MUTATION CLASSES OF CERTAIN QUIVERS WITH POTENTIALS AS DERIVED EQUIVALENCE CLASSES

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ABSTRACT. We characterize the marked bordered unpunctured oriented surfaces with the property that all the Jacobian algebras of the quivers with potentials arising from their triangulations are derived equivalent. These are either surfaces of genus g with b boundary components and one marked point on each component, or the disc with 4 or 5 points on its boundary.

We show that for each such marked surface, all the quivers in the mutation class have the same number of arrows, and the corresponding Jacobian algebras constitute a complete derived equivalence class of finite-dimensional algebras whose members are connected by sequences of Brenner-Butler tilts. In addition, we provide explicit quivers for each of these classes.

We consider also 10 of the 11 exceptional finite mutation classes of quivers not arising from triangulations of marked surfaces excluding the one of the quiver X_7, and show that all the finite-dimensional Jacobian algebras in such class (for suitable choice of potentials) are derived equivalent only for the classes of the quivers $E_6^{(1,1)}$ and X_6.

1. Motivation and Summary of Results

The Bernstein-Gelfand-Ponomarev reflection [10] is an operation on quivers which carries several interpretations. On a combinatorial level, it takes as inputs an acyclic quiver Q and a vertex s of Q which is a source or a sink in Q and outputs a new quiver $\sigma_s Q$. On an algebraic level, it gives rise to a derived equivalence between the path algebras KQ and $K\sigma_s Q$ over any field K. Moreover, when K is algebraically closed, by a result of Happel [21], the path algebras of two acyclic quivers Q and Q' are derived equivalent if and only if Q' can be obtained from Q by performing a finite number of reflections. It is therefore plausible to extend the scope of this operation beyond path algebras of quivers as well as remove the restriction on the vertex to be a sink or a source.

Indeed, a generalization of the combinatorial aspect of reflection is given by the quiver mutation introduced by Fomin and Zelevinsky [19] in their theory of cluster algebras, allowing to mutate a quiver (without loops and 2-cycles) at any vertex. Furthermore, Derksen, Weyman and Zelevinsky [16] have developed the theory of quivers with potentials (QP) and their mutations. The data of a quiver Q and a potential W on it give rise to the Jacobian algebra $\mathcal{P}(Q,W)$ which can be seen as a generalization of the path algebra in the acyclic case.

However, in such generality the algebraic aspect of reflection as a derived equivalence is usually lost, that is, if $\mu_k(Q,W)$ is the mutation of (Q,W) at a vertex k, the Jacobian algebras $\mathcal{P}(Q,W)$ and $\mathcal{P}(\mu_k(Q,W))$ will not be derived equivalent in general. One

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remedy to this situation, provided by Keller and Yang [26], is to replace the Jacobian algebra by the Ginzburg dg-algebra $\Gamma(Q,W)$ which is negatively graded and 3-Calabi-Yau, and then the derived categories of $\Gamma(Q,W)$ and $\Gamma(\mu_k(Q,W))$ are always equivalent.

Another approach is not to replace the Jacobian algebras, but rather restrict attention to mutation classes of QPs possessing desired properties regarding derived equivalence. Following this approach, in this paper we will present mutation classes $\mathcal{Q}$ of connected quivers with potentials having the following two properties:

(δ_1) For any $(Q,W) \in \mathcal{Q}$, the Jacobian algebra $\mathcal{P}(Q,W)$ is finite-dimensional;

(δ_2) For any $(Q,W),(Q',W') \in \mathcal{Q}$, the Jacobian algebras $\mathcal{P}(Q,W)$ and $\mathcal{P}(Q',W')$ are derived equivalent.

Note that by [16], it is enough to check condition (δ_1) for just one member of $\mathcal{Q}$.

The condition (δ_2) is quite restrictive. For example, it is easy to see that while a mutation class of an acyclic connected quiver Q (necessarily with zero potential) always satisfies the condition (δ_1), it will never satisfy the condition (δ_2) unless Q has at most two vertices. Indeed, the Jacobian algebras of the QP in the mutation class of Q are precisely the cluster-tilted algebras of type Q [12, 13]. When Q has at most two vertices, all mutations are BGP reflections so they are derived equivalences, whereas when Q has three or more vertices there exist cluster-tilted algebras which are not hereditary and hence of infinite global dimension [25], thus not derived equivalent to the path algebra KQ.

On the other hand, there are mutation classes satisfying condition (δ_2) but not (δ_1). Indeed, by [26] the property that the Ginzburg dg-algebra has its cohomology concentrated only in degree zero (and hence is quasi-isomorphic to the Jacobian algebra) is preserved under QP mutation. Mutation classes with this property thus satisfy (δ_2), but as their Jacobian algebras are 3-Calabi-Yau, condition (δ_1) does not hold.

Since we deal with mutation classes, it is natural to ask when a single mutation of QP leads to derived equivalence of the corresponding Jacobian algebras. Possible candidates for tilting complexes, studied in this context by Vitoria [35] and Keller-Yang [26, §6], take the following form. For an algebra Λ given by a quiver with relations and a vertex k without loops, consider the complexes

$$T_k^-(\Lambda) = \big(P_k \xrightarrow{f} \bigoplus_{j \to k} P_j\big) \oplus \big(\bigoplus_{i \neq k} P_i\big), \qquad T_k^+(\Lambda) = \big(\bigoplus_{k \to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i \neq k} P_i\big)$$

where P_i denotes the indecomposable projective corresponding to i, the map f (respectively, g) is induced by all the arrows ending (respectively, starting) at k, and the terms P_i for $i \neq k$ lie in degree 0. Each of these complexes is not always tilting, but when it is, it induces a derived equivalence $\mathcal{D}(\Lambda) \xrightarrow{\sim} \mathcal{D}(\operatorname{End} T_k^-(\Lambda))$ (or $\mathcal{D}(\Lambda) \xrightarrow{\sim} \mathcal{D}(\operatorname{End} T_k^+(\Lambda))$) which generalizes the BGP reflection functor and forms a specific instance of a perverse Morita equivalence [33]. In this case we denote the endomorphism algebra by $\mu_k^-(\Lambda)$ (resp. $\mu_k^+(\Lambda)$) and call it the *negative* (resp. *positive*) mutation of the algebra Λ, see [30]. Note that it might well happen that none of the algebra mutations is defined, or that both of them are defined but not isomorphic.

