UNIVERSAL DERIVED EQUIVALENCES OF POSETS OF TILTING MODULES

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Abstract. We show that for two quivers without oriented cycles related by a BGP reflection, the posets of their tilting modules are related by a simple combinatorial construction, which we call flip-flop.

We deduce that the posets of tilting modules of derived equivalent path algebras of quivers without oriented cycles are universally derived equivalent.

1. Introduction

In this note we investigate the combinatorial relations between the posets of tilting modules of derived equivalent path algebras. While it is known that these posets are in general not isomorphic, we show that they are related via a sequence of simple combinatorial constructions, which we call flip-flops.

For two partially ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ and an order preserving function $f : X \to Y$, one can define two partial orders $\leq_+^f$ and $\leq_-^f$ on the disjoint union $X \sqcup Y$, by keeping the original partial orders inside X and Y and setting

$$x \leq_+^f y \iff f(x) \leq_Y y$$
$$y \leq_-^f x \iff y \leq_Y f(x)$$

with no other additional order relations. We say that two posets Z and Z' are related via a *flip-flop* if there exist X, Y and $f : X \to Y$ as above such that $Z \simeq (X \sqcup Y, \leq_+^f)$ and $Z' \simeq (X \sqcup Y, \leq_-^f)$.

Throughout this note, the field k is fixed. Given a (finite) quiver Q without oriented cycles, consider the category of finite-dimensional modules over the path algebra of Q, which is equivalent to the category $\operatorname{rep} Q$ of finite dimensional representations of Q over k, and denote by $\mathcal{T}_Q$ the poset of tilting modules in $\operatorname{rep} Q$ as introduced by [8]. For more information on the partial order on tilting modules see [6], the survey [9] and the references therein.

Let x be a source of Q and let Q' be the quiver obtained from Q by a BGP reflection, that is, by reverting all arrows starting at x. The combinatorial relation between the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ is expressed in the following theorem.

Theorem 1.1. *The posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flip.*

In fact, the subset Y in the definition of a flip-flop can be explicitly described as the set of tilting modules containing the simple at x as direct summand, and we show that it is isomorphic as poset to $\mathcal{T}_{Q \setminus \{x\}}$.

While two posets Z and Z' related via a flip-flop are in general not isomorphic, they are *universally derived equivalent* in the following sense; for any abelian category $\mathcal{A}$, the derived categories of the categories of functors $Z \to \mathcal{A}$ and $Z' \to \mathcal{A}$ are equivalent as triangulated categories, see [7].

For two quivers without oriented cycles Q and Q', we denote $Q \sim Q'$ if Q' can be obtained from Q by a sequence of BGP reflections (at sources or sinks). It is known that the path algebras of Q and Q' are derived equivalent if and only if $Q \sim Q'$, see [5, (I.5.7)], hence by [7, Corollary 1.3] we deduce the following theorem.

Theorem 1.2. *Let Q and Q' be two quivers without oriented cycles whose path algebras are derived equivalent. Then the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are universally derived equivalent.*

The paper is structured as follows. In Section 2 we study the structure of the poset $\mathcal{T}_Q$ with regard to a source vertex x, where the main tool is the existence of an exact functor right adjoint to the restriction $\operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\})$. For the convenience of the reader, we record the dual statements for the case of a sink in Section 3. Building on these results, we analyze the effect of a BGP reflection in Section 4, where a proof of Theorem 1.1 is given. We conclude by demonstrating the theorem on a concrete example in Section 5.

Acknowledgement. I would like to thank Frédéric Chapoton for suggesting a conjectural version of Theorem 1.2 in the case of finite-type quivers and for many helpful discussions.

2. Tilting modules with respect to a source

Let Q be a quiver. For a representation M in $\operatorname{rep} Q$, denote by $M(y)$ the vector space corresponding to a vertex y and by $M(y \to y')$ the linear transformation $M(y) \to M(y')$ corresponding to an edge $y \to y'$ in Q.

Let x be a source in the quiver Q, to be fixed throughout this section.

Lemma 2.1. *The inclusion $j : Q \setminus \{x\} \to Q$ induces a pair (j^{-1}, j_*) of functors*

$$j^{-1} : \operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\}) \qquad j_* : \operatorname{rep}(Q \setminus \{x\}) \to \operatorname{rep} Q$$

such that

$$\operatorname{Hom}_{Q\setminus\{x\}}(j^{-1}M, N) \simeq \operatorname{Hom}_Q(M, j_*N) \tag{2.1}$$

for all $M \in \operatorname{rep} Q$, $N \in \operatorname{rep}(Q \setminus \{x\})$ (that is, j_* is a right adjoint to j^{-1}).

