ON DERIVED EQUIVALENCES OF LINES, RECTANGLES AND TRIANGLES

SEFI LADKANI

Abstract. We present a method to construct new tilting complexes from existing ones using tensor products, generalizing a result of Rickard. The endomorphism rings of these complexes are generalized matrix rings that are "componentwise" tensor products, allowing us to obtain many derived equivalences that have not been observed by using previous techniques.

Particular examples include algebras generalizing the ADE-chain related to singularity theory, incidence algebras of posets and certain Auslander algebras or more generally endomorphism algebras of initial preprojective modules over path algebras of quivers. Many of these algebras are fractionally Calabi-Yau and we explicitly compute their CY dimensions. Among the quivers of these algebras one can find shapes of lines, rectangles and triangles.

Introduction

This work deals with derived equivalences of various rings and algebras. One could argue that the question of derived equivalence has been settled by the seminal result of Rickard [24], stating that for two rings R and S, the derived categories of modules $\mathcal{D}(R)$ and $\mathcal{D}(S)$ are equivalent as triangulated categories if and only if there exists a so-called tilting complex $T \in \mathcal{D}(R)$ such that $\mathrm{End}_{\mathcal{D}(R)}(T) \simeq S$.

Nevertheless, when we are faced with two explicit such rings and want to assess their derived equivalence, it is sometimes notoriously difficult to decide whether a tilting complex exists, and if so, to construct it explicitly.

It is therefore mostly beneficial to have at our disposal techniques to construct tilting complexes. Such techniques have been invented in relation with tilting theory of finite-dimensional algebras and modular representation theory, see for example the books [2] and [16] and the many references therein.

We present a systematic way to construct new tilting complexes from existing ones using tensor products, generalizing a result of Rickard in [25]. In our variation, the resulting endomorphism rings are no longer tensor products as in [25], but rather generalized matrix rings that are "componentwise" tensor products. As the class of such rings is much broader, this allows us to obtain many derived equivalences that have not been observed by using previous techniques.

As an example of our methods, we consider the following three families of algebras. The first, visualized pictorially as *"lines"*, consists of algebras

2010 *Mathematics Subject Classification.* 16E35, 16S50, 16G20, 16G70, 18G15.

This work was supported in part by a European Postdoctoral Institute fellowship.

denoted $A(n, r+1)$ which arise from the linear quiver A_n by taking its path algebra (over some field k) modulo the ideal generated by all the paths of a given length $r+1$. The second, consisting of the tensor products (over k) $kA_r \otimes_k kA_n$ with $r, n \geq 1$, can be visualized as *"rectangles"*, as their quivers are rectangles (with fully commutative relations). Members of these two families have been considered in [21] in connection with derived accessible algebras and spectral methods.

It will be a consequence of our results that for any value of r and n, the line $A(r \cdot n, r+1)$ and the rectangle $kA_r \otimes_k kA_n$ are derived equivalent. As, for $r = 2$ and $n \leq 8$, each algebra $A(n, 3)$ is derived equivalent to the corresponding path algebra in the sequence $A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$ known as the ADE chain [1], one may think of these algebras for $r > 2$ as higher ADE-chains.

The relevance of these algebras to other branches of representation theory is demonstrated by the result of Kussin, Lenzing and Meltzer [18], who have recently shown a relation between the "lines" and the categories of coherent sheaves on weighted projective lines, through the notion of the stable category of vector bundles [21]. In addition, as shown in the same work, and also recently in [7], the stable categories of submodules of nilpotent linear maps studied by Ringel and Schmidmeier [27] are equivalent to bounded derived categories of certain "rectangles".

The third family of algebras that we consider arises as the endomorphism rings of certain initial modules in the preprojective component of path algebras of quivers Q without oriented cycles, in a dual fashion to the terminal modules introduced by Geiss, Leclerc and Schröer [10, §2], and includes also many Auslander algebras of Dynkin quivers and their stable analogues. It will be again a consequence of our results that such algebras are derived equivalent to the tensor product of the path algebra of Q with the path algebra kA_r for a suitable value of r. As the quivers of some of these algebras have a shape of *"triangles"* when Q is the linearly oriented diagram A_n, we obtain a derived equivalence between them and the "rectangles".

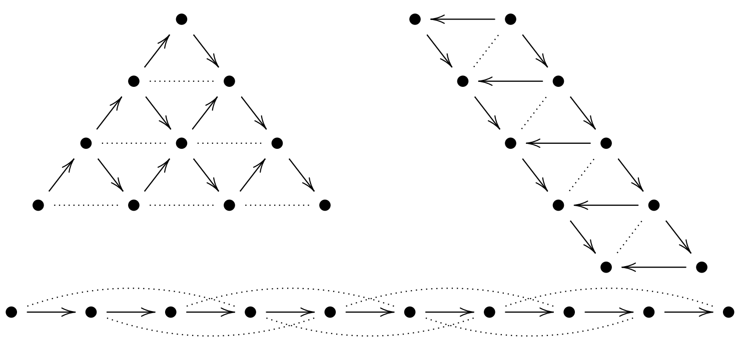

FIGURE 1. Quivers with relations of three derived equivalent algebras: $A(10, 3)$ (*line*), the tensor product $kA_5 \otimes_k kA_2$ (*rectangle*) and the stable Auslander algebra of the path algebra of kA_5 (*triangle*).

One of the simplest examples of derived equivalence between members of these three families is shown in Figure 1, corresponding to the case where $n = 5$ and $r = 2$.

The paper is organized as follows. In Section 1 we present, in detail, our results. The proof of our main Theorem A is given in Section 2. The proofs of its various consequences are the contents of Section 3.

1. The results

1.1. **Tilting complexes for tensor products.** Let A be an algebra over a commutative ring k. As the notion of a tilting complex is central to our investigations, we recall that a complex $T \in \mathcal{D}(A)$ is a *tilting complex* if it has the following two properties:

- T is *exceptional*, that is, $\mathrm{Hom}_{\mathcal{D}(A)}(T, T[r]) = 0$ for all $r \neq 0$;
- T is a *compact generator*, i.e. the smallest triangulated subcategory of $\mathcal{D}(A)$ containing T and closed under isomorphisms and direct summands equals the subcategory of complexes (quasi-)isomorphic to bounded complexes of finitely generated projective A-modules.

Let B be an algebra over k, which is projective as a k-module. Fix a tilting complex U of projective B-modules whose endomorphism algebra is projective as k-module. Then for any tilting complex T of A-modules, a theorem of Rickard [25, Theorem 2.1] tells us that $T \otimes_k U$ is a tilting complex for the tensor product $A \otimes_k B$, with endomorphism algebra which is again a tensor product, namely $\mathrm{End}_{\mathcal{D}(A)} T \otimes_k \mathrm{End}_{\mathcal{D}(B)} U$.

Assume that U decomposes as $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ (the U_i need not be indecomposable). Obviously,

$$T \otimes_k U = (T \otimes_k U_1) \oplus (T \otimes_k U_2) \oplus \cdots \oplus (T \otimes_k U_n).$$

Consider now a variation, where instead of taking just one tilting complex T, we take n of them, say $T_1, T_2, \dots, T_n$, and replace each summand $T \otimes_k U_i$ by $T_i \otimes_k U_i$. Our following Theorem A gives conditions on the T_i which guarantee that the resulting complex is tilting, and moreover computes its endomorphism algebra in terms of those of T_i and U_i.

Theorem A. *Let k be a commutative ring and let A and B be two k-algebras, with B projective as k-module. Let $U_1, U_2, \dots, U_n \in \mathcal{D}(B)$ be complexes bounded from above satisfying:*

(i) *$U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is a tilting complex in $\mathcal{D}(B)$;*
(ii) *The terms U^r of the complex $U = (U^r)$ are projective as k-modules;*
(iii) *The endomorphism k-algebra $\mathrm{End}_{\mathcal{D}(B)}(U)$ is projective as k-module.*

Let $T_1, T_2, \dots, T_n \in \mathcal{D}(A)$ be tilting complexes with the property that for any $1 \leq i, j \leq n$,

$$\mathrm{Hom}_{\mathcal{D}(A)}(T_i, T_j[r]) = 0 \textit{ for any } r \neq 0, \textit{ whenever } \mathrm{Hom}_{\mathcal{D}(B)}(U_i, U_j) \neq 0.$$

Then the complex

$$(T_1 \otimes_k U_1) \oplus (T_2 \otimes_k U_2) \oplus \cdots \oplus (T_n \otimes_k U_n) \tag{1.1}$$

of $(A\otimes_k B)$-modules is a tilting complex in $\mathcal{D}(A\otimes_k B)$, and its endomorphism ring is given by the matrix algebra

$$\begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix} \tag{1.2}$$

where $M_{ij} = \operatorname{Hom}_{\mathcal{D}(A)}(T_j, T_i) \otimes_k \operatorname{Hom}_{\mathcal{D}(B)}(U_j, U_i)$ and the multiplication maps $M_{ij} \otimes_k M_{jl} \to M_{il}$ are given by the obvious compositions.

