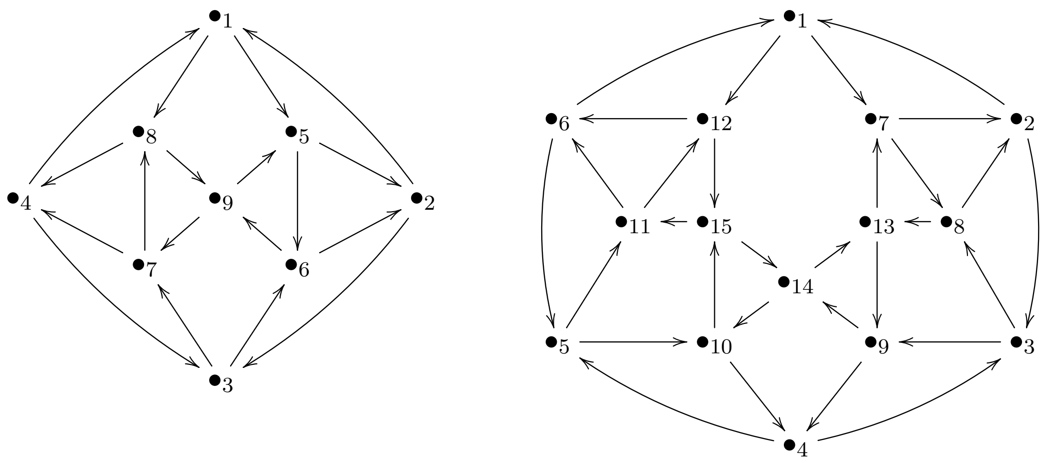


FIGURE 1. Adjacency quivers of triangulations of once-punctured closed surfaces of genus 2 (left) and genus 3 (right).

(a) *There are exactly two arrows starting and two arrows ending at any vertex of Q;*
(b) *The cluster algebra $\mathcal{A}(Q)$ is not Noetherian;*
(c) *The upper cluster algebra $\mathcal{U}(Q)$ is not equal to the cluster algebra $\mathcal{A}(Q)$;*
(d) *Q has no maximal green mutation sequences;*
(e) *There is a non-degenerate potential on Q defining a* Hom*-finite cluster category having cluster-tilting objects that are not reachable from the canonical one;*
(f) *Q does not belong to the class $\mathcal{P}$.*

Therefore each genus $g \geq 1$ gives rise to a finite mutation class consisting of quivers with $6g-3$ vertices that satisfy the properties in the theorem.

Parts (a) and (b) are shown in Section 1, where we also recall some basic properties of the quivers considered in the theorem. In Section 2 we show part (c) by explicitly constructing elements in the upper cluster algebra that do not belong to the cluster algebra, whereas in Section 3 we show parts (d) and (e). Some properties of the class $\mathcal{P}$ of quivers (e.g. uniqueness, rigidity and finite-dimensionality of non-degenerate potentials on them, see Theorem 4.6) are discussed in Section 4, where part (f) of the theorem is shown as a corollary of a stronger statement (Proposition 4.8). Along the way we characterize the quivers belonging to the class $\mathcal{P}$ whose mutation class is finite (Theorem 4.11), and as a consequence deduce an alternative proof of some of the results in [13].

Unlike the once-punctured torus where the mutation class consists of a single quiver, for surfaces of higher genus the mutation classes become very large. A formula for the size of these classes in terms of the genus is given in the paper [2]. Explicit members in each class were provided in [23] (see in particular Figure 1 and Section 3.2 there). In Section 5 we present another construction that produces different explicit members. For genus 2 and 3 the resulting quivers are shown in Figure 1.

Acknowledgement. I would like to thank Arkady Berenstein for useful discussions.

1. Combinatorial properties of the quivers

In this section we explain parts (a) and (b) of the theorem. For background on triangulations and their adjacency quivers we refer to the paper [10] by Fomin, Shapiro and Thurston. In this note we consider only ideal triangulations. If Q is a quiver, we denote by Q_0 its set of vertices and by Q_1 its set of arrows.

Proposition 1.1. *The set of the adjacency quivers of the (ideal) triangulations of a closed surface of genus $g \geq 1$ with one puncture forms a mutation class of quivers. The following statements hold for any quiver Q in this class:*

(a) *Q is connected without oriented cycles of length 2, and for any vertex i of Q, there are exactly two arrows of Q starting at i and two arrows ending at i.*
(b) *There are invertible maps $\phi, \psi \colon Q_1 \to Q_1$ with the following properties:*
  (i) *For any $\alpha \in Q_1$, the set $\{\phi(\alpha), \psi(\alpha)\}$ consists of the two arrows starting at the vertex that α ends at;*
  (ii) *ϕ^3 is the identity on Q_1.*
(c) *For any arrow $\alpha \in Q_1$, the path $\alpha\psi(\alpha)\psi^2(\alpha)\dots\psi^{12g-7}(\alpha)$ is an Eulerian cycle in Q, i.e. it is a cycle that traverses each arrow of Q exactly once.*

Proof. Recall from [10] the compatibility between flips of triangulations and mutations of their adjacency quivers. Now the claim in the preamble follows from the fact that any two triangulations can be connected by a sequence of flips, together with the observation that for a once-punctured closed surface, any triangulation has no self-folded triangles and hence can be flipped at any arc.

The other statements are special case of our results in [22, §2.1] dealing with adjacency quivers of triangulations of arbitrary closed surfaces. The functions ϕ, ψ are denoted there f, g (we changed the notation to avoid confusion with the genus g). Note that the technical condition (T3) on the triangulation needed there is automatically satisfied when there is only one puncture, see [22, Lemma 5.3]. □

Remark 1.2. The mutation classes considered in the proposition were denoted by $\mathcal{Q}_{g,0}$ in our previous work [23]. In fact, it follows from the main results of [23] that the property expressed in part (a) of the proposition actually characterizes these mutation classes.

More precisely, if Q is a connected quiver without oriented 2-cycles such that for any quiver Q' in its mutation class and any vertex i of Q' there are exactly two arrows of Q' starting at i and two arrows ending at i, then Q arises as the adjacency quiver of a triangulation of a once-punctured closed surface.

Now let Q be an adjacency quiver of a triangulation of a once-punctured closed surface. From part (a) of Proposition 1.1 we deduce that the cluster exchange relations in the cluster algebra $\mathcal{A}(Q)$ are homogeneous of degree 2 in the cluster variables. Hence we can repeat the argument of Muller [24, §11.2] and deduce the following statement.