Proof. We shall write the functors j^{-1} and j_* explicitly. For $M \in \operatorname{rep} Q$, define

$$(j^{-1}M)(y) = M(y) \qquad (j^{-1}M)(y \to y') = M(y \to y')$$

for any $y \to y'$ in $Q \setminus \{x\}$. For $N \in \operatorname{rep}(Q \setminus \{x\})$, define

$$(j_*N)(y) = N(y) \qquad (j_*N)(y \to y') = N(y \to y')$$

$$(j_*N)(x) = \bigoplus_{i=1}^{m} N(y_i) \qquad (j_*N)(x \to y_i) = (j_*N)(x) \to N(y_i) \tag{2.2}$$

where $y_1, \dots, y_m$ are the endpoints of the arrows starting at x, $(j_*N)(x) \to N(y_i)$ are the natural projections, and y, y' are in $Q \setminus \{x\}$.

Now (2.1) follows since the maps $M(y_i) \to N(y_i)$ for $1 \le i \le m$ induce a unique map $M(x) \to N(y_1) \oplus \cdots \oplus N(y_m)$ such that the diagrams

$$\begin{tikzcd} M(x) \arrow[r] \arrow[d] & (j_*N)(x) = \bigoplus_{i=1}^m N(y_i) \arrow[d] \\ M(y_i) \arrow[r] & N(y_i) \end{tikzcd}$$

commute for all $1 \le i \le m$. □

Lemma 2.2. *The functor j_* is fully faithful and exact.*

Proof. Observe that $j^{-1}j_*$ is the identity on $\operatorname{rep}(Q \setminus \{x\})$, hence for $N, N' \in \operatorname{rep}(Q \setminus \{x\})$,

$$\operatorname{Hom}_Q(j_*N, j_*N') \simeq \operatorname{Hom}_{Q\setminus\{x\}}(j^{-1}j_*N, N') = \operatorname{Hom}_{Q\setminus\{x\}}(N, N')$$

so that j_* is fully faithful. Its exactness follows from (2.2). □

Denote by $\mathcal{D}^b(Q)$ the bounded derived category $\mathcal{D}^b(\operatorname{rep} Q)$. The exact functors j^{-1} and j_* induce functors

$$j^{-1} : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q \setminus \{x\}) \qquad j_* : \mathcal{D}^b(Q \setminus \{x\}) \to \mathcal{D}^b(Q)$$

with

$$\operatorname{Hom}_{\mathcal{D}^b(Q\setminus\{x\})}(j^{-1}M, N) \simeq \operatorname{Hom}_{\mathcal{D}^b(Q)}(M, j_*N) \tag{2.3}$$

for all $M \in \mathcal{D}^b(Q)$, $N \in \mathcal{D}^b(Q \setminus \{x\})$.

Let S_x be the simple (injective) object of $\operatorname{rep} Q$ corresponding to x.

Lemma 2.3. *The functor j_* identifies* $\operatorname{rep}(Q \setminus \{x\})$ *with the right perpendicular subcategory*

$$S_x^{\perp} = \left\{M \in \operatorname{rep} Q \,:\, \operatorname{Ext}^i(S_x, M) = 0 \text{ for all } i \ge 0\right\} \tag{2.4}$$

of $\operatorname{rep} Q$.

Proof. Observe that $j^{-1}S_x = 0$. Hence by (2.3),

$$\operatorname{Ext}^i_Q(S_x, j_*N) = \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}S_x, N) = 0$$

for all $N \in \operatorname{rep}(Q \setminus \{x\})$.

Conversely, let M be such that $\operatorname{Ext}^i_Q(S_x, M) = 0$ for $i \ge 0$, and let $\varphi : M \to j_*j^{-1}M$ be the adjunction morphism. From $j^{-1}j_*j^{-1}M = j^{-1}M$ we see that $(\ker \varphi)(y) = 0 = (\operatorname{coker} \varphi)(y)$ for all $y \ne x$.

From $0 \to \ker \varphi \to M$ we get

$$0 \to \operatorname{Hom}_Q(S_x, \ker \varphi) \to \operatorname{Hom}_Q(S_x, M) = 0 \tag{2.5}$$

hence $\ker \varphi = 0$. Thus $0 \to M \to j_*j^{-1}M \to \operatorname{coker} \varphi \to 0$ is exact, and from

$$0 = \operatorname{Hom}_Q(S_x, j_*j^{-1}M) \to \operatorname{Hom}_Q(S_x, \operatorname{coker} \varphi) \to \operatorname{Ext}^1_Q(S_x, M) = 0$$

we deduce that $\operatorname{coker} \varphi = 0$, hence $M \simeq j_*j^{-1}M$. □

Lemma 2.4. *The functor j_* takes indecomposables of* $\operatorname{rep}(Q \setminus \{x\})$ *to indecomposables of* $\operatorname{rep} Q$.