Note that in general, the matrix ring (1.2) is not a tensor product of two algebras, but rather a "componentwise" tensor product. Namely, the (i,j)-th entry of (1.2) is the tensor product of the corresponding (i,j)-th entries of the rings $\operatorname{End}_{\mathcal{D}(A)}(T_1 \oplus \dots \oplus T_n)$ and $\operatorname{End}_{\mathcal{D}(B)}(U) = \operatorname{End}_{\mathcal{D}(B)}(U_1 \oplus \dots \oplus U_n)$, both viewed as n-by-n matrix rings whose (i,j)-th entry is $\operatorname{Hom}_{\mathcal{D}(A)}(T_j, T_i)$ and $\operatorname{Hom}_{\mathcal{D}(B)}(U_j, U_i)$, respectively. Hence the theorem can produce many algebras which on first sight might look far from being a tensor product, but nevertheless such a product structure is apparently hidden in their derived category.

The purpose of the technical conditions (ii) and (iii) is just to ensure that the functors $- \otimes_k U_i$ are well-behaved. For example, when k is a field, these conditions are automatically satisfied. They are also satisfied in our main applications, where the ring B, the terms of the complex U and the endomorphism ring $\operatorname{End}_{\mathcal{D}(B)}(U)$ are finitely generated and free as $\mathbb{Z}$-modules.

Theorem A provides us with a machinery to produce many new derived equivalences. Namely, by fixing B and the U_i (usually of combinatorial origin), we obtain for any k-algebra A and tilting complexes $T_1, \dots, T_n$ over A satisfying some compatibility conditions, a derived equivalence between $A \otimes_k B$ and an algebra built from the T_i in a prescribed manner depending only on the U_i. This will be demonstrated in the applications, see Sections 1.2 and 1.3 below.

1.2. **Truncated algebras and ADE chains.** For our first application, recall that the quiver A_n ($n \geq 1$) is the following directed graph on n vertices

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_n \ ,$$

and the path algebra kA_n can be viewed as the k-algebra of upper triangular n-by-n matrices with entries in k. For any k-algebra Λ, the k-algebra $\Lambda \otimes_k kA_n$ thus consists of upper triangular n-by-n matrices with entries in Λ, and we denote it by $\mathcal{T}_n(\Lambda)$.

Theorem B. *Let Λ be a ring and let $T_1, \dots, T_n$ be tilting complexes in $\mathcal{D}(\Lambda)$ satisfying $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_{i+1}[r]) = 0$ for all $1 \leq i < n$ and $r \neq 0$.*

Then the following matrix ring, with Hom *and* End *computed in* $\mathcal{D}(\Lambda)$*,*

$$\begin{pmatrix} \operatorname{End} T_1 & 0 & 0 & \dots & 0 \\ \operatorname{Hom}(T_1, T_2) & \operatorname{End} T_2 & 0 & \dots & \vdots \\ 0 & \operatorname{Hom}(T_2, T_3) & \operatorname{End} T_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \operatorname{Hom}(T_{n-1}, T_n) & \operatorname{End} T_n \end{pmatrix}$$

is derived equivalent to $\mathcal{T}_n(\Lambda)$*.*

Recall that a Λ-module is a *tilting module* if, when viewed as a complex (concentrated in degree 0), it is a tilting complex. The above theorem can be reformulated in the language of iterated tilting, as follows.

Corollary 1.1. *Let* Λ *be a ring. Set* $\Lambda_1 = \Lambda$ *and define, for* $1 \le i < n$, $\Lambda_{i+1} = \operatorname{End}_{\Lambda_i}(Q_i)$ *where* Q_i *is a tilting* Λ_i*-module.*

Then the matrix ring

$$\begin{pmatrix} \Lambda_1 & 0 & 0 & \dots & 0 \\ Q_1 & \Lambda_2 & 0 & \dots & \vdots \\ 0 & Q_2 & \Lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & Q_{n-1} & \Lambda_n \end{pmatrix} \tag{1.3}$$

is derived equivalent to $\mathcal{T}_n(\Lambda)$*.*

As an application, we consider the following two families of algebras over a commutative ring k. The first, consists of algebras denoted $A(n, m+1)$ that are the quotient of the path algebra of A_n by the ideal generated by all the paths of length $m+1$. The piecewise hereditary such algebras (when k is an algebraically closed field) have been investigated in [13].

In the spirit of the philosophy of [21], when studying the derived equivalence class of the algebras $A(n, m+1)$, one should not look at a single such algebra each time, but rather consider them in a sequence, that is, fix m and let the number of vertices n vary. For $m = 1$, the algebra $A(n, 2)$ is derived equivalent to the path algebra of A_n (without relations), as follows from [12, (IV, 6.7)]. Fixing $m = 2$ and setting $n = 1, 2, \dots, 8$, the algebras $A(n, 3)$ are derived equivalent to the path algebras of the quivers in the sequence

$$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$$

known as the *ADE-chain* [1], as observed in [21] using spectral techniques.

The ADE-chain occurs also in our second family, which consists of the algebras $kA_m \otimes_k kA_n$. These can be viewed as the incidence algebras of the products $A_m \times A_n$ of two linear orders, or more pictorially as *"rectangles"*, since they can be identified with the path algebras of the fully commutative rectangle with $m \times n$ vertices, see Figure 2. When $m = 1$, we obviously get back the path algebra kA_n. But for $m = 2$, setting $n = 1, 2, 3, 4$, the algebras $kA_2 \otimes_k kA_n$ are derived equivalent to the path algebras of the quivers in the sequence A_2, D_4, E_6, E_8, so that $kA_2 \otimes_k kA_n$ is derived equivalent to $A(2n, 3)$ for $n \le 4$. The following result generalizes these equivalences and puts them into perspective.

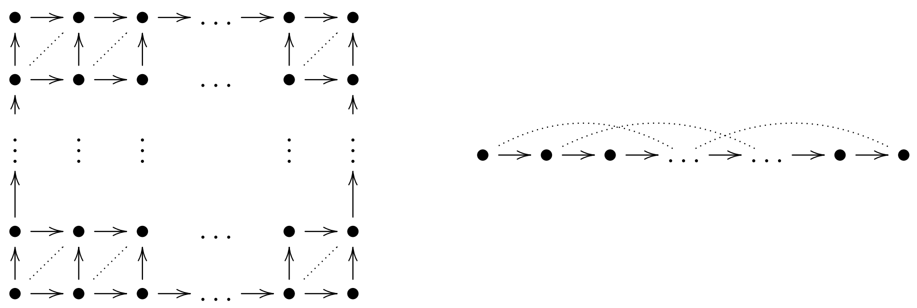

FIGURE 2. Quivers with relations of the "rectangle" (left) and the "line" (right) algebras. The dotted lines indicate the relations.

Corollary 1.2. *Let k be a commutative ring and let $m, n \geq 1$. Then the k-algebras $A(m{\cdot}n, m{+}1)$ and $kA_m \otimes_k kA_n$ are derived equivalent. In particular, $A(m \cdot n, m+1)$ and $A(m \cdot n, n+1)$ are derived equivalent.*

Furthermore, as the derived equivalence actually holds over arbitrary commutative ring k, it can be considered as "universal", depending only on the underlying combinatorics and not on the algebraic data.

For the rest of this subsection, assume that k is a field, and denote by $D = \mathrm{Hom}_k(-, k)$ the usual duality. If Λ is a finite dimensional algebra over k, then $D\Lambda_\Lambda$ is an injective co-generator. When it has finite projective dimension and Λ_Λ has finite injective dimension, the algebra Λ is called *Gorenstein*.

Corollary 1.3. *Let k be a field and let Λ be a finite dimensional k-algebra which is Gorenstein. Then the triangular matrix algebras $\mathcal{T}_n(\Lambda)$ and*

$$\begin{pmatrix} \Lambda & D\Lambda & 0 & \dots & 0 \\ 0 & \Lambda & D\Lambda & \ddots & \vdots \\ \vdots & 0 & \Lambda & \ddots & 0 \\ \vdots & & \ddots & \ddots & D\Lambda \\ 0 & \dots & \dots & 0 & \Lambda \end{pmatrix}$$

are derived equivalent.

The algebra appearing in the corollary is known as the $(n-1)$-*replicated algebra* of Λ and it has connections with $(n-1)$-cluster-categories, see [3]. It is not clear a-priori why replicated algebras of two derived equivalent Gorenstein algebras should also be derived equivalent. But, in view of Corollary 1.3, as tensor products behave well with respect to derived equivalences, this is also the case for replicated algebras:

Corollary 1.4. *The n-replicated algebras of two derived equivalent, finite dimensional, Gorenstein algebras are also derived equivalent.*

1.3. **Endomorphism algebras.** Another application of Theorem A concerns the endomorphism algebras of direct sums of tilting complexes. We start with the following result on "configurations" of tilting complexes according to the poset $\{1 < 2 < \cdots < n\}$. It is possible to formulate analogous results for general posets, but such generality is not needed for our purposes.