Proposition 1.3. *Let Q be an adjacency quiver of a triangulation of a closed surface with one puncture.*

(a) *$\mathcal{A}(Q)$ admits a non-negative grading such that all cluster variables have degree 1.*
(b) *$\mathcal{A}(Q)$ is infinitely generated and non-Noetherian.*

2. On the upper cluster algebra

In this section we prove part (c) of the theorem. We follow an idea communicated to me by Berenstein [4] on angles in order to explicitly construct an element in the upper cluster algebra that does not belong to the cluster algebra.

We start by recalling, in the language of ice quivers [15, §4], the notion of cluster algebra of geometric type with skew-symmetric exchange matrix [3, 11, 12]. An *ice quiver* is a quiver $\widetilde{Q}$ without oriented cycles of length 2 (hence also without loops) together with a subset of vertices called *frozen*. We shall assume that the set of vertices is $\widetilde{Q}_0 = \{1, 2, \dots, m\}$, and the subset of frozen ones is $\{n+1, \dots, m\}$ for some $n \leq m$. A *seed* $(\mathbf{x}, \widetilde{Q})$ is a pair consisting of an ice quiver $\widetilde{Q}$ together with a tuple $\mathbf{x} = (x_1, x_2, \dots, x_m)$ of elements in the field $\mathbb{Q}(u_1, u_2, \dots, u_m)$ which freely generate it.

Let $1 \leq k \leq n$ (i.e. k is not frozen). The *mutation* of the seed $(\mathbf{x}, \widetilde{Q})$ at k is the seed $(\mathbf{x}', \widetilde{Q}')$ where $\widetilde{Q}' = \mu_k(\widetilde{Q})$ is the quiver obtained from $\widetilde{Q}$ by mutation at k and $\mathbf{x}' = (x'_1, \dots, x'_m)$ is defined by setting $x'_i = x_i$ for $i \neq k$ and

$$x'_k \cdot x_k = \prod_{\substack{\text{arrows} \\ k \to i}} x_i + \prod_{\substack{\text{arrows} \\ j \to k}} x_j \tag{2.1}$$

where the arrows are considered in the quiver $\widetilde{Q}$. The frozen vertices of $\widetilde{Q}'$ are the same as those in $\widetilde{Q}$.

Let $\widetilde{Q}$ be an ice quiver and $\mathbf{x} = (x_1, \dots, x_m)$ a sequence of m indeterminates. The *upper cluster algebra* $\mathcal{U}(\widetilde{Q})$ consists of the elements in $\mathbb{Q}(x_1, \dots, x_m)$ that can be written as Laurent polynomials with integer coefficients in the elements of any cluster $\mathbf{x}'$ appearing in a seed $(\mathbf{x}', \widetilde{Q}')$ that can be obtained from the seed $(\mathbf{x}, \widetilde{Q})$ by a sequence of mutations at non-frozen vertices. Define the algebra $\mathcal{A}(\widetilde{Q})$ to be the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \dots, x_m)$ generated by the elements of these clusters. It is a cluster algebra of geometric type according to the definition in [11, §5], however in later references [3, 12] the *cluster algebra* is defined as $\mathcal{A}(\widetilde{Q})[x_{n+1}^{-1}, \dots, x_m^{-1}]$. Obviously, the two definitions coincide when there are no frozen vertices ("no coefficients").

We work in a slightly more general setting than is actually needed. Let (S, M) be a marked bordered oriented surface and let n be the number of arcs in any triangulation of (S, M). If S is not closed, in this section it would be convenient to think of the boundary segments of (S, M), which are sides of triangles in any triangulation of (S, M), as arcs labeled $n+1, n+2, \dots, m$ for some $m > n$.

For a triangulation T of (S, M) denote by $\widetilde{Q}_T$ the *extended adjacency quiver* of T defined similarly to the ordinary adjacency quiver Q_T but taking into account also the boundary segments. We think of $\widetilde{Q}_T$ as an ice quiver on m vertices labeled $1, 2, \dots, m$ where the vertices corresponding to the boundary segments, labeled $n+1, \dots, m$, are frozen and the full subquiver on the non-frozen vertices is Q_T.

By abuse of notation, a *seed* $(\mathbf{x}, T)$ is a pair consisting of a triangulation T of (S, M) together with a tuple $x = (x_1, x_2, \dots, x_m)$ of elements in the field $\mathbb{Q}(u_1, u_2, \dots, u_m)$ which freely generate it. Let $1 \leq k \leq n$ and assume that the arc labeled k can be flipped. The *mutation* of the seed $(\mathbf{x}, T)$ is the seed $\mu_k(\mathbf{x}, T) = (\mathbf{x}', \mu_k(T))$ where $\mu_k(T)$ is the triangulation obtained from T by flipping the arc labeled k and $\mathbf{x}' = (x'_1, \dots, x'_m)$

FIGURE 2. Gluing a triangle and adding a marked point p for a boundary segment γ.

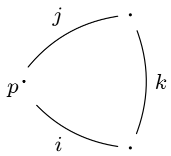
is defined according to the usual rule of seed mutation (2.1), where the arrows are considered in the quiver $\widetilde{Q}_T$.

The next lemma shows that the compatibility between flips of triangulations and mutations of their adjacency quivers [10] holds also for extended adjacency quivers.

Lemma 2.1. *$\widetilde{Q}_{\mu_k(T)} = \mu_k(\widetilde{Q}_T)$, hence the map defined by $(\mathbf{x}, T) \mapsto (\mathbf{x}, \widetilde{Q}_T)$ is compatible with seed mutations.*

Proof. Consider the marked surface $(\widetilde{S}, \widetilde{M})$ obtained from (S, M) by gluing to S a small triangle near each boundary segment and adding as marked point its inner vertex, see Figure 2. Topologically, S and $\widetilde{S}$ have the same genus and the same number of boundary components, and the number of points in $\widetilde{M}$ on each boundary component is twice as that of M. By construction, the arcs of a triangulation T of (S, M) together with the boundary segments naturally form a triangulation $\widetilde{T}$ of $(\widetilde{S}, \widetilde{M})$, the map $T \to \widetilde{T}$ is compatible with flips of arcs of T, and the extended adjacency quiver of T is just the adjacency quiver of $\widetilde{T}$. $\square$

Definition 2.2. Let $(\mathbf{x}, T)$ be a seed such that T has no self-folded triangles. Consider a triangle of T with sides labeled i, j, k and a marked point $p \in M$ as in the following picture



We define the (commutative) *angle* from i to j as

$$\measuredangle_p(i, j) = \frac{x_k}{x_i x_j} \tag{2.2}$$

(in the notation we suppress the dependency of this quantity on the seed $(\mathbf{x}, T)$).