Given a Jacobian algebra $\mathcal{P}(Q,W)$ and a vertex k, there are now two notions of mutation that we may consider. The first is mutation of quivers with potentials leading to the Jacobian algebra $\mathcal{P}(\mu_k(Q,W))$, whereas the second is given by the negative and positive algebra mutations of $\mathcal{P}(Q, W)$. Roughly speaking, the mutation is *good* if these two notions are compatible. More precisely, the mutation of (Q, W) at the vertex k is *good* if $\mu_k^-(\mathcal{P}(Q, W)) \simeq \mathcal{P}(\mu_k(Q, W))$ or $\mu_k^+(\mathcal{P}(Q, W)) \simeq \mathcal{P}(\mu_k(Q, W))$.

By definition, a good mutation implies the derived equivalence of the corresponding Jacobian algebras, known in the physics literature as Seiberg duality. Hence one is motivated to consider the following property which implies the condition (δ_2).

(δ_3) For any $(Q, W) \in \mathcal{Q}$, the mutation at any vertex k of Q is good. Furthermore, if both algebra mutations of $\mathcal{P}(Q, W)$ at k are defined, they are isomorphic.

It will turn out that all the mutation classes possessing properties (δ_1) and (δ_2) which we will present have also the stronger property (δ_3). Note that for these classes an answer to Question 12.2 in [16] can be given in a very explicit way, namely at any vertex k the (unique) algebra mutation of $\mathcal{P}(Q, W)$ coincides with $\mathcal{P}(\mu_k(Q, W))$.

Motivated by algorithmic applications, e.g. [30, §5.3], we consider the additional finiteness condition:

(δ_4) $\mathcal{Q}$ consists of a finite number of quivers.

According to Felikson, Shapiro and Tumarkin [17], the connected quivers whose mutation class is finite are either those arising from triangulations of bordered oriented surfaces with marked points as introduced by Fomin, Shapiro and Thurston [18], or they are mutation equivalent to one of 11 exceptional quivers, or they are acyclic with 2 vertices and $r \geq 3$ arrows between them.

1.1. **Mutation classes from triangulations of bordered surfaces with marked points.** For quivers arising from such triangulations, potentials have been defined by Labardini-Fragoso [28]. In this paper we further restrict our attention to the case of *no punctures*, that is, the marked points lie on the boundary of the surface. The associated potentials are then sums of oriented 3-cycles (*triangles*) and the resulting Jacobian algebras are the gentle algebras studied by Assem, Brüstle, Charbonneau-Jodoin and Plamondon [2].

We start by characterizing the configurations of marked points yielding mutations classes satisfying the property (δ_2).

Theorem 1. *Let S be a surface with boundary ∂S and $M \subset \partial S$ a finite set of marked points with at least one point in any connected component of ∂S.*

Then the mutation class of quivers with potentials corresponding to the triangulations of (S, M) satisfies condition (δ_2) if and only if either:

- *M contains exactly one point from each connected component of ∂S, or*
- *S is a disc and M consists of 4 or 5 points.*

In view of this theorem, any $g \geq 0$ and $b \geq 1$ such that $(g, b) \neq (0, 1)$ give rise to a mutation class of QPs which we will denote by $\mathcal{Q}_{g,b}$ having the required properties (δ_1) and (δ_2). Namely, take S to be a bordered surface of genus g with b boundary components, M to be a set of b points containing one point from each boundary component and consider all the quivers with potentials arising from the triangulations of (S, M). Note that the case $(0, 1)$ corresponding to the disc is excluded as it has no triangulations.

We denote by $\mathcal{T}_{g,b}$ the class of Jacobian algebras of the QPs in $\mathcal{Q}_{g,b}$. As these are finite-dimensional gentle algebras, one can consider the derived invariant developed by

(g, b)	Vertices	Arrows	Triangles	Size of $\mathcal{Q}_{g,b}$
(0,2)	2	2	0	1
(1,1)	4	7	2	1
(0,3)	6	9	2	6
(1,2)	8	14	4	56
(0,4)	10	16	4	140
(2,1)	10	19	6	105
(1,3)	12	21	6	3236

TABLE 1. The numbers of vertices, arrows and triangles for quivers in $\mathcal{Q}_{g,b}$ and the size of $\mathcal{Q}_{g,b}$ for small values of (g, b).

Avella-Alaminos and Geiss [5] given as a function $\mathbb{N}^2 \to \mathbb{N}$ which we write as a finite sum $\sum_{i=1}^{r} c_i(n_i, m_i)$ where $c_i > 0$ and $(n_1, m_1), \dots, (n_r, m_r)$ are distinct elements of $\mathbb{N}^2$.

Further properties of the mutation class $\mathcal{Q}_{g,b}$ and the corresponding algebras $\mathcal{T}_{g,b}$ are elaborated in the next theorem.

Theorem 2. *Let $(g, b) \neq (0, 1)$ and let $\mathcal{Q}_{g,b}$ be the class of quivers with potentials arising from triangulations of a bordered surface of genus g with b boundary components and b marked points, one at each boundary component. Let $\mathcal{T}_{g,b}$ be the corresponding class of Jacobian algebras.*

(a) *$\mathcal{Q}_{g,b}$ is a mutation class satisfying conditions (δ_1), (δ_2), (δ_3) and (δ_4).*

(b) *Any quiver in $\mathcal{Q}_{g,b}$ has n vertices, e arrows and t triangles, where*

$$n = 6(g-1) + 4b, \qquad e = 2n - b = 12(g-1) + 7b, \qquad t = 4(g-1) + 2b.$$

(c) *For any algebra $\Lambda \in \mathcal{T}_{g,b}$,*
   (i) *the determinant of its Cartan matrix is 2^t,*
   (ii) *its Avella-Alaminos-Geiss invariant is $b(1, 1) + t(0, 3)$.*