Proof. Let N be an indecomposable representation of $Q \setminus \{x\}$, and assume that $j_*N = M_1 \oplus M_2$. Then $N \simeq j^{-1}j_*N = j^{-1}M_1 \oplus j^{-1}M_2$, hence we may assume that $j^{-1}M_2 = 0$.

Thus $M_2 = S_x^n$ for some $n \geq 0$. But j_*N belongs to the right perpendicular subcategory $S_x^{\perp}$ which is closed under direct summands, hence $n = 0$ and $M_2 = 0$. □

Recall that $T \in \operatorname{rep} Q$ is a *tilting module* if $\operatorname{Ext}^i(T, T) = 0$ for all $i > 0$, and the direct summands of T generate $\mathcal{D}^b(Q)$ as a triangulated category. If T is basic, the latter condition can be replaced by the condition that the number of indecomposable summands of T equals the number of vertices of Q.

For a tilting module T, define

$$T^{\perp} = \left\{ M \in \operatorname{rep} Q \,:\, \operatorname{Ext}^i(T, M) = 0 \text{ for all } i > 0 \right\}$$

and set $T \leq T'$ if $T^{\perp} \supseteq T'^{\perp}$. By [6], $T \leq T'$ if and only if $\operatorname{Ext}^i_Q(T, T') = 0$ for all $i > 0$.

Denote by $\mathcal{T}_Q$ the set of basic tilting modules of $\operatorname{rep} Q$, and by $\mathcal{T}_Q^x$ the subset of $\mathcal{T}_Q$ consisting of all tilting modules which have S_x as direct summand.

Lemma 2.5. *$\mathcal{T}_Q^x$ is an open subset of $\mathcal{T}_Q$, that is, if $T \in \mathcal{T}_Q^x$ and $T \leq T'$, then $T' \in \mathcal{T}_Q^x$.*

Proof. Let $T \in \mathcal{T}_Q^x$ and $T' \in \mathcal{T}_Q$ such that $T \leq T'$. Then $T' \in T^{\perp}$, and in particular $\operatorname{Ext}^i(S_x, T') = 0$ for $i > 0$. Since S_x is injective, it follows that $\operatorname{Ext}^i(T', S_x) = 0$ for $i > 0$, hence if $T' \notin \mathcal{T}_Q^x$, then $S_x \oplus T'$ would also be a basic tilting module, contradiction to the fact that the number of indecomposable summands of a basic tilting module equals the number of vertices of Q. □

Proposition 2.6. *Let T be a tilting module in $\operatorname{rep} Q$. Then $j^{-1}T$ is a tilting module of $\operatorname{rep}(Q \setminus \{x\})$.*

Proof. We consider two cases. First, assume that T contains S_x as direct summand. Write $T = S_x^n \oplus T'$ with $n > 0$, where T' does not have S_x as direct summand. Then $j^{-1}T = j^{-1}T'$ and $T' \in S_x^{\perp}$, hence $j_*j^{-1}T' = T'$ and

$$\begin{aligned} \text{(2.6)} \quad \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T, j^{-1}T) &= \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T', j^{-1}T') \\ &= \operatorname{Ext}^i_Q(T', j_*j^{-1}T') = \operatorname{Ext}^i_Q(T', T') = 0 \end{aligned}$$

Now assume that T does not contain S_x as direct summand, and let $\varphi : T \to j_*j^{-1}T$ be the adjunction morphism. Then $\operatorname{Hom}_Q(S_x, T) = 0$ and similarly to (2.5), we deduce that $\ker \varphi = 0$. Observe that $\operatorname{coker} \varphi = S_x^n$ for some $n \geq 0$ is injective, hence from the exact sequence $0 \to T \to j_*j^{-1}T \to \operatorname{coker} \varphi \to 0$ we get for $i > 0$,

$$\text{(2.7)} \qquad 0 = \operatorname{Ext}^i(T, T) \to \operatorname{Ext}^i(T, j_*j^{-1}T) \to \operatorname{Ext}^i(T, \operatorname{coker} \varphi) = 0$$

therefore $\operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}T, j^{-1}T) = \operatorname{Ext}^i_Q(T, j_*j^{-1}T) = 0$ for $i > 0$.

