Homework #4 in Algebraic Structures 2

Problem 4.1. Prove Lemma 4.1.

Problem 4.2. Prove Lemma 4.3.

Problem 4.3. Let p be a prime, and let $\zeta \in \mathbb{C}$ be such that $\zeta^p = 1$ but $\zeta \neq 1$. Let $L = \mathbb{Q}(\zeta) \subset \mathbb{C}$.

(a) Find $[L : \mathbb{Q}]$. *Hint:* Use problem 3.3(b).
(b) Let $\sigma \in Gal(L/\mathbb{Q})$ be an automorphism. Show that there exists a number $\alpha(\sigma)$ such that $0 < \alpha(\sigma) < p$ and $\sigma(\zeta) = \zeta^{\alpha(\sigma)}$.
(c) Show that the map $\alpha : Gal(L/\mathbb{Q}) \to (\mathbb{Z}/p\mathbb{Z})^\times$ defined by $\sigma \mapsto \alpha(\sigma)$ is a homomorphism of groups.
(d) Deduce that $Gal(L/\mathbb{Q})$ is isomorphic to the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$.

Problem 4.4. Let K be a field of charactristic 2. We have seen that any quadrtic extension L of K has either the form

$$L = K[t]/(t^2 - a), a \in K$$

or the form

$$L = K[t]/(t^2 - t - a), a \in K$$

. Find the groups $\mathrm{Gal}(L/K)$ in both cases .

Problem 4.5. Find the group $\mathrm{Gal}(L/\mathbb{Q})$ where $L := \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Problem 4.6. Let n be a natural number, K a field such that the equation $t^n = 1$ has n distinct roots in $K, a \in K$ an element such that the polynomial $p(t) := t^n - a \in K[t]$ is irreducible,$L := K[t]/(p(t))$. Show that

a) the extension $L \supset K$ is normal,
b) the Galois group $\mathrm{Gal}(L/K)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Hint: Let $1, \zeta, \dots, \zeta^{n-1}$ be the n-th roots of unity in K, and let α be the image of t in L. Show that each automorphism $\sigma \in Gal(L/K)$ defines a number $i(\sigma)$ such that $0 \leq i(\sigma) < n$ and $\sigma(\alpha) = \alpha \cdot \zeta^{i(\sigma)}$. Then show that the map $i : Gal(L/K) \to \mathbb{Z}/n\mathbb{Z}$ defined by taking σ to $i(\sigma)$ is an isomorphism of groups.

Problem 4.7. a) Show that the the polynomial $t^5 - 2 \in \mathbb{Q}[t]$ is irreducible.

Let $L := \mathbb{Q}[t]/(t^5 - 2)$ and M be a splitting field of $t^5 - 2$ over $\mathbb{Q}$.
a) Show that the equation $t^5 = 1$ has a solution in M,
b) Find $[M : \mathbb{Q}]$,
c) Prove that the extension $L \supset \mathbb{Q}$ is not normal.