(d) *Any two algebras in $\mathcal{T}_{g,b}$ are connected by a sequence of algebra mutations, that is, if $\Lambda, \Lambda' \in \mathcal{T}_{g,b}$ then there exist algebras $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_m = \Lambda'$ in $\mathcal{T}_{g,b}$ and a sequence of vertices $k_1, k_2, \dots, k_m$ such that for any $1 \leq j \leq m$ we have $\Lambda_j = \mu^-_{k_j}(\Lambda_{j-1})$ or $\Lambda_j = \mu^+_{k_j}(\Lambda_{j-1})$.*

(e) *$\mathcal{T}_{g,b}$ is closed under derived equivalence, that is, if $\Lambda \in \mathcal{T}_{g,b}$ and Λ' is derived equivalent to Λ, then also $\Lambda' \in \mathcal{T}_{g,b}$.*

(f) *When $(g, b) \neq (g', b')$, the classes $\mathcal{Q}_{g,b}$ and $\mathcal{Q}_{g',b'}$ (as well as $\mathcal{T}_{g,b}$ and $\mathcal{T}_{g',b'}$) are disjoint.*

In particular we see that for any even positive integer n there exists at least one mutation class of QPs with the properties (δ_1), (δ_2), (δ_3), (δ_4) whose quivers have n vertices. For small number of vertices, details for these classes are given in Table 1 and representative quivers from each class are shown in Figure 1. To make our results more explicit we will outline a procedure to draw these quivers for any (g, b) in Section 3.

The theorem implies that each of the classes $\mathcal{Q}_{g,b}$ (and $\mathcal{T}_{g,b}$) enjoys two remarkable properties. First, all the quivers in $\mathcal{Q}_{g,b}$ have the same number of arrows. This is quite rare for an arbitrary mutation class (which is then necessarily finite). Second, not only that all the algebras in $\mathcal{T}_{g,b}$ are derived equivalent, but $\mathcal{T}_{g,b}$ is closed under derived

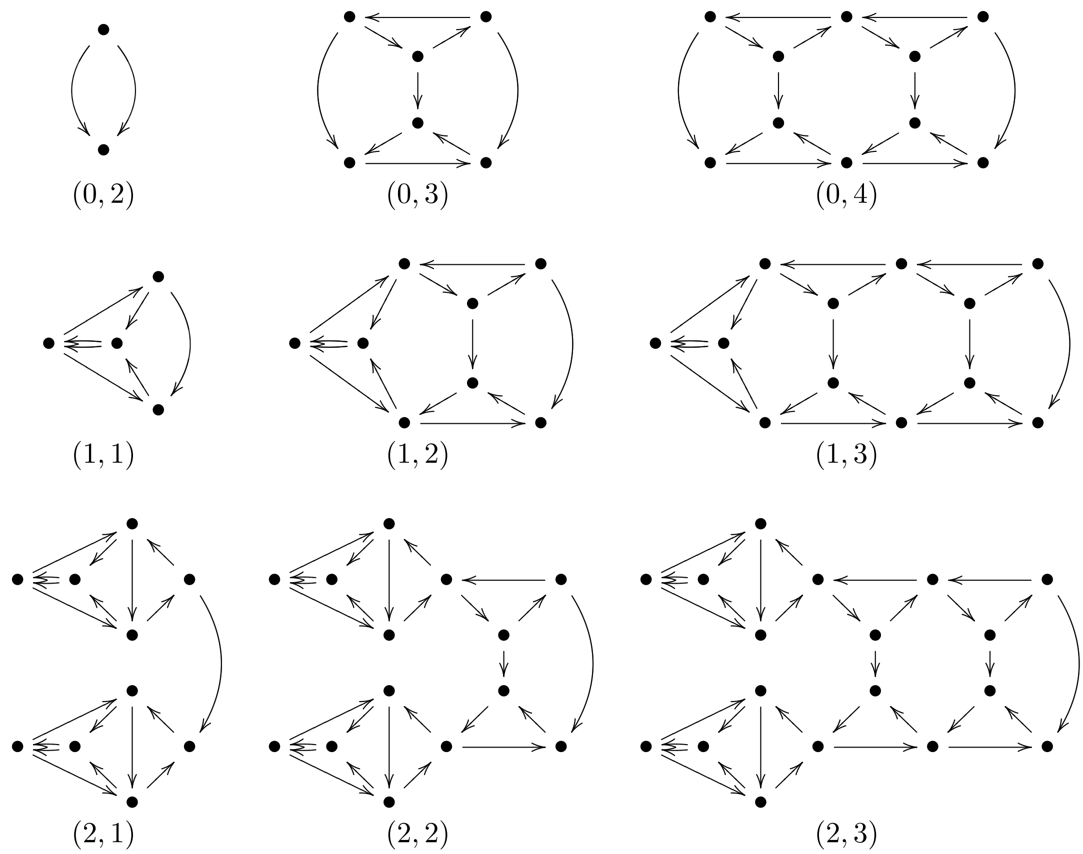


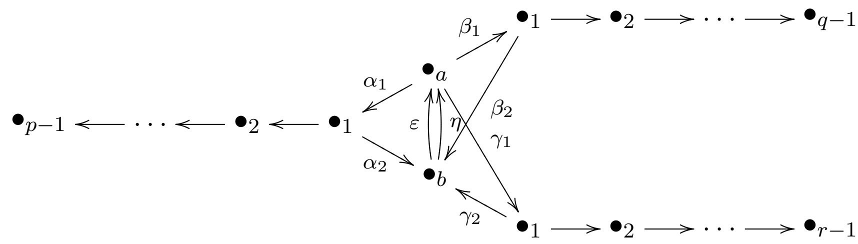
FIGURE 1. Representative quivers in each $\mathcal{Q}_{g,b}$ for small values of (g, b). The potentials are sums of the oriented 3-cycles (chosen such that any arrow belongs to at most one cycle).

equivalence and hence constitutes an entire derived equivalence class of algebras which can be explicitly described using QP and their mutations.