Let $p \in M$ be a marked point and let $i_1, i_2, \dots, i_r$ be the arcs incident to p arranged in a counterclockwise order. If p lies on the boundary of S then i_1, i_r (which might coincide) are boundary segments and the *sum of angles at* p is defined as

$$\measuredangle_p(\mathbf{x}, T) = \measuredangle_p(i_1, i_2) + \measuredangle_p(i_2, i_3) + \cdots + \measuredangle_p(i_{r-1}, i_r),$$

whereas if p is a puncture, we define the sum of angles to be

$$\measuredangle_p(\mathbf{x}, T) = \measuredangle_p(i_1, i_2) + \measuredangle_p(i_2, i_3) + \cdots + \measuredangle_p(i_{r-1}, i_r) + \measuredangle_p(i_r, i_1).$$

Proposition 2.3 (Preservation of angles)**.** *Let $(\mathbf{x}, T)$ be a seed such that T has no self-folded triangles. Assume that in the quiver $\widetilde{Q}_T$ there are exactly two arrows starting at the vertex k and two arrows ending there. Then $\mu_k(T)$ does not have self-folded triangles and $\measuredangle_p(\mathbf{x}, T) = \measuredangle_p \mu_k(\mathbf{x}, T)$ for any $p \in M$.*

Proof. It suffices to consider the contributions of angles inside the quadrilateral where the flip of k takes place. This quadrilateral looks like the left picture below

where the indices i_1, i_2, j_1, j_2 of the side arcs satisfy $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$ due to our assumption on the vertex k in $\widetilde{Q}_T$.

The effect of flip is shown in the right picture, so that $\widetilde{Q}_{\mu_k(T)}$ is equal to the mutation $\mu_k(\widetilde{Q}_T)$, in this quiver the in-degree and out-degree of the vertex k are equal to 2 and no self-folded triangles have been created in $\mu_k(T)$.

By symmetry it is enough to consider the contribution of angles at a marked point p as shown. Indeed, dividing the cluster exchange relation

$$x_k x'_k = x_{i_1} x_{i_2} + x_{j_1} x_{j_2}$$

by $x_{i_1} x_{j_1} x'_k$ we deduce that

$$\measuredangle_p(i, j) = \frac{x_k}{x_{i_1} x_{j_1}} = \frac{x_{i_2}}{x_{j_1} x'_k} + \frac{x_{j_2}}{x_{i_1} x'_k} = \measuredangle_p(k, j_1) + \measuredangle_p(i_1, k).$$

□

Now let (S, M) be a marked surface and assume that either:

- All points of M lie on the boundary of S ("no punctures"); or
- S is closed and $|M| = 1$.

In this case, any triangulation of (S, M) does not have self-folded triangles and the in-degree and out-degree of any non-frozen vertex $1 \leq k \leq n$ in its extended adjacency quiver are both equal to 2. This has several consequences for a triangulation T:

1. Proposition 2.3 applies.
2. If $\mathbf{x}'$ is a cluster in any (usual) seed $(\mathbf{x}', \widetilde{Q}')$ mutable from $(\mathbf{x}, \widetilde{Q}_T)$ at non-frozen vertices, there exists a triangulation T' of (S, M) such that $(\mathbf{x}', T')$ is a seed.
3. The exchange relations (2.1) are homogeneous of degree 2, hence the algebra $\mathcal{A}(\widetilde{Q}_T)$ admits a non-negative grading such that the elements of each cluster have degree 1.

Proposition 2.4. *Let $p \in M$ be a marked point and $(\mathbf{x}, T)$ be any seed.*

(a) *$\measuredangle_p(\mathbf{x}, T)$ is a Laurent polynomial in $x_1, \dots, x_m$;*
(b) *$\measuredangle_p(\mathbf{x}, T)$ is invariant under seed mutations;*
(c) *$\measuredangle_p(\mathbf{x}, T) \in \mathcal{U}(\widetilde{Q}_T)$;*
(d) *$\measuredangle_p(\mathbf{x}, T)$ is homogeneous of degree -1, hence it does not belong to $\mathcal{A}(\widetilde{Q}_T)$.*

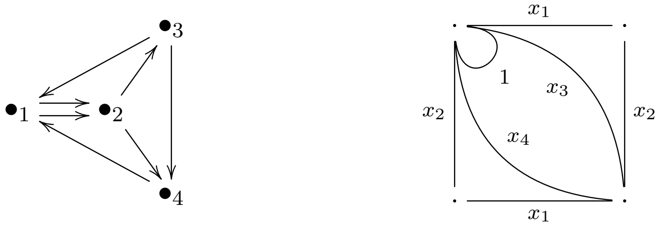
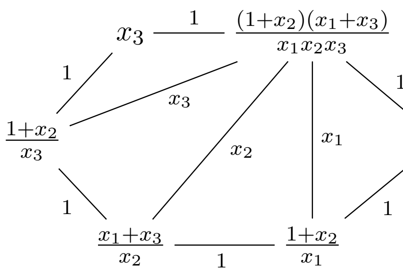


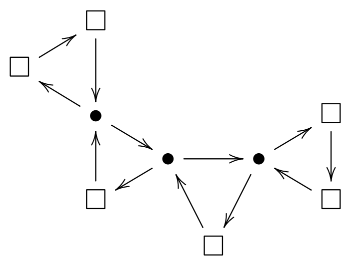
FIGURE 3. The quiver $Q_{1,1}$ together with a corresponding triangulation of the torus with one boundary component and one marked point.

Proof. Claims (a) and (d) are immediate from Definition 2.2, claim (b) follows from Proposition 2.3 and claim (c) follows from (a) and (b). □

By setting the variables $x_{n+1}, \dots, x_m$ (corresponding to the boundary segments) to 1, we obtain elements in the upper cluster algebra of $\mathcal{U}(Q_T)$ (without coefficients). We illustrate this in two examples.

Example 2.5. Consider a triangulation of a disc with 6 marked points (i.e. a hexagon) shown in the left picture. Its extended adjacency quiver is shown in the right picture, where we indicated the frozen vertices by □. On each arc we write the variable and at each marked point we write the corresponding sum of angles, which is a member of the upper cluster algebra $\mathcal{U}(A_3)$.





In this case the upper cluster algebra and the cluster algebra coincide as follows from [3].