Theorem C. *Let Λ be a ring, and let $T_1, \dots, T_n$ be tilting complexes in $\mathcal{D}(\Lambda)$ satisfying $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_j[r]) = 0$ for all $1 \le i < j \le n$ and $r \neq 0$.*

Then the matrix ring (with Hom *and* End *computed in $\mathcal{D}(\Lambda)$)*

$$\begin{pmatrix} \operatorname{End} T_1 & 0 & 0 & \dots & 0 \\ \operatorname{Hom}(T_1,T_2) & \operatorname{End} T_2 & 0 & \dots & 0 \\ \operatorname{Hom}(T_1,T_3) & \operatorname{Hom}(T_2,T_3) & \operatorname{End} T_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}(T_1,T_n) & \operatorname{Hom}(T_2,T_n) & \operatorname{Hom}(T_3,T_n) & \dots & \operatorname{End} T_n \end{pmatrix}$$

is derived equivalent to $\mathcal{T}_n(\Lambda)$.

Remark 1.5. When $n = 2$, Theorems B and C coincide, and in view of the equivalent Corollary 1.1, they state that when Λ is a ring and T is a tilting Λ-module, the two rings

$$\begin{pmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Lambda & 0 \\ {}_{\operatorname{End} T}T_{\Lambda} & \operatorname{End} T \end{pmatrix}$$

are derived equivalent. This is a special case of Theorem 4.5 in [20].

If T is a tilting complex for a ring Λ, then its endomorphism ring is derived equivalent to $\Lambda = \mathcal{T}_1(\Lambda)$. The next corollary shows that more generally, under certain compatibility conditions, if $T_1, \dots, T_n$ are tilting complexes, then the endomorphism ring of their sum is derived equivalent to $\mathcal{T}_n(\Lambda)$.

Corollary 1.6. *Let Λ be a ring and let $T_1, \dots, T_n$ be tilting complexes in $\mathcal{D}(\Lambda)$ such that:*

(i) $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_j[r]) = 0$ *for all $i < j$ and $r \neq 0$,*
(ii) $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(T_j, T_i) = 0$ *for all $i < j$.*

Then $\operatorname{End}_{\mathcal{D}(\Lambda)}(T_1 \oplus \cdots \oplus T_n)$ and $\mathcal{T}_n(\Lambda)$ are derived equivalent.

We deduce the following result concerning endomorphism algebras arising from an auto-equivalence of the derived category. Denote by $\operatorname{per} \Lambda$ the full subcategory of perfect complexes in $\mathcal{D}(\Lambda)$, which consists of all the complexes that are isomorphic in $\mathcal{D}(\Lambda)$ to bounded ones with finitely generated projective terms.

Corollary 1.7. *Let Λ be a ring and $F : \operatorname{per} \Lambda \xrightarrow{\sim} \operatorname{per} \Lambda$ an auto-equivalence. Let $e_1 < e_2 < \cdots < e_n$ be an increasing sequence of integers and denote by $\Delta = \{e_j - e_i \,:\, 1 \le i < j \le n\}$ the set of its (positive) differences. Assume that for any $d \in \Delta$,*

(i) $\mathrm{H}^r(F^d \Lambda) = 0$ *for all $r \neq 0$;*
(ii) $\mathrm{H}^0(F^{-d} \Lambda) = 0$.

Then $\operatorname{End}_{\mathcal{D}(\Lambda)}(F^{e_1}\Lambda \oplus F^{e_2}\Lambda \oplus \cdots \oplus F^{e_n}\Lambda)$ is derived equivalent to $\mathcal{T}_n(\Lambda)$.

In particular, for any sequence of n consecutive integers, the conditions (i) and (ii) need to be checked for $d = 0, 1, \dots, n-1$.

1.4. **Fractionally Calabi-Yau algebras.** Throughout this section, k denotes a field. Let $\mathcal{T}$ be a triangulated k-category with finite dimensional Hom-sets. Recall that a *Serre functor* on $\mathcal{T}$ is an auto-equivalence $\nu : \mathcal{T} \to \mathcal{T}$ with bifunctorial isomorphisms

$$\operatorname{Hom}_{\mathcal{T}}(X, Y) \xrightarrow{\simeq} D \operatorname{Hom}_{\mathcal{T}}(Y, \nu X)$$

for $X, Y \in \mathcal{T}$, see [5].

By a *fraction* we mean a pair (d, e) of integers with $e \geq 1$. By abuse of notation we write a fraction also in the traditional way as $\frac{d}{e}$, but one should keep in mind that the common factors cannot always be canceled. We say that a triangulated k-category $\mathcal{T}$ with shift functor [1] and Serre functor ν is *fractionally Calabi-Yau of dimension* $\frac{d}{e}$ (or $\frac{d}{e}$*-CY* in short) if $\nu^e \simeq [d]$, see [17]. Obviously, being $\frac{d}{e}$-CY implies being $\frac{\ell d}{\ell e}$-CY for any $\ell \geq 1$.

For a finite dimensional algebra Λ over k, denote by $\operatorname{mod} \Lambda$ the category of finite dimensional right Λ-modules and by $\mathcal{D}^b(\operatorname{mod} \Lambda)$ its bounded derived category. We say that Λ is *fractionally CY* if $\mathcal{D}^b(\operatorname{mod} \Lambda)$ is. An interesting class of such algebras is provided by [23, Theorem 4.1], namely the path algebra of any Dynkin quiver is $\frac{h-2}{h}$-CY, where h is the Coxeter number of the corresponding Dynkin diagram. In particular, kA_n is $\frac{n-1}{n+1}$-CY. More generally, a connection between the fractionally CY property and n-representation finiteness is outlined in the recent paper [14].

Fractionally CY algebras behave well with respect to tensor products. Indeed, define the sum of two fractions $\frac{d_1}{e_1}$ and $\frac{d_2}{e_2}$ as the fraction $\frac{d}{e}$, where e is the least common multiple of e_1, e_2 and d is set such that $\frac{d}{e} = \frac{d_1}{e_1} + \frac{d_2}{e_2}$ as rational numbers. Now, if A is α-CY and B is β-CY, then $A \otimes_k B$ is $(\alpha + \beta)$-CY, provided that it has finite global dimension. This happens, for example, when the field k is perfect, or when k is arbitrary and $B = kA_n$. Thus, starting with two such algebras, Theorem A can be used to construct many new fractionally CY algebras, all of CY-dimension $\alpha + \beta$, which are not necessarily tensor products of algebras. A particular case is the following:

Corollary 1.8. *Let Λ be λ-CY. Then $\mathcal{T}_n(\Lambda)$ is $(\lambda + \frac{n-1}{n+1})$-CY, hence the algebras in Theorem B, Corollaries 1.1 (and in particular 1.3), 1.6 and 1.7 are all $(\lambda + \frac{n-1}{n+1})$-CY.*

The next corollary is an immediate consequence of Corollary 1.2. For $m = 2$, it provides another explanation for the CY-dimensions computed in [18].

Corollary 1.9. *The line $A(n \cdot m, m + 1)$ is $(\frac{m-1}{m+1} + \frac{n-1}{n+1})$-CY.*

1.5. **Path algebras of quivers and Auslander algebras.** The results of Section 1.3 can be applied in particular for preprojective components of path algebras of quivers, a setting which we now recall, see e.g. [26]. Let k denote an algebraically closed field throughout this section. Let Q be a (finite) quiver without oriented cycles with set of vertices Q_0, and denote by kQ the path algebra of Q. The indecomposable projectives of kQ are in one-to-one correspondence with the vertices $x \in Q_0$, and we denote them by $\{P_x\}_{x \in Q_0}$.

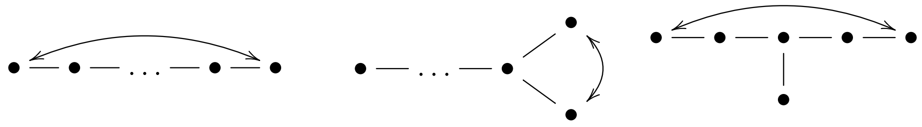

FIGURE 3. Automorphisms of the Dynkin diagrams A_n, D_{2n+1} and E_6.

The category $\mathcal{D}^b(\operatorname{mod} kQ)$ admits a Serre functor ν given by $\nu = - \overset{\mathbf{L}}{\otimes}_{kQ} D(kQ)$, see [12]. The Auslander-Reiten translation τ on $\operatorname{mod} kQ$ can be written as $\tau = H^0 \circ \nu[-1]$, giving a bijection between the non-projective and non-injective indecomposables of $\operatorname{mod} kQ$, with inverse $\tau^- = H^0 \circ \nu^-[1]$ where ν^- is the inverse of ν.

Corollary 1.10. *Let $r \geq 0$ such that $\tau^{-r} P_x \neq 0$ for all $x \in Q_0$. Then the algebras*

$$\operatorname{End}_{kQ}\left(kQ \oplus \tau^{-1} kQ \oplus \cdots \oplus \tau^{-r} kQ\right) \quad \textit{and} \quad kQ \otimes_k kA_{r+1} = \mathcal{T}_{r+1}(kQ)$$

are derived equivalent.

For a finite dimensional k-algebra Λ with a finite number of isomorphism classes of indecomposables in $\operatorname{mod} \Lambda$ (i.e. Λ is of *finite representation type*), the *Auslander algebra* of Λ is defined as $\operatorname{Aus}(\Lambda) = \operatorname{End}_\Lambda(M)$ where M is the sum of the (non-isomorphic) indecomposable Λ-modules, see [4].