Explicit descriptions of all the algebras derived equivalent to a given algebra Λ are quite rare since by Rickard theorem [32] any endomorphism ring of a tilting complex over Λ gives rise to an algebra derived equivalent to Λ, and it is usually hard to control all the possible tilting complexes. Some notable instances of algebras Λ where such explicit descriptions have been obtained include the path algebras of quivers of Dynkin types A [3] and D [24] as well as affine type $\tilde{A}$ [4]. Other instances are the classes of Brauer tree algebras with fixed numerical parameters (number of edges, multiplicity of the exceptional vertex) which are closed under derived equivalence and moreover any two algebras in a class can be connected by a sequence of algebra mutations, see [27, §5].

1.2. **Exceptional mutation classes.** We now turn our attention to the 11 exceptional quivers whose mutation classes are finite. Of these, E_6, E_7, E_8, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ are acyclic and hence their mutation classes will not satisfy condition (δ_2). In order to deal with $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$, we consider the more general QP given by the quiver

$Q_{p,q,r}$:



and the potential $W_{p,q,r} = \varepsilon(\alpha_1\alpha_2 + \beta_1\beta_2) + \eta(\alpha_1\alpha_2 + \gamma_1\gamma_2)$ for some $p, q, r \geq 2$. The quivers $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ coincide with the quivers $Q_{3,3,3}$, $Q_{2,4,4}$, $Q_{2,3,6}$, respectively.

Similar diagrams appear in singularity theory as Coxeter-Dynkin diagrams [31, §3.9]. In fact, the Jacobian algebra $\mathcal{P}(Q_{p,q,r}, W_{p,q,r})$ can be realized as the endomorphism algebra of a cluster-tilting object in the cluster category of the weighted projective line $\mathbb{X}_{p,q,r}$ [6]. As already observed by Barot and Geiss [7], the quivers $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ can thus be realized by the tubular cluster categories corresponding to the tubular weights $(3,3,3)$, $(2,4,4)$, $(2,3,6)$ respectively. Moreover, the quiver $E_6^{(1,1)}$ which turns out to be of particular interest with regard to derived equivalence is also realized by the stable category of modules over the preprojective algebra of Dynkin type D_4 studied by Geiss-Leclerc-Schröer [20].

Theorem 3. *Let $p, q, r \geq 2$ and let $\mathcal{Q}_{p,q,r}$ be the mutation class of $(Q_{p,q,r}, W_{p,q,r})$ defined above. Then:*

(a) *$\mathcal{Q}_{p,q,r}$ satisfies condition (δ_2) if and only if $(p,q,r) = (3,3,3)$.*

(b) *The class $\mathcal{Q}_{3,3,3}$ corresponding to the quiver $E_6^{(1,1)}$ consists of 49 QPs and satisfies conditions (δ_1), (δ_2), (δ_3), (δ_4).*

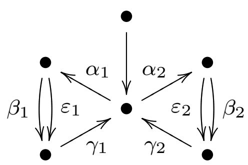
(c) *Furthermore, the Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$ of any $(Q, W) \in \mathcal{Q}_{3,3,3}$ has the following properties:*

  (i) *For any vertex k, one and only one of the algebra mutations $\mu_k^-(\Lambda)$, $\mu_k^+(\Lambda)$ is defined;*

  (ii) *The determinant of the Cartan matrix of Λ is 4 and its Coxeter polynomial is $(x^2+1)^4$.*

Finally we have to deal with the quivers X_6 and X_7 discovered by Derksen and Owen [15]. While for X_7 we could not find a potential whose Jacobian algebra is finite-dimensional, for X_6 we have the following result.

Theorem 4. *Consider the quiver X_6 and the potential W_6 as given below:*



, $$W_6 = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_1\varepsilon_1\gamma_1\alpha_2\varepsilon_2\gamma_2.$$

Then the mutation class of (X_6, W_6) *consists of* 5 *QPs and satisfies conditions* (δ_1), (δ_2), (δ_3), (δ_4). *The Cartan determinant of the corresponding Jacobian algebras is* 4 *and their Coxeter polynomial is* $(x-1)^6$.

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2. On the proofs

2.1. **Good mutations for gentle Jacobian algebras.** Consider a gentle algebra $\Lambda = KQ/I$ given as a quiver with relations over an algebraically closed field. Since the only relations are zero-relations (of length 2), the algebra Λ has a basis consisting of all the non-zero paths.

A *maximal non-zero path* (known also as a non-trivial permitted thread) is a path $\alpha_1\alpha_2\dots\alpha_m$ between some vertices i and j which is non-zero in Λ and is maximal with this property, that is, for any arrow β ending at i and arrow γ starting at j we have $\beta\alpha_1 = 0$ and $\alpha_m\gamma = 0$ in Λ. Any arrow is contained in a unique maximal non-zero path.

Similarly, a *maximal zero path* (known also as a non-trivial forbidden thread) is a path $\alpha_1\alpha_2\dots\alpha_m$ in Q such that $\alpha_s\alpha_{s+1} = 0$ in Λ for all $1 \leq s < m$ which is maximal with this property. For further details we refer the reader to [5].

We start by characterizing when algebra mutations are defined in terms of maximal paths. In addition, the next proposition implies that algebra mutations of gentle algebras without loops coincide with Brenner-Butler tilts [11].

Proposition 2.1. *Let* $\Lambda = KQ/I$ *be gentle and let* k *be a vertex of* Q *without loops.*

(a) $\mu_k^-(\Lambda)$ *is defined if and only if no maximal non-zero path starts at* k.
(b) $\mu_k^+(\Lambda)$ *is defined if and only if no maximal non-zero path ends at* k.
(c) $\mu_k^-(\Lambda)$ *is defined if and only if the Brenner-Butler tilting module of* Λ *at* k *is defined.*
(d) $\mu_k^+(\Lambda)$ *is defined if and only if the Brenner-Butler tilting module of* Λ^{op} *at* k *is defined.*

Proof. Follows from [30, Prop. 2.3]. □

Let (Q, W) be a QP arising from a triangulation of a bordered unpunctured oriented surface, so that its Jacobian algebra is gentle. The next proposition characterizes the vertices k for which mutations are good in terms of their neighborhoods. The *neighborhood* of a vertex k is the full subquiver on the set consisting of k and all vertices which are targets of arrows starting at k or sources of arrows ending at k. By a *triangle* in a quiver arising from a triangulation we mean an oriented 3-cycle bound by radical square zero relations. Such triangles are in bijection with the internal triangles of the triangulation.