Example 2.6. The quiver $Q_{1,1}$ shown in Figure 3 is the adjacency quiver of any triangulation of the torus with one boundary component and one marked point. For the triangulation shown in Figure 3, the element of Proposition 2.4 (after specializing $x_5 = 1$) is given by

$$\frac{x_1^2 + x_2^2 + x_3^2}{x_1x_2x_3} + \frac{x_1^2 + x_2^2 + x_4^2}{x_1x_2x_4} + \frac{1 + x_3^2 + x_4^2}{x_3x_4} \in \mathcal{U}(Q_{1,1}).$$

Now let S be a closed surface and $|M| = 1$. In this case there are no frozen vertices (i.e. $\widetilde{Q}_T = Q_T$ for any triangulation). Since there is only one puncture, we may omit the reference to that puncture and rewrite the sum of angles as

$$\measuredangle(\mathbf{x}, T) = \sum_{\text{triangles } \{i,j,k\}} \left(\frac{x_k}{x_ix_j} + \frac{x_i}{x_jx_k} + \frac{x_j}{x_kx_i} \right) = \sum \frac{x_i^2 + x_j^2 + x_k^2}{x_ix_jx_k}. \tag{2.3}$$

Corollary 2.7. *Let Q be an adjacency quiver of a triangulation of a once-punctured closed surface. Then $\mathcal{A}(Q) \neq \mathcal{U}(Q)$.*

Proof. We adapt the proofs in [3, 24].

Consider the sum $\mu = \measuredangle(\mathbf{x}, T)$ of all the angles in the triangles of a triangulation T whose adjacency quiver is Q. By Proposition 2.4, $\mu \in \mathcal{U}(Q)$ but $\mu \notin \mathcal{A}(Q)$. □

Remark 2.8. In the case of the torus, the element μ is twice the element $\frac{x^2+y^2+z^2}{xyz}$ considered in [24].

Example 2.9. Consider the triangulation of a once-punctured surface of genus 2 shown in Figure 6. Its adjacency quiver is shown in Figure 1 and the element μ is

$$\frac{x_1^2 + x_2^2 + x_5^2}{x_1x_2x_5} + \frac{x_2^2 + x_3^2 + x_6^2}{x_2x_3x_6} + \frac{x_3^2 + x_4^2 + x_7^2}{x_3x_4x_7} + \frac{x_4^2 + x_1^2 + x_8^2}{x_4x_1x_8} + \frac{x_5^2 + x_6^2 + x_9^2}{x_5x_6x_9} + \frac{x_7^2 + x_8^2 + x_9^2}{x_7x_8x_9}.$$

3. On maximal green sequences

In this section we prove parts (d) and (e) of the theorem. We will not define maximal green mutation sequences here, instead we refer the reader to the surveys [14, 16] by Keller and to the paper [5] by Brüstle, Dupont and Pérotin initiating the study of such sequences. For the basic notions on quivers with potentials that will be needed, we refer to the paper [8] by Derksen, Weyman and Zelevinsky.

Throughout this section we fix a quiver Q which is an adjacency quiver of a triangulation of a once-punctured closed surface of some genus $g \geq 1$. Let ϕ, ψ be as in Proposition 1.1. Define an equivalence relation $\sim$ on the set of arrows Q_1 by setting $\alpha \sim \alpha'$ if $\alpha' = \phi^r(\alpha)$ for some integer r (in fact it suffices to consider $r \in \{0, 1, 2\}$ as ϕ^3 is the identity on Q_1).

Consider the following two potentials on Q:

$$W_0 = \sum_{\alpha \in Q_1/\sim} \alpha\phi(\alpha)\phi^2(\alpha)$$

$$W_1 = \sum_{\alpha \in Q_1/\sim} \alpha\phi(\alpha)\phi^2(\alpha) - \beta\psi(\beta)\psi^2(\beta)\dots\psi^{12g-7}(\beta)$$

where the sum is taken over representatives of the $\sim$-equivalence classes of arrows and β is any arrow. Since ϕ^3 is the identity on Q_1, taking other representatives results in cyclically equivalent terms, hence these potentials are well defined.

Proposition 3.1. *The potentials W_0 and W_1 are non-degenerate. The Jacobian algebra $\mathcal{P}(Q, W_0)$ is infinite-dimensional whereas $\mathcal{P}(Q, W_1)$ is finite-dimensional.*

Proof. By [22, §2.2] we know that W_1 is the potential associated by Labardini [20] to a triangulation whose adjacency quiver is Q. It consists of two summands; the first is the sum of all 3-cycles in Q corresponding to the triangles, and the second is the cycle "around" the puncture. We have shown the finite-dimensionality of its Jacobian algebra in [22]. The non-degeneracy of W_1 follows by combining the fact that all triangulations do not have self-folded triangles together with the compatibility between flips and mutations [20].

The potential W_0, obtained from W_1 by omitting the cycle around the puncture, was introduced in our previous work [23, §4.3], where the relevant results are explained. □

We note that a similar statement appears in [13, Proposition 9.13].

Corollary 3.2. *Q has no maximal green sequences.*

Proof. By the previous proposition, Q has a non-degenerate potential whose Jacobian algebra is infinite-dimensional. The result now follows from Proposition 8.1 in [5]. $\square$

Let $\mathcal{C}$ be the generalized cluster category associated to the quiver with potential (Q, W_1) whose Jacobian algebra is finite-dimensional. By the results of Amiot [1], it is a Hom-finite 2-Calabi-Yau triangulated category with a canonical cluster-tilting object Γ whose endomorphism algebra is $\mathcal{P}(Q, W_1)$. We denote by Σ the suspension in $\mathcal{C}$.

We can repeat the argument of Plamondon [25, Example 4.3] to deduce that there are cluster-tilting objects in $\mathcal{C}$ that are not reachable from Γ via finite sequences of mutations.

Proposition 3.3. *The cluster-tilting object $\Sigma\Gamma$ in $\mathcal{C}$ is not reachable by a finite sequence of mutations from the canonical cluster-tilting object Γ.*

Proof. The quiver with potential (Q, W_1) is non-degenerate, hence in any iterated mutation of quivers with potentials, the underlying quiver is the iterated quiver mutation of Q. If follows from [17] that the quiver Q_U of the endomorphism algebra of a cluster-tilting object in $\mathcal{C}$ obtained from the canonical one by iterated mutation does not have oriented cycles of length 2 and it belongs to the mutation class of Q.