By Gabriel [8], the quivers whose path algebra is of finite representation type are precisely those whose underlying graph is a Dynkin diagram of type A, D or E. For such a quiver, let

$$r_x = \max\left\{r \geq 0 \,:\, \tau^{-r} P_x \neq 0\right\}$$

denote the size of the τ^--orbit of P_x in $\operatorname{mod} kQ$. Following [14], we call the algebra kQ *homogeneous* if $r_x = r$ does not depend on x.

By considering the Auslander-Reiten quiver of kQ, as given in [9, §6.5] or [26, §2.2], it is known that kQ is homogeneous if and only if the orientation is invariant under the automorphism of the underlying Dynkin diagram as drawn in Figure 3. Namely, for A_n and E_6 it is the reflection around the vertical axis at the middle, for D_{2n+1} it is the reflection around the horizontal axis, and for the other diagrams it is the identity, see also [14, Prop 3.2]. These diagram automorphisms appear also in the theory of finite twisted groups of Lie type, see e.g. [6, §4.4]. We call such invariant orientations *symmetric*. Note that there are no symmetric orientations on the diagram A_{2n} and any orientation on D_{2n}, E_7 and E_8 is symmetric.

Corollary 1.11. *Let Q be a Dynkin quiver with a symmetric orientation. Then the Auslander algebra* $\operatorname{Aus}(kQ)$ *is derived equivalent to an incidence algebra of a poset as indicated in the following table,*

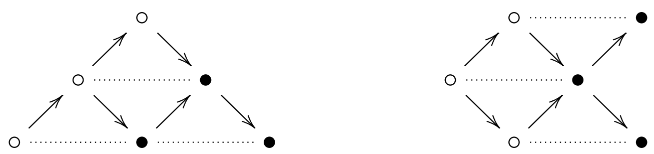

FIGURE 4. Quivers with relations of the Auslander algebras of the diagram A_3 with linear (left) and bipartite (right) orientations.

Diagram	*Orientation*	*Derived type of Auslander algebra*	*CY-dimension*
A_{2n+1}	*symmetric*	$A_{2n+1} \times A_{n+1}$	$\frac{2^{\varepsilon} n(2n+3)}{2^{\varepsilon}(n+1)(n+2)} = \frac{2n}{2n+2} + \frac{n}{n+2}$
D_{2n}	*any*	$D_{2n} \times A_{2n-1}$	$\frac{(2n-2)(4n-1)}{(2n-1)2n} = \frac{2n-2}{2n-1} + \frac{2n-2}{2n}$
D_{2n+1}	*symmetric*	$D_{2n+1} \times A_{2n}$	$\frac{(4n-2)(4n+1)}{4n(2n+1)} = \frac{4n-2}{4n} + \frac{2n-1}{2n+1}$
E_6	*symmetric*	$E_6 \times A_6$	$\frac{130}{84} = \frac{10}{12} + \frac{5}{7}$
E_7	*any*	$E_7 \times A_9$	$\frac{152}{90} = \frac{8}{9} + \frac{8}{10}$
E_8	*any*	$E_8 \times A_{15}$	$\frac{434}{240} = \frac{14}{15} + \frac{14}{16}$

where in the first row ε is 0 *when n is even and* 1 *otherwise. In particular, all these Auslander algebras are fractionally Calabi-Yau.*

One should keep in mind that while the derived category $\mathcal{D}^b(\operatorname{mod} kQ)$ is independent on the orientation of Q, this is no longer true for the derived category $\mathcal{D}^b(\operatorname{mod} \operatorname{Aus}(kQ))$. Indeed, even in the simplest example of the diagram A_3, the Auslander algebra corresponding to the linear orientation is derived equivalent to kD_6, while that of the same diagram but with alternating orientation (which is symmetric) is derived equivalent to kE_6 (or equivalently, $kA_2 \otimes_k kA_3$). The quivers with relations of the two Auslander algebras are shown in Figure 4, where we used white dots to indicate the vertices corresponding to the indecomposable projective kQ-modules.

A nicer picture is obtained if, instead of considering $\operatorname{Aus}(kQ)$, one considers the *stable* Auslander algebra, which in our case can be defined as $\underline{\operatorname{Aus}}(kQ) = \operatorname{End}(M)$, where M is the sum of all indecomposable non-projective kQ-modules. Looking again at Figure 4, we see that while the Auslander algebras are not derived equivalent, when restricting to the stable part (the black dots) we get two derived equivalent algebras of type A_3.

This holds in general, as shown by the theorem below, which follows from recent results of [15], see also [11, Prop. A.2]. An alternative approach will be presented in the forthcoming paper [19].

Theorem. *Let Q and Q' be two orientations of a Dynkin diagram. Then $\underline{\operatorname{Aus}}(kQ)$ and $\underline{\operatorname{Aus}}(kQ')$ are derived equivalent.*

Using this result, we deduce the following.

Corollary 1.12. *Let Q be a Dynkin quiver whose underlying graph is not A_{2n}. Then the stable Auslander algebra of kQ is derived equivalent to an incidence algebra of a poset as indicated in the following table.*

Diagram	*Derived type of stable Auslander algebra*	*CY-dimension*
A_{2n+1}	$A_{2n+1} \times A_n$	$\frac{4n-2}{2n+2} = \frac{2n}{2n+2} + \frac{n-1}{n+1}$
D_{2n}	$D_{2n} \times A_{2n-2}$	$\frac{4n-5}{2n-1} = \frac{2n-2}{2n-1} + \frac{2n-3}{2n-1}$
D_{2n+1}	$D_{2n+1} \times A_{2n-1}$	$\frac{8n-6}{4n} = \frac{4n-2}{4n} + \frac{2n-2}{2n}$
E_6	$E_6 \times A_5$	$\frac{18}{12} = \frac{10}{12} + \frac{4}{6}$
E_7	$E_7 \times A_8$	$\frac{15}{9} = \frac{8}{9} + \frac{7}{9}$
E_8	$E_8 \times A_{14}$	$\frac{27}{15} = \frac{14}{15} + \frac{13}{15}$

In particular, all these stable Auslander algebras are fractionally Calabi-Yau.

Finally, we obtain the following connection between some "triangles" and "rectangles". An example (for $n = 3$) is shown in Figure 5.

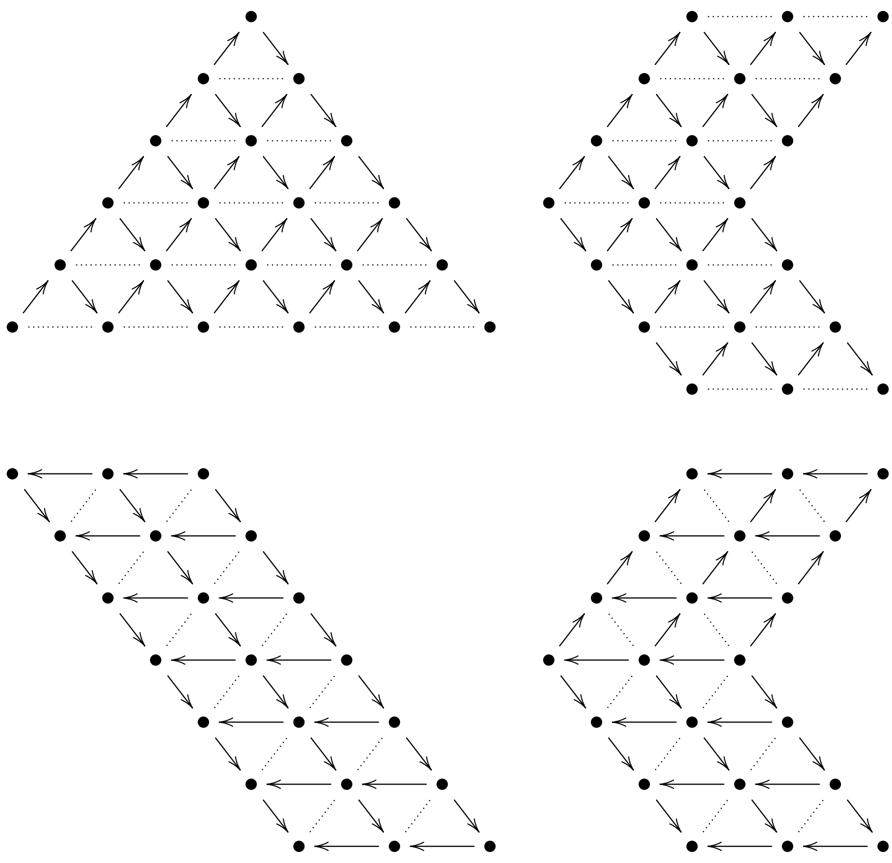

FIGURE 5. Quivers with relations of four derived equivalent algebras. Starting at the upper left and going clockwise: the triangle $\underline{\mathrm{Aus}}(kA_7)$ with linear orientation on A_7; $\underline{\mathrm{Aus}}(kA_7)$ with a symmetric orientation on A_7; $kA_7 \otimes_k kA_3$ with a symmetric orientation (on A_7); and the rectangle $kA_7 \otimes_k kA_3$ (with linear orientation).