To show that the direct summands of $j^{-1}T$ generate $\mathcal{D}^b(Q \setminus \{x\})$, it is enough to verify that for any $y \in Q \setminus \{x\}$, the corresponding projective

P_y in $\operatorname{rep}(Q \setminus \{x\})$ has a resolution with objects from $\operatorname{add} j^{-1}T$. Indeed, let $y \in Q \setminus \{x\}$ and consider the projective $\widetilde{P}_y$ of $\operatorname{rep} Q$. Applying the exact functor j^{-1} on an $\operatorname{add} T$-resolution of $\widetilde{P}_y$ gives the required $\operatorname{add} j^{-1}T$-resolution of $P_y = j^{-1}\widetilde{P}_y$. □

Note that $j^{-1}T$ may not be basic even if T is basic. Write $\operatorname{basic}(j^{-1}T)$ for the module obtained from $j^{-1}T$ by deleting duplicate direct summands. Then $\operatorname{basic}(j^{-1}T)$ is a basic tilting module with $\operatorname{basic}(j^{-1}T)^{\perp} = (j^{-1}T)^{\perp}$. It follows by the adjunction (2.3) that for $N \in \operatorname{rep}(Q \setminus \{x\})$,

$$N \in (j^{-1}T)^{\perp} \iff j_*N \in T^{\perp}$$

Corollary 2.7. *The map* $\pi_x : T \mapsto \operatorname{basic}(j^{-1}T)$ *is an order-preserving function* $(\mathcal{T}_Q, \leq) \to (\mathcal{T}_{Q\setminus\{x\}}, \leq)$.

Proof. Let $T \leq T'$ and consider $N \in (j^{-1}T')^{\perp}$. Then $j_*N \in T'^{\perp} \subseteq T^{\perp}$, hence $N \in (j^{-1}T)^{\perp}$, so that $j^{-1}T \leq j^{-1}T'$. □

Let N, N' be objects of $\operatorname{rep}(Q \setminus \{x\})$ with $\operatorname{Ext}^i_{Q\setminus\{x\}}(N, N') = 0$ for all $i > 0$. By the adjunctions (2.3),

$$\begin{aligned}
&\operatorname{Ext}^i_Q(j_*N, j_*N') \simeq \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}j_*N, N') = \operatorname{Ext}^i_{Q\setminus\{x\}}(N, N') = 0\\
&\operatorname{Ext}^i_Q(S_x, j_*N') \simeq \operatorname{Ext}^i_{Q\setminus\{x\}}(j^{-1}S_x, N') = 0\\
&\operatorname{Ext}^i_Q(j_*N, S_x) = 0
\end{aligned}$$

where the last equation follows since S_x injective. Hence

$$\operatorname{Ext}^i_Q(S_x \oplus j_*N, S_x \oplus j_*N') = 0 \text{ for all } i > 0 \tag{2.8}$$

Corollary 2.8. *Let T be a basic tilting module in* $\operatorname{rep}(Q \setminus \{x\})$. *Then* $S_x \oplus j_*T$ *is a basic tilting module in* $\operatorname{rep} Q$.

Proof. Indeed, $\operatorname{Ext}^i_Q(S_x \oplus j_*T, S_x \oplus j_*T) = 0$ for $i > 0$, by (2.8).

Let n be the number of vertices of Q. Since T is a basic tilting module for $Q \setminus \{x\}$, it has $n-1$ indecomposable summands, hence by Lemmas 2.3 and 2.4, j_*T decomposes into $n-1$ indecomposable summands. It follows that $S_x \oplus j_*T$ is a tilting module. □

Corollary 2.9. *The map* $\iota_x : T \mapsto S_x \oplus j_*T$ *is an order preserving function* $(\mathcal{T}_{Q\setminus\{x\}}, \leq) \to (\mathcal{T}^x_Q, \leq)$.

Proof. Let $T \leq T'$ in $\mathcal{T}_{Q\setminus\{x\}}$. Then $\operatorname{Ext}^i_{Q\setminus\{x\}}(T, T') = 0$ for all $i > 0$ and the claim follows from (2.8). □

Proposition 2.10. *We have*

$$\pi_x \iota_x(T) = T$$

for all $T \in \mathcal{T}_{Q\setminus\{x\}}$. *In addition,*

$$T \leq \iota_x \pi_x(T)$$

for all $T \in \mathcal{T}_Q$, *with equality if and only if* $T \in \mathcal{T}^x_Q$.

In particular we see that ι_x induces a retract $\iota_x\pi_x$ of $\mathcal{T}_Q$ onto $\mathcal{T}^x_Q$ and an isomorphism of posets between $\mathcal{T}_{Q\setminus\{x\}}$ and $\mathcal{T}^x_Q$.