We use the notion of index from [6] and write the index of an object Y in $\mathcal{C}$ with respect to such a cluster-tilting object U in $\mathcal{C}$ as $\operatorname{ind}_U Y = \sum_{i=1}^n y_i[U_i]$ in the split Grothendieck group of $\operatorname{add} U$, where $U_1, \dots, U_n$ are the non-isomorphic indecomposable summands of U. If U' is the cluster-tilting object obtained from U by exchanging the summand U_k and $\operatorname{ind}_{U'} Y = \sum_{i=1}^n y'_i[U'_i]$ is the corresponding index, then by [6] the coefficients y'_i are obtained from y_i according to the mutation rule

$$y'_i = \begin{cases} -y_k & \text{if } i = k, \\ y_i + r[y_k]_+ & \text{if } i \neq k \text{ and there are } r \text{ arrows } i \to k \text{ in } Q_U, \\ y_i - r[-y_k]_+ & \text{if } i \neq k \text{ and there are } r \text{ arrows } k \to i \text{ in } Q_U \end{cases}$$

where $[y]_+ = \max(0, y)$. By part (a) of Proposition 1.1, the in-degree and out-degree of any vertex of Q_U are equal to 2 and hence

$$y'_1 + \dots + y'_n = -y_k + \sum_{i \neq k} y_i + 2\left([y_k]_+ - [-y_k]_+\right) = y_k + \sum_{i \neq k} y_i = y_1 + \dots + y_n$$

so that the sum of the coefficients appearing in the index is constant for all the cluster-tilting objects U reachable from the canonical one Γ.

Now write $\Gamma = \Gamma_1 \oplus \dots \oplus \Gamma_n$. Obviously

$$\operatorname{ind}_\Gamma \Sigma\Gamma = \sum_{i=1}^n (-1) \cdot [\Gamma_i] \qquad\qquad \operatorname{ind}_{\Sigma\Gamma} \Sigma\Gamma = \sum_{i=1}^n 1 \cdot [\Sigma\Gamma_i]$$

so the corresponding sums of coefficients are $-n$ and n, respectively. Hence $\Sigma\Gamma$ is not reachable from Γ by a sequence of mutations. $\square$

Remark 3.4. Corollary 3.2 could be also be deduced from Proposition 3.3 as in the proof of [5, Proposition 2.21]. In fact, Proposition 3.3 implies the stronger statement that Q does not have a reddening sequence as defined in [16].

Remark 3.5. Consider the graph of (isomorphism classes of basic) cluster-tilting objects in $\mathcal{C}$, where edges correspond to mutations. In the course of the above proof we have seen that this graph has at least two connected components, one containing Γ and another containing $\Sigma\Gamma$. It is interesting to compare this with Proposition 7.10 of [10] asserting that for a once-punctured closed surface, the tagged arc complex and its dual graph have two connected components.

4. On the class $\mathcal{P}$

We describe a procedure of inductively building quivers from simpler ones while keeping various properties regarding the possible potentials on them.

Definition 4.1. A *triangular extension* [1, §3.3] of two quivers Q' and Q'' is any quiver Q obtained from the disjoint union of Q' and Q'' by adding some new arrows (possibly none) from vertices of Q' to vertices of Q''.

Definition 4.2. Let $\mathcal{Q}$ be a set of quivers. Denote by $\langle \mathcal{Q} \rangle$ the smallest set of quivers that contains $\mathcal{Q}$ and is closed under performing quiver mutations and triangular extensions.

Definition 4.3. Let $\bullet$ be the quiver with one vertex and no arrows. The class $\mathcal{P}$ of Kontsevich and Soibelman [18, §8.4] is $\langle \bullet \rangle$. In particular, all quivers that are mutation equivalent to ones without oriented cycles are in $\mathcal{P}$.

Remark 4.4. Let Q be a triangular extension of two quivers Q' and Q''. If W is a potential on Q, let $W|_{Q'}$ be the *restriction* of W to Q' consisting of all the terms of W whose arrows lie entirely in Q'. By abuse of notation we identify the complete path algebra of Q' with its image in the complete path algebra of Q, so we can think of $W|_{Q'}$ as a potential on Q' or on Q as needed. We define $W|_{Q''}$ similarly.

We observe the following:

(a) Any cycle in Q is already a cycle contained in Q' or in Q''.

(b) If W is a potential on Q, then $W = W|_{Q'} + W|_{Q''}$. Hence for any arrow $\alpha \in Q_1$,

$$\partial_\alpha W = \begin{cases} \partial_\alpha \left(W|_{Q'}\right) & \text{if } \alpha \in Q'_1, \\ \partial_\alpha \left(W|_{Q''}\right) & \text{if } \alpha \in Q''_1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) If W', W'' are rigid potentials on Q', Q'' respectively, then $W = W' + W''$ is a rigid potential on Q. Indeed, we have to verify that any cycle in Q is cyclically equivalent to an element in the Jacobian ideal of W and this follows from the previous claims since $W' = W|_{Q'}$ and $W'' = W|_{Q''}$.

Proposition 4.5. *Let* (P) *be any of the following properties of a quiver Q:*

(P1) *Any non-degenerate potential on Q has finite-dimensional Jacobian algebra;*
(P2) *Any non-degenerate potential on Q is rigid;*
(P3) *A non-degenerate potential on Q is unique up to right equivalence;*
(P4) *There is a rigid potential on Q;*

and let $\mathcal{Q}$ be a set of quivers. If each quiver in $\mathcal{Q}$ has property (P)*, then all quivers in $\langle \mathcal{Q} \rangle$ have property* (P)*.*

Proof. We need to verify that each of the properties is preserved under quiver mutations and triangular extensions.

First, we assume property (P) for a quiver Q and show it for a mutation $\mu_k(Q)$. Let W a non-degenerate potential on $\mu_k(Q)$. Then the corresponding mutation of the quiver with potential $(\mu_k(Q), W)$ is (Q, W') for some non-degenerate potential W' on Q.

The following claims can be found in [8].

(P1) The Jacobian algebra of $(\mu_k(Q), W)$ is finite-dimensional if that of (Q, W') is.
(P2) $(\mu_k(Q), W)$ is rigid if (Q, W') is.
(P3) If W_1, W_2 are two non-degenerate potentials on $\mu_k(Q)$, then the corresponding potentials W_1', W_2' on Q are right equivalent if and only if W_1, W_2 are.
(P4) If (Q, W) is rigid, then its mutation is rigid and is of the form $(\mu_k(Q), W')$ since rigid potentials are non-degenerate.

Now let Q be a triangular extension of Q' and Q'' and assume that property (P) holds for Q' and Q''. We show that it also holds for Q. Let W be a potential on Q. Then by Remark 4.4, $W = W' + W''$ with $W' = W|_{Q'}$, $W'' = W|_{Q''}$. If W is non-degenerate, then the potentials W', W'' are also non-degenerate [20, Corollary 22].

(P1) Follows from [1, Prop. 3.7].
(P2) Follows from Remark 4.4.
(P3) If W_1', W_2' are right equivalent potentials on Q' and W_1'', W_2'' are right equivalent potentials on Q'', then $W_1' + W_1''$ and $W_2' + W_2''$ are right equivalent potentials on Q. The right equivalence is obtained by "gluing" the right equivalences on Q' and Q'', defining it to be the identity on all the arrows from Q' to Q''.
(P4) Follows from Remark 4.4.