Corollary 1.13. *The Auslander algebra corresponding to the linear orientation on A_{2n} is derived equivalent to $kA_{2n+1} \otimes_k kA_n$.*

Remark 1.14. It turns out that the fractions appearing in Corollaries 1.11 and 1.12 as the CY-dimensions of the corresponding (stable) Auslander algebras are the best possible, i.e. common factors cannot be canceled. This can be seen for example by considering the Coxeter polynomials of the two Dynkin quivers comprising the derived type (see [21, §1.1]) and examining the orders of products of pairs of roots, one from each polynomial.

2. Proof of Theorem A

In this section we prove Theorem A, using calculations involving tensor products of complexes. For the convenience of the reader, we give a rather detailed account, see also [16, Chapter 6] for a similar treatment.

Let k be a commutative ring and let A be a k-algebra. Our conventions are that modules are right modules and $\otimes$ (without subscript) will always mean tensor product over k. Denote by $\operatorname{Mod} A$ the category of all (right) A-modules, and by $\operatorname{Proj} A$ and $\operatorname{proj} A$ the full subcategories of projective and finitely-generated projective modules, respectively.

Let B be another k-algebra. If M is an A-module and N is a B-module, then $M \otimes N$ is a module over the k-algebra $A \otimes B$.

Lemma 2.1. *Let $P \in \operatorname{proj} A$ and $Q \in \operatorname{proj} B$. Then*

(a) $P \otimes Q \in \operatorname{proj}(A \otimes B)$.
(b) *For any $M \in \operatorname{Mod} A$ and $N \in \operatorname{Mod} B$, we have a natural isomorphism*

$$\operatorname{Hom}_A(P, M) \otimes \operatorname{Hom}_B(Q, N) \to \operatorname{Hom}_{A\otimes B}(P \otimes Q, M \otimes N).$$

Proof. Both claims are true for $P = A$ and $Q = B$, and pass to finite sums and direct summands. □

For an additive category $\mathcal{A}$, denote by $\mathcal{C}(\mathcal{A})$ the category of complexes over $\mathcal{A}$, by $\mathcal{C}^-(\mathcal{A})$ its full subcategory consisting of complexes bounded above and by $\mathcal{C}^b(\mathcal{A})$ the subcategory of bounded complexes. We abbreviate $\mathcal{C}(\operatorname{Mod} A)$, $\mathcal{C}^-(\operatorname{Mod} A)$, $\mathcal{C}^b(\operatorname{Mod} A)$ by $\mathcal{C}(A)$, $\mathcal{C}^-(A)$, $\mathcal{C}^b(A)$ and denote by $\mathcal{D}(A)$, $\mathcal{D}^-(A)$, $\mathcal{D}^b(A)$ the corresponding derived categories.

Recall [22, §V.9] that if $K = (K^p)$ and $L = (L^q)$ are complexes of k-modules, their *tensor product* $K \otimes L$ is the complex whose terms are

$$(K \otimes L)^n = \bigoplus_{p+q=n} K^p \otimes L^q$$

with the differential defined on each piece $K^p \otimes L^q$ by

$$d_{K\otimes L}(x \otimes y) = d_K(x) \otimes y + (-1)^p x \otimes d_L(y).$$

If $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(B)$, then $X \otimes Y \in \mathcal{C}(A \otimes B)$. When $Y \in \mathcal{C}^-(B)$ and the terms of Y are projective as k-modules, the functor $- \otimes Y : \mathcal{C}(A) \to \mathcal{C}(A \otimes B)$ is exact, hence induces a triangulated functor from $\mathcal{D}(A)$ to $\mathcal{D}(A \otimes B)$, which restricts to $\mathcal{D}^-(A) \to \mathcal{D}^-(A \otimes B)$.

If T and X are complexes of A-modules, the complex $\mathrm{Hom}_A^\bullet(T,X)$ is defined by

$$\mathrm{Hom}_A^\bullet(T,X)^n = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_A(T^p, X^{n+p}), \tag{2.1}$$

with the differential taking $f = (f^p : T^p \to X^{n+p})_{p\in\mathbb{Z}}$ to df, whose p-th component is

$$(df)^p = d_X^{n+p} f^p - (-1)^n f^{p+1} d_T^p.$$

We have $\mathrm{H}^r\big(\mathrm{Hom}_A^\bullet(T,X)\big) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(T, X[r])$ where $\mathcal{K}(A)$ is the homotopy category of complexes.

Lemma 2.2. *Let $P \in \mathcal{C}^b(\mathrm{proj}\, A)$, $Q \in \mathcal{C}^b(\mathrm{proj}\, B)$ and $X \in \mathcal{C}^-(\mathrm{Mod}\, A)$, $Y \in \mathcal{C}^-(\mathrm{Mod}\, B)$. Then the natural map*

$$\mathrm{Hom}_A^\bullet(P,X) \otimes \mathrm{Hom}_B^\bullet(Q,Y) \to \mathrm{Hom}_{A\otimes B}^\bullet(P\otimes Q, X\otimes Y) \tag{2.2}$$

is an isomorphism.

Proof. Since P, Q are bounded and X, Y are bounded above, the complexes in both sides of (2.2) are bounded above, and the product in (2.1) can be replaced by a direct sum.

The n-th term of the complex in the left hand side of (2.2) is the sum

$$\bigoplus_{i,j,p,q=n-p} \mathrm{Hom}_A(P^i, X^{p+i}) \otimes \mathrm{Hom}_B(Q^j, Y^{q+j})$$

with the differential

$$d(f\otimes g) = d_X f \otimes g - (-1)^p f d_P \otimes g + (-1)^p f \otimes d_Y g - (-1)^{p+q} f \otimes g d_Q$$

for $f \in \mathrm{Hom}_A(P^i, X^{p+i})$, $g \in \mathrm{Hom}_B(Q^j, Y^{q+j})$.

Similarly, the n-th term of the complex in the right hand side of (2.2) is the sum

$$\bigoplus_{i,j,p,q=n-p} \mathrm{Hom}_A(P^i \otimes Q^j, X^{p+i} \otimes Y^{q+j})$$

with the differential

$$\begin{aligned} d(f\otimes g) &= d_{X\otimes Y}(f\otimes g) - (-1)^{p+q}(f\otimes g) d_{P\otimes Q} \\ &= d_X f \otimes g + (-1)^{p+i} f \otimes d_Y g - (-1)^{p+q}\big(f d_P \otimes g + (-1)^i f \otimes g d_Q\big) \end{aligned}$$

for $f\otimes g \in \mathrm{Hom}_{A\otimes B}(P^i \otimes Q^j, X^{p+i} \otimes Y^{q+j})$.

The two complexes are now isomorphic by using the isomorphism of Lemma 2.1 and mapping $f\otimes g \mapsto (-1)^{iq} f \otimes g$, where $f \in \mathrm{Hom}_A(P^i, X^{p+i})$ and $g \in \mathrm{Hom}_B(Q^j, Y^{q+j})$. $\square$

Lemma 2.3. *Let $K \in \mathcal{C}^-(\mathrm{Mod}\, k)$, and let $g : L' \to L$ be a map of complexes in $\mathcal{C}^-(\mathrm{Proj}\, k)$ which is a quasi-isomorphism. Then*

$$1_K \otimes g : K \otimes L' \to K \otimes L$$

is also a quasi-isomorphism.

Proof. Choose a projective resolution of K, that is, a quasi-isomorphism $f : K' \to K$ with $K' \in \mathcal{C}^-(\operatorname{Proj} k)$. Then in the following commutative square

$$\begin{tikzcd} K' \otimes L' \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"] & K' \otimes L \arrow[d, "f\otimes 1"] \\ K \otimes L' \arrow[r, "1\otimes g"] & K \otimes L \end{tikzcd}$$

the vertical maps and the top horizontal map are quasi-isomorphisms. It follows that the bottom horizontal map is also a quasi-isomorphism. □

Corollary 2.4. *Let $X, X' \in \mathcal{C}^-(\operatorname{Mod} A)$ and let $Y' \to Y$ be a map of complexes in $\mathcal{C}^-(\operatorname{Mod} B)$ which is a quasi-isomorphism. Assume that the terms of Y and Y' are projective as k-modules. If $X \simeq X'$ in $\mathcal{D}(A)$, then $X \otimes Y \simeq X' \otimes Y'$ in $\mathcal{D}(A \otimes B)$.*

Proof. We have $X' \otimes Y' \simeq X \otimes Y' \xrightarrow{\simeq} X \otimes Y$, where the first isomorphism follows from the exactness of $- \otimes Y'$ and the second from Lemma 2.3. □

The following lemma is a variation on the Künneth formula [22, §V.10], relating the cohomology of a tensor product of complexes with the tensor product of the cohomologies.

Lemma 2.5. *Let $K \in \mathcal{C}^-(\operatorname{Mod} k)$ and $L \in \mathcal{C}(\operatorname{Proj} k)$. Assume that the cohomology of L is concentrated in degree 0 and that $\mathrm{H}^0(L)$ is projective. Then the natural map*

$$\mathrm{H}^\bullet(K) \otimes \mathrm{H}^\bullet(L) \to \mathrm{H}^\bullet(K \otimes L)$$

is an isomorphism.