Proposition 2.2. *Let* (Q, W) *be a QP arising from a triangulation of a bordered unpunctured oriented surface. Then:*

(a) *The mutation of* (Q, W) *at a vertex is good if and only if its neighborhood is not one of the following*

where $\bullet$ *denotes the vertex.*

(b) *If the mutation of* (Q, W) *at a vertex* k *is good and both the negative and positive algebra mutations of* $\mathcal{P}(Q, W)$ *at* k *are defined, then they are isomorphic.*

(c) *A good mutation preserves the numbers of arrows and triangles in the quiver.*

Proof. To assess whether a mutation is good or not, *a-priori* one needs to check that an algebra mutation of $\mathcal{P}(Q, W)$ at k is defined as well as to verify that it is isomorphic to $\mathcal{P}(\mu_k(Q, W))$. Compared to the former, the latter verification is much harder, so we have developed criteria and algorithms to test for good mutations using only the data whether the relevant algebra mutations are defined or not [30, §5], building on the notion of $\mathcal{D}$-split sequences of Hu and Xi [23].

Note that these criteria were formulated for cluster-tilting objects in 2-Calabi-Yau categories, but by [1] and [12], the Jacobian algebras $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_k(Q, W))$ can be realized as endomorphism algebras of neighboring cluster-tilting objects in a 2-Calabi-Yau triangulated category.

In the case of QP arising from triangulations of unpunctured bordered surfaces, the only relations lie in oriented 3-cycles, so in order to determine whether an algebra mutation is defined at some vertex, it is enough to consider its neighborhood. As the number of possible neighborhoods is finite, they can be checked on a computer. Examples of similar checks can be seen in [9, §3]. $\square$

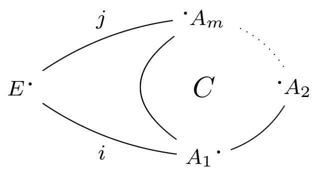
Proposition 2.3. *Let* (Q, W) *be a QP arising from a triangulation of a bordered unpunctured oriented surface. If the mutation at the vertex* k *is not good, then the Jacobian algebras* $\mathcal{P}(Q, W)$ *and* $\mathcal{P}(\mu_k(Q, W))$ *are not derived equivalent.*

Proof. The determinant of the Cartan matrix of such gentle Jacobian algebra is 2^t where t denotes the number of 3-cycles with radical square zero relations, as computed by Holm [22]. The result now follows by observing that mutations which are not good change the number of triangles in the quiver by 1. $\square$

These results allow for a description of the derived equivalence classes of the gentle algebras arising from triangulations of a given marked unpunctured surface in terms of the properties of the corresponding triangulations, generalizing the derived equivalence classifications of cluster-tilted algebras of Dynkin type A [14] and affine type $\tilde{A}$ [8]. This is a subject of further investigations.

2.2. **A necessary condition for derived equivalence.** Let S be a surface of genus $g \geq 0$ with $b \geq 1$ boundary components. Let M be a set of marked points on the boundary, with at least one point on each component.

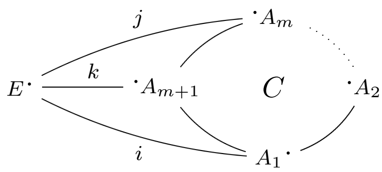
Lemma 2.4. *Assume that* $(g, b) \neq (0, 1)$*. Then there exists a triangulation of* (S, M) *containing, for any boundary component* C *with marked points* $A_1, A_2, \dots, A_m$*, two distinct arcs* i *and* j *starting at* A_1 *and* A_m *respectively having a common endpoint* E *(which* *might be on C) as in the following picture.*



Proof. By induction on the number of marked points in M. When this number is minimal, that is, M contains exactly one point from each component, such triangulations can be explicitly constructed, see Section 3.

Suppose we have constructed such triangulation for M and let $M' = M \cup \{A_{m+1}\}$ where A_{m+1} is a new marked point on a boundary component C. Then we obtain a triangulation for M' with the required property by adding the arc k connecting E and A_{m+1} as in the following picture.

(2.1)



$\square$

Proposition 2.5. *Assume that $(g, b) \neq (0, 1)$. If the mutation class of the QP arising from the triangulations of (S, M) satisfies condition (δ_2), then M contains exactly one point from each boundary component of S.*

Proof. Assume that M has $m+1$ points $A_1, \dots, A_m, A_{m+1}$ on a boundary component C for some $m \geq 1$. By applying Lemma 2.4 for $M \setminus \{A_{m+1}\}$ and then the inductive step in its proof, we get a triangulation having the arcs i, j, k as in (2.1). Let (Q, W) denote the quiver with potential corresponding to this triangulation. Then the neighborhood of k in Q is one of

$$\begin{tikzcd} & \bullet_k \arrow[dr] & \\ \bullet_i \arrow[ur] & & \bullet_j \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} & \bullet_k \arrow[dr] & \\ \bullet_i \arrow[ur] \arrow[rr] & & \bullet_j \end{tikzcd}$$

hence by Proposition 2.2 the mutation of (Q, W) at k is not good. By Proposition 2.3, the Jacobian algebras $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_k(Q, W))$ are not derived equivalent. $\square$

Remark 2.6. If $(g, b) = (0, 1)$, that is, S is a disc, then M has at least 4 points on its boundary. Denoting the number of points by $n+3$ for some $n \geq 1$, it is well known that the quivers arising from the triangulations of (S, M) are precisely those in the mutation class of the Dynkin quiver A_n. Such class will satisfy condition (δ_2) only for $n = 1, 2$.