Proof. If $T \in \mathcal{T}_{Q\setminus\{x\}}$, then $j^{-1}(S_x \oplus j_*T) = j^{-1}j_*T = T$, hence $\pi_x\iota_x(T) = \mathrm{basic}(T) = T$.

Let $T \in \mathcal{T}_Q$. Then $\mathrm{Ext}^i_Q(T, S_x) = 0$ for $i > 0$. Moreover, by the argument in the proof of Proposition 2.6 (see (2.6) and (2.7)), $\mathrm{Ext}^i_Q(T, j_*j^{-1}T) = 0$. It follows that $S_x \oplus j_*j^{-1}T \in T^\perp$, thus $T \le \iota_x\pi_x(T)$.

If $T = \iota_x\pi_x(T)$, then obviously T has S_x as summand, so that $T \in \mathcal{T}^x_Q$. Conversely, if $T \in \mathcal{T}^x_Q$, then $T = S_x \oplus T'$ with $T' \in S_x^\perp$, and by Lemma 2.3, $T' = j_*j^{-1}T'$, hence $\iota_x\pi_x(T) = S_x \oplus j_*j^{-1}T' = S_x \oplus T' = T$. $\square$

Corollary 2.11. *Let $X = \mathcal{T}_Q \setminus \mathcal{T}^x_Q$ and $Y = \mathcal{T}^x_Q$. Define $f : X \to Y$ by $f = \iota_x\pi_x$. Then $\mathcal{T}_Q \simeq (X \sqcup Y, \le^f_+)$.*

Proof. Let $T \in X$ and $T' \in Y$. If $T \le T'$, then by the previous proposition,

$$f(T) = \iota_x\pi_x(T) \le \iota_x\pi_x(T') = T'$$

hence $T \le T'$ in $\mathcal{T}_Q$ if and only if $f(T) \le T'$ in $\mathcal{T}^x_Q$. $\square$

3. Tilting modules with respect to a sink

Now let Q' be the quiver obtained from Q by reflection at the source x. For the convenience of the reader, we record, without proofs, the analogous (dual) results for this case.

Lemma 3.1. *The inclusion $i : Q \setminus \{x\} \to Q'$ induces a pair $(i_!, i^{-1})$ of functors*

$$i^{-1} : \mathrm{rep}\, Q' \to \mathrm{rep}(Q \setminus \{x\}) \qquad i_! : \mathrm{rep}(Q \setminus \{x\}) \to \mathrm{rep}\, Q'$$

such that

$$\mathrm{Hom}_{\mathrm{rep}(Q\setminus\{x\})}(N, i^{-1}M) \simeq \mathrm{Hom}_{\mathrm{rep}\, Q}(i_!N, M)$$

for all $M \in \mathrm{rep}\, Q$, $N \in \mathrm{rep}(Q \setminus \{x\})$ (that is, $i_!$ is a left adjoint to i^{-1}).

Proof. For $M \in \mathrm{rep}\, Q'$, define

$$(i^{-1}M)(y) = M(y) \qquad (i^{-1}M)(y \to y') = M(y \to y')$$

for any $y \to y'$ in $Q \setminus \{x\}$. For $N \in \mathrm{rep}(Q \setminus \{x\})$, define

$$(i_!N)(y) = N(y) \qquad (i_!N)(y \to y') = N(y \to y')$$
$$(i_!N)(x) = \bigoplus_{l=1}^{m} N(y_l) \qquad (i_!N)(y_l \to x) = N(y_l) \to (i_!N)(x)$$

where $y_1, \dots, y_m$ are the starting points of the arrows ending at x, $N(y_l) \to (i_!N)(x)$ are the natural inclusions, and y, y' are in $Q \setminus \{x\}$. $\square$

Lemma 3.2. *The functor $i_!$ is fully faithful and exact.*

Let S'_x be the simple (projective) object of $\mathrm{rep}\, Q'$ corresponding to x.

Lemma 3.3. *The functor $i_!$ identifies $\mathrm{rep}(Q\setminus\{x\})$ with the left perpendicular subcategory*

$${}^\perp S'_x = \left\{ M \in \mathrm{rep}\, Q' \,:\, \mathrm{Ext}^i(M, S'_x) = 0 \textit{ for all } i \ge 0 \right\}$$

of $\mathrm{rep}\, Q'$.

Lemma 3.4. *The functor $i_!$ takes indecomposables of* $\operatorname{rep}(Q \setminus \{x\})$ *to indecomposables of* $\operatorname{rep} Q'$.

Denote by $\mathcal{T}_{Q'}^x$ the subset of $\mathcal{T}_{Q'}$ consisting of all tilting modules which have S'_x as direct summand.