Proof. Let d_L denote the differential of L. Then from the exact sequence

$$\ker d_L^0 \to \mathrm{H}^0(L) \to 0$$

and the assumption that $\mathrm{H}^0(L)$ is projective, we deduce the existence of a map $s : \mathrm{H}^0(L) \to \ker d_L^0 \hookrightarrow L^0$, such that the map of complexes

$$\begin{tikzcd} L' : & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{H}^0(L) \arrow[r] \arrow[d, "s"] & 0 \arrow[r] \arrow[d] & \dots \\ L : & \dots \arrow[r] & L^{-1} \arrow[r] & L^0 \arrow[r] & L^1 \arrow[r] & \dots \end{tikzcd}$$

is a quasi-isomorphism.

By Lemma 2.3, the induced map $K \otimes L' \to K \otimes L$ is a quasi-isomorphism, hence

$$\begin{aligned} \mathrm{H}^\bullet(K) \otimes \mathrm{H}^\bullet(L) &= \mathrm{H}^\bullet(K) \otimes \mathrm{H}^0(L) \simeq \mathrm{H}^\bullet(K \otimes \mathrm{H}^0(L)) \\ &= \mathrm{H}^\bullet(K \otimes L') \simeq \mathrm{H}^\bullet(K \otimes L) \end{aligned}$$

where the exactness of $- \otimes \mathrm{H}^0(L)$ implies the isomorphism at the first row. □

Lemma 2.6. *Assume that B is projective as a k-module and let $Q \in \operatorname{proj} B$. Then:*

(a) *$Q \in \operatorname{Proj} k$.*

(b) $\operatorname{Hom}_B(Q,N) \in \operatorname{Proj} k$ *for any* $N \in \operatorname{Mod} B$ *which is projective as a* k*-module.*

Proof. Both claims are true for $Q = B$. Now the first passes to (arbitrary) sums and direct summands, while the second passes to finite sums and direct summands. □

For a complex $T \in \mathcal{D}(A)$, let $\langle \operatorname{add} T \rangle$ denote the smallest triangulated subcategory of $\mathcal{D}(A)$ closed under isomorphisms and taking direct summands. We say that T *generates* $\langle \operatorname{add} T \rangle$. In particular, A generates $\operatorname{per} A$, the full subcategory of *perfect complexes* in $\mathcal{D}(A)$, which consists of all the complexes that are isomorphic (in $\mathcal{D}(A)$) to bounded ones with finitely generated projective terms.

Proposition 2.7. *Assume that B is projective as a k-module. Let $T \in \operatorname{per} A$, $U \in \operatorname{per} B$, $T' \in \mathcal{D}^-(A)$ and $U' \in \mathcal{D}^-(B)$. Assume that:*

(i) $\operatorname{Hom}_{\mathcal{D}(B)}(U, U'[r]) = 0$ *for all* $r \neq 0$,
(ii) *The terms of U and U' are projective as k-modules,*
(iii) $\operatorname{Hom}_{\mathcal{D}(B)}(U, U')$ *is projective as k-module,*

Then $T \otimes U \in \operatorname{per}(A \otimes B)$ and for any $r \in \mathbb{Z}$,

$$\operatorname{Hom}_{\mathcal{D}(A\otimes B)}\big(T \otimes U, (T' \otimes U')[r]\big) \simeq \operatorname{Hom}_{\mathcal{D}(A)}(T, T'[r]) \otimes \operatorname{Hom}_{\mathcal{D}(B)}(U, U')$$

Proof. There exist $P \in \mathcal{C}^b(\operatorname{proj} A)$ which is isomorphic to T in $\mathcal{D}(A)$ and $Q \in \mathcal{C}^b(\operatorname{proj} B)$ which is isomorphic to U in $\mathcal{D}(B)$. By Lemma 2.1, $P \otimes Q \in \mathcal{C}^b(\operatorname{proj}(A \otimes B))$. Since the isomorphism between Q and U in $\mathcal{D}(B)$ can be represented by a "roof" of quasi-isomorphisms in $\mathcal{C}(B)$

$$\begin{tikzcd} & \widetilde{Q} \arrow[dl] \arrow[dr] & \\ Q & & U \end{tikzcd}$$

with $\widetilde{Q} \in \mathcal{C}^-(\operatorname{Proj} B)$, we deduce from Corollary 2.4 that $P \otimes Q$ is isomorphic to $T \otimes U$ in $\mathcal{D}(A \otimes B)$, as Q, $\widetilde{Q}$ and U are in $\mathcal{C}^-(\operatorname{Proj} k)$ by Lemma 2.6(a) and the assumption (ii).

Therefore $T \otimes U$ is perfect and

$$\operatorname{Hom}_{\mathcal{D}(A\otimes B)}\big(T \otimes U, (T' \otimes U')[r]\big) = \operatorname{Hom}_{\mathcal{D}(A\otimes B)}\big(P \otimes Q, (T' \otimes U')[r]\big) = \operatorname{Hom}_{\mathcal{K}(A\otimes B)}\big(P \otimes Q, (T' \otimes U')[r]\big) = \mathrm{H}^r\big(\operatorname{Hom}^\bullet_{A\otimes B}(P \otimes Q, T' \otimes U')\big)$$

where $\mathcal{K}(A \otimes B)$ is the homotopy category of $\mathcal{C}(A \otimes B)$.

Now, by Lemma 2.2,

$$\operatorname{Hom}^\bullet_{A\otimes B}(P \otimes Q, T' \otimes U') \simeq \operatorname{Hom}^\bullet_A(P, T') \otimes \operatorname{Hom}^\bullet_B(Q, U').$$

Consider the complex of k-modules $\operatorname{Hom}^\bullet_B(Q, U')$. By Lemma 2.6(b), its terms are projective. Moreover, as

$$\mathrm{H}^r\big(\operatorname{Hom}^\bullet_B(Q, U')\big) \simeq \operatorname{Hom}_{\mathcal{D}(B)}(U, U'[r]),$$

hypotheses (i) and (iii) imply that its cohomology is concentrated in degree zero and the zeroth cohomology is projective as k-module. Thus, by

Lemma 2.5,

$$\begin{aligned}\mathrm{H}^r\big(\mathrm{Hom}^\bullet_A(P,T')\otimes \mathrm{Hom}^\bullet_B(Q,U')\big) &\simeq \mathrm{H}^r\big(\mathrm{Hom}^\bullet_A(P,T')\big)\otimes \mathrm{H}^0\big(\mathrm{Hom}^\bullet_B(Q,U')\big)\\ &\simeq \mathrm{Hom}_{\mathcal{D}(A)}(T,T'[r])\otimes \mathrm{Hom}_{\mathcal{D}(B)}(U,U')\end{aligned}$$

and the claim follows. □

Lemma 2.8. *Assume that B is projective as a k-module. Let $U = U_1\oplus\cdots\oplus U_n$ be a complex in $\mathcal{D}^-(B)$ that generates* $\operatorname{per} B$ *whose terms are projective as k-modules, and let $T_1,\dots,T_n$ be complexes in* $\operatorname{per} A$ *such that each T_i generates* $\operatorname{per} A$. *Then the complex $(T_1\otimes U_1)\oplus(T_2\otimes U_2)\oplus\cdots\oplus(T_n\otimes U_n)$ generates* $\operatorname{per}(A\otimes B)$.

Proof. Each of the summands $T_i\otimes U_i$ lies in $\operatorname{per}(A\otimes B)$ by Proposition 2.7. Moreover, by the argument at the beginning of the proof of that proposition, we may (and will) assume that T_i and U_i are bounded complexes of finitely generated projective modules.

Now $A\in\langle\operatorname{add} T_i\rangle$, since T_i generates $\operatorname{per} A$, hence $A\otimes U_i\in\langle\operatorname{add}(T_i\otimes U_i)\rangle$, as $-\otimes U_i$ is exact and commutes with taking direct summands.

Therefore, it is enough to show that in $\mathcal{K}^b(\operatorname{proj}(A\otimes B))$,

$$A\otimes B\in\langle\operatorname{add}(A\otimes U_1),\dots,\operatorname{add}(A\otimes U_n)\rangle.$$

This follows from the facts that $B\in\langle\operatorname{add}(U_1\oplus\cdots\oplus U_n)\rangle$, as U generates $\operatorname{per} B$, and $A\otimes - : \mathcal{K}^b(\operatorname{proj} B)\to\mathcal{K}^b(\operatorname{proj}(A\otimes B))$ is an exact functor. □

Now we have all the ingredients to complete the proof of Theorem A.

Proof of Theorem A. Since the terms of U_i are projective as k-modules and $T_i\in\operatorname{per} A$, we may assume that $T_i\in\mathcal{D}^-(A)$ by replacing it with a suitable quasi-isomorphic complex.