2.3. **Sufficiency and further properties.** We fix $(g, b) \neq (0, 1)$. Recall that $\mathcal{Q}_{g,b}$ denotes the mutation class consisting of the QP arising from triangulations of a marked surface of genus g with b boundary components and a marked point on each component, and $\mathcal{T}_{g,b}$ denotes the class of the corresponding (gentle) Jacobian algebras.

Lemma 2.7. *A quiver in $\mathcal{Q}_{g,b}$ has n vertices, e arrows and t triangles, where*

$$n = 6(g-1) + 4b, \qquad e = 12(g-1) + 7b, \qquad t = 4(g-1) + 2b.$$

Proof. The claim on the number of vertices follows from [18]. Since each boundary component contains exactly one marked point, any triangulation consists of t internal triangles and b non-internal triangles, one for each boundary component (which becomes one of its sides).

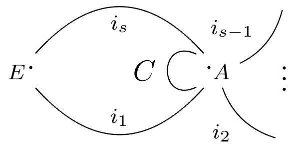
Fix a triangulation corresponding to the quiver. We count in two ways the pairs (γ, Δ) where Δ a triangle and γ is an arc which is one of its sides. On the one hand, for every internal triangle Δ there are 3 such arcs γ and for every non-internal one there are 2 such arcs, giving us a total of $3t + 2b$ pairs. On the other hand, each arc is a side of exactly two triangles so that the total number of such pairs is $2n$. From the equality $2n = 3t + 2b$ we deduce the formula for t.

Finally, each internal triangle gives rise to a 3-cycle in the quiver and hence to three arrows whereas each non-internal one gives rise to one arrow. Thus $e = 3t + b$ and we get the formula for e as well. □

For a gentle algebra Λ, denote its Avella-Alaminos-Geiss derived invariant [5] by $\phi(\Lambda)$.

Lemma 2.8. *Let $\Lambda \in \mathcal{T}_{g,b}$. Then $\phi(\Lambda) = b(1,1) + t(0,3)$ where $t = 4(g-1) + 2b$.*

Proof. Let C be a boundary component and A the marked point on C. In the triangulation corresponding to Λ, let $i_1, i_2, \ldots, i_s$ denote the arcs passing through A traversed in an anti-clockwise order as in the following picture.



Then in the quiver with relations of Λ the path $i_1 \to i_2 \to \cdots \to i_s$ is a maximal non-zero path whereas the arrow $i_1 \to i_s$ induced by the non-internal triangle containing the arcs i_1, i_s is a maximal zero path.

Thus, each component C contributes $(1,1)$ to $\phi(\Lambda)$. In addition, each internal triangle in the triangulation yields an oriented 3-cycle with radical square zero relations, thus contributes $(0,3)$ to $\phi(\Lambda)$. □

Proposition 2.9. *Let Λ be a gentle algebra with $\phi(\Lambda) = b(1,1) + t(0,3)$ for some $b > 0$ and $t \geq 0$. Then:*

(a) *All zero-relations lie in radical square zero oriented 3-cycles.*
(b) *For any vertex v, the subquiver formed by v and the arrows incident to v is not one of the following:*

Proof.

(a) Any zero relation not in a radical square zero oriented cycle would yield a contribution of some (n, m) with $m > 1$ to $\phi(\Lambda)$. In addition, a radical square zero oriented cycle of length m yields a contribution of $(0, m)$.
(b) An inspection of each of the four cases reveals a contribution of (n, m) with $n > 1$ to $\phi(\Lambda)$.

□

Corollary 2.10. *Let Λ be a gentle algebra with $\phi(\Lambda) = b(1,1) + t(0,3)$ for some $b > 0$ and $t \geq 0$. Then $\Lambda \simeq \mathcal{P}(Q, W)$ for some (Q, W) arising from a triangulation of a marked unpunctured surface. Moreover, all the mutations of (Q, W) are good.*

Proof. The first assertion follows from Prop. 2.9(a) and [2, Prop. 2.8]. The second follows from Prop. 2.9(b) together with the characterization of good mutations in Proposition 2.2. □

Proof of Theorem 2.

(a) Since it is known that conditions (δ_1) and (δ_4) hold, we only need to show condition (δ_3) which will then imply (δ_2). Indeed, let $(Q, W) \in \mathcal{Q}_{g,b}$. By Lemma 2.8, $\phi(\mathcal{P}(Q, W)) = b(1, 1) + t(0, 3)$. Now the claim follows by part (b) of Proposition 2.9 together with Proposition 2.2.
(b) Follows from Lemma 2.7.
(c) Follows from [22] and Lemma 2.8.
(d) Λ and Λ' are connected by a sequence of QP mutations which are good by property (δ_3) of $\mathcal{Q}_{g,b}$ and hence induce the required sequence of algebra mutations. Note that these are also Brenner-Butler tilts according to Proposition 2.1.
(e) Let Λ' be derived equivalent to an algebra in $\mathcal{T}_{g,b}$. By a result of Schröer and Zimmermann [34], the class of gentle algebras is closed under derived equivalence and hence Λ' is also gentle with the same derived invariant $\phi(\Lambda') = b(1, 1)+t(0, 3)$.

  By Corollary 2.10, $\Lambda' \simeq \mathcal{P}(Q, W)$ for some (Q, W) arising from a triangulation of a bordered unpunctured surface (S, M) and moreover all mutations are good. Since any mutation $\mu_k(Q, W)$ leads to a derived equivalent Jacobian algebra, applying again Corollary 2.10 we see that all mutations of $\mu_k(Q, W)$ are good as well. Applying mutations repeatedly we deduce that the mutation class of (Q, W) satisfies condition (δ_2), and hence by the results of Section 2.2 (S, M) is either a disc with 4 or 5 marked points or a surface of genus g' with b' boundary components and a marked point on each component. The first case is impossible since $\phi(KA_n) = (n + 1, n - 1)$ [5, §7]. Thus we are in the second case and by Lemma 2.8 we must have $b' = b$ and $g' = g$ so that $\Lambda' \in \mathcal{T}_{g,b}$.
(f) Clear.