□

Theorem 4.6. *Any quiver in class $\mathcal{P}$ has a unique non-degenerate potential (up to right equivalence) which is rigid and its Jacobian algebra is finite-dimensional.*

Proof. The quiver $\bullet$ trivially satisfies the properties (P1), (P3) and (P4), since the only potential is zero. □

Definition 4.7. Define a partial order $<$ on the set of quivers as follows:

$$Q' < Q \quad \text{if } |Q_0'| < |Q_0|, \text{ or } |Q_0'| = |Q_0| \text{ and } |Q_1'| < |Q_1|.$$

For a quiver Q, let $Q^<$ be the set of all quivers smaller than Q in the order just defined. The quivers in $Q^<$ can be thought to be simpler than Q, as they either have less vertices or the same number of vertices but less arrows.

The next proposition shows that adjacency quivers of triangulations of once-punctured closed surfaces cannot be built from simpler quivers by using the operations defining the class $\mathcal{P}$.

Proposition 4.8. *Let Q be an adjacency quiver of a triangulation of a once-punctured closed surface. Then $Q \notin \langle Q^< \rangle$.*

Proof. Q is not mutation equivalent to any quiver in $Q^<$ since all members in the mutation class of Q have the same number of arrows [23]. Moreover, any mutation of Q is not a triangular extension of any two smaller quivers since it has an Eulerian cycle (cf. Prop. 1.1) and hence there exists a path between any two vertices. $\square$

As a corollary we obtain part (f) of the theorem.

Corollary 4.9. *An adjacency quiver of a triangulation of a once-punctured closed surface does not belong to class $\mathcal{P}$.*

Remark 4.10. We could also deduce this corollary by combining Theorem 4.6 with Proposition 3.1.

There is no general procedure to determine if a given quiver belongs to the class $\mathcal{P}$. However, if the mutation class of the quiver is finite, a naive algorithm would be to enumerate on the members of that class, search for quivers which are triangular extensions of two smaller ones and then apply the algorithm recursively for the smaller quivers. The (connected) quivers whose mutation class is finite were classified by Felikson, Shapiro and Tumarkin [9]. Apart from quivers with two vertices and some $r \geq 3$ arrows from one vertex to the other, such quivers either arise as adjacency quivers of triangulations of marked surfaces, or they belong to 11 exceptional mutation classes. The next theorem shows that most of the quivers with finite mutation class belong to the class $\mathcal{P}$.

Theorem 4.11. *A connected quiver whose mutation class is finite belongs to the class $\mathcal{P}$ if and only if it is* ***not*** *one of the following:*

(a) *An adjacency quiver of a triangulation of a closed surface; or*
(b) *The quiver $Q_{1,1}$ shown in Figure 3; or*
(c) *A member of the mutation class of the quiver X_7.*

Applying Theorem 4.6, we deduce the following.

Corollary 4.12. *A quiver whose mutation class is finite and none of its connected components belong to the above families* (a)*,* (b) *or* (c) *has a unique non-degenerate potential (up to right equivalence) which is rigid and its Jacobian algebra is finite-dimensional.*

Remark 4.13. The uniqueness of potentials for these quivers has also been shown by Geiss, Labardini and Schröer in [13] using other techniques. In that work the authors also show the uniqueness, up to weak right equivalence, of potentials for many mutation classes in the family (a) and construct two inequivalent potentials on $Q_{1,1}$.

Remark 4.14. Corollary 4.12 does not give the potential explicitly. An explicit construction for adjacency quivers of triangulations has been given by Labardini [20]. For the exceptional mutation classes that are not acyclic, potentials can be found in our work [21].

The rest of this section is devoted to the proof of Theorem 4.11. We start by recording the following observation.

Lemma 4.15. *If Q has a sink i and $Q \setminus \{i\}$ belongs to $\mathcal{P}$, then Q belongs to $\mathcal{P}$.*

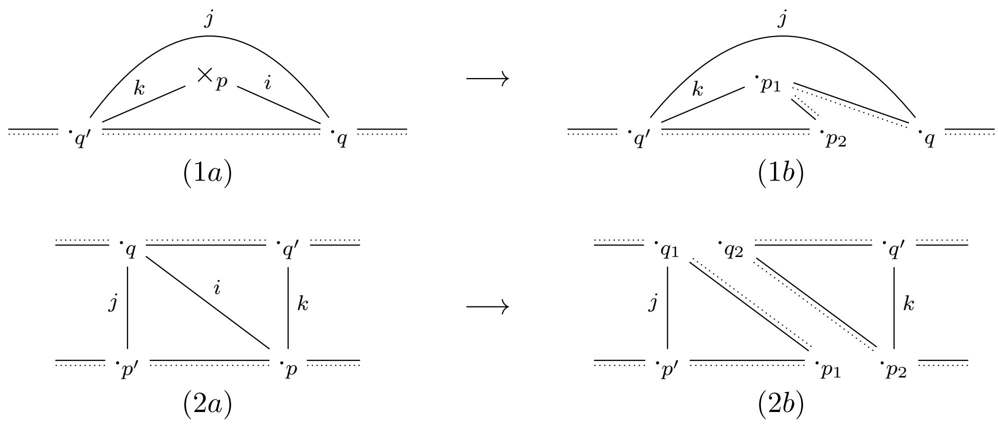


FIGURE 4. Reduction by cutting along the arc i at a puncture p (top row) and at two distinct boundary points p, q (bottom row).

Checking which of the 11 exceptional finite mutation classes belongs to $\mathcal{P}$ is routine using Lemma 4.15 (or can be done on a computer). In particular, the quivers E_n, $\widetilde{E}_n$, $E_n^{1,1}$ for $n = 6, 7, 8$ and X_6 belong to $\mathcal{P}$, but X_7 does not (the latter two quivers were introduced in [7]).

For adjacency quivers of triangulations with at least two marked points, we adapt the arguments of Muller in [24, §10]. Denote by $\mathcal{S}$ the set of marked surfaces (S, M) which are not closed and have at least two marked points.

