SUMMER SCHOOL: UNIQUE CONTINUATION AND INVERSE PROBLEMS

1. Unique Continuation and Carleman Estimates (9 talks)

In the first part of the summer school we deal with unique continuation questions for PDE. In particular, we study the relation between unique continuation properties and Carleman inequalities.

Introduction (3 talks): L^2 Carleman estimates.

(1) **Daniel Campos:**
Lerner, Nicolas. "Carleman Inequalities" (https://webusers.imj-prg.fr/~nicolas.lerner/m2carl.pdf): Sections 1.1.3, 1.1.4, 1.2.1-1.2.3 and 1.3.3 (Lemma 1.3.4).
The author motivates and deduces L^2 Carleman estimates. He illustrates the significance of the commutator in proving a Carleman estimate. Take Sections 1.1.1-1.1.3 as background material.

Additional background reading: Wolff, Thomas H. "Recent work on sharp estimates in second-order elliptic unique continuation problems." Journal of Geometric Analysis 3.6 (1993): 621-650.

(2) **Max Hallgren:**
Escauriaza, Luis, Seregin, Gregory, and ??ver??k, Vladimir. "Backward uniqueness for parabolic equations." Archive for rational mechanics and analysis 169.2 (2003): 147-157.
The authors prove a backward uniqueness result for the heat equation in domains exterior to balls. To this end they rely on the two Carleman estimates for the heat equation.

(3.) **Diana Stan:**
Escauriaza, Luis, Kenig, Carlos, Ponce, Gustavo and Vega, Luis. "The sharp Hardy uncertainty principle for Schr??dinger evolutions." Duke Mathematical Journal 155.1 (2010): 163-187.
The authors prove a sharp version of the Hardy uncertainty principle by deriving a Carleman estimate for Schr??dinger evolutions.

L^p Carleman estimates and the Osculation Technique (3 talks).

(4) **Zihui He:**
SUCP, Talk 1: Koch, Herbert, and Tataru, Daniel. "Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients." Communications on Pure and Applied Mathematics 54.3 (2001): 339-360.
The authors prove sharp L^p Carleman estimates for potentials and gradient potentials. They work in the set-up of Lipschitz metrics. Based on the Carleman estimates, they prove the strong unique continuation property for the corresponding second order elliptic operators.
Presentation: Present Section 4 for Δ and derive the SUCP for potentials $V \in L^{\frac{n}{2}}$. Take the L^p bounds for eigenfunctions as given.

(5) **Adolfo Arroyo-Rabasa:**
Wolff, Thomas H. "A property of measures in $\mathbb{R}^N$ and an application to unique continuation." Geometric & Functional Analysis GAFA 2.2 (1992): 225-284.
The author introduces an osculation technique to overcome the problem of the nonexistence of uniform Carleman estimates for the gradient at critical regularities. He applies this to prove the weak unique continuation property at critical regularities for the gradient.
For the presentation: Focus on the presentation of Theorem 1, discuss Lemmas 1 and 1' (without proof); discuss in detail the proof of Theorem 1 based on the necessary ingredients in Sections 6, 7 (do not prove Theorems 2-4).

(6.) **Immanuel Zachhuber:**
SUCP, Talk 2: Koch, Herbert, and Tataru, Daniel. "Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients." Communications on Pure and Applied Mathematics 54.3 (2001): 339-360.
Present Sections 5-7, take the Wolff result for granted. Only consider the case of Δ (do not consider variable coefficients!).

Additional background reading on the SUCP: Wolff, Thomas H. "Recent work on sharp estimates in second-order elliptic unique continuation problems." Journal of Geometric Analysis 3.6 (1993): 621-650.

Counterexamples (3 talks).

(7) **Julian Weigt:**
Koch, Herbert, and Tataru, Daniel. "Sharp counterexamples in unique continuation for second order elliptic equations." Journal fur die Reine und Angewandte Mathematik (2002): 133-146.
The article constructs counterexamples to the weak unique continuation property for potentials and gradient potentials, thus establishing the sharpness of the corresponding known weak unique continuation results on the scale of L^p spaces.

With background reading: Kenig, Carlos E., and Nikolai Nadirashvili. "A counterexample in unique continuation." Mathematical Research Letters 7.5/6 (2000): 625-630.