□

Proof of Theorem 1. For the implications in one direction, combine Proposition 2.5 and Remark 2.6. The other direction follows from Theorem 2. □

2.4. **The exceptional quivers.** Consider the quiver with potential $(Q_{p,q,r}, W_{p,q,r})$ of Section 1.2. By computing the Cartan matrix of the Jacobian algebra $\mathcal{P}(Q_{2,2,2}, W_{2,2,2})$ we see that its determinant equals 4, hence the same is true for any of the algebras $\mathcal{P}(Q_{p,q,r}, W_{p,q,r})$ obtained by gluing the linear quivers A_{p-1}, A_{q-1} and A_{r-1}.

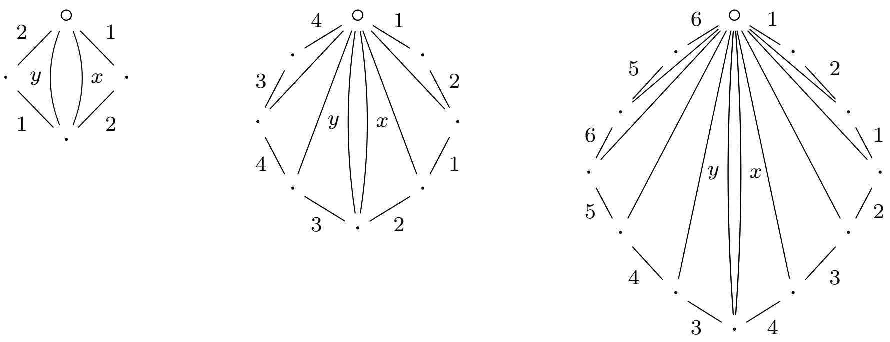
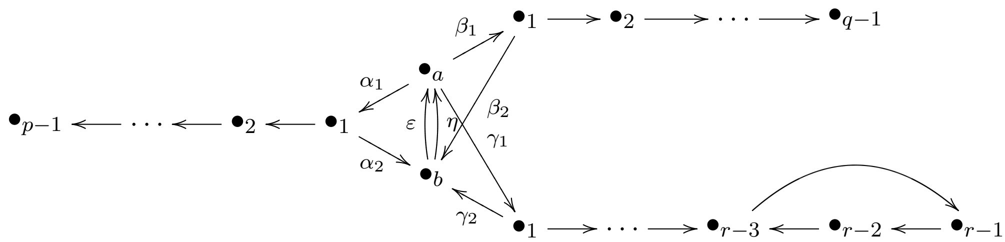


FIGURE 2. Triangulations of bordered surfaces of genus g with one hole, denoted by $\circ$, for $g = 1, 2, 3$. Edges having the same label are identified.

If at least one of p, q, r is greater than 3, say $r > 3$, then by performing mutation at the vertex labeled $r-2$ we get the Jacobian algebra $\mathcal{P}(Q'_{p,q,r}, W'_{p,q,r})$ with the quiver



and potential $W'_{p,q,r} = W_{p,q,r} + \Delta$, where Δ is the new 3-cycle in $Q'_{p,q,r}$. The Cartan matrix of this algebra has determinant 8, as can be seen by direct calculation or invoking [29]. It follows that $\mathcal{P}(Q_{p,q,r}, W_{p,q,r})$ and $\mathcal{P}(Q'_{p,q,r}, W'_{p,q,r})$ are not derived equivalent and hence $\mathcal{Q}_{p,q,r}$ does not satisfy condition (δ_2).

We are left with the case where $p, q, r \leq 3$. The mutation classes $\mathcal{Q}_{2,2,2}$, $\mathcal{Q}_{2,2,3}$ and $\mathcal{Q}_{2,3,3}$ coincide with the mutation classes of the acyclic quivers $\tilde{D}_4$, $\tilde{D}_5$ and $\tilde{E}_6$, respectively, hence they cannot satisfy condition (δ_2) either.

For the class $\mathcal{Q}_{3,3,3}$ as well as for the mutation class of (X_6, W_6), one computes the Jacobian algebras in these finite mutation classes and applies the algorithms developed in [30].

3. QUIVERS

In this section we provide a recipe to produce explicit quivers in each of the classes $\mathcal{Q}_{g,b}$ introduced in Section 1.1.

3.1. **The case $g > 0$ and $b = 1$.** We draw the fundamental polygon with $4g$ edges labeled $1, 2, 1, 2, \dots, 2g-1, 2g, 2g-1, 2g$ corresponding to a surface of genus g, and put the single hole inside the polygon near one of its vertices which serves as the marked point (recall that all vertices of the polygon get identified on the surface). Then there exist triangulations as shown in Figure 2.

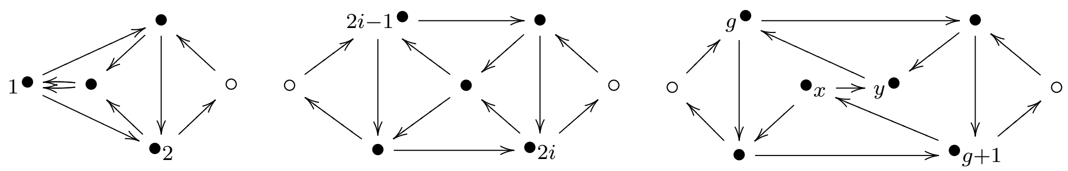


FIGURE 3. Three building blocks for a quiver in $\mathcal{Q}_{g,1}$. Gluing points are marked with $\circ$.