The claim that the complex $V=\bigoplus_{i=1}^n(T_i\otimes U_i)$ is perfect, exceptional and that its endomorphism algebra is isomorphic to the one in (1.2), is a direct consequence of Proposition 2.7. Finally, by Lemma 2.8, V generates $\operatorname{per}(A\otimes B)$. □

3. Proof of Theorems B, C and their applications

3.1. **The path algebra of A_n.** We quickly review some relevant facts concerning the quiver A_n. Let R be a ring, and denote by RA_n the ring of upper-triangular n-by-n matrices with entries in R. More abstractly, it is the free R-module on the basis $\{e_{ij}\}_{1\le i\le j\le n}$, with the multiplication given by the rules $re_{ij}=e_{ij}r$ for $r\in R$ and $e_{ij}e_{pq}$ equals e_{iq} if $p=j$ and zero otherwise. When R is commutative, RA_n is known as the *path algebra* of A_n over R, and also as the incidence algebra over R of the linearly ordered poset on $\{1,2,\dots,n\}$.

The category of (right) RA_n-modules is equivalent to the category of diagrams of R-modules of the shape A_n. In other words, a module M over RA_n can be described as a diagram

$$M(1)\to M(2)\to\cdots\to M(n)$$

of R-modules, and a morphism of RA_n-modules is just a morphism of diagrams. Namely, $M(i)=Me_{ii}$ and the map $M(i)\to M(i+1)$ is given by the right action of $e_{i,i+1}$.

Consider the following RA_n-modules, for $1 \leq i \leq n$,

$$\begin{array}{ll} P_i: & 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow R \xrightarrow{1_R} R \xrightarrow{1_R} \dots \xrightarrow{1_R} R \\ S_i: & 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \\ I_i: & R \xrightarrow{1_R} \dots \xrightarrow{1_R} R \xrightarrow{1_R} R \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \end{array}$$

where $S_i(i) = R$. These modules are free as R-modules, and from the adjunction

$$\mathrm{Hom}_{RA_n}(P_i, M) \simeq \mathrm{Hom}_R(R, M(i)) = M(i) \tag{3.1}$$

we see that P_i are projective RA_n-modules. In fact, $RA_n = \bigoplus_{i=1}^n P_i$. Note that when R is a field, P_i, I_i and S_i are the indecomposable projective, injective and simple corresponding to the vertex i.

We have $S_n = P_n$, $S_1 = I_1$, $I_n = P_1$, and there are short exact sequences

$$\begin{array}{lr} 0 \to P_{i+1} \to P_i \to S_i \to 0 & 1 \leq i < n, \\ 0 \to P_{i+1} \to P_1 \to I_i \to 0 & 1 \leq i < n, \\ 0 \to S_i \to I_i \to I_{i-1} \to 0 & 1 < i \leq n. \end{array} \tag{3.2}$$

Proposition 3.1.

(a) *$P_1 \oplus \dots \oplus P_n$ is a tilting complex in $\mathcal{D}(RA_n)$ with endomorphism ring RA_n.*

(b) *$I_1 \oplus \dots \oplus I_n$ is a tilting complex in $\mathcal{D}(RA_n)$ with endomorphism ring RA_n. In particular,*

$$\mathrm{Hom}_{\mathcal{D}(RA_n)}(I_i, I_j) \simeq \mathrm{Hom}_{\mathcal{D}(RA_n)}(P_i, P_j) = \begin{cases} R & \text{if } j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

(c) *$S_1 \oplus S_2[1] \oplus \dots \oplus S_n[n-1]$ is a tilting complex in $\mathcal{D}(RA_n)$ and*

$$\mathrm{Hom}_{\mathcal{D}(RA_n)}(S_i[i-1], S_j[j-1]) = \begin{cases} R & \text{if } j - i \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first claim is obvious. For the others, note that the short exact sequences in (3.2) show that

$$\operatorname{per} RA_n = \langle \mathrm{add}(P_1 \oplus \dots \oplus P_n) \rangle = \langle \mathrm{add}(S_1 \oplus \dots \oplus S_n) \rangle = \langle \mathrm{add}(I_1 \oplus \dots \oplus I_n) \rangle,$$

hence each of the complexes is prefect and generates $\operatorname{per} RA_n$. The proof that each complex is exceptional and the computation of the morphism spaces follow easily from (3.2) and the adjunction (3.1). $\square$

Remark 3.2. It would have been sufficient to have the above discussion for $R = \mathbb{Z}$. The results for general R would then follow from Theorem 2.1 of [25] (or Theorem A).

3.2. **Theorem B and its corollaries.**

Proof of Theorem B. Let $A = \Lambda$, $B = \mathbb{Z}A_n$ and set $U_i = S_i[i-1]$ for $1 \leq i \leq n$. The result now follows from Proposition 3.1(c) and Theorem A. $\square$

Proof of the equivalence of Theorem B and Corollary 1.1. We first assume the theorem and prove its corollary. Let Λ_i and Q_i be as in Corollary 1.1. Set $T_1 = \Lambda$. For $1 \leq i < n$, let F_i be the equivalence

$$F_i = - \overset{\mathbf{L}}{\otimes}_{\Lambda_{i+1}} Q_{i\Lambda_i} : \mathcal{D}(\Lambda_{i+1}) \xrightarrow{\sim} \mathcal{D}(\Lambda_i)$$

taking Λ_{i+1} to Q_i, and set $T_{i+1} = F_1 F_2 \cdot \ldots \cdot F_i(\Lambda_{i+1})$. Then $T_1, \ldots, T_n$ are tilting complexes in $\mathcal{D}(\Lambda)$,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_{i+1}[r]) &= \mathrm{Hom}_{\mathcal{D}(\Lambda)}(F_1 \ldots F_{i-1}\Lambda_i, F_1 \ldots F_{i-1}F_i\Lambda_{i+1}[r]) \\
&= \mathrm{Hom}_{\mathcal{D}(\Lambda_i)}(\Lambda_i, F_i\Lambda_{i+1}[r]) = \mathrm{Hom}_{\mathcal{D}(\Lambda_i)}(\Lambda_i, Q_i[r]) \\
&= \begin{cases} Q_i & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and $\mathrm{End}_{\mathcal{D}(\Lambda)}(T_{i+1}) \simeq \mathrm{End}_{\mathcal{D}(\Lambda_{i+1})}(\Lambda_{i+1}) = \Lambda_{i+1}$. The result now follows from the theorem.

Conversely, assume Corollary 1.1 and let $T_1, \ldots, T_n$ be as in Theorem B. Set $\Lambda_i = \mathrm{End}_{\mathcal{D}(\Lambda)}(T_i)$ and let G_i be the equivalence

$$G_i = \mathbf{R}\mathrm{Hom}(T_i, -) : \mathcal{D}(\Lambda) \xrightarrow{\sim} \mathcal{D}(\Lambda_i)$$

taking T_i to Λ_i. Consider $Q_i = G_i(T_{i+1}) \in \mathcal{D}(\Lambda_i)$ for $1 \leq i < n$. Then Q_i is a tilting complex in $\mathcal{D}(\Lambda_i)$ with endomorphism ring Λ_{i+1}, and from

$$\mathrm{Hom}_{\mathcal{D}(\Lambda_i)}(\Lambda_i, Q_i[r]) = \mathrm{Hom}_{\mathcal{D}(\Lambda_i)}(G_iT_i, G_iT_{i+1}[r]) \simeq \mathrm{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_{i+1}[r])$$

we see that it is isomorphic (in $\mathcal{D}(\Lambda_i)$) to a tilting Λ_i-module. So Corollary 1.1 can be applied to get the required statement. □

Proof of Corollary 1.2. Let $m, n \geq 1$, set $\Lambda = kA_m$ and consider the k-module $Q = \mathrm{Hom}_k(\Lambda, k)$. It is free with basis elements $\{\varphi_{ji}\}_{1 \leq i \leq j \leq m}$ defined by $\varphi_{ji}(e_{pq}) = \delta_{ip}\delta_{jq}$, and has a natural Λ-bimodule structure given by $(\lambda\varphi\lambda')(x) = \varphi(\lambda' x \lambda)$ for $\lambda, \lambda' \in \Lambda$ and $\varphi \in Q$. Writing this explicitly for the basis elements, we have

(3.3)
$$\varphi_{ji}e_{pq} = \begin{cases} \varphi_{jq} & \text{if } p = i \text{ and } q \leq j, \\ 0 & \text{otherwise,} \end{cases} \qquad e_{pq}\varphi_{ji} = \begin{cases} \varphi_{pi} & \text{if } p \geq i \text{ and } q = j, \\ 0 & \text{otherwise.} \end{cases}$$

We can identify $I_i = \bigoplus_{j \leq i} k\varphi_{ij}$, so that as a right Λ-module, $Q = \bigoplus_{i=1}^n I_i$ is a tilting module by Proposition 3.1(b). Moreover, the left Λ-action on Q coincides with that via $\mathrm{End}_\Lambda(Q_\Lambda) \simeq \Lambda$ given in that proposition.