Lemma 4.16. *Let $(S, M) \in \mathcal{S}$ and assume that it is not an unpunctured disc. Then there exists a triangulation of (S, M) whose adjacency quiver Q has a sink i and moreover the quiver $Q \setminus \{i\}$ is the adjacency quiver of triangulation of a marked surface $(S', M') \in \mathcal{S}$.*

Proof. We follow the proof of Theorem 10.6 in [24]. There are three reduction cases. In each case we find an arc i and a triangulation of (S, M) containing it such that in its adjacency quiver Q the corresponding vertex i is a sink and $Q \setminus \{i\}$ is the adjacency quiver of a triangulation of the marked surface (S', M') obtained by cutting along the arc i, which still belongs to $\mathcal{S}$. We demonstrate this in Figure 4.

If (S, M) has a puncture p, then there are arcs i, j, k as in picture (1a) (it is possible that for the marked points on the boundary, $q = q'$). The surface (S', M') is shown in picture (1b).

If (S, M) has at least two boundary components, then there is an arc i connecting marked points on distinct boundary components. Find arcs j, k as in picture (2a) (it is possible that these arcs coincide). The surface (S', M') is shown in picture (2b).

If (S, M) has one boundary component with at least two marked points and the genus of S is not zero, then there is some arc i which connects distinct marked points on the boundary component of S such that cutting along i does not disconnect S. We proceed as in the previous case. $\square$

Lemma 4.17. *An adjacency quiver of a marked surface in $\mathcal{S}$ belongs to the class $\mathcal{P}$.*

FIGURE 5. Inserting a boundary component.

Proof. If (S, M) is an unpunctured disc, an adjacency quiver is mutation equivalent to a Dynkin quiver of type A_n, hence belongs to $\mathcal{P}$. Otherwise, we proceed by induction on the number of arcs in a triangulation of (S, M), the induction step being an application of Lemma 4.16 and Lemma 4.15. $\square$

In order to complete the proof of the "if" part of Theorem 4.11, it remains to consider surfaces of genus $g \geq 1$ with exactly one boundary component and one marked point on that component. We denote the mutation class of the corresponding adjacency quivers by $\mathcal{Q}_{g,1}$, in agreement with the notation in our papers [21, 23].

Lemma 4.18. *A quiver in $\mathcal{Q}_{g,1}$ belongs to the class $\mathcal{P}$ if and only if $g > 1$.*

Proof. The quiver $Q_{1,1}$ is not a triangular extension and its mutation class consists of a single element, hence it is not in $\mathcal{P}$. From now on assume $g > 1$.

Let $(S', \{p'\})$ be a surface of genus $g - 1$ with one boundary component γ' and a marked point p' on γ'. Let $(S'', \{p''\})$ be a torus with one boundary component γ'' and a marked point p'' on γ''. Gluing these surfaces along γ' and γ'', identifying the marked point p' with p'', we get a closed surface $(S, \{p\})$ of genus g with one puncture p which is the image of p' (and p'').

Two triangulations T' of $(S', \{p'\})$ and T'' of $(S'', \{p''\})$ yield a triangulation T of $(S, \{p\})$ by taking the arcs of T' and T'' together with the arc γ which is the image in S of γ' (or γ''). Hence the adjacency quiver Q_T is obtained from the disjoint union of the *extended* adjacency quivers (cf. Section 2) $Q' = \widetilde{Q}_{T'}$ and $Q'' = \widetilde{Q}_{T''}$ by identifying the two frozen vertices corresponding to the boundary segments γ_1 in Q' and γ_2 in Q''.

The arc γ is contained in two triangles of T as in Figure 5, one arising from S' and the other from S''. By cutting out a disc containing p within one of the triangles and adding an arc "parallel" to γ we get a triangulation of a surface of genus g with one boundary component and one marked point. Denoting the parallel arcs by γ' and γ'', the adjacency quiver Q of this triangulation is obtained from the disjoint union of Q' and Q'' by adding a single arrow $\gamma' \to \gamma''$, hence it is a triangular extension of Q' and Q''.

The argument in the proof of Lemma 2.1 shows that the extended adjacency quivers Q' and Q'' are (usual) adjacency quivers for surfaces with two marked points, hence by Lemma 4.17 they belong to $\mathcal{P}$. It follows that $Q \in \mathcal{P}$ as well. $\square$

We complete the proof of the "only if" part of Theorem 4.11 by considering closed surfaces and applying an algebraic argument generalizing Corollary 4.9 to arbitrary number of punctures.

Lemma 4.19. *An adjacency quiver of a triangulation of a closed surface does not belong to the class $\mathcal{P}$.*

Proof. If the surface is not a sphere with 4 or 5 punctures, the potential associated to the triangulation by Labardini [20] has recently been shown by him to be non-degenerate [19]. However, by [22] it is not rigid and the claim follows from Theorem 4.6.

The mutation classes of adjacency quivers of triangulations of a sphere with 4 or 5 punctures consist of 4 and 26 quivers, respectively. One checks on a computer that for each of these quivers, any two vertices i and j are connected by a path from i to j as well as a path from j to i. Therefore these quivers cannot be triangular extensions. Since any quiver in the mutation class is not a triangular extension, we get the claim. $\square$

We conclude with a few remarks.

Remark 4.20. By combining Lemma 4.18 with [24, Theorem 10.5] we see that there are quivers in class $\mathcal{P}$ whose cluster algebra is not locally acyclic as defined in [24]. Hence the two notions do not coincide.

Remark 4.21. A counting argument shows that each quiver in $\mathcal{Q}_{g,1}$ does not have any sinks or sources. Indeed, the number of arrows is one less than twice the number of vertices and the in-degree and out-degree of any vertex are bounded by 2. From Lemma 4.18 we deduce that there are quivers in $\mathcal{P}$ that are not mutation equivalent to quivers with a sink or a source.

This could be made more formal as follows. Consider the class $\mathcal{P}'$ of quivers defined similarly to the class $\mathcal{P}$, except that we only allow triangular extensions of two quivers where one of them is a point (in analogy with one-point extensions and co-extensions). By construction, any quiver in $\mathcal{P}'$ is mutation equivalent to a quiver with a sink or a source. We therefore get a sequence of strict inclusions

$$\{\text{quivers that are mutation-equivalent to acyclic ones}\} \subsetneqq \mathcal{P}' \subsetneqq \mathcal{P}.$$

The proof of Theorem 4.11 shows that a connected quiver whose mutation class is finite belongs to the class $\mathcal{P}'$ if and only if it belongs to $\mathcal{P}$ and is not a member of a class $\mathcal{Q}_{g,1}$ for some $g > 1$.

5. Explicit construction of some quivers

In order to make our results more concrete, we present explicit constructions of quivers appearing in the main theorem. Such constructions were already presented in our previous work [23, §3.2] yielding quivers having some block structure but with double arrows. Here we present another procedure yielding quivers without double arrows.