(8) **Mihajlo Cekic:**
Mandache, Niculae. "On a counterexample concerning unique continuation for elliptic equations in divergence form." Mathematical Physics, Analysis and Geometry 1.3 (1998): 273-292.
The author constructs counterexamples to the unique continuation property for low regularity metrics thus showing that on a H??lder scale Lipschitz metrics are optimal.

With additional background in: Miller, Keith. "Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with H??lder continuous coefficients." Archive for Rational Mechanics and Analysis 54.2 (1974): 105-117.

(9.) **Joao Pedro Ramos:**
Bourgain, Jean, and Wolff, Thomas H. "A remark on gradients of harmonic functions in dimension$\geq$ 3." Colloquium Mathematicae. Vol. 1. No. 60-61. 1990.
The authors construct a harmonic function in the upper half-plane $\mathbb{R}^3_+$ which is $C^{1,\alpha}$

regular up to the boundary such that it and its gradient vanish simultaneously on a subset of the boundary $\mathbb{R}^2 \times \{0\}$ of the upper half plane of positive measure (in 2D this is impossible).

With background material:

- *Wolff, T. "Counterexamples with harmonic gradients in $\mathbb{R}^3$." Essays in Honor of Elias M. Stein, Princeton Math. Ser 42 (1995): 321-384.*
- *Beliaev, Dmitri B., and Victor P. Havin. "On the uncertainty principle for M. Riesz potentials." Arkiv f??r Matematik 39.2 (2001): 223-243.*

2. Inverse Problems: The Calderón Problem (8 talks)

In the second part of the summer school we deal the Calderón problem, as a model inverse problem in which ideas from unique continuation play an important role. In the Calderón problem, one seeks to recover the conductivity of a medium by current and voltage measurements at the boundary of the medium. Mathematically this amounts to the question, whether the Dirichlet-to-Neumann map encodes enough information to recover the metric of the conductivity equation. An overview article on these questions is

Uhlmann, Gunther. "Electrical impedance tomography and Calder??n's problem." Inverse problems 25.12 (2009): 123011.

Formally, for $n \geq 3$, the nD Calderón problem is overdetermined. This indicates that one can hope for good uniqueness properties.

Boundary recovery (1 talk):

(10) **Yi-Hsuan Lin:**
Kohn, Robert, and Vogelius, Michael. "Determining conductivity by boundary measurements." Communications on Pure and Applied Mathematics 37.3 (1984): 289-298.
The article deals with the recovery of the conductivity at the boundary. This relies on the construction of highly oscillatory solutions.

CGO, Uniqueness and Limiting Carleman Weights (3 talks):

(11) **Wiktoria Zaton:**
Sylvester, John, and Uhlmann, Gunther. "A global uniqueness theorem for an inverse boundary value problem." Annals of mathematics (1987): 153-169.
This seminal result exploits complex geometric optics solutions in order to derive uniqueness for the Calderón problem under sufficient regularity assumptions.

(12/13) Kenig, Carlos E., Johannes Sj??strand, and Gunther Uhlmann. "The Calder??n problem with partial data." Annals of mathematics (2007): 567-591.
The authors prove uniqueness for the Calderón problem with partial data. Their construction of suitable CGO solutions relies on suitable Carleman estimates.
The presentation is split into two parts:

- (12) **Marco Fraccaroli:**
  Talk 1: *Present the Carleman estimates from sections 2,3 and derive the existence of CGO solutions in section 4.*
- (13.) **Gennady Uraltsev:**
  Talk 2: *Present sections 5-7.*

Look also at Chapter 5 in
http://users.jyu.fi/~salomi/lecturenotes/calderon_lectures.pdf
for information on partial data; in particular the Hahn-Banach argument is explained in detail there.

Stability (2 talks):

(14) **Dimitrije Cicmilovic:**
Alessandrini, Giovanni. “Stable determination of conductivity by boundary measurements.” Applicable Analysis 27.1-3 (1988): 153-172.
This article establishes logarithmic stability estimates for the Calderón problem.