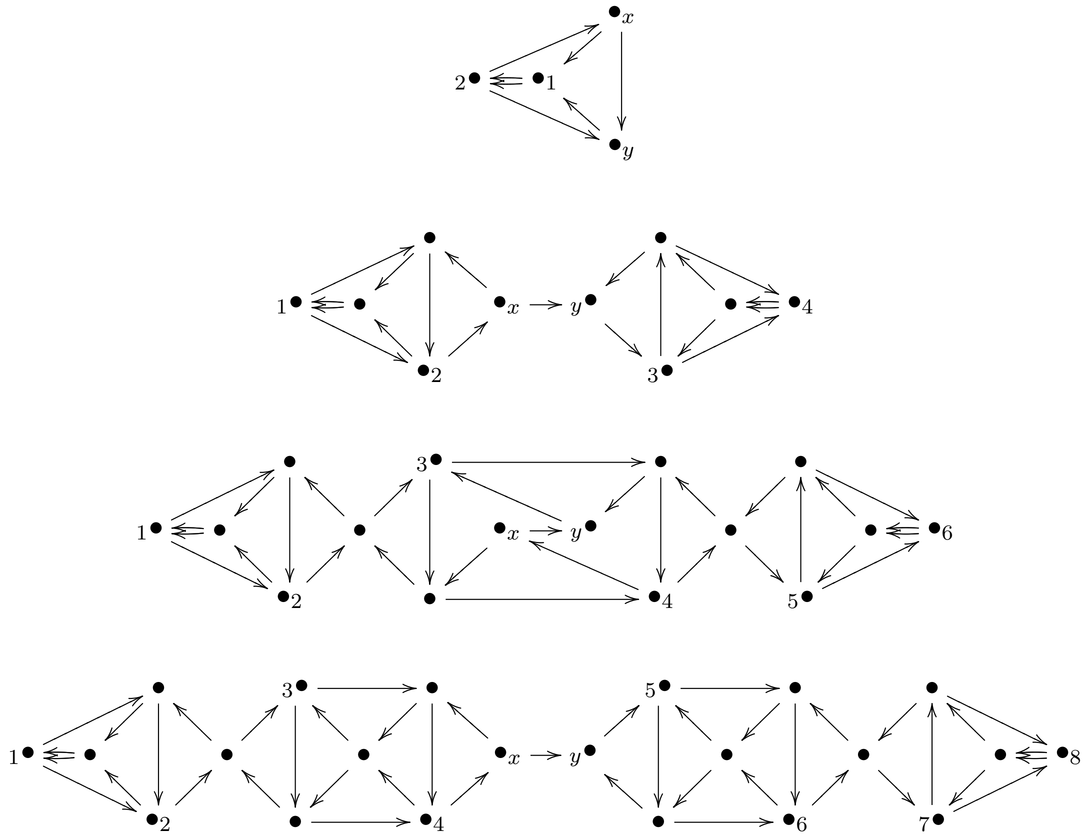


FIGURE 4. Quivers in $\mathcal{Q}_{g,1}$ for $g = 1, 2, 3, 4$.

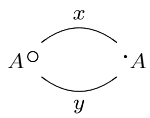
The corresponding quivers are then built by gluing three kinds of building blocks shown in Figure 3, as demonstrated in Figure 4 for $g \leq 4$. Each block corresponds to a pair of consecutive labels $2i - 1, 2i$ of edges in the polygon, and it is constructed by considering all the triangles adjacent to these edges.

The left block corresponds to the initial and terminal pairs of labels $\{1, 2\}$ and $\{2g - 1, 2g\}$. We have drawn the picture only for one pair, as it is symmetric (and isomorphic) for the other pair. The middle one corresponds to all the intermediate pairs $\{2i - 1, 2i\}$

where $1 < i < g$ and $g \neq 2i - 1$, whereas the right one arises from the middle pair $\{g, g+1\}$ when g is odd and involves the non-inner triangle containing the arcs x and y.

3.2. The case $g > 0$ and $b > 1$. The only triangle in the above triangulations which is not inner is the "middle" one

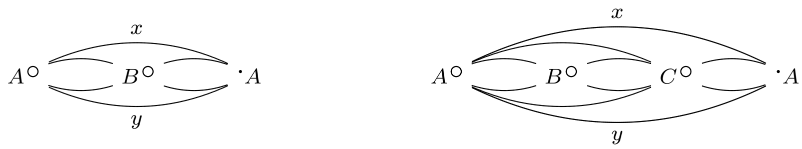
(3.1)



where the marked point is indicated by the letter A and $\circ$ is the hole.

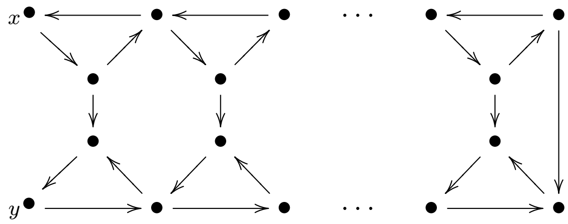
When there is more than one hole, we may arrange the other holes inside this triangle and refine the triangulation to pass through the additional marked points $B, C, \dots$ as in the following pictures

(3.2)



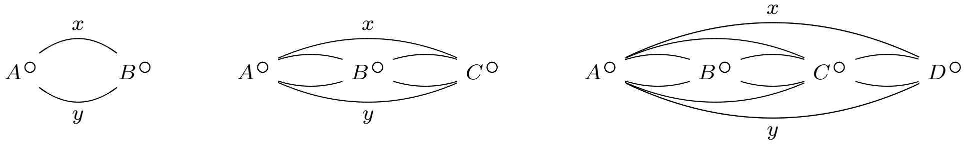
A quiver in $\mathcal{Q}_{g,b}$ is thus obtained from our representative in $\mathcal{Q}_{g,1}$ by replacing the single arrow $\bullet_x \longrightarrow \bullet_y$ corresponding to the picture (3.1) by the quiver

(3.3)



with $2(b-1)$ oriented triangles. The b vertical arrows of this quiver correspond to the b triangles which are not inner in the triangulation.

3.3. The case $g = 0$ and $b > 1$. This case is quite similar to the previous one. In fact, by replacing the copy of the marked point $\cdot_A$ in the pictures (3.1) and (3.2) by an additional hole we get triangulations of the sphere with b holes and b marked points, one at each hole, which are shown below for $b = 2, 3, 4$.



The corresponding quiver in $\mathcal{Q}_{0,b}$ is obtained from the one in (3.3) with $2(b-2)$ oriented triangles by adding an arrow from x to y coming from the "external" triangle consisting of the edges x, y and the boundary of the rightmost hole, see for example the top row of Figure 1.

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