Lemma 3.5. *$\mathcal{T}_{Q'}^x$ is a closed subset of $\mathcal{T}_{Q'}$, that is, if $T \in \mathcal{T}_{Q'}^x$ and $T' \leq T$, then $T' \in \mathcal{T}_{Q'}^x$.*

Proposition 3.6. *Let T be a tilting module in* $\operatorname{rep} Q'$. *Then $i^{-1}T$ is a tilting module of* $\operatorname{rep}(Q \setminus \{x\})$.

Corollary 3.7. *The map $\pi'_x : T \mapsto \operatorname{basic}(i^{-1}T)$ is an order-preserving function $(\mathcal{T}_{Q'}, \leq) \to (\mathcal{T}_{Q\setminus\{x\}}, \leq)$.*

Lemma 3.8. *Let T be a basic tilting module in* $\operatorname{rep}(Q \setminus \{x\})$. *Then $S'_x \oplus i_!T$ is a basic tilting module of* $\operatorname{rep} Q'$.

Corollary 3.9. *The map $\iota'_x : T \mapsto S'_x \oplus i_!T$ is an order preserving function $(\mathcal{T}_{Q\setminus\{x\}}, \leq) \to (\mathcal{T}_{Q'}^x, \leq)$.*

Proposition 3.10. *We have*

$$\pi'_x \iota'_x(T) = T$$

for all $T \in \mathcal{T}_{Q\setminus\{x\}}$. In addition,

$$T \geq \iota'_x \pi'_x(T)$$

for all $T \in \mathcal{T}_{Q'}$, with equality if and only if $T \in \mathcal{T}_{Q'}^x$.

Corollary 3.11. *Let $X' = \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x$ and $Y' = \mathcal{T}_{Q'}^x$. Define $f' : X' \to Y'$ by $f' = \iota'_x \pi'_x$. Then $\mathcal{T}_{Q'} \simeq (X' \sqcup Y', \leq_-^{f'})$.*

4. Tilting modules with respect to reflection

Let $F : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q')$ be the BGP reflection defined by the source x. For the convenience of the reader, we describe F explicitly following [4, (IV.4, Exercise 6)] (see also [7]).

Observe that a complex of representations of Q can be described as a collection of complexes K_y of finite-dimensional vector spaces for the vertices y of Q, together with morphisms $K_y \to K_{y'}$ for the arrows $y \to y'$ in Q. Given such data, let $y_1, \dots, y_m$ be the endpoints of the arrows of Q starting at x, and define a collection $\{K'_y\}$ of complexes by

$$K'_x = \operatorname{Cone}\Big(K_x \to \bigoplus_{i=1}^m K_{y_i}\Big) \tag{4.1}$$

$$K'_y = K_y \qquad\qquad y \in Q \setminus \{x\}$$

with the morphisms $K'_y \to K'_{y'}$ identical to $K_y \to K_{y'}$ for $y \to y'$ in $Q \setminus \{x\}$, and the natural inclusions $K'_{y_i} = K_{y_i} \to \operatorname{Cone}(K_x \to \bigoplus K_{y_j}) = K'_x$ for the reversed arrows $y_i \to x$ in Q'.

This definition can be naturally extended to give a functor $\widetilde{F}$ from the category of complexes over $\operatorname{rep} Q$ to the complexes over $\operatorname{rep} Q'$, which induces the triangulated equivalence F. The action of F on complexes is given, up to quasi-isomorphism, by (4.1).

Lemma 4.1 ([1]). *F induces a bijection between the indecomposables of* $\operatorname{rep} Q$ *other than S_x and the indecomposables of* $\operatorname{rep} Q'$ *other than S'_x.*

Proof. If M is an indecomposable of $\operatorname{rep} Q$, then FM is indecomposable of $\mathcal{D}^b(Q')$ since F is a triangulated equivalence.

Now let $M \neq S_x$ be an indecomposable of $\operatorname{rep} Q$. The map $M(x) \to \bigoplus_{i=1}^m M(y_i)$ must be injective, otherwise one could decompose $M = S_x^n \oplus N$ for some $n > 0$ and N. Using (4.1) we see that FM is quasi-isomorphic to the stalk complex supported on degree 0 that can be identified with $M' \in \operatorname{rep} Q'$, given by

$$\begin{aligned} M'(x) &= \operatorname{coker}\Big(M(x) \to \bigoplus_{i=1}^m M(y_i)\Big) \\ M'(y) &= M(y) \qquad\qquad y \in Q \setminus \{x\} \end{aligned} \tag{4.2}$$

$\square$

Note also that from (4.1) it follows that $FS_x = S'_x[1]$.