The "rectangle" algebra is $kA_m \otimes kA_n = \Lambda \otimes kA_n$. The lower triangular matrix algebra in (1.3) is isomorphic to the following upper triangular one.

$$\begin{pmatrix}
\Lambda_n & Q_{n-1} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \Lambda_3 & Q_2 & 0 \\
\vdots & \ldots & 0 & \Lambda_2 & Q_1 \\
0 & \ldots & 0 & 0 & \Lambda_1
\end{pmatrix}.$$

Hence, applying Corollary 1.1 with $Q_i = Q$ and $\Lambda_i = \Lambda$, we get that $\Lambda \otimes kA_n$ is derived equivalent to the matrix algebra

$$\begin{pmatrix} \Lambda & Q & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \Lambda & Q & 0 \\ \vdots & \dots & 0 & \Lambda & Q \\ 0 & \dots & 0 & 0 & \Lambda \end{pmatrix}$$

having a basis $\{e_{ij}^{(s)}\}_{1 \leq i \leq j \leq n, 1 \leq s \leq n} \cup \{\varphi_{ji}^{(s)}\}_{1 \leq i \leq j \leq n, 1 \leq s < n}$ corresponding to the copies of Λ and Q, where the multiplication $e_{ij}^{(s)} e_{pq}^{(s)}$ is the usual one, $e_{pq}^{(s)} \varphi_{ji}^{(s)}$ and $\varphi_{ji}^{(s)} e_{pq}^{(s+1)}$ are given by (3.3), and all other products are zero.

Finally, "line" algebra $A(n{\cdot}m, m{+}1)$, which is kA_{nm} modulo the ideal generated by all the paths of length $m+1$, has a k-basis $\{\varepsilon_{ij}\}_{1 \leq i \leq j \leq \min(nm, i+m)}$ where the product $\varepsilon_{ij} \varepsilon_{pq}$ equals e_{iq} if $p = j$ and $q \leq i + m$, and zero otherwise. One can then directly verify, using (3.3), that the k-linear map defined by

$$e_{ij}^{(s)} \mapsto \varepsilon_{(s-1)m+i,(s-1)m+j} \qquad\qquad \varphi_{ji}^{(s)} \mapsto \varepsilon_{(s-1)m+j, sm+i}$$

is an isomorphism of algebras, compare similar calculations in [28] in connection with repetitive algebras. $\square$

Proof of Corollary 1.3. This is immediate from Corollary 1.1, since when Λ is Gorenstein, $D\Lambda$ is a tilting module and $\operatorname{End}_\Lambda(D\Lambda) \simeq \Lambda$. $\square$

3.3. **Theorem C and its corollaries.**

Proof of Theorem C. Let $A = \Lambda$, $B = \mathbb{Z}A_n$ and set $U_i = P_{n+1-i}$ for $1 \leq i \leq n$. The result now follows from Proposition 3.1(a) and Theorem A. $\square$

Proof of Corollary 1.6. Observe that $\operatorname{End}_{\mathcal{D}(\Lambda)}(T_1 \oplus \dots \oplus T_n)$ is isomorphic to the matrix ring

$$\begin{pmatrix} \operatorname{End} T_1 & \operatorname{Hom}(T_2, T_1) & \operatorname{Hom}(T_3, T_1) & \dots & \operatorname{Hom}(T_n, T_1) \\ \operatorname{Hom}(T_1, T_2) & \operatorname{End} T_2 & \operatorname{Hom}(T_3, T_2) & \dots & \operatorname{Hom}(T_n, T_2) \\ \operatorname{Hom}(T_1, T_3) & \operatorname{Hom}(T_2, T_3) & \operatorname{End} T_3 & \dots & \operatorname{Hom}(T_n, T_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}(T_1, T_n) & \operatorname{Hom}(T_2, T_n) & \operatorname{Hom}(T_3, T_n) & \dots & \operatorname{End} T_n \end{pmatrix}$$

and the condition (ii) guarantees that all the terms above the main diagonal vanish. The result now follows from Theorem C. $\square$

Proof of Corollary 1.7. Let $T_i = F^{e_i}\Lambda$ for $1 \leq i \leq n$. Then

$$\operatorname{Hom}_{\mathcal{D}(\Lambda)}(T_i, T_j[r]) \simeq \operatorname{Hom}_{\mathcal{D}(\Lambda)}(\Lambda, F^{e_j - e_i}\Lambda[r]) \simeq \mathrm{H}^r(F^{e_j - e_i}\Lambda),$$

so that conditions (i) and (ii) match the corresponding ones in Corollary 1.6. The result now follows from that corollary. $\square$

3.4. **Path algebras of quivers and Auslander algebras.**

Proof of Corollary 1.10. Consider the autoequivalence $F = \nu^{-}[1]$ on the derived category $\mathcal{D}^b(\mathrm{mod}\, kQ)$. By our hypotheses, $F^i(kQ) = \tau^{-i}kQ$ for $0 \leq i \leq r$ are concentrated in degree zero, and moreover $\mathrm{H}^0(F^{-i}(kQ)) = \tau^i kQ = 0$ for $1 \leq i \leq r$. The result now follows from Corollary 1.7 with $n = r+1$ and the sequence $0, 1, \dots, r$. □

Proof of Corollary 1.11. The corresponding path algebras are homogeneous, so that their Auslander algebras are of the form $\mathrm{End}_{kQ}(kQ \oplus \cdots \oplus \tau^{-r}kQ)$ for some r determined in [9]. Now apply Corollary 1.10. □

Proof of Corollary 1.12. Each of the diagrams in the table admits at least one orientation whose path algebra is homogeneous. Now the result follows from Corollary 1.10 and the fact that the derived equivalence class of the stable Auslander algebra does not depend on the orientation. □

Proof of Corollary 1.13. The Auslander algebra of the linear orientation on A_{2n} is isomorphic to the stable Auslander algebra of the linear orientation on A_{2n+1}. Now use Corollary 1.12. □

References

1. *The ADE chain*, Workshop at Bielefeld, October 2008.
2. Lidia Angeleri Hügel, Dieter Happel, and Henning Krause (eds.), *Handbook of tilting theory*, London Mathematical Society Lecture Note Series, vol. 332, Cambridge University Press, Cambridge, 2007.
3. I. Assem, T. Brüstle, R. Schiffler, and G. Todorov, *m-cluster categories and m-replicated algebras*, J. Pure Appl. Algebra **212** (2008), no. 4, 884–901.
4. M. Auslander, *Representation dimension of artin algebras*, Queen Mary College, Mathematics Notes, University of London (1971), 1–179.
5. A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.
6. Roger Carter, Graeme Segal, and Ian Macdonald, *Lectures on Lie groups and Lie algebras*, London Mathematical Society Student Texts, vol. 32, Cambridge University Press, Cambridge, 1995, With a foreword by Martin Taylor.
7. Xiao-Wu Chen, *The stable monomorphism category of a Frobenius category*, Math. Res. Lett. **18** (2011), no. 1, 125–137.
8. Peter Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. 6 (1972), 309.
9. ———, *Auslander-Reiten sequences and representation-finite algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 831, Springer, Berlin, 1980, pp. 1–71.
10. Christof Geiss, Bernard Leclerc, and Jan Schröer, *Cluster algebra structures and semicanoncial bases for unipotent groups*, preprint at `arXiv:math/0703039`.
11. ———, *Auslander algebras and initial seeds for cluster algebras*, J. Lond. Math. Soc. (2) **75** (2007), no. 3, 718–740.
12. Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
13. Dieter Happel and Uwe Seidel, *Piecewise hereditary Nakayama algebras*, Algebr. Represent. Theory **13** (2010), no. 6, 693–704.
14. Martin Herschend and Osamu Iyama, *n-representation-finite algebras and twisted fractionally Calabi-Yau algebras*, Bull. London Math. Soc. **43** (2011), no. 3, 449–466.
15. Osamu Iyama and Steffen Oppermann, *Stable categories of $(n+1)$-preprojective algebras*, preprint at `arXiv:0912.3412`.

16. Steffen König and Alexander Zimmermann, *Derived equivalences for group rings*, Lecture Notes in Mathematics, vol. 1685, Springer-Verlag, Berlin, 1998, With contributions by Bernhard Keller, Markus Linckelmann, Jeremy Rickard and Raphaël Rouquier.
17. Maxim Kontsevich, *Course at the ENS*, Paris, 1998.
18. Dirk Kussin, Helmut Lenzing, and Hagen Meltzer, *Nilpotent operators and weighted projective lines*, preprint at `arXiv:1002.3797`.
19. Sefi Ladkani, *Derived equivalences and mutation sequences for Auslander algebras*, in preparation.
20. ———, *Derived equivalences of triangular matrix rings arising from extensions of tilting modules*, Algebr. Represent. Theory **14** (2011), no. 1, 57–74.
21. Helmut Lenzing and José Antonio de la Peña, *Spectral analysis of finite dimensional algebras and singularities*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 541–588.
22. Saunders Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963.
23. Jun-ichi Miyachi and Amnon Yekutieli, *Derived Picard groups of finite-dimensional hereditary algebras*, Compositio Math. **129** (2001), no. 3, 341–368.
24. Jeremy Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
25. ———, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.
26. Claus Michael Ringel, *On algorithms for solving vector space problems. II. Tame algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 831, Springer, Berlin, 1980, pp. 137–287.
27. Claus Michael Ringel and Markus Schmidmeier, *Invariant subspaces of nilpotent linear operators. I*, J. Reine Angew. Math. **614** (2008), 1–52.
28. Jan Schröer, *On the quiver with relations of a repetitive algebra*, Arch. Math. (Basel) **72** (1999), no. 6, 426–432.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Current address: Mathematical Institute of the University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `sefil@math.uni-bonn.de`