Recall that a closed surface of genus $g \geq 1$ can be obtained by taking the fundamental polygon with $4g$ sides labeled $1, 2, 1, 2, \ldots, 2g-1, 2g, 2g-1, 2g$ and identifying sides having the same label (with appropriate orientations that will not be relevant here). Under this identification, all the vertices of the polygon are being mapped to the same point. Thus, any triangulation of this $4g$-gon gives rise to a triangulation of a once-punctured closed surface of genus g by identifying the puncture with that common point and adding the $2g$ arcs corresponding to the distinct sides of the $4g$-gon.

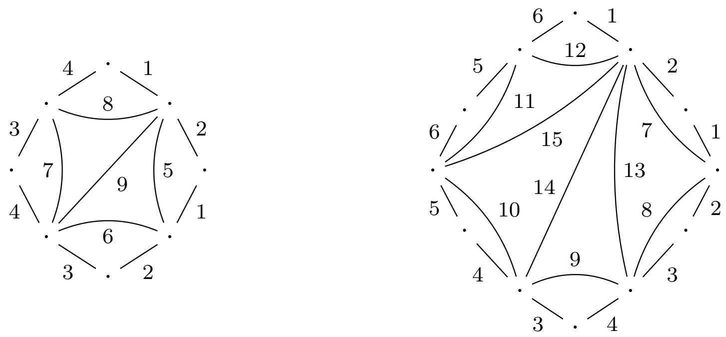
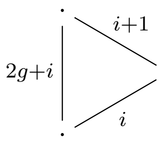


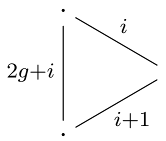
FIGURE 6. Triangulations of closed surfaces of genus g with one puncture, for $g = 2, 3$. Arcs having the same label are identified.

From now on assume that $g \geq 2$. First we add $2g$ new arcs labeled $2g+1, 2g+2, \dots, 4g$ such that each arc $2g+i$ is the side of a triangle as shown below:

(5.1)



$(1 \leq i < 2g \text{ odd})$



$(1 < i \leq 2g \text{ even})$

These new arcs encircle a $2g$-gon inside the fundamental $4g$-gon, and any triangulation of this inner $2g$-gon (consisting of additional $2g-3$ arcs) yields a triangulation of the surface with $6g-3$ arcs. Its adjacency quiver will not contain double arrows, since by our choice of triangles in (5.1), for any two arcs there is at most one triangle having both of them as sides.

In particular, we can choose a triangulation of the inner $2g$-gon whose adjacency quiver is a linearly oriented Dynkin quiver A_{2g-3} and get a triangulation of the once-punctured surface of genus $g \geq 2$ whose adjacency quiver has $6g-3$ vertices numbered $1, 2, \dots, 6g-3$ with the arrows

$$\begin{array}{llll} i \to 2g+i & i \to 2g+(i-1) & 2g+i \to i+1 & (1 \leq i < 2g \text{ odd}) \\ i \to i-1 & i \to i+1 & 2g+i \to i & (1 < i \leq 2g \text{ even}) \end{array}$$

(here $i-1$ and $i+1$ are computed "modulo $2g$" to take values in the range $[1, 2g]$, i.e. if $i = 1$ then $i-1$ is $2g$ and if $i = 2g$ then $i+1$ is 1) together with the arrows

$$\begin{array}{cllc} 2g+2 \to 4g+1 & 4g+1 \to 2g+3 & 4g+1 \to 2g+1 & \\ 2g+i+1 \to 4g+i & 4g+i \to 2g+i+2 & 4g+i \to 4g+i-1 & (2 \leq i \leq 2g-3) \\ 4g-1 \to 4g & 2g+1 \to 2g+2 & 4g \to 6g-3 & \end{array}$$

corresponding to the chosen triangulation of the inner $2g$-gon.

Examples of these triangulations for surfaces of genus 2 and 3 are shown in Figure 6. The corresponding adjacency quivers are those appearing in Figure 1.

References

1. Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 6, 2525–2590.
2. Roland Bacher and Alina Vdovina, *Counting 1-vertex triangulations of oriented surfaces*, Discrete Math. **246** (2002), no. 1-3, 13–27, Formal power series and algebraic combinatorics (Barcelona, 1999).
3. Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52.
4. Arkady Berenstein and Vladimir Retakh, *Noncommutative clusters*, 2013.
5. Thomas Brüstle, Grégoire Dupont, and Matthieu Pérotin, *On maximal green sequences*, Int. Math. Res. Not. (2013).
6. Raika Dehy and Bernhard Keller, *On the combinatorics of rigid objects in 2-Calabi-Yau categories*, Int. Math. Res. Not. (2008), no. 11, Art. ID rnn029, 17.
7. Harm Derksen and Theodore Owen, *New graphs of finite mutation type*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 139, 15.
8. Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations. I. Mutations*, Selecta Math. (N.S.) **14** (2008), no. 1, 59–119.
9. Anna Felikson, Michael Shapiro, and Pavel Tumarkin, *Skew-symmetric cluster algebras of finite mutation type*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 4, 1135–1180.
10. Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Math. **201** (2008), no. 1, 83–146.
11. Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic).
12. ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164.
13. Christof Geiß, Daniel Labardini-Fragoso, and Jan Schröer, *The representation type of Jacobian algebras*, `arXiv:1308.0478`.
14. Bernhard Keller, *On cluster theory and quantum dilogarithm identities*, Representations of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 85–116.
15. ———, *Cluster algebras and derived categories*, Derived categories in algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 123–183.
16. ———, *Quiver mutation and combinatorial DT-invariants*, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), Discrete Math. Theor. Comput. Sci. Proc., AS, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013.
17. Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. **226** (2011), no. 3, 2118–2168.
18. Maxim Kontsevich and Yan Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, `arXiv:0811.2435`.
19. Daniel Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions*, `arXiv:1206.1798`.
20. ———, *Quivers with potentials associated to triangulated surfaces*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 3, 797–839.
21. Sefi Ladkani, *Mutation classes of certain quivers with potentials as derived equivalence classes*, `arXiv:1102.4108`.
22. ———, *On Jacobian algebras from closed surfaces*, `arXiv:1207.3778`.
23. ———, *Which mutation classes of quivers have constant number of arrows?*, `arXiv:1104.0436`.
24. Greg Muller, *Locally acyclic cluster algebras*, Adv. Math. **233** (2013), 207–247.
25. Pierre-Guy Plamondon, *Generic bases for cluster algebras from the cluster category*, Int. Math. Res. Not. (2013), no. 10, 2368–2420.

Mathematical Institute of the University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

URL: `http://www.math.uni-bonn.de/people/sefil`

E-mail address: `sefil@math.uni-bonn.de`