Corollary 4.2. *$j^{-1}T = i^{-1}FT$ for all $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$.*

Proof. This follows from (4.2), since T does not have S_x as summand. $\square$

Corollary 4.3. *F induces an isomorphism of posets $\rho : \mathcal{T}_Q \setminus \mathcal{T}_Q^x \to \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x$.*

Proof. For $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$, define $\rho(T) = FT$. Observe that if T has n indecomposable summands, so does FT. Moreover, if $T, T' \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$, then $\operatorname{Ext}^i_{Q'}(FT, FT') \simeq \operatorname{Ext}^i_Q(T, T')$, hence $\rho(T) \in \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x$ and $\rho(T) \leq \rho(T')$ if $T \leq T'$. $\square$

Corollary 4.4. *We have a commutative diagram*

$$\begin{tikzcd}
 & \mathcal{T}_Q \setminus \mathcal{T}_Q^x \arrow[rr, "\rho", "\simeq"'] \arrow[dl, "f"'] \arrow[dr, "\pi_x"] & & \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x \arrow[dl, "\pi'_x"'] \arrow[dr, "f'"] & \\
\mathcal{T}_Q^x & & \mathcal{T}_{Q\setminus\{x\}} \arrow[ll, "\iota_x", "\simeq"'] \arrow[rr, "\iota'_x", "\simeq"'] & & \mathcal{T}_{Q'}^x
\end{tikzcd}$$

Proof. We have to show the commutativity of the middle triangle, that is, $\pi_x = \pi'_x\rho$. Indeed, let $T \in \mathcal{T}_Q \setminus \mathcal{T}_Q^x$. Then $\pi_x(T) = \operatorname{basic}(j^{-1}T)$, $\pi'_x\rho(T) = \operatorname{basic}(i^{-1}FT)$ and the claim follows from Corollary 4.2. $\square$

Theorem 4.5. *The posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.*

Proof. Use Corollaries 2.11, 3.11 and 4.4. $\square$

5. Example

Consider the following two quivers Q and Q' whose underlying graph is the Dynkin diagram A_4. The quiver Q' is obtained from Q by reflection at the source 4.

$$Q : \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longleftarrow \bullet_4 \qquad Q' : \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4$$

For $1 \leq i \leq j \leq 4$, denote by ij the indecomposable representation of Q (or Q') supported on the vertices $i, i+1, \ldots, j$.