(15) **Lisa Onkes:**
Mandache, Niculae. “Exponential instability in an inverse problem for the Schr??dinger equation.” Inverse Problems 17.5 (2001): 1435-1444.
This article establishes the optimality of the logarithmic estimates due to Alessandrini, in that it shows that there has to be an exponential instability. In particular, this shows the high degree of ill-posedness of the Calderón problem.

Recovery (2 talks):

(16,17) Nachman, Adrian I. ”Reconstructions from boundary measurements.” Annals of Mathematics 128.3 (1988): 531-576.
The author provides a reconstruction scheme for the potential and the conductivity matrix.
The talk is split into two parts:
(16) **Itamar Oliveira:**
Talk 1: *Sections 1,2,3.*
(17.) **Alex Amenta:**
Talk 2: *Sections 4,5,6.*
In both talks you may take the results on layer potentials, which are proved in the appendix as given.

3. 2D Unique Continuation and 2D Inverse Problems (7 talks)

Finally in the last part of the summer school we deal with specifically two-dimensional settings: In two dimensions very strong tools from complex analysis are available. Hence, both for unique continuation as well as for the Calderón problem there are very strong results and techniques available. Many of them rely on reducing the problems to settings in which quasiconformal maps play an important role.

2D Unique Continuation and Quasiconformal Maps (3 talks).

(18) **Maria Angeles Garcia-Ferrero:**
Kenig, Carlos, Silvestre, Luis and Wang, Jenn-Nan: “On Landis??? conjecture in the plane.” Communications in Partial Differential Equations 40.4 (2015): 766-789.
In this article the authors prove a quantitative version of Landis’ conjecture. In order to achieve this, they first rewrite the problem as a Beltrami system and then rely on Carleman estimates for this reduced problem. This is necessary as a direct application of a Carleman estimate would not distinguish between the real and complex settings. Due to examples of Meshkov, it is however known that Landis’ conjecture fails in the complex case.

(19) **Gael Yomgne Diebou:**
Quasiconformal maps: Present Theorems 5.5.1, 8.6.1, 8.6.2 in Astala, Iwaniec, Martin: "Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane".
This talk focuses on uniqueness properties of solutions to nonlinear Beltrami equations and the Stoilow factorization.

(20) **Constantin Bilz:**
Alessandrini, Giovanni. "Strong unique continuation for general elliptic equations in 2D." Journal of Mathematical Analysis and Applications 386 (2012): 669-676.
The author proves a strong unique continuation result in two dimensions which holds under very weak conditions, e.g. L^∞ *conditions for the metric. The argument relies on special two-dimensional techniques, e.g. on quasiconformal mappings in the plane.*
Present the article but also explain the proof of Theorem 3.1 (c.f. Proof of Theorem 18.5.1 in Astala, Iwaniec, Martin: "Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane").

The 2D Calderón Problem (4 talks). The $2D$ Calderón problem is a (in terms of dimension-counting) formally determined inverse problem. Hence, its treatment requires very strong techniques and tools. Techniques from inverse scattering and complex analysis play an important role.

(21) **Olli Saari:**
The Calderón problem with L^∞ conductivity.
This talk deals with the two-dimensional Calderón problem with L^∞ *conductivity. Inverse scattering is an important tool here.*
Present the content of chapter 18 (p.490-513) in Astala, Iwaniec, Martin: Elliptic partial differential equations and quasiconformal mappings in the plane. Treat background material from Section 8 as given, i.e. do not present the proofs of this.

(22,23,24) Nachman, Adrian I., Regev, Idan, Tataru, Daniel, A Nonlinear Plancherel Theorem with Applications to Global Well-Posedness for the Defocusing Davey-Stewartson Equation and to the Inverse Boundary Value Problem of Calderon https://arxiv.org/pdf/1708.04759.
The article views the Calderón problem in relation to an inverse scattering problem associated with a completly integrable system. It proves uniqueness under the assumptions that the logarithm of the conductivity is in H^1.
The presentation is split into two talks:
(22) **Gianmarco Brocchi:**
Talk 1: sections 2, 3 up to Lemma 3.7.
(23) **Xian Liao:**
Talk 2: profile decomposition and section 4.
(24.) **Pavel Zorin-Kranich:**
Talk 3: section 6, reduction of the inverse problem to Theorem 1.10 and explaining the relation between the problems dealt with in the article (c.f. introduction).