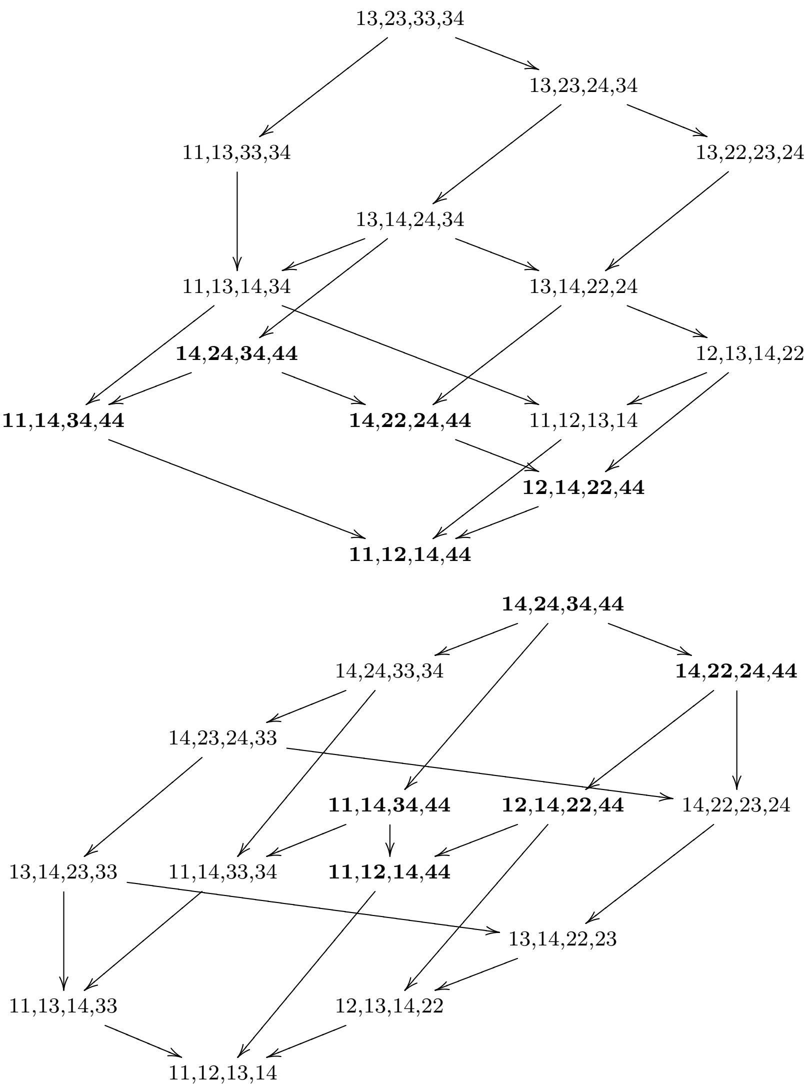

FIGURE 1. Hasse diagrams of the posets $\mathcal{T}_Q$ (top) and $\mathcal{T}_{Q'}$ (bottom).

Figure 1 shows the Hasse diagrams of the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$, where we used bold font to indicate the tilting modules containing the simple 44 as summand. The subsets $\mathcal{T}_Q^4$ and $\mathcal{T}_{Q'}^4$ of tilting modules containing 44 are isomorphic to the poset of tilting modules of the quiver A_3 with the linear orientation.

Note that $\mathcal{T}_Q$ was computed in [8, Example 3.2], while $\mathcal{T}_{Q'}$ is a Tamari lattice and the underlying graph of its Hasse diagram is the 1-skeleton of the Stasheff associhedron of dimension 3, see [2, 3].

Figure 2 shows the values of the functions π_4 and π'_4 on $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$, respectively. The functions $f : \mathcal{T}_Q \setminus \mathcal{T}_Q^4 \to \mathcal{T}_Q^4$ and $f' : \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^4 \to \mathcal{T}_{Q'}^4$ can then be easily computed.

Finally, the isomorphism $\rho : \mathcal{T}_Q \setminus \mathcal{T}_Q^4 \to \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^4$ is induced by the BGP reflection at the vertex 4, whose effect on the indecomposables (excluding

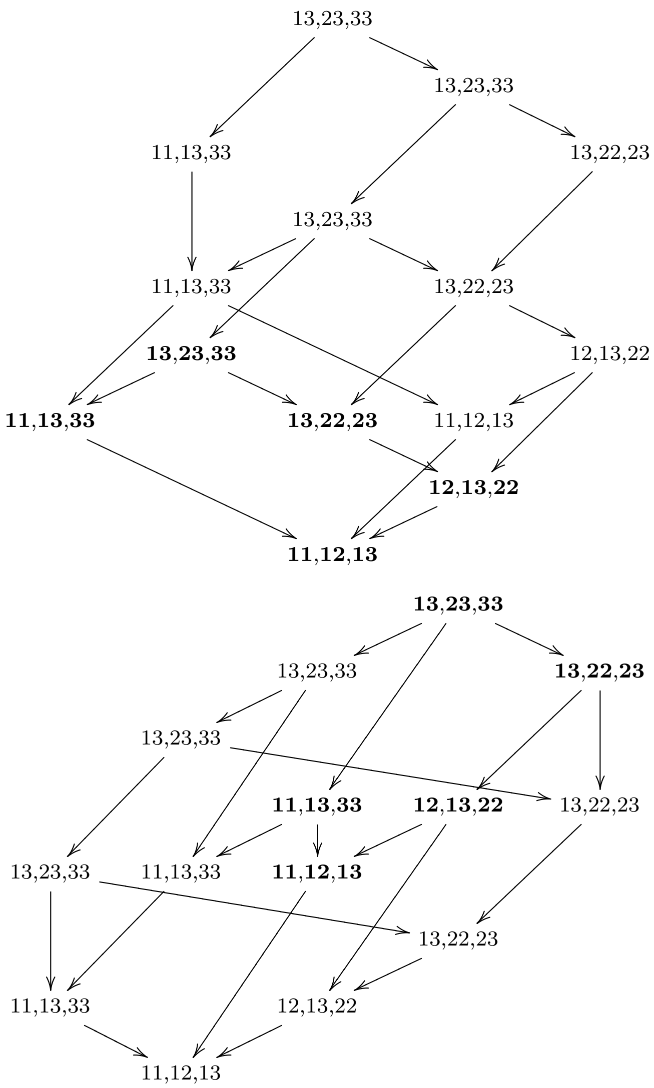

FIGURE 2. The functions π_4, π'_4 on $\mathcal{T}_Q$, $\mathcal{T}_{Q'}$.

44) is given by

$$11 \leftrightarrow 11 \qquad 12 \leftrightarrow 12 \qquad 13 \leftrightarrow 14 \qquad 22 \leftrightarrow 22 \qquad 23 \leftrightarrow 24 \qquad 33 \leftrightarrow 34$$

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